SYDNEY
BOYS
HIGH
SCHOOL

NESA Number:

Name:

Maths Class:

A B 1 2 S

2023

YEAR 12
TERM 2
HSC Task #3

Mathematics Extension 2

General Instructions

- Working Time – 65 minutes
- Write using non-erasable black pen
- NESA approved calculators may be used
- Marks may **NOT** be awarded for messy or badly arranged work
- Unless otherwise stated, all answers should be left in simplified, exact form
- In Questions 11–17, show ALL relevant mathematical reasoning and/or calculations

Total Marks:

Section I – 10 marks (Personal Reference sheet)

Leave your Personal Reference Sheet in this booklet.

Section II – 10 marks (pages 2–4)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section III – 30 marks (pages 5–7)
- Attempt Questions 11–17
- Allow about 50 minutes for this section

Examiner:

PSP

Marks	
Section I	/10
Section II	/10
Section III	/30
Total	/50

Section II

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is a primitive of $\dfrac{e^x}{1+e^{2x}}$?

A. $\log_e\left(1+e^{2x}\right)$

B. $\dfrac{1}{2}\log_e\left(1+e^{2x}\right)$

C. $\tan^{-1} e^{2x}$

D. $\tan^{-1} e^{x}$

2 Which of the following is a correct primitive of $\dfrac{1}{\sqrt{4-9x^2}}$?

A. $-\dfrac{1}{6}\cos^{-1}\dfrac{3x}{2}+C$

B. $-\dfrac{1}{3}\cos^{-1}\dfrac{3x}{2}+C$

C. $-\dfrac{2}{3}\cos^{-1}\dfrac{3x}{2}+C$

D. $-\dfrac{1}{3}\cos^{-1}\dfrac{2x}{3}+C$

3 $\displaystyle\int 1-\cos 10x\,dx$ is equivalent to which of the following?

A. $\dfrac{1}{2}\displaystyle\int \sin^2 5x\,dx$

B. $\dfrac{1}{2}\displaystyle\int \cos^2 5x\,dx$

C. $2\displaystyle\int \sin^2 5x\,dx$

D. $2\displaystyle\int \cos^2 5x\,dx$

4 The expression $\dfrac{ax+b}{(2x-1)^2(x+1)}$ has partial fraction form $\dfrac{1}{x+1}-\dfrac{2}{2x-1}-\dfrac{1}{(2x-1)^2}$.

What are the values of a and b, where a and b are non-zero real constants?

A. $a = 16, b = 7$

B. $a = 7, b = 16$

C. $a = -7, b = 2$

D. $a = 2, b = -7$

5 Which of the following is a primitive of $\dfrac{x-1}{x^2+4}$?

A. $2\ln\left(x^2+4\right) - 2\arctan x$

B. $\dfrac{1}{2}\ln(2x) - \arctan x$

C. $2\ln\left(x^2+4\right) - \arctan\left(\dfrac{x}{2}\right)$

D. $\dfrac{1}{2}\ln\left(x^2+4\right) - \dfrac{1}{2}\arctan\left(\dfrac{x}{2}\right)$

6 If $\dfrac{d}{dx}(x\sin x) = \sin x + x\cos x$ then which of the following is $\dfrac{1}{k}\displaystyle\int x\cos x\,dx$ equivalent to?

A. $\dfrac{1}{k}x\sin x - \displaystyle\int \sin x\,dx$

B. $\dfrac{1}{k}\left(x\sin x - \displaystyle\int \sin x\,dx\right)$

C. $\dfrac{1}{k}(x\sin x - \sin x)$

D. $\dfrac{1}{k}\left(\displaystyle\int x\sin x\,dx - \int \sin x\,dx\right)$

7 Which of the following is equivalent to $\displaystyle\int_a^b x^3 e^{2x^4}\,dx$, where $a,\ b \in \mathbb{R}$?

A. $\displaystyle\int_{a^4}^{b^4} e^{2u}\,du$

B. $\displaystyle\int_a^b e^u\,du$

C. $\dfrac{1}{4}\displaystyle\int_{a^4}^{b^4} e^{2u}\,du$

D. $\dfrac{1}{8}\displaystyle\int_{8a^3}^{8b^3} e^u\,du$

8 If $f(x)$ is a non-zero odd function with period π, which of the following statements is false?

A. $\displaystyle\int_0^{2\pi} f(x)\,dx = 0$

B. $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx = 2\int_0^{\frac{\pi}{2}} f(x)\,dx$

C. $\displaystyle\int_0^{\pi} f(x)\,dx = -\int_0^{\pi} f(-x)\,dx$

D. $\displaystyle\int_{\alpha}^{\alpha+\pi} f(x)\,dx = \int_0^{\pi} f(x)\,dx$, for any real number α.

9 The substitution $t = \tan\frac{1}{2}\theta$ is used to find $\int \sec\theta\, d\theta$.

Which of the following gives the correct expression for the required integral?

A. $\displaystyle\int \frac{1}{2(1-t^2)}\, dt$

B. $\displaystyle\int \frac{2}{1-t^2}\, dt$

C. $\displaystyle\int \frac{2t}{1-t^2}\, dt$

D. $\displaystyle\int \frac{4t}{(1+t^2)^2}\, dt$

10 The function $f(x)$ is odd and continuous.

Given that $\displaystyle\int_0^a f(x)\, dx = b$, what is the value of $\displaystyle\int_0^a \left[f(x-a) - f(a-x) \right] dx$?

A. 0

B. b

C. $2b$

D. $-2b$

Section III – Part A

Label your Answer Booklet as Section III Part A

Part A – 15 marks
Attempt Questions 11 – 14

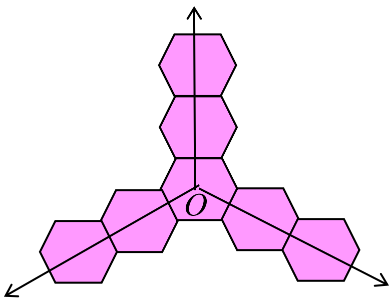
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (2 mark)

Find $\displaystyle\int \frac{e^{3x}+8}{e^{x}+2}\,dx$. **2**

Question 12 (3 marks)

Segments of three lines in a horizontal plane meet at a point O. **3**
A flat shape is formed by placing identical hexagons along these segments of lines.
The shape will continue to grow along these three segments of lines.
The first hexagon is placed at O.
An example is the flat shape in the diagram which is formed by 7 hexagons.



Let S_n be the number of sides on the perimeter of the shape formed by n hexagons.

Use mathematical induction to show that $S_n = 4n + 2$, where $n \geq 1$.

Question 13 (3 marks)

Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} x\sin x\,dx$. **3**
Leave your answer in simplified exact form.

Section A continues on page 6

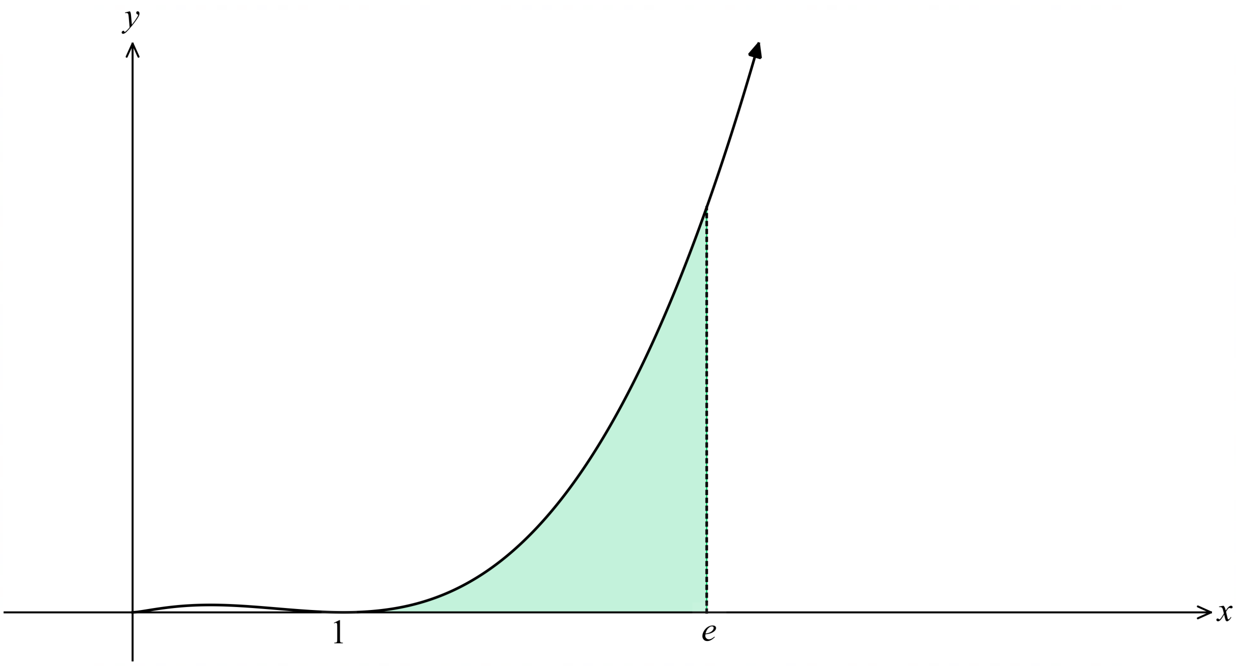
Question 14 (7 marks)

Let $I_n = \int_1^e x^2(\ln x)^n \, dx,\ n \geq 0$

(a) Find the value of I_0. **1**

(b) Show that $I_n = \frac{1}{3}\left(e^3 - nI_{n-1}\right)$, where $n \geq 1$. **3**

(c) The following diagram shows the graph of $y = x(\ln x)^2,\ x > 0$. **3**



The shaded region is rotated though 2π radians about the x-axis.

Find the volume of the solid formed.

End of Section A

Section III – Part B

Label your Answer Booklet as Section III Part B

Part A – 15 marks
Attempt Questions 15 – 17

Your responses should include relevant mathematical reasoning and/or calculations.

Question 15 (5 marks)

Find $\displaystyle\int \frac{\sin x \cos x}{\sin^2 x + 6\sin x + 8}\, dx$. **5**

Question 16 (4 marks)

The function f is defined, for $x > 1$, by

$$f(x) = \int_1^x \sqrt{\frac{t-1}{t+1}}\, dt.$$

Do NOT attempt to evaluate this integral.

(a) Show that **3**

$$2f\left(\frac{x+2}{2}\right) = \int_0^x \sqrt{\frac{u}{u+4}}\, du.$$

(b) Evaluate in terms of f **3**

$$\int_1^2 \frac{u^2}{\sqrt{u^2+4}}\, du.$$

Question 17 (6 marks)

Find $\displaystyle\int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\, dx$. **4**

End of paper

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2023

YEAR 12
TERM 3
ASSESSMENT TASK 3

Mathematics Extension 2

Sample Solutions

MC Answers

1	D	**6**	B
2	B	**7**	C
3	C	**8**	B
4	C	**9**	B
5	D	**10**	D

Section II Multiple Choice Solutions

1 Which of the following is a primitive of $\dfrac{e^x}{1+e^{2x}}$?

A. $\log_e\left(1+e^{2x}\right)$

B. $\dfrac{1}{2}\log_e\left(1+e^{2x}\right)$

C. $\tan^{-1}e^{2x}$

D. $\tan^{-1}e^x$

u-sub with $u = e^x$: $\dfrac{e^x}{1+e^{2x}} = \dfrac{e^x}{1+\left(e^x\right)^2}$

2 Which of the following is a correct primitive of $\dfrac{1}{\sqrt{4-9x^2}}$?

A. $-\dfrac{1}{6}\cos^{-1}\dfrac{3x}{2}+C$

B. $-\dfrac{1}{3}\cos^{-1}\dfrac{3x}{2}+C$

C. $-\dfrac{2}{3}\cos^{-1}\dfrac{3x}{2}+C$

D. $-\dfrac{1}{3}\cos^{-1}\dfrac{2x}{3}+C$

$$\frac{1}{\sqrt{4-9x^2}}=\frac{1}{\sqrt{9\left(\frac{4}{9}-x^2\right)}}=\frac{1}{3}\frac{1}{\sqrt{\left(\frac{2}{3}\right)^2-x^2}}$$

3 $\int 1-\cos 10x\,dx$ is equivalent to which of the following?

A. $\dfrac{1}{2}\int \sin^2 5x\,dx$

B. $\dfrac{1}{2}\int \cos^2 5x\,dx$

C. $2\int \sin^2 5x\,dx$

D. $2\int \cos^2 5x\,dx$

$\sin^2 x = \dfrac{1}{2}(1-\cos 2x)$

4 The expression $\dfrac{ax+b}{(2x-1)^2(x+1)}$ has partial fraction form $\dfrac{1}{x+1}-\dfrac{2}{2x-1}-\dfrac{1}{(2x-1)^2}$.

What are the values of a and b, where a and b are non-zero real constants?

A. $a = 16,\ b = 7$

B. $a = 7,\ b = 16$

C. $a = -7,\ b = 2$

D. $a = 2,\ b = -7$

Substitute $x = 0$: $\dfrac{b}{(-1)^2(1)}=\dfrac{1}{1}-\dfrac{2}{-1}-\dfrac{1}{(-1)^2}\Rightarrow b=2$

5 Which of the following is a primitive of $\dfrac{x-1}{x^2+4}$?

A. $2\ln\left(x^2+4\right) - 2\arctan x$

B. $\dfrac{1}{2}\ln(2x) - \arctan x$

C. $2\ln\left(x^2+4\right) - \arctan\left(\dfrac{x}{2}\right)$

D. $\dfrac{1}{2}\ln\left(x^2+4\right) - \dfrac{1}{2}\arctan\left(\dfrac{x}{2}\right)$

$$\frac{x-1}{x^2+4} = \frac{1}{2}\times\frac{2x}{x^2+4} - \frac{1}{x^2+4}$$

6 If $\dfrac{d}{dx}(x\sin x) = \sin x + x\cos x$ then which of the following is $\dfrac{1}{k}\displaystyle\int x\cos x\,dx$ equivalent to?

A. $\dfrac{1}{k}x\sin x - \displaystyle\int \sin x\,dx$

B. $\dfrac{1}{k}\left(x\sin x - \displaystyle\int \sin x\,dx\right)$

C. $\dfrac{1}{k}(x\sin x - \sin x)$

D. $\dfrac{1}{k}\left(\displaystyle\int x\sin x\,dx - \int \sin x\,dx\right)$

$$x\cos x = \frac{d}{dx}(x\sin x) - \sin x \Rightarrow \frac{1}{k}\int x\cos x\,dx = \frac{1}{k}\int \frac{d}{dx}(x\sin x) - \sin x\,dx$$

7 Which of the following is equivalent to $\displaystyle\int_a^b x^3 e^{2x^4}\,dx$, where $a,\ b \in \mathbb{R}$?

A. $\displaystyle\int_{a^4}^{b^4} e^{2u}\,du$

B. $\displaystyle\int_a^b e^u\,du$

C. $\dfrac{1}{4}\displaystyle\int_{a^4}^{b^4} e^{2u}\,du$

D. $\dfrac{1}{8}\displaystyle\int_{8a^3}^{8b^3} e^u\,du$

u-sub with $u = x^4$

8 If $f(x)$ is a non-zero odd function with period π, which of the following statements is false?

A. $\displaystyle\int_0^{2\pi} f(x)\ dx = 0$

B. $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} f(x)\ dx = 2\int_0^{\frac{\pi}{2}} f(x)\ dx$

NOTE:
For multiple choice: consider $f(x) = \sin 2x$

C. $\displaystyle\int_0^{\pi} f(x)\ dx = -\int_0^{\pi} f(-x)\ dx$

D. $\displaystyle\int_{\alpha}^{\alpha+\pi} f(x)\ dx = \int_0^{\pi} f(x)\ dx$, for any real number α.

B is false as this is true for EVEN functions

A is true:

$$\int_0^{2\pi} f(x)\,dx = \int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx$$

Now $\displaystyle\int_{\pi}^{2\pi} f(x)\,dx = \int_0^{\pi} f(2\pi - x)\,dx = \int_0^{\pi} f(\pi+\pi+(-x))\,dx = \int_0^{\pi} f(-x)\,dx = -\int_0^{\pi} f(x)\,dx$

C is true by definition of an ODD function

D is true by definition of a periodic function

9 The substitution $t = \tan\frac{1}{2}\theta$ is used to find $\int \sec\theta \, d\theta$.

Which of the following gives the correct expression for the required integral?

A. $\displaystyle\int \frac{1}{2(1-t^2)}\, dt$

B. $\displaystyle\int \frac{2}{1-t^2}\, dt$

C. $\displaystyle\int \frac{2t}{1-t^2}\, dt$

D. $\displaystyle\int \frac{4t}{(1+t^2)^2}\, dt$

$t = \tan\frac{1}{2}\theta \Rightarrow \theta = 2\tan^{-1} t$

$\therefore d\theta = \dfrac{2}{1+t^2}\, dt$

10 The function $f(x)$ is odd and continuous.

Given that $\displaystyle\int_0^a f(x)\, dx = b$, what is the value of $\displaystyle\int_0^a \left[f(x-a) - f(a-x) \right] dx$?

A. 0

B. b

C. $2b$

D. $-2b$

$f(x-a) = -f(a-x)$

Also $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

Question 11 **2 marks**

Find $\displaystyle\int \frac{e^{3x}+8}{e^x+2}\,dx$

$$= \int \frac{(e^x)^3 + 2^3}{e^x+2}\,dx$$

$$= \int \frac{(e^x+2)\left((e^x)^2 - 2e^x + 2^2\right)}{e^x+2}\,dx$$

$$= \int \left(e^{2x} - 2e^x + 4\right)dx$$

$$= \frac{e^{2x}}{2} - 2e^x + 4x + C$$

<u>Comment</u>: If pupils tried anything other than algebraic manipulation, it generally did not succeed. Most pupils got this question correct.

$$(e^x+2)^3 = e^{3x} + 6e^{2x} + 12e^x + 8$$

Hence, $\displaystyle I = \int \frac{(e^x+2)^3 - 6e^x(e^x+2)}{e^x+2}\,dx$

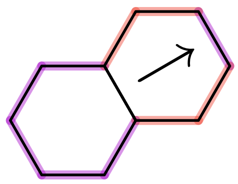
$$= \int \left(e^{2x} + 4e^x + 4 - 6e^x\right)dx$$

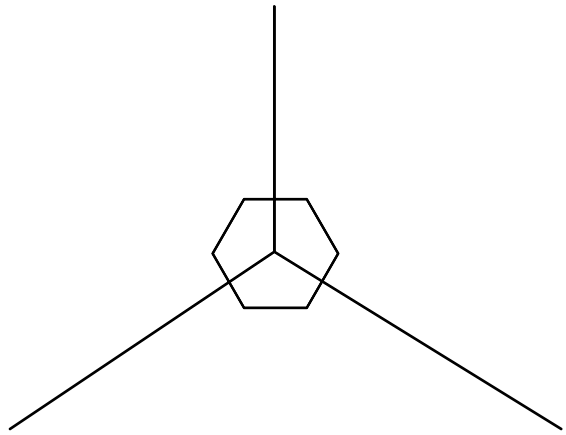
$$= \int \left(e^{2x} - 2e^x + 4\right)dx$$

$$= \frac{e^{2x}}{2} - 2e^x + 4x + C$$

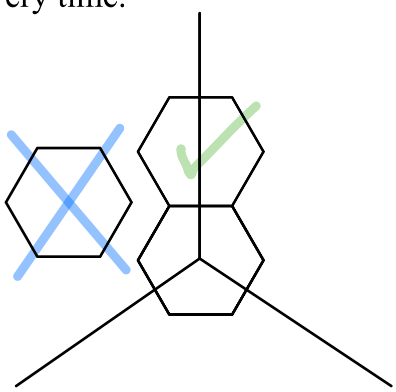
Question 12 **3 marks**

An essential part of answering this question is explaining satisfactorily why the formula is $S_n = 4n + 2, n \geq 1.$





When a hexagon is placed adjacent to another, the number of sides goes from 6 to 10. One might consider it as the sides where the new hexagons are placed next to each other are no longer part of the perimeter. Or, one might say the side of the original is pushed to the opposite side of the new one and we only need consider that there are 4 new sides every time.



A concern may be that a hexagon might be placed in such a way that more than one side gets displaced, however one can see from the diagram that this is not possible, since the hexagons must be placed along the lines.

Proof:

1. Prove true for $n = 1$.

$$\begin{aligned} S_1 &= 4 \times 1 + 2 \\ &= 6 \end{aligned}$$

Hence, true for $n = 1$.

2. Assume true for $n = k$. $S_k = 4k + 2$

3. Prove true for $n = k + 1$.

$$\begin{aligned} S_{k+1} &= 4\left(k+1\right) + 2 \\ &= 4k + 2 + 4 \\ &= S_k + 4 \end{aligned}$$

Hence, true for $n = k + 1$.

* * In this step, you must have a statement, that conveys why four is added each time. Those who simply state four is added each time will not get full marks. An example that will get rewarded: When a new hexagon is added to a hexagon they both lose a side and the new hexagon adds 5 sides. Hence the new sides to the structure totals 4.

4. The statement, the number of sides on the perimeter of hexagons placed adjacently on three lines radiating from a point equally is $S_n = 4n + 2, \forall n \geq 1,$ is true for $n = 1$ and true for $n = k + 1$, given $n = k$ is true. Hence the statement is true for all $n \geq 1$.

Comment: A point of contention will be satisfactorily arguing why the given equation works. The equation obviously adds 4 for every subsequent hexagon. Pupils need to explain in a way that adds meaning, that is, something the equation doesn't already tell me.
Of NOTE, is that, some pupils are not using the assumption to prove the last step. If the assumption is not used, you have not proven the statement by use of induction, and full marks cannot be awarded if the question requires induction to be used.

Question 13 **3 marks**

Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} x \sin x \, dx$

$$= -x\cos x\Big|_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \cos x \, dx$$

$$= \sin x\Big|_0^{\frac{\pi}{6}} - \frac{\pi}{6} \times \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} - \frac{\pi\sqrt{3}}{12}$$

<u>Comment</u>: A few pupils chose the wrong function to be the derivative. The wrong choice becomes self-evident, but most did not come to that realisation. Otherwise, this was well done.

Question 14

Let $I_n = \int_1^e x^2 (\ln x)^n \, dx, n \geq 0, \, n \in \mathbb{Z}.$

Find the value of I_0. **1 mark**

$$I_0 = \int_1^e x^2 dx$$
$$= \left. \frac{x^3}{3} \right|_1^e$$
$$= \frac{e^3 - 1}{3}$$

Show $I_n = \frac{1}{3}\left(e^3 - nI_{n-1}\right), \forall n \geq 1, \, n \in \mathbb{Z}.$ **3 marks**

Let $f'(x) = 3x^2 \Rightarrow f(x) = x^3$

$$g(x) = \frac{(\ln x)^n}{3} \Rightarrow g'(x) = \frac{n}{3x} (\ln x)^{n-1}$$

$$I_n = \left. \frac{x^3 (\ln x)^n}{3} \right|_1^e - \frac{n}{3} \int_1^e x^2 (\ln x)^{n-1} dx$$
$$= \frac{e^3}{3} - \frac{n}{3} I_{n-1}$$
$$= \frac{1}{3}\left(e^3 - nI_{n-1}\right)$$

Find the volume of the rotation of $y = x(\ln x)^2$, $1 < x < e$, through 2π radians. **3 marks**

$$
\begin{aligned}
V &= \pi \int_1^e y^2 dx, \quad \text{where} \quad y = x(\ln x)^2 \\
&= \pi \int_1^e x^2 (\ln x)^4 dx \\
&= \pi I_4 \\
&= \frac{\pi}{3}\left(e^3 - 4I_3\right) \\
&= \frac{\pi}{3}\left(e^3 - \frac{4}{3}\left(e^3 - 3I_2\right)\right) \\
&= \frac{\pi}{9}\left(3e^3 - 4\left(e^3 - \frac{3}{3}\left(e^3 - 2I_1\right)\right)\right) \\
&= \frac{\pi}{9}\left(3e^3 - 4\left(e^3 - \left(e^3 - \frac{2}{3}\left(e^3 - I_0\right)\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
V &= \frac{\pi}{27}\left(9e^3 - 4\left(3e^3 - \left(3e^3 - 2\left(e^3 - \frac{1}{3}\left(e^3 - 1\right)\right)\right)\right)\right) \\
&= \frac{\pi}{81}\left(27e^3 - 4\left(9e^3 - \left(9e^3 - 2\left(3e^3 - \left(e^3 - 1\right)\right)\right)\right)\right) \\
&= \frac{\pi}{81}\left(27e^3 - 4\left(9e^3 - \left(9e^3 - 2\left(2e^3 + 1\right)\right)\right)\right) \\
&= \frac{\pi}{81}\left(27e^3 - 4\left(9e^3 - \left(5e^3 - 2\right)\right)\right) \\
&= \frac{\pi}{81}\left(27e^3 - 4\left(4e^3 + 2\right)\right) \\
&= \frac{\pi}{81}\left(11e^3 - 8\right) \quad \text{cubic units}
\end{aligned}
$$

OR

$$
\begin{aligned}
\pi I_4 &= \frac{\pi}{3}\left(e^3 - 4I_3\right) \\
&= \frac{\pi}{9}\left(12I_2 - e^3\right) \\
&= \frac{\pi}{9}\left(8I_1 - 3e^3\right) \\
&= \frac{\pi}{27}\left(e^3 - 8I_0\right) \\
&= \frac{\pi}{81}\left(11e^3 - 8\right)
\end{aligned}
$$

<u>Comment</u>: Parts i and ii were done with aplomb. The algebra in part iii brought nearly half the cohort undone.
I sympathise... no, really!

15) $I = \int \frac{\sin x \cos x}{\sin^2 x + 6\sin x + 8}\,dx$

let $u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$I = \int \frac{u.\cancel{\cos x}}{u^2+6u+8} \cdot \frac{du}{\cancel{\cos x}}$

$= \int \frac{u}{(u+4)(u+2)}\,du$

$\frac{u}{(u+4)(u+2)} = \frac{a}{u+4} + \frac{b}{u+2}$

$u \equiv a(u+2) + b(u+4)$

let $u = -2$

$-2 = b(-2+4)$

$2b = -2$

$b = -1$

let $u = -4$

$-4 = a(-4+2)$

$-2a = -4$

$a = 2$

$I = \int \left(2 \cdot \frac{1}{u+4} - \frac{1}{u+2}\right) du$

$= 2\ln|u+4| - \ln|u+2| + c$

$= 2\ln|\sin x + 4| - \ln|\sin x + 2| + C$

COMMENT

There are other forms that the answer could take.

$$I = \ln \left| \frac{(\sin x + 4)^2}{\sin x + 2} \right| + C$$

$$I = \frac{1}{2} \ln | \sin^2 x + 6 \sin x + 8 | + \frac{3}{2} \ln \left| \frac{\sin x + 4}{\sin x + 2} \right| + C$$

Given that $I = \int \cos x . f(\sin x) \, dx$

The substitution $u = \sin x$ is a good starting point.

Given that the question is worth 5 marks, you would expect another technique like partial fractions (to make up the rest of the marks).

Most students could see the substitution and that the answer would involve a logarithm.

Students that got stuck tended to complete the square instead of factorising the denominator.

Question 16

a) $2f\left(\frac{x+2}{2}\right) = 2\int_1^{\frac{x+2}{2}} \sqrt{\frac{t-1}{t+1}}\, dt$

let $t = \frac{u+2}{2}$

$\frac{dt}{du} = \frac{1}{2}$

$dt = \frac{1}{2} du$

limit change

when $t=1$

$\frac{u+2}{2} = 1$

$u+2=2$

$u=0$

when $t = \frac{x+2}{2}$

$\frac{u+2}{2} = \frac{x+2}{2}$

$u = x$

$$= 2\int_0^x \sqrt{\frac{\frac{u+2}{2}-1}{\frac{u+2}{2}+1} \times \frac{2}{2}} \cdot \frac{1}{2}\, du$$

$$= \int_0^x \sqrt{\frac{u+2-2}{u+2+2}}\, du$$

$$= \int_0^x \sqrt{\frac{u}{u+4}}\, du$$

b) $I = \int_1^2 \frac{u^2}{\sqrt{u^2+4}}\, du$

let $u = \sqrt{w}$

$\frac{du}{dw} = \frac{1}{2\sqrt{w}}$

$du = \frac{dw}{2\sqrt{w}}$

limit change

when $u=1$, $w=1$

$u=2$, $w=4$

$$I = \int_1^4 \frac{w}{\sqrt{w+4}} \cdot \frac{dw}{2\sqrt{w}}$$

$$= \frac{1}{2}\int_1^4 \sqrt{\frac{w}{w+4}}\,dw$$

$$= \frac{1}{2}\int_0^4 \sqrt{\frac{w}{w+4}}\,dw - \frac{1}{2}\int_0^1 \sqrt{\frac{w}{w+4}}\,dw$$

$$= f\left(\frac{4+2}{2}\right) - f\left(\frac{1+2}{2}\right)$$

$$= f(3) - f(1.5)$$

COMMENT:

This was a very different style of question. Arguably the most challenging on the paper. Only a few students were awarded full marks.

Many students struggled with the different pronumerals and how they varied with respect to each other.

eg. To find $f\left(\frac{x+2}{2}\right)$ many students did not simply replace x with $\frac{x+2}{2}$.

Most students did not make any progress towards writing $\int_1^2 \frac{u^2}{\sqrt{u^2+4}}\,du$ in terms of function values of f. Students should have used their time in manipulating the integrand into a previous form rather than using another strategy to evaluate.

17) $I = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\, dx$

Note: $\frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}$ is defined for

$x < -1$ or $x \geq 1$

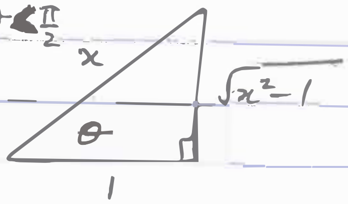
$$I = \int \frac{1}{x^2} \cdot \sqrt{\frac{x-1}{x+1}} \cdot \sqrt{\frac{x-1}{x-1}}\, dx$$

$$I = \int \frac{x-1}{x^2\sqrt{x^2-1}}\, dx$$

* if $\boxed{x \geq 1}$ $\sqrt{(x-1)^2} = x-1$

let $x = \sec\theta$ ie $0 \leq \theta < \frac{\pi}{2}$

$\frac{dx}{d\theta} = \sec\theta \tan\theta$



$dx = \sec\theta \tan\theta\, d\theta$

$$I = \int \frac{\sec\theta - 1}{\sec^2\theta\sqrt{\sec^2\theta - 1}} \cdot \sec\theta \tan\theta\, d\theta$$

$$= \int \frac{\sec\theta - 1}{\sec\theta \cancel{\sqrt{\tan^2\theta}}} \cdot \cancel{\tan}\theta\, d\theta$$

* $\sqrt{\tan^2\theta} = \tan\theta$ since $0 \leq \theta < \frac{\pi}{2}$

$$= \int \left(1 - \frac{1}{\sec\theta}\right) d\theta$$

$$= \int (1 - \cos\theta)\, d\theta$$

$$= \theta - \sin\theta + C$$

$$= \sec^{-1}x - \frac{\sqrt{x^2-1}}{x} + C$$

COMMENT

There are other forms that the answer could take

$$I = \cos^{-1}\left(\frac{1}{x}\right) - \frac{\sqrt{x^2-1}}{x} + C$$

$$I = \tan^{-1}\left(\sqrt{x^2-1}\right) - \frac{\sqrt{x^2-1}}{x} + C$$

$$I = \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right) - \frac{\sqrt{x^2-1}}{x} + C$$

$$I = -\sin^{-1}\left(\frac{1}{x}\right) - \frac{\sqrt{x^2-1}}{x} + C$$

However, these are all for $x \geq 1$

What about if $x < -1$?

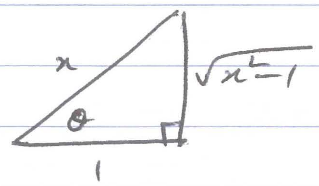
$$I = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\sqrt{\frac{x-1}{x-1}} \cdot dx$$

$$I = \int \frac{1-x}{x^2\sqrt{x^2-1}}\, dx$$ * if $\boxed{x<-1}$ $\sqrt{(x-1)^2} = 1-x$

let $x = \sec\theta$ ie $\frac{\pi}{2} < \theta < \pi$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta$$

$$dx = \sec\theta\tan\theta\, d\theta$$



$$I = \int \frac{1-\sec\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}} \cdot \sec\theta\tan\theta\, d\theta$$

$$= \int \frac{1-\sec\theta}{\sec\theta\sqrt{\tan^2\theta}} \cdot \tan\theta\, d\theta$$

$$= \int \frac{1-\sec\theta}{\sec\theta(-\tan\theta)} \cdot \tan\theta\, d\theta$$ * $\sqrt{\tan^2\theta} = -\tan\theta$ since $\frac{\pi}{2} < \theta < \pi$

$$= \int \left(\frac{\sec\theta - 1}{\sec\theta}\right) d\theta$$

$$= \int \left(1 - \frac{1}{\sec\theta}\right) d\theta$$

$$= \int (1 - \cos\theta)\, d\theta$$

$$= \theta - \sin\theta + C$$

$$= \sec^{-1}x - \left(-\frac{\sqrt{x^2-1}}{x}\right) + C$$

$$= \sec^{-1}x + \frac{\sqrt{x^2-1}}{x} + C$$

Note: since $\frac{\pi}{2} < \theta < \pi$
$\sin\theta > 0$
since $x < 0$
$\sin\theta = -\frac{\sqrt{x^2-1}}{x}$.

There are other forms that the answer could take when $x < -1$.

$$I = \cos^{-1}\left(\frac{1}{x}\right) + \frac{\sqrt{x^2-1}}{x} + C$$

$$I = \tan^{-1}\left(-\sqrt{x^2-1}\right) + \frac{\sqrt{x^2-1}}{x} + C$$

$$I = \sin^{-1}\left(-\frac{\sqrt{x^2-1}}{x}\right) + \frac{\sqrt{x^2-1}}{x} + C$$

Most students were able to make some progress towards an answer whether it be by rationalising the numerator or making a trig substitution or some other technique.

From a marking point of view, students were not required to consider the negative case.