SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class:
12MX2 ___

2022

YEAR 12
TERM 2
HSC Task #3

Mathematics Extension 2

General Instructions

- Working Time – 65 minutes
- Write using black pen
- NESA approved calculators may be used
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–17, show ALL relevant mathematical reasoning and/or calculations

Total Marks: **Section I – 10 marks (Personal Reference sheet)**

Leave your Personal Reference Sheet in this booklet.

Section II – 10 marks (pages 2–5)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section III – 30 marks (pages 6–17)
- Attempt Questions 11–18
- Allow about 45 minutes for this section

Examiner:

PSP

Marks	
Section I	/10
Section II	/10
Section III	/30
Total	/50

Section II

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 In order to evaluate $\int_{-2}^{2} \left| 1 - x^2 \right| dx$, which of the following will help to provide the correct answer?

A. $\int_{-2}^{2} \left| 1 - x^2 \right| dx = 0$

B. $\int_{-2}^{2} \left| 1 - x^2 \right| dx = 2\int_{0}^{2} \left(1 - x^2\right) dx$

C. $\int_{-2}^{2} \left| 1 - x^2 \right| dx = 2\int_{0}^{1} \left(1 - x^2\right) dx + 2\int_{1}^{2} \left(x^2 - 1\right) dx$

D. $\int_{-2}^{2} \left| 1 - x^2 \right| dx = \left| \int_{-2}^{2} \left(1 - x^2\right) dx \right|$

2 Find $\int \frac{1}{\sqrt{12 + 4x - x^2}}\, dx$

A. $2\arccos\left(\frac{x}{2}\right) - \frac{\sqrt{4 + x^2}}{3} + c$

B. $2\arcsin\left(\frac{x}{2}\right) - \frac{\sqrt{9 - x^2}}{2} + c$

C. $\arccos\left(\frac{x - 2}{3}\right) + c$

D. $\arcsin\left(\frac{x - 2}{4}\right) + c$

3 What is the indefinite integral of $\int \frac{\sec^2(\ln x)}{x}\,dx$?

A. $\tan(\cos x)+C$

B. $\sec(\ln x)+C$

C. $\tan(\ln x)+C$

D. $\sec(\cos x)+C$

4 Which of the following integrals is equivalent to $\int_0^{\sqrt{3}} \frac{\ln(\arctan x)}{1+x^2}\,dx$, using a suitable substitution?

A. $\int_0^{\frac{\pi}{3}} \log_e u\,du$

B. $\int_0^{\sqrt{3}} \log_e u\,du$

C. $\int_0^{\frac{\pi}{3}} \frac{\log_e u}{1+\tan^2 u}\,du$

D. $\int_0^{\sqrt{3}} \frac{\log_e u}{1+\tan^2 u}\,du$

5 Which of the following integrals is equivalent to

$$\int_{-2}^{-1} \frac{(x^2-1)^{\frac{3}{2}}}{x}\,dx$$

after applying the substitution $x=\sec\theta$?

A. $\int_{\frac{\pi}{3}}^{\pi} \tan^4\theta\,d\theta$

B. $\int_{\frac{2\pi}{3}}^{\pi} \tan^4\theta\,d\theta$

C. $\int_{\frac{4\pi}{3}}^{\pi} \tan^4\theta\,d\theta$

D. $\int_{\frac{5\pi}{3}}^{\pi} \tan^4\theta\,d\theta$

6 Which of the following is an expression for $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx$ after the substitution $t = \tan\frac{x}{2}$?

A. $\int_0^1 \frac{1}{1+2t}\, dt$

B. $\int_0^1 \frac{2}{1+2t}\, dt$

C. $\int_0^1 \frac{1}{(1+t)^2}\, dt$

D. $\int_0^1 \frac{2}{(1+t)^2}\, dt$

7 Which of the following expressions is equal to $\int x^2 e^{-x}\, dx$?

A. $-x^2e^{-x} + \int 2xe^{-x}\, dx$

B. $-2xe^{-x} - \int 2xe^{-x}\, dx$

B. $-x^2e^{-x} - \int 2xe^{-x}\, dx$

C. $-2xe^{-x} + \int 2xe^{-x}\, dx$

8 Which integral is necessarily equal to $\int_{-a}^{a} f(x)\, dx$?

A. $\int_0^a \left[f(x) - f(-x)\right] dx$

B. $\int_0^a \left[f(x) - f(a-x)\right] dx$

C. $\int_0^a \left[f(x-a) + f(-x)\right] dx$

D. $\int_0^a \left[f(x-a) + f(a-x)\right] dx$

9 Given that $x^2 - 2x + 2$ is a factor of $x^4 + 4$, what is the value of $\int_{-1}^{1} \frac{(x+1)^2 + 1}{x^4 + 4}\, dx$?

A. $\tan^{-1}\left(\frac{1}{2}\right)$ B. $\tan^{-1}(2)$ C. $\frac{\pi}{4}$ D. $\frac{\pi}{2}$

10 By considering the graph of $y = \sin^{-1}\sqrt{x}$, or otherwise, what is the value of $\int_{0}^{1} \sin^{-1}\sqrt{x}\, dx$?

A. π B. $\frac{\pi}{2}$ C. $\frac{\pi}{4}$ D. $\frac{\pi}{8}$

NESA Number									

YEAR 12 Mathematics Extension 2

TERM 2 HSC TASK #3

Part A

Section III

Part A – 15 marks
Attempt Questions 11 – 15

Answer each question in the spaces provided. Blank pages are provided at the end of the Part to allow rewriting of a question.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (1 mark)

Find $\int \frac{3}{\sqrt{2-x}}\,dx$. **1**

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Question 12 (2 marks)

Find $\int \frac{x+3}{\sqrt{2x-x^2}}\,dx$ **2**

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Question 13 (4 marks)

(a) Show that $\int \sec x \, dx = \ln\left| \sec x + \tan x \right| + C$. **1**

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(b) By using a substitution of the form $x = a\tan\theta$, where $a \in \mathbb{R}$, find $\int \frac{1}{\sqrt{9+4x^2}} \, dx$. **3**

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Question 14 (3 marks)

Evaluate $\int_0^{\frac{\pi}{6}} 2x\sec^2 2x\, dx$. **3**

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Question 15 (5 marks)

(a) What are the values of A, B, and C for which the following identity is true? **3**

$$\frac{5x^2-x+1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

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(b) Hence, find $\int \frac{5x^2-x+1}{x\left(x^2+1\right)}\,dx$. **2**

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End of Section A

Use this space to re-write any question. Clearly indicate that you have done so.

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YEAR 12 Mathematics Extension 2

TERM 2 HSC TASK #3

Part B

Section III

Part B **15 marks**
Attempt Questions 16 – 18

Answer each question in the spaces provided. Two blank pages are provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 16 (9 marks)

(a) (i) Using an appropriate substitution, prove that **2**

$$\int_0^{2a} f(x)\,dx = \int_{-a}^{a} f(a-x)\,dx$$

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(ii) Hence show that $\int_0^1 \sqrt{x(1-x)}\,dx = \dfrac{\pi}{8}$. **2**

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Question 16 continues on page 16

Question 16 (continued)

(b) (i) If $I_n = \int_0^1 \sqrt{x}(1-x)^{\frac{n}{2}}\,dx$, where $n = 0, 1, 2, \ldots$, show that **3**

$$I_n = \frac{n}{n+3} I_{n-2}\text{, for } n \geq 2\,.$$

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(ii) Hence evaluate $\int_0^1 \sqrt{x}(1-x)^{\frac{5}{2}}\,dx$. **2**

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Question 17 (3 marks)

Let n be a positive integer. **3**

You may use the fact that $0 < \left(\frac{n}{n+1}\right)^n \leq \frac{1}{2}$. Do NOT prove.

Prove by mathematical induction that $(2n-1)! \leq n^{2n-1}$.

Question 18 (3 marks)

Using the following identity, or otherwise, **3**

$$\left(\frac{\log_e x-1}{1+\left(\log_e x\right)^2}\right)^2 = -\frac{2\log_e x}{\left(1+\left(\log_e x\right)^2\right)^2}+\frac{1}{1+\left(\log_e x\right)^2} \quad \text{(Do NOT prove)}$$

find $\displaystyle\int_0^1 \left(\frac{\log_e x-1}{1+\left(\log_e x\right)^2}\right)^2 dx$.

End of paper

Use this space to re-write any question. Clearly indicate that you have done so.

Use this space to re-write any question. Clearly indicate that you have done so.

Use this space to re-write any question. Clearly indicate that you have done so.

SYDNEY
BOYS
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2022

YEAR 12
TERM 2
ASSESSMENT TASK 3

Mathematics Extension 2

Sample Solutions

NOTE: Some of you may be disappointed with your mark.

This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown 'working' does not justify that your solution is worth any marks.

Put Your NAME and CLASS on the cover page

MC Answers

1	C	**6**	D
2	D	**7**	A
3	C	**8**	D
4	A	**9**	B
5	C	**10**	C

Section II MC Solutions

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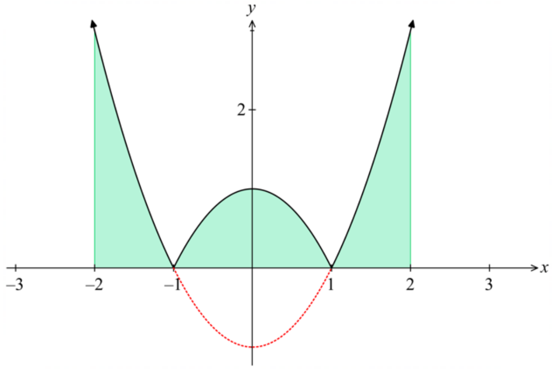
1 In order to evaluate $\int_{-2}^{2} \left| 1-x^2 \right| dx$, which of the following will help to provide the correct answer?



A. $\int_{-2}^{2} \left| 1-x^2 \right| dx = 0$

B. $\int_{-2}^{2} \left| 1-x^2 \right| dx = 2\int_{0}^{2} \left(1-x^2\right) dx$

C. $\int_{-2}^{2} \left| 1-x^2 \right| dx = 2\int_{0}^{1} \left(1-x^2\right) dx + 2\int_{1}^{2} \left(x^2-1\right) dx$

D. $\int_{-2}^{2} \left| 1-x^2 \right| dx = \left| \int_{-2}^{2} \left(1-x^2\right) dx \right|$

2 Find $\int \frac{1}{\sqrt{12+4x-x^2}} dx$

A. $2\arccos\left(\frac{x}{2}\right) - \frac{\sqrt{4+x^2}}{3} + c$

B. $2\arcsin\left(\frac{x}{2}\right) - \frac{\sqrt{9-x^2}}{2} + c$

C. $\arccos\left(\frac{x-2}{3}\right) + c$

D. $\arcsin\left(\frac{x-2}{4}\right) + c$

$$\int \frac{1}{\sqrt{12+4x-x^2}} dx = \int \frac{1}{\sqrt{16-(x-2)^2}} dx$$

3 What is the indefinite integral of $\int \frac{\sec^2(\ln x)}{x}\,dx$?

A. $\tan(\cos x)+C$

B. $\sec(\ln x)+C$

C. $\tan(\ln x)+C$

D. $\sec(\cos x)+C$

$$\int \frac{\sec^2(\ln x)}{x}\,dx = \int \frac{1}{x}\sec^2(\ln x)\,dx = \int \sec^2(\ln x)\,d\ln x$$

4 Which of the following integrals is equivalent to $\int_0^{\sqrt{3}} \frac{\ln(\arctan x)}{1+x^2}\,dx$, using a suitable substitution?

A. $\int_0^{\frac{\pi}{3}} \log_e u\,du$

B. $\int_0^{\sqrt{3}} \log_e u\,du$

C. $\int_0^{\frac{\pi}{3}} \frac{\log_e u}{1+\tan^2 u}\,du$

D. $\int_0^{\sqrt{3}} \frac{\log_e u}{1+\tan^2 u}\,du$

Looking at the upper and lower bounds, the answer must be A or C.

$$\int \frac{\ln(\arctan x)}{1+x^2}\,dx = \int \frac{1}{1+x^2}\ln(\arctan x)\,dx$$

$$= \int \ln u\,du$$

5 Which of the following integrals is equivalent to

$$\int_{-2}^{-1} \frac{\left(x^2-1\right)^{\frac{3}{2}}}{x}\, dx$$

after applying the substitution $x = \sec\theta$?

A. $\int_{\frac{\pi}{3}}^{\pi} \tan^4\theta\, d\theta$

B. $\int_{\frac{2\pi}{3}}^{\pi} \tan^4\theta\, d\theta$

C. $\int_{\frac{4\pi}{3}}^{\pi} \tan^4\theta\, d\theta$

D. $\int_{\frac{5\pi}{3}}^{\pi} \tan^4\theta\, d\theta$

$\int_{-2}^{-1} \frac{\left(x^2-1\right)^{\frac{3}{2}}}{x}\, dx < 0$ for $(-2,-1)$.

B is out as $\int_{\frac{2\pi}{3}}^{\pi} \tan^4\theta\, d\theta > 0$.

A & D are out as there is an asymptote at $x = \frac{\pi}{3}$ and $x = \frac{3\pi}{2}$ respectively.

Given $\frac{4\pi}{3} > \pi$ then $\int_{\frac{4\pi}{3}}^{\pi} \tan^4\theta\, d\theta < 0$.

6 Which of the following is an expression for $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx$ after the substitution $t = \tan\frac{x}{2}$?

A. $\int_0^1 \frac{1}{1+2t}\, dt$

B. $\int_0^1 \frac{2}{1+2t}\, dt$

C. $\int_0^1 \frac{1}{(1+t)^2}\, dt$

D. $\int_0^1 \frac{2}{(1+t)^2}\, dt$

$t = \tan\frac{x}{2} \Rightarrow x = 2\arctan t \Rightarrow dx = \frac{2}{1+t^2}\, dt$

With the "2", it must be B or D.

$\frac{1}{1+\sin x} = \frac{1}{1+\frac{2t}{1+t^2}} = \frac{1+t^2}{(t+1)^2}$.

7 Which of the following expressions is equal to $\int x^2 e^{-x}\, dx$?

A. $-x^2 e^{-x} + \int 2xe^{-x}\, dx$

B. $-2xe^{-x} - \int 2xe^{-x}\, dx$

B. $-x^2 e^{-x} - \int 2xe^{-x}\, dx$

C. $-2xe^{-x} + \int 2xe^{-x}\, dx$

$$\int x^2 e^{-x}\, dx = \int \frac{d}{dx}\left(-e^{-x}\right) x^2\, dx$$

$$= -e^{-x}x^2 - \int -e^{-x} \times \frac{d}{dx}\left(x^2\right) dx$$

8 Which integral is necessarily equal to $\int_{-a}^{a} f(x)\, dx$?

A. $\int_0^a \left[f(x) - f(-x)\right] dx$

B. $\int_0^a \left[f(x) - f(a-x)\right] dx$

C. $\int_0^a \left[f(x-a) + f(-x)\right] dx$

D. $\int_0^a \left[f(x-a) + f(a-x)\right] dx$

$$\int_{-a}^{a} f(x)\, dx = \int_0^a f(x)\, dx + \int_{-a}^{0} f(x)\, dx$$

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

$$\int_{-a}^{0} f(x)\, dx = \int_0^a f(-x)\, dx = \int_0^a f(-(a-x))\, dx$$

9 Given that x^2-2x+2 is a factor of x^4+4, what is the value of $\int_{-1}^{1}\frac{(x+1)^2+1}{x^4+4}\,dx$?

A. $\tan^{-1}\left(\frac{1}{2}\right)$ B. $\tan^{-1}(2)$ C. $\frac{\pi}{4}$ D. $\frac{\pi}{2}$

$$x^4+4=\left(x^4+4x^2+4\right)-4x^2$$

$$=\left(x^2+2\right)^2-4x^2$$

$$=\left(x^2+2-2x\right)\left(x^2+2+2x\right)$$

$$=\left(x^2-2x+2\right)\left(x^2+2x+2\right)$$

Or just use long division!

$$\frac{(x+1)^2+1}{x^4+4}=\frac{x^2+2x+2}{\left(x^2-2x+2\right)\left(x^2+2x+2\right)}$$

$$=\frac{1}{x^2-2x+2}$$

$$=\frac{1}{(x-1)^2+1}$$

$$\int_{-1}^{1}\frac{(x+1)^2+1}{x^4+4}\,dx=\int_{-1}^{1}\frac{1}{(x-1)^2+1}\,dx$$

$$=\left[\arctan(x-1)\right]_{-1}^{1}$$

$$=0-\arctan(-2)$$

$$=\arctan(2)$$

10 By considering the graph of $y = \sin^{-1}\sqrt{x}$, or otherwise, what is the value of $\int_0^1 \sin^{-1}\sqrt{x}\, dx$?

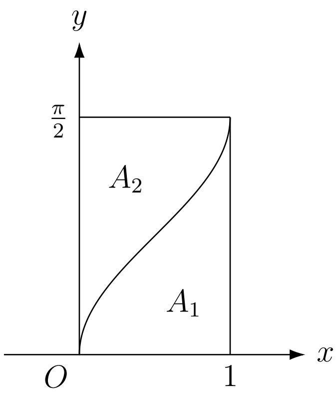
A. π B. $\frac{\pi}{2}$ C. $\frac{\pi}{4}$ D. $\frac{\pi}{8}$

Since $y = \sin^{-1}\sqrt{x}$ is increasing in the domain $0 < x < 1$ (noting its actual domain is $0 \le x \le 1$), the range must be $0 \le y \le \frac{\pi}{2}$.

Let A_1 be the area bounded by the curve $y = \sin^{-1}\sqrt{x}$, the x-axis and $x = 1$. Let A_2 be the area bounded by the curve $y = \sin^{-1}\sqrt{x}$, the y-axis and $y = \frac{\pi}{2}$.

$$\begin{aligned} A_1 + A_2 &= \int_0^1 y\, dx + \int_0^{\frac{\pi}{2}} x\, dy \\ &= \int_0^1 \sin^{-1}\sqrt{x}\, dx + \int_0^{\frac{\pi}{2}} \sin^2 y\, dy \end{aligned}$$

However, the sum of the areas A_1 and A_2 form a rectangle of side lengths 1 and $\frac{\pi}{2}$.



By inspection and deduction of the available options, the most likely answer is (B). This can be formally shown as follows

$$\begin{aligned} \int_0^1 \sin^{-1}\sqrt{x}\, dx + \int_0^{\frac{\pi}{2}} \sin^2 y\, dy &= \frac{\pi}{2} \times 1 \\ \Rightarrow \int_0^1 \sin^{-1}\sqrt{x}\, dx &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin^2 y\, dy \\ &= \frac{\pi}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 2y - 1)\, dy \\ &= \frac{\pi}{2} + \frac{1}{2}\left[\frac{1}{2}\sin 2y - y\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4} \end{aligned}$$

Section III Part A

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (1 mark)

Find $\int \frac{3}{\sqrt{2-x}}\,dx$. **1**

Using the reverse chain rule:

$$-3\int -1\times(2-x)^{-\frac{1}{2}}\,dx = -3\times 2\sqrt{2-x}+C$$

$$= -6\sqrt{2-x}+C$$

NOTE: There were too many pupils who stated that the primitive of $\frac{1}{\sqrt{u}}$ is $-\sqrt{u}$ and thus ending up with a positive answer, since they did everything else correctly.

Pupils should be referring to the reference sheet's version of the Reverse Chain Rule to get their answer.

Question 12 (2 marks)

Find $\displaystyle\int \frac{x+3}{\sqrt{2x-x^2}}\,dx$ **2**

Let $I = \displaystyle\int \frac{x+3}{\sqrt{2x-x^2}}\,dx$.

By using algebraic manipulation, we can get:

$$I = \int \frac{1}{2}\left(\frac{-(2-2x)+8}{\sqrt{2x-x^2}}\right)dx$$

$$= -\frac{1}{2}\int\left(\frac{2-2x}{\sqrt{2x-x^2}} - \frac{8}{\sqrt{2x-x^2}}\right)dx$$

$$= 4\int \frac{1}{\sqrt{2x-x^2}}\,dx - \frac{1}{2}\int \frac{2-2x}{\sqrt{2x-x^2}}\,dx$$

$$= 4\int \frac{1}{\sqrt{1-(x-1)^2}}\,dx - \int \frac{1}{2}\times\frac{2-2x}{\sqrt{2x-x^2}}\,dx$$

Let $f(x) = x-1$, then $f'(x) = 1$
Let $u = 2x-x^2$, then $u' = 2-2x$.
Let $g(u) = \frac{1}{2}u^{-\frac{1}{2}}$ then its primitive is $G(u) = u^{\frac{1}{2}}$.

$$I = 4\int \frac{f'(x)}{\sqrt{1-(f(x))^2}}\,dx - \int u' \times g(u)\,dx$$

$$= 4\sin^{-1}(f(x)) - G(u) + C$$

$$= 4\sin^{-1}(x-1) - \sqrt{2x-x^2} + C$$

NOTE: Answering this question with efficiency required the algebraic manipulation shown here. Pupils who did not get that right in the first place rarely (if at all) got the question correct. Once the algebraic manipulation was done, the reference sheet would provide the appropriate solution.

Question 13 (4 marks)

(a) Show that $\int \sec x \, dx = \ln| \sec x + \tan x | + C$. **1**

Let $I = \int \sec x \, dx$

$$= \int \frac{(\sec x + \tan x)\sec x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$$

Let $u = \sec(x) + \tan(x)$, then $u' = \sec x \tan x + \sec^2 x$.

$$I = \int \frac{u'}{u} dx$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

NOTE: I was generous with this, but ideally would have liked to see some sort of statement that referred to the standard form in the reference sheet of

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C.$$

(b) By using a substitution of the form $x = a\tan\theta$, where $a \in \mathbb{R}$, find $\int \frac{1}{\sqrt{9+4x^2}}\,dx$. **3**

Let $I = \int \frac{1}{\sqrt{9+4x^2}}\,dx$

$$= \frac{1}{2}\int \frac{\frac{2}{3}}{\sqrt{1+\left(\frac{2x}{3}\right)^2}}\,dx$$

Let $x = \frac{3}{2}\tan\theta$, then $\frac{dx}{d\theta} = \frac{3}{2}\sec^2\theta$ and $dx = \frac{3}{2}\sec^2\theta\,d\theta$

$$I = \frac{1}{2}\int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}\,d\theta$$

$$= \frac{1}{2}\int \sec\theta\,d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

Now $\tan\theta = \frac{2}{3}x$ and using the ratios of a right-angle triangle $\sec\theta = \frac{\sqrt{9+4x^2}}{3}$.

$$I = \frac{1}{2}\ln\left|\frac{\sqrt{9+4x^2}}{3} + \frac{2x}{3}\right| + C$$

$$= \frac{1}{2}\ln\left|\sqrt{9+4x^2} + 2x\right| + C$$

NOTE: Some pupils used the substitution $x = a\tan\theta$ and should have realised at some point that a must be $\frac{3}{2}$, but invariably didn't.

Others used $x = 3\tan\theta$ and ended up with $3\sqrt{1+4\tan^2\theta}$ in the denominator but miraculously found the solution in the next couple of lines with no justification.

Some students produced the final answer with trigonometric functions in it, which should have been simplified and turned back into expressions in x.
Students who did this could NOT receive full marks.

Question 14 (3 marks)

Evaluate $\int_0^{\frac{\pi}{6}} 2x\sec^2 2x\,dx$. **3**

Let $I = \int_0^{\frac{\pi}{6}} 2x\sec^2 2x\,dx$

Let $u' = 2\sec^2 2x$ and $v = x$, then $u = \tan 2x$ and $v' = 1$.

$$
\begin{aligned}
I &= \left[x\tan 2x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \tan 2x\,dx \\
&= \left(\frac{\pi}{6}\tan\frac{\pi}{3} - 0\right) - \left[-\frac{\ln\left|\cos 2x\right|}{2}\right]_0^{\frac{\pi}{6}} \\
&= \frac{\pi\sqrt{3}}{6} + \left(\frac{\ln\left(\cos\frac{\pi}{3}\right) - \ln\left(\cos 0\right)}{2}\right) \\
&= \frac{\pi\sqrt{3}}{6} + \left(\frac{\ln\left(\frac{1}{2}\right) - \ln\left(1\right)}{2}\right) \\
&= \frac{\pi\sqrt{3}}{6} - \frac{\ln 2}{2}
\end{aligned}
$$

NOTE: Pupils may have used slightly different equations for the parts but must have $I = \left[x\tan 2x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \tan 2x\,dx$ as the result.

Many pupils had a multiple of this and/or somehow swapped $2x$ for x as the argument of the tangent function.
As per the last question, pupils that left the solution in terms of some sort of trigonometric function (cosine) could not get full marks.

Many pupils stated that the primitive of $\tan 2x$ is $-\ln\left(\cos 2x\right)$ forgetting the half that should be in front of it.

Question 15 (5 marks)

(a) What are the values of A, B, and C for which the following identity is true? **3**

$$\frac{5x^2 - x + 1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$\text{RHS} = \frac{A\left(x^2+1\right) + x\left(Bx + C\right)}{x\left(x^2+1\right)}$$

$$= \frac{\left(A+B\right)x^2 + Cx + A}{x\left(x^2+1\right)}$$

Equating coefficients in the numerators for LHS with RHS:

$A + B = 5$, $C = -1$ and $A = 1 \Rightarrow B = 4$

NOTE: This was generally well done and the question was generous in the amount of marks considering how easy it is.
If pupils did not get it correct, it was usually an arithmetic error.
Considering the marks given the question, a simple error ended up costing students a whole mark.
More serious errors could not earn many marks at all.

(b) Hence, find $\displaystyle\int \frac{5x^2 - x + 1}{x\left(x^2+1\right)}\,dx$. **2**

From part (a): $\displaystyle I = \int \left(\frac{1}{x} + \frac{4x-1}{x^2+1}\right)dx$

$$= \ln|x| + 2\int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$= \ln|x| + 2\ln\left(x^2+1\right) - \tan^{-1} x + C$$

NOTE: This question was a simple algebraic manipulation and then use of the reference sheet.
If a pupil got the wrong numbers from part (a), this was considered an error carried forward (ecf) for the whole question.

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YEAR 12 Mathematics Extension 2

TERM 2 HSC TASK #3

Part B

Section III

Part B 15 marks
Attempt Questions 16 – 18

Answer each question in the spaces provided. Two blank pages are provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 16 (9 marks)

(a) (i) Using an appropriate substitution, prove that **2**

$$\int_0^{2a} f(x)\,dx = \int_{-a}^{a} f(a-x)\,dx$$

$LHS = \int_a^{2a} f(x)\,dx$

let $u = a - x$ change limits

$\frac{du}{dx} = -1$ $x = 0$ $x = 2a$

$dx = -du$ $u = a$ $u = -a$

$= \int_a^{-a} f(a-u) \cdot -du$

$= \int_{-a}^{a} f(a-u)\,du$

$= \int_{-a}^{a} f(a-x)\,dx = RHS$

(ii) Hence show that $\int_0^1 \sqrt{x(1-x)}\,dx = \frac{\pi}{8}$. **2**

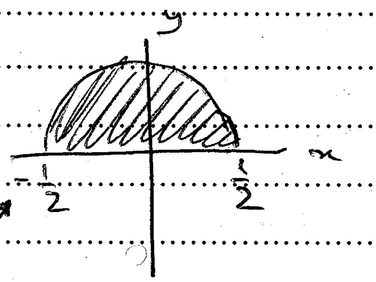
$LHS = \int_0^1 \sqrt{x(1-x)}\,dx$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{(\frac{1}{2}-x)(1-(\frac{1}{2}-x))}\,dx$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{(\frac{1}{2}-x)(\frac{1}{2}+x)}\,dx$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1}{4} - x^2}\,dx$

this represents



$= \frac{1}{2}\left(\pi\left(\frac{1}{2}\right)^2\right)$

Question 16 continues on page 16

$= \frac{\pi}{8}$

$= RHS$

Question 16 (continued)

(b) (i) If $I_n = \int_0^1 \sqrt{x}(1-x)^{\frac{n}{2}}\, dx$, where $n = 0, 1, 2, \ldots$, show that **3**

$$I_n = \frac{n}{n+3} I_{n-2}, \text{ for } n \geq 2.$$

$I_n = \int_0^1 \sqrt{x}(1-x)^{\frac{n}{2}} dx$

$u = (1-x)^{\frac{n}{2}}$ $\quad v' = \sqrt{x} = x^{\frac{1}{2}}$

$u' = \frac{n}{2}(1-x)^{\frac{n}{2}-1}$ $\quad v = \frac{2}{3}x^{3/2}$

$= \frac{2}{3}x\sqrt{x}$

$I_n = \left.\frac{2}{3}x\sqrt{x}(1-x)^{\frac{n}{2}}\right]_0^1 + \frac{n}{3}\int_0^1 x\sqrt{x}(1-x)^{\frac{n-2}{2}} dx$

$= \cancel{\frac{2}{3}1\sqrt{1}(1-1)^{\frac{n}{2}} - \frac{2}{3}0\sqrt{0}(1-0)^{n/2}} - \frac{n}{3}\int_0^1 (-x+1-1)\sqrt{x}(1-x)^{\frac{n-2}{2}} dx$

$= -\frac{n}{3}\int_0^1 (1-x)\sqrt{x}(1-x)^{\frac{n-2}{2}} dx + \frac{n}{3}\int_0^1 \sqrt{x}(1-x)^{\frac{n-2}{2}} dx$

$= -\frac{n}{3}\int_0^1 \sqrt{x}(1-x)^{\frac{n}{2}} dx + \frac{n}{3}\int_0^1 \sqrt{x}(1-x)^{\frac{n-2}{2}} dx$

$I_n = -\frac{n}{3} I_n + \frac{n}{3} I_{n-2}$

$3I_n = -nI_n + nI_{n-2}$

$I_n(n+3) = nI_{n-2}$

$I_n = \frac{n}{n+3} I_{n-2}$

(ii) Hence evaluate $\int_0^1 \sqrt{x}(1-x)^{\frac{5}{2}}\, dx$. **2**

$\int_0^1 \sqrt{x}(1-x)^{5/2} dx = I_5$

$= \frac{5}{5+3} I_3$

$= \frac{5}{8}\left(\frac{3}{3+3} I_1\right)$

$= \frac{5}{16}\int_0^1 \sqrt{x}(1-x)^{\frac{1}{2}} dx$

$= \frac{5}{16}\int_0^1 \sqrt{x(1-x)}\, dx$

$= \frac{5}{16}\left(\frac{\pi}{8}\right)$ from a)(ii)

$= \frac{5\pi}{128}$

Question 17 (3 marks)

Let n be a positive integer. **3**

You may use the fact that $0 < \left(\frac{n}{n+1}\right)^n \leq \frac{1}{2}$. Do NOT prove.

Prove by mathematical induction that $(2n-1)! \leq n^{2n-1}$.

Prove true for $n=1$

$LHS = (2(1)-1)!$ $\quad$ $RHS = (1)^{2(1)-1}$

$= 1!$ $\quad$ $= 1^1$

$= 1$ $\quad$ $= 1$

$LHS \leq RHS$

$\therefore$ true for $n=1$

Assume true for $n=k$, $k \in \mathbb{Z}^+$

$(2k-1)! \leq k^{2k-1}$

Prove true for $n=k+1$

ie $(2k+1)! \leq (k+1)^{2k+1}$

$LHS = (2k+1)!$

$= (2k+1)(2k)(2k-1)!$

$\leq (2k+1)(2k)\,k^{2k-1}$

$= 2(2k+1).k^{2k}$

$= 2(2k+1).k^k.k^k$

Note: $\left(\frac{n}{n+1}\right)^n \leq \frac{1}{2}$

$\frac{n^n}{(n+1)^n} \leq \frac{1}{2}$

$n^n \leq \frac{1}{2}(n+1)^n$ for integral n

$\leq 2(2k+1).\frac{1}{2}(k+1)^k.\frac{1}{2}(k+1)^k$

$= \frac{(2k+1)}{2}(k+1)^{2k}$

$\leq \frac{2k+2}{2}(k+1)^{2k}$

$= \frac{\cancel{2}(k+1)}{\cancel{2}}(k+1)^{2k}$

$= (k+1)^{2k+1}$

$= RHS$

$\therefore$ true for $n=k+1$

$\therefore$ true by induction for positive integers n

Question 18 (3 marks)

Using the following identity, or otherwise, **3**

$$\left(\frac{\log_e x-1}{1+(\log_e x)^2}\right)^2 = -\frac{2\log_e x}{\left(1+(\log_e x)^2\right)^2}+\frac{1}{1+(\log_e x)^2} \quad \text{(Do NOT prove)}$$

find $\displaystyle\int_0^1 \left(\frac{\log_e x-1}{1+(\log_e x)^2}\right)^2 dx$.

OTHERWISE

$$\int_a^1 \left(\frac{\log_e x-1}{1+(\log_e x)^2}\right)^2 dx$$

$$= \int_a^1 \frac{(\log_e x)^2-2(\log_e x)+1}{(1+(\log_e x)^2)^2}\, dx$$

$$= \int_a^1 \frac{(1+(\log_e x)^2).1 - x.2(\log_e x).\frac{1}{x}}{(1+(\log_e x)^2)^2}\, dx$$

"Reverse Quotient Rule"

$$\int \frac{vu'-uv'}{v^2}\,dx = \frac{u}{v}+C$$

$$= \left[\frac{x}{1+(\log_e x)^2}\right]_a^1$$

$$= \frac{1}{1+(\cancel{\log_e 1})^2} - \frac{a}{1+(\log_e a)^2}$$

$$= 1 - \frac{a}{1+(\log_e a)^2}$$

$$\therefore \lim_{a\to 0}\int_a^1\left(\frac{\log_e x-1}{1+(\log_e x)^2}\right)^2 dx = \lim_{a\to 0}\left(1-\frac{a}{1+(\log_e a)^2}\right)$$

$$= 1-0$$

$$= 1$$

End of paper

Use this space to re-write any question. Clearly indicate that you have done so.

USING IDENTITY

$$\int_a^1 \left(\frac{\log_e x - 1}{1+(\log_e x)^2}\right)^2 dx$$

$$= \int_a^1 \left(-\frac{2\log_e x}{(1+(\log_e x)^2)^2} + \frac{1}{1+(\log_e x)^2}\right) dx$$

$$= \int_a^1 \frac{-2\log_e x}{(1+(\log_e x)^2)^2}\,dx + \int_a^1 \frac{1}{1+(\log_e x)^2}\,dx$$

$$\text{Let } I = \int_a^1 -\frac{2\log_e x}{(1+(\log_e x)^2)^2}\,dx$$

$$= -\int_a^1 x \cdot \frac{1}{x}\, 2\log_e x \cdot \left(1+(\log_e x)^2\right)^{-2} dx$$

let $u = x$, $v' = \frac{2\log_e x}{x}\left(1+(\log_e x)^2\right)^{-2}$

$u' = 1$, $v = \left(1+(\log_e x)^2\right)^{-1} \cdot -1$

$$= -\left[\left[-\frac{x}{1+(\log_e x)^2}\right]_a^1 + \int_a^1 \frac{1}{1+(\log_e x)^2}\,dx\right]$$

$$= \left[\frac{x}{1+(\log_e x)^2}\right]_a^1 - \int_a^1 \frac{1}{1+(\log_e x)^2}\,dx$$

$$\therefore \int_a^1 \left(\frac{\log_e x - 1}{1+(\log_e x)^2}\right)^2 dx = \left[\frac{x}{1+(\log_e x)^2}\right]_a^1 - \cancel{\int_a^1 \frac{1}{1+(\log_e x)^2}\,dx} + \cancel{\int_a^1 \frac{1}{1+(\log_e x)^2}\,dx}$$

$$= \frac{1}{1+(\log_e 1)^2} - \frac{a}{1+(\log_e a)^2}$$

Use this space to re-write any question. Clearly indicate that you have done so.

$$= 1 - \frac{a}{1+(\log_e a)^2}$$

$$\therefore \lim_{a \to 0} \int_a^1 \left(\frac{\log_e x - 1}{1+(\log_e x)^2} \right)^2 dx = \lim_{a \to 0} \left(1 - \frac{a}{1+(\log_e a)^2} \right)$$

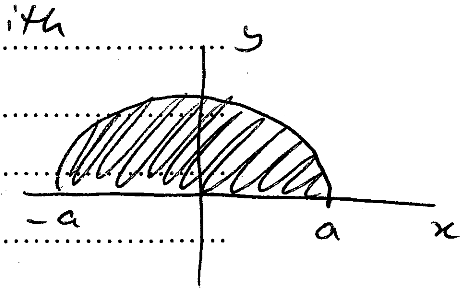
$$= 1 - 0$$

$$= 1$$

COMMENTS

16) a) i) Students should be familiar with the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ and the substitution needed to establish it.
This is no different.

a) ii) Students should be familiar with $\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$ representing



$= \frac{1}{2}\pi(a)^2$

Sadly the vast majority did not make the connection.

b) i) Most students were able to apply integration by parts to get
$I_n = \frac{n}{3}\int_0^1 x\sqrt{x}(1-x)^{\frac{n-2}{2}}dx$
However, not many wrote
$x = -(1-x-1)$ to get I_n & I_{n-2}.

Use this space to re-write any question. Clearly indicate that you have done so.

b)ii) Unfortunately, not many students realised that $I_1 = \frac{\pi}{8}$ (from a)ii))

17) Most struggled with the induction when it came to using the fact that $0 < \left(\frac{n}{n+1}\right)^n \leq \frac{1}{2}$.

Not many completed the proof.

18) This is an improper integral as the integrand is not defined for $x = 0$. Again, very few made significant progress.