Sydney Girls High School

YEAR 12 HSC ASSESSMENT TASK 2 for 2020

MATHEMATICS EXTENSION 2

Reading time: 5 minutes

Writing time: 60 minutes

Topic: Proof; Vectors (Partial topic)

General Instructions:

- There are Five (5) Questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- White-out or correction tape is NOT to be used.
- A correct answer without working will be awarded a maximum of 1 mark.
- A Mathematics Reference Sheet is provided.

Student Name :__________________ **Teacher Name :**__________________

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Question 1 **Marks**

a) Write the negation of the statement: *A car weighs less than 2 tonnes*. **1**

b) Translate the following statement into everyday language: **1**

$$\forall\, a \in \mathbb{Z}^+ ,\exists\, x \in \mathbb{R} \text{ such that } x = \sqrt{a}.$$

c) Write the contrapositive of the following statement: **1**

if a number is negative, then it does not have a square root.

d) Consider the statement *P*: If *n* is divisible by 5, then the final digit of *n* is 5.

i) State the converse of statement *P* and call it *Q*. **1**

ii) Comment on whether $P \Leftrightarrow Q$ and justify your answer. **1**

e) Given $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ find in component form, in terms of *i*, *j* and *k*:

i) $2\underset{\sim}{b} - \underset{\sim}{a}$; **1**

ii) the vector in the same direction as $\underset{\sim}{a}$ that is 4 units in length. **2**

f) Determine the logical value of the following statement and justify your answer. **2**

$$\forall\, x \in \mathbb{R}, \exists\, y \in \mathbb{R} \text{ such that } xy = 6.$$

g) By considering the contrapositive, prove that if $a^2 - 2a + 7$ is even, then *a* is odd. **2**

End of Question 1

Question 2 (Start a New Page) **Marks**

a) Given $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$, find:

i) $\underset{\sim}{a} \bullet \underset{\sim}{b}$; **1**

ii) $|\underset{\sim}{a}|$; **1**

iii) Hence, use the dot product to find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to the nearest degree. **2**

b) Prove by mathematical induction that $2^n > n^2$ where $n \in \mathbb{Z},\ n \geq 5$. **3**

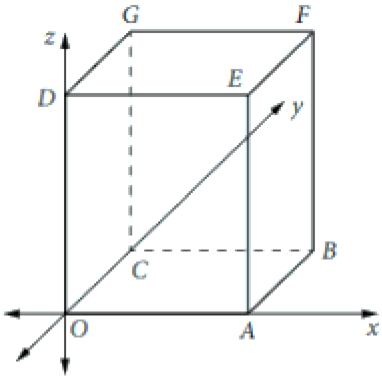
c) Prove by mathematical induction that $3^n + 7^n$ is divisible by 10, if n is an odd integer. **3**

d) Use a proof by contradiction to prove that the sum of a rational and an irrational number is irrational. **2**

End of Question 2

Question 3 (Start a New Page) **Marks**

a) *OABCDEFG* is a rectangular prism where *O* is the origin. $OA = OD = 4$ units and $OC = 2$ units.



Determine:

i) the vectors for $\overrightarrow{AB}$ and $\overrightarrow{BF}$. **2**

ii) the position vectors for *B* and *F*. **2**

iii) the size of angle $\angle FOB$, to the nearest degree. **2**

b) Show by calculation that the points $A(-1, 4, -5)$, $B(1, 1, 0)$ and $C(5, -5, 10)$ are collinear. **2**

c) If x and y are positive real numbers, prove that:

i) $x + \dfrac{1}{x} \geq 2.$ **2**

ii) $(x + y)\left(\dfrac{1}{x} + \dfrac{1}{y}\right) \geq 4.$ **2**

End of Question 3

Question 4 (Start a New Page) **Marks**

a) If $T_1 = 6$ and $T_{n+1} = T_n + 2^n + 4$, use mathematical induction to prove that $T_n = 2^n + 4n$ for positive integers n. **3**

b) The position vectors of the points A, B, C and D, respectively, are:

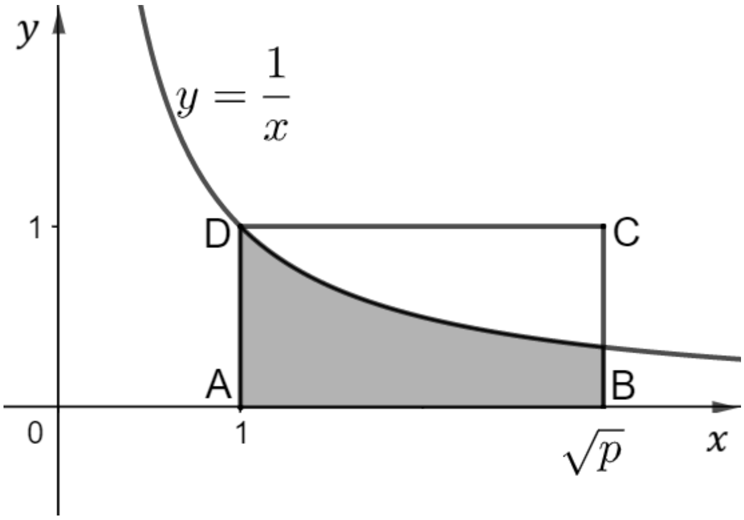
$\overrightarrow{OA} = -3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\overrightarrow{OB} = -\underset{\sim}{i} + 3\underset{\sim}{j}$,

$\overrightarrow{OC} = 2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$, $\overrightarrow{OD} = 3\underset{\sim}{k}$.

i) Show that $\overrightarrow{AB} = \overrightarrow{DC}$. **2**

ii) What shape is $ABCD$? Justify your answer. **1**

c)



Consider the curve $y = \frac{1}{x}$, $x > 0$, as shown in the diagram above.

i) Explain why $\int_1^{\sqrt{p}} \frac{1}{x}\,dx < \sqrt{p} - 1$, where $p > 1$. **1**

ii) Hence show that $0 < \ln p < 2\sqrt{p} - 2$. **2**

iii) Hence deduce that $\lim_{p \to \infty} \frac{\ln p}{p} = 0$. **1**

d) Without the use of a calculator, prove $\sqrt[8]{8!} < \sqrt[9]{9!}$. **2**

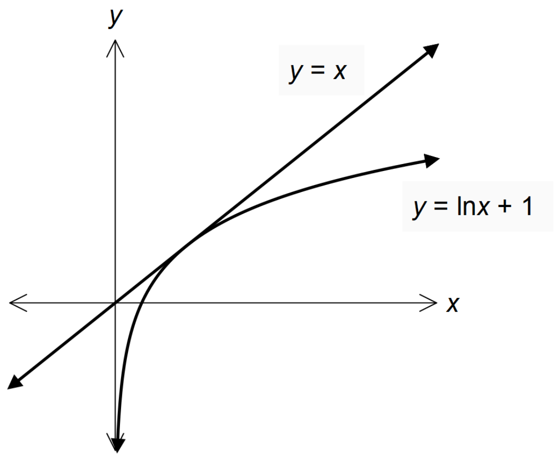
End of Question 4

Question 5 (Start a New Page) **Marks**

a) Prove by mathematical induction that: $\sum_{r=1}^{n}(-1)^{r-1}r^2=(-1)^{n-1}\left[\frac{n(n+1)}{2}\right]$ **3**

b) Find a unit vector which is perpendicular to both $\underset{\sim}{a}=2\underset{\sim}{i}+4\underset{\sim}{j}-3\underset{\sim}{k}$ and $\underset{\sim}{b}=-4\underset{\sim}{i}-5\underset{\sim}{j}+3\underset{\sim}{k}$. Give your answer in component form. **3**

c)



The graphs of $y=x$ and $y=lnx+1$ are shown. **3**

Use a calculus based approach to show that $\forall\ x>0,\ x\geq lnx+1$.

d) The area of a triangle is given by Heron's formula as $A=\sqrt{s(s-a)(s-b)(s-c)}$ **3**

where a, b and c are the lengths of the sides and $s=\frac{1}{2}(a+b+c)$.

Given that $\sqrt{ab}\leq\frac{a+b}{2}$ and $ab+bc+ca\leq a^2+b^2+c^2$, or otherwise, show that

$A\leq\frac{a^2+b^2+c^2}{6}$.

End of Question 5

End of Task

Q1

a) All cars do not weigh less than 2 tonnes.

b) For all possible integer a, there exist a real number x such that $x = \sqrt{a}$.

c) If a number has a square root then it is positive.

d) i) If the final digit of n is 5 then it is divisible by 5.

ii) $P \Leftrightarrow Q$: false. Consider 20. 20 is divisible by 5 but the last digit is zero (Not 5).

e) i) $$2\underset{\sim}{b} - \underset{\sim}{a} = 2\begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 0 \\ -10 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -7 \end{pmatrix}$$

$$= 3\underset{\sim}{i} - 4\underset{\sim}{j} - 7\underset{\sim}{k}$$

$\boxed{Q1}$

e) ii) $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \dfrac{i + 4j - 3k}{\sqrt{1^2 + 4^2 + (-3)^2}}$

$$\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{26}} \left(\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}\right)$$

Vector in same direction as $\underset{\sim}{a}$ with 4 units in length $= \dfrac{4}{\sqrt{26}}\left(\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}\right)$

f) The statement is false

Eg if $x = 0$ $\therefore xy \neq 6$.

g) <u>Contrapositive</u>:

If a is even then $a^2 - 2a + 7$ is odd.

Let $a = 2k$, $k \in \mathbb{Z}^+$

$$\begin{aligned} a^2 - 2a + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 2(2k^2 - 2k + 3) + 1 \\ &= 2(M) + 1 \quad \therefore \text{odd} \end{aligned}$$

$\therefore$ Contrapositive is true

Thus original is true.

Question 2 a)

$\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

i) $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$= (1 \times 1) + (-1 \times -2) + (2 \times 3)$

$= 1 + 2 + 6$

$\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = 9$ ✓ (1)

ii) $|\underset{\sim}{a}| = \sqrt{1^2 + (-1)^2 + 2^2}$

$\therefore |\underset{\sim}{a}| = \sqrt{6}$ ✓ (1)

iii) $|\underset{\sim}{b}| = \sqrt{1^2 + (-2)^2 + 3^2}$

$|\underset{\sim}{b}| = \sqrt{14}$

$\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$9 = \sqrt{6} . \sqrt{14} \cos\theta$ ✓

$\cos\theta = \dfrac{9}{\sqrt{6}.\sqrt{14}}$

$\theta = \cos^{-1} \dfrac{9}{\sqrt{84}}$

$\theta = 11^\circ$ (nearest degree) ✓ (2)

* Any numerical errors made in parts i) and ii) were carried on for part iii)

Question 2b)

Show by induction $2^n > n^2$, $n \geq 5$, $n \in \mathbb{Z}$

For $n=5$: $LHS = 2^n = 2^5 = 32$ $\quad$ $RHS = n^2 = 5^2 = 25$

$LHS > RHS$

$\therefore$ true for $n=5$. ✓

Assume $2^k > k^2$, $k \geq 5$, $k \in \mathbb{Z}$

RTP: $2^{k+1} > (k+1)^2$

Proof:
$$\begin{aligned} LHS &= 2^{k+1} \\ &= 2(2^k) \\ &> 2(k^2) \text{ (by assumption)} ✓ \\ &= k^2 + (k^2) \\ &> k^2 + (2k+1) \text{ (for } k \geq 5) \\ &= (k+1)^2 \end{aligned}$$

*Justification was required at this step of proof.

[Since: $(k-1)^2 > 0$
$k^2 - 2k + 1 > 0$
$k^2 > 2k - 1$
i.e. $k^2 > 2k+1$].

$\therefore 2^{k+1} > (k+1)^2$

$\therefore$ true for $n = k+1$. ✓

Hence, true for $n=k+1$ iff true for $n=k$.
Since true for $n=5$, true $\forall n \geq 5$ by Principle of Mathematical Induction.

3.

* A proof where $LHS - RHS > 0$ was used:

$$\begin{aligned} & LHS - RHS \\ &= 2^{k+1} - (k+1)^2 \\ &= 2(2^k) - (k+1)^2 \\ &> 2k^2 - k^2 - 2k - 1 \text{ (by assumption)} ✓ \\ &= k^2 - 2k - 1 \end{aligned}$$

*justification required here.

$$\begin{aligned} &= (k-1)^2 - 2, \text{ since } k \geq 5 \\ &> 0 \end{aligned}$$

[$\Rightarrow k \geq 5$
$(k-1) \geq 4$
$(k-1)^2 \geq 16$
$\therefore (k-1)^2 \geq 2$] ✓

i.e. $2^{k+1} - (k+1)^2 > 0$

$\therefore 2^{k+1} > (k+1)^2$.

Question 2c)

Show by induction that $3^n + 7^n$ is divisible by 10, if n is odd, $n \in \mathbb{Z}$

For $n=1$: $3^1 + 7^1 = 10$, which is divisible by 10.

$\therefore$ true for $n=1$.

Assume: $3^k + 7^k = 10M$ $(M \in \mathbb{Z},\ k \text{ is odd})$ ✓

i.e. $3^k = 10M - 7^k$.

RTP: $3^{k+2} + 7^{k+2} = 10N$ $(N \in \mathbb{Z})$

Proof: $\text{LHS} = 3^{k+2} + 7^{k+2}$

$= 3^2 . (3^k) + 7^2 . (7^k)$ ✓

$= 9(10M - 7^k) + 49(7^k)$ (by assumption)

$= 90M - 9(7^k) + 49(7^k)$

$= 90M + 40(7^k)$

$= 10(9M + 4(7^k))$

$= 10N$ (since $(9M + 4(7^k)) \in \mathbb{Z}$)

$= \text{RHS}$.

$\therefore$ true for $n = k+1$. ✓

$\therefore$ By Principle of Mathematical Induction, $3^n + 7^n$ is divisible by 10, $\forall$ odd, $n \in \mathbb{Z}$.

3

* Mostly completed well ☺

Question 2d)

Let a be a rational number and b be an irrational number

★ It was common to state the irrational number as a surd. Eg $\sqrt{6}$. This caused confusion.

Assume by way of contradiction that the sum $(a+b)$ is rational.

★ Students needed to state a clear Assumption statement for improvement.

That is:

If $a = \frac{p}{q}$ and $(a+b) = \frac{r}{s}$

(where $p, q, r, s \in \mathbb{Z}$, $HCF(p,q)=1$, $q \neq 0$ and $HCF(r,s)=1$, $s \neq 0$).

Then $\frac{p}{q} + b = \frac{r}{s}$

① mark at this point was awarded.

Make b the subject:

$$b = \frac{r}{s} - \frac{p}{q}$$

$$b = \frac{qr - ps}{sq}$$

$\therefore$ b is rational. That is, b can be written in the form $\frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$, $HCF(m,n)=1$.

But b is irrational (given).

$\therefore$ Contradiction.

That is, the sum of a rational and irrational number is irrational.

②.

★ The contradiction (Conclusion) was written well by the students who had a clear assumption.

Q3

a) i) $\overrightarrow{AB} = \overrightarrow{OC}$ (from the diagram)

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$\overrightarrow{BF} = \overrightarrow{OD}$ (from the diagram)

$$\overrightarrow{BF} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

ii) $$\overrightarrow{OB} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OF} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

iii) $$\cos\theta = \frac{\overrightarrow{OB} \cdot \overrightarrow{OF}}{|\overrightarrow{OB}| \cdot |\overrightarrow{OF}|}$$

$$\cos\theta = \frac{16 + 4}{\sqrt{20} \cdot \sqrt{36}}$$

$$\theta = 42^\circ$$

b) $A(-1, 4, -5)$ $B(1, 1, 0)$ $C(5, -5, 10)$

$$\overrightarrow{AB} = \begin{pmatrix} 1+1 \\ 1-4 \\ 0+5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 5-1 \\ -5-1 \\ 10-0 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 10 \end{pmatrix}$$

$\overrightarrow{BC} = 2\overrightarrow{AB}$

Thus $\overrightarrow{AB} \parallel \overrightarrow{BC}$ and these two vectors have common point B.

Therefore, A, B, C are Collinear.

Q3

c) i) $x + \frac{1}{x} \geqslant 2$

$$\left(\sqrt{x} - \sqrt{\frac{1}{x}}\right)^2 \geqslant 0$$

$$x - 2 + \frac{1}{x} \geqslant 0$$

$x + \frac{1}{x} \geqslant 2$ as required.

ii) $(x-y)^2 \geqslant 0$

$$x^2 - 2xy + y^2 \geqslant 0$$

$$x^2 + y^2 \geqslant 2xy$$

Divide by xy

$$\frac{x}{y} + \frac{y}{x} \geqslant 2$$

$$1 + \frac{x}{y} + \frac{y}{x} + 1 \geqslant 4$$

$(x+y)\left(\frac{1}{x} + \frac{1}{y}\right) \geqslant 4$ as required.

QUESTION 4

(a) Step 1: Prove true for $n=1$.

Given: $T_1 = 6$

By formula $T_1 = 2^1 + 4\times 1$

$= 6$

$\therefore$ Formula works for $n=1$

Step 2: Assume true for $n=k$ where $k \in \mathbb{Z}^+$.

i.e. assume $T_k = 2^k + 4k$.

Step 3: Prove true for $n=k+1$

i.e. prove $T_{k+1} = 2^{k+1} + 4(k+1)$.

$$\begin{aligned} LHS &= T_{k+1} \\ &= T_k + 2^k + 4 \quad \text{(by known formula)} \\ &= 2^k + 4k + 2^k + 4 \quad \text{(using assumed formula)} \\ &= 2\times 2^k + 4k + 4 \\ &= 2^{k+1} + 4(k+1) \\ &= RHS \end{aligned}$$

If true for $n=k$, then true for $n=k+1$.

Since true for $n=1$, true by mathematical induction for all positive integers n.

Generally, well done by most. However, some students got confused with the assumption and assumed the known formula as part of Step 2.

Also, Step 1 - prove for $n=1$ not $n=2$.

The proof in Step 3 needs to show all key steps (including the highlighted part).

Question 4 (continued)

(b) (i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= -\underset{\sim}{i} + 3\underset{\sim}{j} - (-3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$

$= 2\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD}$

$= 2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k} - (3\underset{\sim}{k})$

$= 2\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$

$\therefore \overrightarrow{AB} = \overrightarrow{DC}$

(ii) Since $\overrightarrow{AB} = \overrightarrow{DC}$ then $|\overrightarrow{AB}| = |\overrightarrow{DC}|$

and $\overrightarrow{AB} \parallel \overrightarrow{DC}$.

$\therefore$ ABCD is a parallelogram
(one pair of sides is both equal and parallel)

(i) Most students showed effectively that $\overrightarrow{AB} = \overrightarrow{DC}$.
However, some students just wrote down the vector of $\overrightarrow{AB}$ and $\overrightarrow{DC}$ without showing how they got their answer. Students were expected (for 2 marks) to show the highlighted lines.

(ii) Not particularly well done. Students did not identify the correct type of quadrilateral and many students did not clearly justify why it was a parallelogram.
Whilst some students found the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$, this was not the most efficient approach.

Question 4 (continued)

(c) (i) Clearly shaded area < area of rectangle ABCD

Shaded area $= \int_1^{\sqrt{p}} \frac{1}{x}\,dx$

Area of rectangle $= (\sqrt{p}-1)\times 1$
$= \sqrt{p}-1$

$\therefore \int_1^{\sqrt{p}} \frac{1}{x}\,dx < \sqrt{p}-1$

Students needed to relate the inequality to area. Students should show explicit working for the area of ABCD.

(ii) Using (i) result:

$$\left[\ln x\right]_1^{\sqrt{p}} < \sqrt{p}-1$$
$$\ln\sqrt{p} - \ln 1 < \sqrt{p}-1$$
$$\ln p^{\frac{1}{2}} - 0 < \sqrt{p}-1$$
$$\frac{1}{2}\ln p < \sqrt{p}-1$$
$$\ln p < 2\sqrt{p}-2$$

Since $p>1$, $\ln p > 0$. $\therefore\ 0 < \ln p < 2\sqrt{p}-2$

Most students successfully integrated and evaluated and used log laws. However, students did not communicate in general why the zero value was part of the integral.

(iii) Using (ii) result:

$$\frac{0}{p} < \frac{\ln p}{p} < \frac{2\sqrt{p}}{p} - \frac{2}{p}$$

i.e. $$0 < \frac{\ln p}{p} < \frac{2}{\sqrt{p}} - \frac{2}{p}$$

Since $\lim_{p\to\infty}\left(\frac{2}{\sqrt{p}}-\frac{2}{p}\right)=0$ then $\lim_{p\to\infty}\frac{\ln p}{p} = 0$.

Many students did not make use of (ii) where $p>1$ result and were not able to provide a convincing argument for the result of interest.

Question 4 (continued)

(d) RTP: $\sqrt[8]{8!} < \sqrt[9]{9!}$

$$(8!)^9 < (9!)^8$$

$$8!^8 \times 8! < 8!^8 \times 9^8$$

Proof: $1 < 9,\ 2 < 9, \ldots\ldots, 8 < 9$

Hence $1 \times 2 \times \ldots \times 8 < 9 \times 9 \times \ldots \times 9$

i.e. $8! < 9^8$

$$(8!)^8 \times 8! < (8!)^8 \times 9^8$$

$$(8!)^9 < (8! \times 9)^8$$

$$(8!)^9 < (9!)^8$$

$$(8!)^{\frac{9}{72}} < (9!)^{\frac{8}{72}}$$

$$\therefore \sqrt[8]{8!} < \sqrt[9]{9!}$$

Many students did not make any significant progress in this question. Ensure that all steps are clearly shown.

Question 5 a)

$$\sum_{r=1}^{n} (-1)^{r-1} r^2 = (-1)^{n-1}\left[\frac{n(n+1)}{2}\right], \quad n \geq 1$$

$$S(n) = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} n^2 = (-1)^{n-1}\left[\frac{n(n+1)}{2}\right]$$

For $n=1$: LHS $= (-1)^{1-1}(1)^2$ $\quad$ RHS $= (-1)^{1-1}\left[\frac{1(1+1)}{2}\right]$

$= 1$ $\quad\quad$ $= \frac{2}{2} = 1$

* Substitution of $n=1$ in LHS and RHS needs to be shown.

$\therefore$ LHS = RHS. True for $n=1$. ✓

Assume true for $n=k$, $k \geq 1$:

$$S(k): 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k-1} k^2 = (-1)^{k-1}\left[\frac{k(k+1)}{2}\right].$$

Prove true for $n = k+1$. That is prove:

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k-1} k^2 + (-1)^k (k+1)^2 = (-1)^k\left[\frac{(k+1)(k+2)}{2}\right]$$

$$\begin{aligned}
LHS &= S_k + T_{k+1} \\
&= (-1)^{k-1}\left[\frac{k(k+1)}{2}\right] + (-1)^k (k+1)^2 \quad ✓ \\
&= (-1)^k \cdot (-1)^{-1} \cdot \left(\frac{k(k+1)}{2}\right) + (-1)^k (k+1)(k+1) \\
&= (-1)^k \cdot \frac{(k+1)}{2}\left[-k + 2(k+1)\right] \\
&= (-1)^k \cdot \frac{(k+1)}{2}(-k + 2k + 2) \quad ✓ \\
&= (-1)^k \left[\frac{(k+1)(k+2)}{2}\right] \\
&= RHS.
\end{aligned}$$

* Carefully shown factorisation was required for the final mark.

3.

$\therefore$ $S(k+1)$ is true.

Since $S(k+1)$ is true iff $S(k)$ is true.

Then by the Principle of Mathematical Induction $S(n)$ is true for all positive integers $n \geq 1$.

Question 5 b)

$\underset{\sim}{a} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -4\underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k}$

Let $\underset{\sim}{w} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ be a unit vector that is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Using the fact that $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ when $\underset{\sim}{a} \perp \underset{\sim}{b}$

$\underset{\sim}{a} \cdot \underset{\sim}{w} = 0$: $\quad 2x + 4y - 3z = 0 \quad \cdots (1)$

$\underset{\sim}{b} \cdot \underset{\sim}{w} = 0$: $\quad -4x - 5y + 3z = 0 \quad \cdots (2)$ ✓

$\underset{\sim}{w}$ is a unit vector: $\sqrt{x^2 + y^2 + z^2} = 1$

$\therefore x^2 + y^2 + z^2 = 1 \quad \cdots (3)$

Solving simultaneously:

(1) + (2) : $\quad -2x - y = 0$

$\therefore y = -2x \quad \cdots (4)$

(1) × 2 + (2) : $\quad 3y - 3z = 0$ ✓

$3y = 3z$

$\therefore y = z \Rightarrow z = -2x \quad \cdots (5)$

Sub (4) and (5) into (3): $\quad x^2 + (-2x)^2 + (-2x)^2 = 1$

$9x^2 = 1$

$x = \pm \frac{1}{3}$

when $x = \frac{1}{3}$: $\quad \underset{\sim}{w} = \frac{1}{3}\underset{\sim}{i} - \frac{2}{3}\underset{\sim}{j} - \frac{2}{3}\underset{\sim}{k}$

$= \frac{1}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}\right) = \frac{1}{3}\begin{pmatrix}1\\-2\\-2\end{pmatrix}$.

when $x = -\frac{1}{3}$: $\quad \underset{\sim}{w} = -\frac{1}{3}\underset{\sim}{i} + \frac{2}{3}\underset{\sim}{j} + \frac{2}{3}\underset{\sim}{k}$

$= -\frac{1}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}\right) = -\frac{1}{3}\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ✓

* Marked generously.
* A mark was lost if a unit vector was not given as a final answer.
* Correct process but incorrect vector was 2 marks.

3

Question 5c)

Let $f(x) = x$ $\quad$ $g(x) = \ln x + 1$

$f(1) = 1$ $\quad$ $g(1) = \ln 1 + 1 = 1$

$\Rightarrow$ $f(x)$ is a tangent to $g(x)$ at $x = 1$

$\Rightarrow$ point of intersection is $(1, 1)$. ✓

Now $f'(x) = 1$ $\quad$ $g'(x) = \frac{1}{x}$.

Consider cases:

For $x > 1$: $\quad f'(x) > g'(x)$

Since both functions start at $y = 1$

$$f(x) > g(x)$$

$$\therefore \quad x > \ln x + 1.$$ ✓

For $0 < x < 1$: $\quad f'(x) < g'(x)$

since $g(x)$ is increasing faster than $f(x)$ but they both arrive at $y = 1$ (when $x = 1$) then

$$g(x) < f(x)$$

$$\ln x + 1 < x$$

i.e. $\quad x > \ln x + 1$. ✓

Hence $\quad x \geqslant \ln x + 1 \quad \forall x > 0$.

3

* The majority of students only considered the case for $x > 1$, but mistakenly assumed that it held $\forall x > 0$.

* Finding the point of intersection of the two functions was important for the solution.

Q5 c) Alternate Solution.

Consider $h(x) = x - \ln x - 1$

$$h'(x) = 1 - \frac{1}{x}$$

$$h''(x) = \frac{1}{x^2}$$

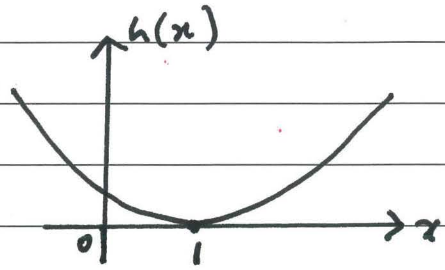
Since $h''(x) > 0 \ \forall x$, $h(x)$ is concave-up.

For minimum turning point $h'(x) = 0$:

$$1 - \frac{1}{x} = 0$$

$$\frac{x-1}{x} = 0$$

$$\therefore x = 1$$



when $x = 1$, $h(1) = 1 - \ln(1) - 1$

$h(1) = 0$.

Hence minimum turning point is $h(1) = 0$

$\Rightarrow (1, 0)$

That is: $h(x) \geq 0$ for $x \in [(-\infty, 1] \cup [1, \infty)]$

and $h'(x) > 0$, $x > 0$.

$\therefore \ x - \ln x - 1 \geq 0$

$x \geq \ln x + 1 \ \forall x > 0$.

3

Typo

Question 5 d).

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{1}{2}(a+b+c) \quad \Rightarrow \therefore 2s = (a+b+c)$$

Given $\sqrt{ab} \leq \dfrac{a+b}{2}$

Applying AM/GM inequality:

$$\sqrt{s(s-a).(s-b)(s-c)} \leq \frac{s(s-a)+(s-b)(s-c)}{2}$$

$$\leq \frac{s^2-as+s^2-bs-cs+bc}{2}$$

$$\leq \frac{2s^2-s(a+b+c)+bc}{2}$$

$$\leq \frac{2s^2-2s^2+bc}{2}$$

$$\therefore A \leq \frac{bc}{2} \quad \checkmark$$

Similarly $A \leq \dfrac{ca}{2}$ and $A \leq \dfrac{ab}{2}$ ✓

Adding $3A = \dfrac{ab+bc+ca}{2}$

$$A = \frac{ab+bc+ca}{6}$$

But $ab+bc+ca \leq a^2+b^2+c^2$

$$\therefore A \leq \frac{a^2+b^2+c^2}{6} \quad \checkmark$$

3

* Few students applied the AM/GM inequality and achieved full marks!

Question 5 d) Alternate Solution

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{\tfrac{1}{2}(a+b+c)\cdot\tfrac{1}{2}(a+b+c-2a)\cdot\tfrac{1}{2}(a+b+c-2b)\cdot\tfrac{1}{2}(a+b+c-2c)}$$

$$= \sqrt{\tfrac{1}{2}(b+c+a)\cdot\tfrac{1}{2}(b+c-a)\cdot\tfrac{1}{2}(a+c-b)\cdot\tfrac{1}{2}(a+b-c)}$$

$$= \sqrt{\left(\tfrac{1}{2}\right)^4(b+c+a)(b+c-a)(a+c-b)(a-(c-b))}$$

$$= \sqrt{\left(\tfrac{1}{2}\right)^4\left((b+c)^2-a^2\right)\left(a^2-(c-b)^2\right)}$$

$$= \sqrt{\left(\tfrac{1}{2}\right)^4\left(b^2+2bc+c^2-a^2\right)\left(a^2-c^2+2bc-b^2\right)}$$

$$\leq \frac{1}{4}\left(\frac{\cancel{b^2}+2bc+\cancel{c^2}-\cancel{a^2}+\cancel{a^2}-\cancel{c^2}+2bc-\cancel{b^2}}{2}\right).$$

$$\leq \frac{1}{4}\cdot\frac{4bc}{2}$$

$$\therefore A \leq \frac{bc}{2}$$

Continue as per previous method.

* Many students started with this approach to the question, and were awarded at least 1 mark.