Sydney Girls High School

LABOR OMNIA VINCIT

MATHEMATICS EXTENSION 2

YEAR 12

Assessment Task 2

2021

Reading time: 5 minutes

Writing time: 60 minutes

Total marks: 50

Topics: Complex Numbers and Vectors

General Instructions:

- There are four (4) questions.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write using a black or blue pen.
- White-out or correction tape is NOT to be used.
- A Mathematics reference sheet is provided.

Student Name:__________________________ **Teacher Name:**____________________

Question 1 [13 marks]

(a) The points A and B have the position vectors $2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ and $4\underset{\sim}{i} - 3\underset{\sim}{j} - 5\underset{\sim}{k}$ respectively. Find,

(i) the midpoint of AB [1]

(ii) $\overrightarrow{AB}$ [1]

(iii) $\left|\overrightarrow{AB}\right|$ [1]

(b) Find the monic polynomial of smallest degree with real coefficients such that $2 + 3i$ and 5 are zeros. [2]

(c) Use vectors to show that $A(-1, 7, 11)$, $B(1, 3, 5)$ and $C(4, -3, -4)$ are collinear. [2]

(d) For $z = -\sqrt{3} + i$.

(i) Find $|z|$ and $\arg(z)$. [2]

(ii) Hence evaluate z^9 and express your answer in simplest form. [2]

(e) Find the angle between the vector $\begin{bmatrix} \sqrt{5} \\ \sqrt{21} \\ -\sqrt{2} \end{bmatrix}$ and the y-axis. [2]

End of Question 1

Question 2 [13 marks]

Start on a NEW page.

(a)

(i) Find the vector equation of the interval between $P(5, -8, 3)$ and $Q(-2, 3, -1)$ in the form $\underset{\sim}{r} = \underset{\sim}{p} + \lambda \underset{\sim}{d}$ where $0 \leq \lambda \leq 1$. [2]

(ii) Find the point R on the line $\underset{\sim}{r}$ when $\lambda = 4$. [1]

(iii) Hence, find the ratio $PQ : QR$. [1]

(b) Find the three cube roots of $2 - 2i$ in exponential form. [2]

(c) Two lines have vector equations,

$$\underset{\sim}{r_1} = \begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \quad \underset{\sim}{r_2} = \begin{bmatrix} 5 \\ 1 \\ 12 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$$

(i) Show that these lines are parallel and distinct. [2]

(ii) Let A be the point on $\underset{\sim}{r_1}$ when $\lambda = 0$ and B be the point on $\underset{\sim}{r_2}$ when $\mu = 0$, find the vector $\overrightarrow{AB}$. [1]

(iii) Find the projection of $\overrightarrow{AB}$ onto the direction vector of $\underset{\sim}{r_1}$. [2]

(iv) Hence, find the distance between the lines $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$. [2]

End of Question 2

Question 3 [12 marks]

Start on a NEW page.

(a)

(i) Given that $e^{i\theta} = \cos\theta + i\sin\theta$ and n is an integer show that, [1]

$$e^{in\theta} + e^{-in\theta} = 2\cos n\theta$$

(ii) Let $\omega = e^{i\frac{2\pi}{5}}$, show that $1, \omega, \omega^{-1}, \omega^2, \omega^{-2}$ are the five zeros of the expression $z^5 - 1$. [1]

(iii) Prove that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. [2]

(iv) Using parts (i) and (ii) deduce that a factorisation is [2]

$$z^5 - 1 = (z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$$

(v) Another factorisation is $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$, hence conclude that [3]

$$\sin\frac{\pi}{10}\cos\frac{4\pi}{5} = -\frac{1}{4}$$

(b) Find the Cartesian equation and its domain, of the curve with vector equation, [2]

$$\underset{\sim}{r} = 2^t\underset{\sim}{i} + (t^2+1)\underset{\sim}{j} \text{ where } t \in (-\infty, \infty).$$

(c) Find the value of θ, such that $-\pi < \theta \leq \pi$, that maximises the value of $\left|e^{i\theta} - 3 - \sqrt{3}i\right|$. [1]

End of Question 3

Question 4 [12 marks]

Start on a NEW page.

(a) Consider the equation $(z-1)^4+(z+1)^4=0$.

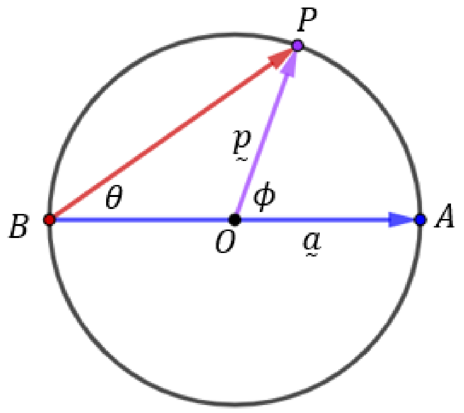
(i) Show that the equation can be written as $w^4+1=0$ by letting $w=\frac{z-1}{z+1}$. [1]

(ii) Find the four values of θ so that $w=\text{cis}\,\theta$ is a solution to the equation $w^4+1=0$. [3]

(iii) Given that θ takes one of the solution values, by equating the two expressions for w from part (i) and (ii), write z in terms of θ. [1]

(iv) Hence, by expressing z in terms of $\frac{\theta}{2}$ and then letting θ take its four values write the four solutions of $(z-1)^4+(z+1)^4=0$ in terms of **cot**angents. [3]

(b) O is the centre and AB is a diameter of the circle with radius of length r. Let $\overrightarrow{OA}=\underset{\sim}{a}$, $\overrightarrow{OP}=\underset{\sim}{p}$, $\angle ABP=\theta$ and $\angle AOP=\phi$.



(i) Find $\overrightarrow{BP}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{p}$. [1]

(ii) By considering the two dot products $\underset{\sim}{a}\bullet\underset{\sim}{p}$ and $\overrightarrow{BA}\bullet\overrightarrow{BP}$, prove that $\phi=2\theta$. [3]

End of Question 4

End of Task

X2 HSC TASK 2, 2021 : SOLUTIONS

Question 1

(a) (i) Let M be the midpoint of AB.

$$\vec{OM} = \frac{\vec{OA} + \vec{OB}}{2}$$

$$= \frac{1}{2}(2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k} + 4\underset{\sim}{i} - 3\underset{\sim}{j} - 5\underset{\sim}{k})$$

$$= 3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}$$

$$\therefore \quad M = (3, -1, -4)$$

The question asked for the midpoint (not a vector) so the answer should reflect that. $3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}$ was awarded the mark but $(3,-1,-4)$ is the expected form of the answer.

(ii) $$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= 4\underset{\sim}{i} - 3\underset{\sim}{j} - 5\underset{\sim}{k} - (2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k})$$

$$= 2\underset{\sim}{i} - 4\underset{\sim}{j} - 2\underset{\sim}{k}$$

(iii) $$|\vec{AB}| = \sqrt{2^2 + (-4)^2 + (-2)^2}$$

$$= \sqrt{24}$$

i.e. $|\vec{AB}| = 2\sqrt{6}$

(ii) and (iii) mostly well done.

Q1 (continued)

(b) Since the coefficients are real, $2-3i$ must also be a root.

$\therefore$ Polynomial $= (z-5)\underline{(z-(2+3i))(z-(2-3i))}$

↳ sum of roots $= 4$

product of roots $= 4+9$

$= 13$

$= (z-5)(z^2-4z+13)$

$= z^3-5z^2-4z^2+20z+13z-65$

$= z^3-9z^2+33z-65$ (monic as required)

Many students did not recognise that complex roots come in conjugate pairs when the coefficients are all real. The final answer should have shown a polynomial with real coefficients.

(c) $\vec{AB} = \vec{OB} - \vec{OA}$

$= \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} -1 \\ 7 \\ 11 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}$

$\vec{AC} = \vec{OC} - \vec{OA}$

$= \begin{bmatrix} 4 \\ -3 \\ -4 \end{bmatrix} - \begin{bmatrix} -1 \\ 7 \\ 11 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ -10 \\ -15 \end{bmatrix}$

$= \frac{5}{2}\begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}$

$\vec{AC} = \frac{5}{2}\vec{AB}$ $\therefore \vec{AC} \parallel \vec{AB}$

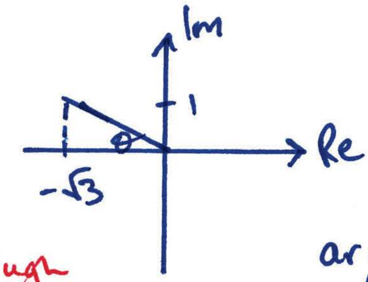
Since $\vec{AC}$ and $\vec{AB}$ have A as a common point, A, B and C are collinear.

The reasoning provided by many students was on the minimal side and should have included better explanation. Some students found the equation of line AB and showed that C satisfied the equation.

Q1

(d) (i) $|z| = \sqrt{(\sqrt{3})^2 + 1^2}$

$|z| = 2$



$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$

$\arg(z) = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

Generally, well done though some students made errors for the argument, particulaly writing $-\frac{\pi}{6}$ as the argument. A quick diagram will help to prevent this mistake.

(ii) $z^9 = \left(2 \operatorname{cis} \frac{5\pi}{6}\right)^9$

$= 2^9 \operatorname{cis} \frac{45\pi}{6}$

$= 512 \operatorname{cis} \frac{3\pi}{2}$

$\therefore z^9 = -512i$

A number of students did not simplify at the end.

(e) A vector in the direction of the y-axis is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore \angle$ between y-axis and the given vector $= \cos^{-1}\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$

$= \cos^{-1}\left(\frac{\sqrt{5}\times 0 + \sqrt{21}\times 1 + (-\sqrt{2})\times 0}{\sqrt{(\sqrt{5})^2 + (\sqrt{21})^2 + (-\sqrt{2})^2} \times 1}\right)$

$= \cos^{-1}\left(\frac{\sqrt{21}}{\sqrt{28}}\right)$

$= \cos^{-1}\frac{\sqrt{3}}{2}$

$= \frac{\pi}{6}$ or 30°

Most students successfully calculated the angle. However, some did not really know how to begin.

Question 2 [13 marks]

Start on a NEW page.

(a)

(i) Find the vector equation of the interval between $P(5,-8,3)$ and $Q(-2,3,-1)$ in the form $\underset{\sim}{r} = \underset{\sim}{p} + \lambda \underset{\sim}{d}$ where $0 \le \lambda \le 1$. [2]

(ii) Find the point R on the line $\underset{\sim}{r}$ when $\lambda = 4$. [1]

(iii) Hence, find the ratio $PQ : QR$. [1]

i) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 5 \\ -8 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} -7 \\ 11 \\ -4 \end{bmatrix}$ ✓

$= \underset{\sim}{d}$

$\underset{\sim}{r} = \underset{\sim}{p} + \lambda \underset{\sim}{d}$

$\underset{\sim}{r} = \begin{bmatrix} 5 \\ -8 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -7 \\ 11 \\ -4 \end{bmatrix}$ ✓

OR

$\underset{\sim}{r} = (5-7\lambda)\underset{\sim}{i} + (-8+11\lambda)\underset{\sim}{j} + (3-4\lambda)\underset{\sim}{k}$

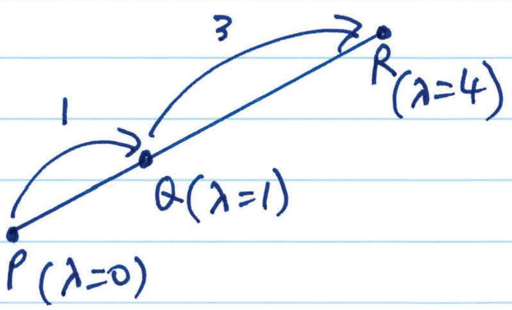
where $0 \le \lambda \le 1$

ii) when $\lambda = 4$: $R = (5 - 4\times 7,\ -8 + 4\times 11,\ 3 - 4\times 4)$

$\underline{\underline{R = (-23,\ 36,\ -13)}}$ ✓

* R is a point - Many students gave their answer as a column vector!

iii)



$PQ : QR$

$\underline{\underline{1 : 3}}$ ✓

* Simplest approach was to think of λ and its value at each point P, Q, R.

Question 2 - Continued.

(b) Find the three cube roots of $2 - 2i$ in exponential form. [2]

$|2-2i| = \sqrt{4+4} = 2\sqrt{2} = 2^{3/2}$

$\arg(2-2i) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\therefore 2-2i = 2^{3/2} e^{(-\pi/4 + 2k\pi)i}$

Let $z = re^{i\theta}$ be a cube root.

then $z^3 = 2-2i$

gives: $r^3 e^{i3\theta} = 2^{3/2} e^{(-\pi/4 + 2k\pi)i}$ (by De Moivres).

Equating moduli + arguments:

$r^3 = 2^{3/2}$ $\qquad$ $3\theta = -\frac{\pi}{4} + 2k\pi$

$\therefore r = \sqrt{2}$ $\qquad$ $\theta = -\frac{\pi}{12} + \frac{2k\pi}{3}$ ✓

Finding the modulus was fairly tricky for students.

using principal arguments:

$k=0: \theta = -\frac{\pi}{12}$

$k=1: \theta = \frac{7\pi}{12}$

$k=-1: \theta = -\frac{9\pi}{12} = -\frac{3\pi}{4}$.

$\therefore$ roots are:

$z = \sqrt{2} e^{-\pi/12\, i}$ or $z = \sqrt{2} e^{7\pi/12\, i}$ or $z = \sqrt{2} e^{-3\pi/4\, i}$.

✓.

Question 2 – continued

(c) Two lines have vector equations,

label lines

$$\underset{\sim}{r_1} = \begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \quad \underset{\sim}{r_2} = \begin{bmatrix} 5 \\ 1 \\ 12 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$$

$r_1 = a_1 + \lambda d_1 \quad r_2 = a_2 + \mu d_2$

(i) Show that these lines are parallel and distinct. [2]

(ii) Let A be the point on $\underset{\sim}{r_1}$ when $\lambda = 0$ and B be the point on $\underset{\sim}{r_2}$ when $\mu = 0$, find the vector $\overrightarrow{AB}$. [1]

i) Comparing direction vectors:

$$d_2 = \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} = -2\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = -2d_1$$

since $d_2 = -2d_1$, lines are parallel.

Test when $\lambda = 0$ to find a consistent value for μ, when a_1 lies on r_2:

$9 = 5 + 4\mu \Rightarrow \mu = 1$

$2 = 1 + 4\mu \Rightarrow \mu = \frac{1}{4}$

$-5 = 12 - 2\mu \Rightarrow \mu = 8\frac{1}{2}$.

since there is an inconsistent value for μ, then the lines are distinct.

*Simply stating that the lines began at a different point was not sufficient to show that the lines were distinct.

ii) $A = (9, 2, -5)$

$B = (5, 1, 12)$

$$\overrightarrow{AB} = \begin{bmatrix} 5 \\ 1 \\ 12 \end{bmatrix} - \begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix}$$

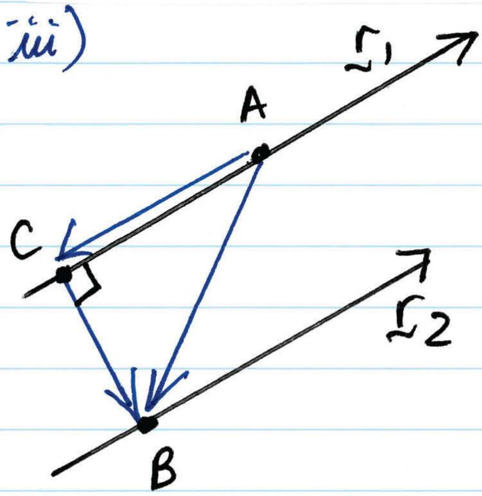
$$\therefore \overrightarrow{AB} = \begin{bmatrix} -4 \\ -1 \\ 17 \end{bmatrix}$$

Question 2 – Continued.

(iii) Find the projection of $\overrightarrow{AB}$ onto the direction vector of $\underset{\sim}{r}_1$. [2]

(iv) Hence, find the distance between the lines $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$. [2]

iii)



$$\text{proj}_{\underset{\sim}{d}_1} \overrightarrow{AB}$$

$$= \frac{(-2,-2,1)\cdot(-4,-1,17)}{|(-2,-2,1)|} \times \begin{bmatrix}-2\\-2\\1\end{bmatrix}$$

$$= \frac{8+2+17}{4+4+1} \times \begin{bmatrix}-2\\-2\\1\end{bmatrix}$$

$$= \frac{27}{9}\begin{bmatrix}-2\\-2\\1\end{bmatrix}$$

$$= 3\begin{bmatrix}-2\\-2\\1\end{bmatrix}$$

$$= \begin{bmatrix}-6\\-6\\3\end{bmatrix}$$

iv) Let distance between $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ be $|\overrightarrow{CB}|$

$$|\overrightarrow{CB}| = |\overrightarrow{AB} - \text{proj}_{\underset{\sim}{d}_1}\overrightarrow{AB}|$$

$$= \left|\begin{bmatrix}-4\\-1\\17\end{bmatrix} - \begin{bmatrix}-6\\-6\\3\end{bmatrix}\right|$$

$$= \left|\begin{bmatrix}2\\5\\14\end{bmatrix}\right|$$

$$= \sqrt{2^2+5^2+14^2}$$

$$= \sqrt{225}$$

$$\therefore |d| = 15$$

* Vector subtraction had to include the answer from part iii) for this mark.

Question 3 (12 marks)

(a) (i) $e^{in\theta} + e^{-in\theta} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ **1**

$$= \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}$$
$$= 2\cos n\theta$$

(ii) If k is an integer, then $(\omega^k)^5 - 1 = \left(\left(e^{i\frac{2\pi}{5}}\right)^5\right)^k - 1$ **1**

$$= (e^{2\pi i})^k - 1$$
$$= 1^k - 1$$
$$= 0.$$

Therefore ω^k is a zero of $z^5 - 1$ for all integers k, so in particular, $1, \omega, \omega^{-1}, \omega^2, \omega^{-2}$ are five zeroes of $z^5 - 1$.

(iii) From (ii), w^k is a zero of $z^5 - 1$ for all integers k.
Hence $1, \omega, \omega^2, \omega^3, \omega^4$ are five zeroes of $z^5 - 1$. **2**

The sum of the roots of $z^5 - 1 = 0$ is $-\dfrac{b}{a} = -\dfrac{0}{1} = 0$.

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$$

(iv) From (i), $\omega + \omega^{-1} = 2\cos\dfrac{2\pi}{5}$ and $\omega^2 + \omega^{-2} = 2\cos\dfrac{4\pi}{5}$. **2**

From (ii), $z^5 - 1$ can be factorised as

$$\begin{aligned} z^5 - 1 &= (z-1)(z-\omega)(z-\omega^{-1})(z-\omega^2)(z-\omega^{-2}) \\ &= (z-1)\left(z^2 - (\omega+\omega^{-1})z + 1\right)\left(z^2 - (\omega^2+\omega^{-2})z + 1\right) \\ &= (z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right) \end{aligned}$$

(v) Comparing the two factorisations for $z^5 - 1$ gives **3**

$$\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right) = z^4 + z^3 + z^2 + z + 1$$

Equating coefficients of z^2:

$$\begin{aligned} 1 + 4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} + 1 &= 1 \\ \cos\frac{2\pi}{5}\cos\frac{4\pi}{5} &= -\frac{1}{4} \\ \sin\left(\tfrac{\pi}{2} - \tfrac{2\pi}{5}\right)\cos\frac{4\pi}{5} &= -\frac{1}{4} \\ \sin\frac{\pi}{10}\cos\frac{4\pi}{5} &= -\frac{1}{4} \end{aligned}$$

(b) From $\underset{\sim}{r} = 2^t\underset{\sim}{i} + (t^2+1)\underset{\sim}{j}$ we have **2**

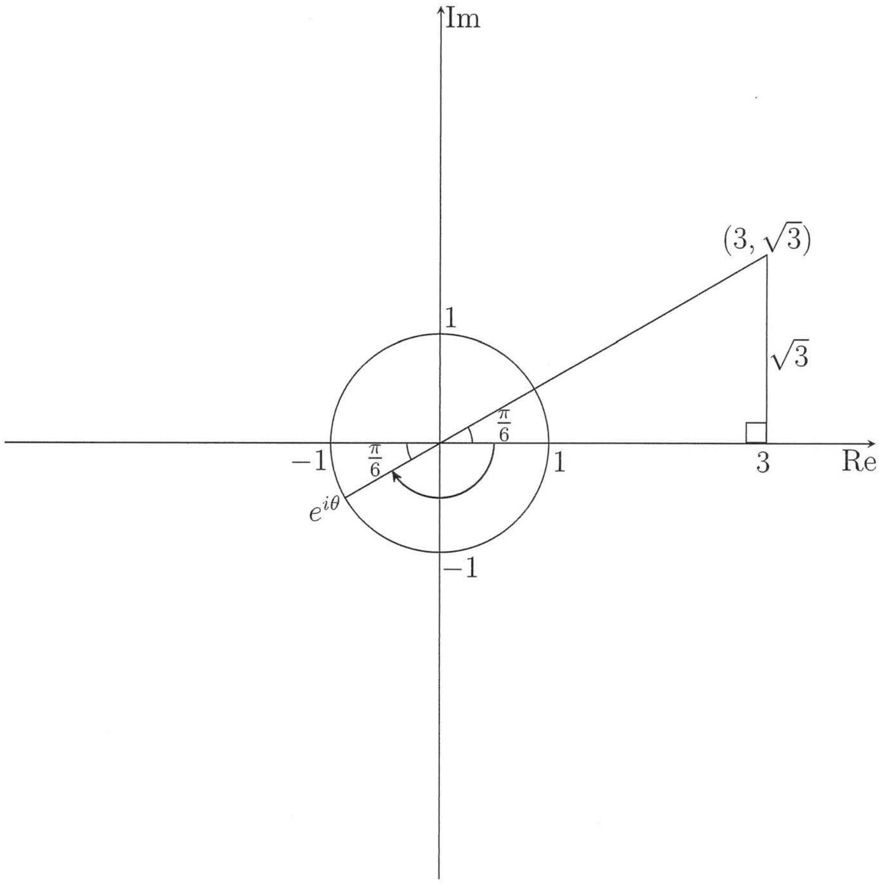
$$x = 2^t \qquad (1)$$
$$y = t^2 + 1 \qquad (2)$$

From (1), $t = \log_2 x$. Substitute this into (2):

$$y = (\log_2 x)^2 + 1\,\checkmark$$

Since $x = 2^t > 0$, the domain of the curve is $x > 0$. ✓

(c) Since $|e^{i\theta}| = 1$, the expression $|e^{i\theta} - 3 - \sqrt{3}|$ can be interpreted as the distance on the Argand diagram from $(3, \sqrt{3})$ to a point on the unit circle. **1**



This distance is maximised when $e^{i\theta}$ lies in the third quadrant on the line joining $(3, \sqrt{3})$ and the origin, as shown. The angle that this line makes with the positive real axis is $\tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$. The principal argument of $e^{i\theta}$ is $-(\pi - \frac{\pi}{6}) = -\frac{5\pi}{6}$. ✓

Notes on Question 3

(a)(i) This part was done very well. Students must include a step which shows use of $\cos(-n\theta) = \cos n\theta$ and $\sin(-n\theta) = -\sin n\theta$.

(a)(ii) There were two successful approaches, namely substitution or by finding the fifth roots of unity. Most responses were successful.

(a)(iii) In order to make progress, students either had to note that $w^{-1} = w^4$ and $w^{-2} = w^3$, or use the sum of the roots of $z^5 - 1 = 0$. Attempting to compute the sum directly resulted in $1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5}$, but students were unable to prove that the exact value of this expression was zero. (Note that using a calculator is insufficient, since the calculator only provides very accurate approximations. For example, your calculator will say $e^{10^{-10}} = 0$, which is false).

(a)(iv) Students should pay close attention to questions which begin "using parts (i) and (ii)" or say "hence". Marks can often be awarded for mundane uses of previous parts. For example, the result of (ii) is that $1, w, w^{-1}, w^2, w^{-2}$ are the zeroes of $z^5 - 1$, which in turn implies that $z^5 - 1$ can be factorised as $(z-1)(z-w)(z-w^{-1})(z-w^2)(z-w^{-2})$. This simple observation was worth 1 mark, but many students missed this. The remaining mark was given for *explicitly* showing how (i) was used to complete the factorisation.

(a)(v) A number of plausible approaches were awarded marks including sum of roots, substituting $z = 1$, or equating coefficients of z or z^3. However, the only approaches which were completely successful involved substituting $z = i$ or equating coefficients of z^2. To obtain the final mark, students must show why $\sin\frac{\pi}{10} = \cos\frac{2\pi}{5}$ using complementary angles.

(b) Note that $(\log_2 x)^2$ is NOT the same as $2\log_2 x$. This was a pervasive error which suggests that students need to revise logarithms to clarify their understanding. Students also need to take care when using interval notation as there was some confusion between open brackets (which *exclude* endpoints) and closed brackets (which *include* endpoints).

(c) This part was extremely challenging with almost no correct responses. For more practice with this style of question, study the solution above and then try Q9 from last year's Ext 2 HSC paper (2020).

Question 4

(a) (i) $(z-1)^4 + (z+1)^4 = 0$

Divide through by $(z+1)^4$.

$$\frac{(z-1)^4}{(z+1)^4} + \frac{(z+1)^4}{(z+1)^4} = \frac{0}{(z+1)^4}$$

$$\left(\frac{z-1}{z+1}\right)^4 + 1 = 0$$

Let $w = \frac{z-1}{z+1}$. Hence, $w^4 + 1 = 0$.

Better solutions started with the known equation in terms of z and manipulated it to the equation $w^4+1=0$. Many solutions could have been structured more effectively and clearly.

(ii) $(\operatorname{cis}\theta)^4 + 1 = 0$

$$\operatorname{cis} 4\theta = -1$$
$$= \operatorname{cis}(\pi + 2k\pi)$$

$4\theta = \pi + 2k\pi$ where $k = 0, 1, 2, 3$

$\theta = \frac{\pi + 2k\pi}{4}$ $\quad \therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\left(\text{or } \theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\right)$

Some students did not convert -1 to $\operatorname{cis}\pi$.

All four solutions were needed.

The final answers should have been θ values not w values (such as $\operatorname{cis}\frac{\pi}{4}, \ldots$).

Qn. 4 (continued)

(a) (iii) $w = \dfrac{z-1}{z+1}$

$w(z+1) = z-1$

$wz + w = z-1$

$w+1 = z - zw$

$z(1-w) = 1+w$

$\therefore \quad z = \dfrac{1+\text{cis}\,\theta}{1-\text{cis}\,\theta}$

This question was an algebraic manipulation one that students should have been able to do. However, there were quite a number of students who did not see what was needed.

(iv) using (iii) $z = \dfrac{1+\overset{\cos(2\times\frac{\theta}{2})}{\cos\theta} + i\overset{\sin(2\times\frac{\theta}{2})}{\sin\theta}}{1-\cos\theta - i\sin\theta}$

$= \dfrac{1+2\cos^2\frac{\theta}{2} - 1 + i\times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-(1-2\sin^2\frac{\theta}{2}) - i\times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \dfrac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)} \times \dfrac{\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}{\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}$

$= \cot\frac{\theta}{2}\left(\dfrac{\cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} + i\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)}{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}\right)$

$= \cot\frac{\theta}{2} \times \frac{i}{1}$

$z = i\cot\frac{\theta}{2}$

$\therefore$ The four solutions are $z = i\cot\frac{\pi}{8},\ i\cot\frac{3\pi}{8},\ i\cot\frac{5\pi}{8},\ i\cot\frac{7\pi}{8}$

Many students found this challenging and made minimal progress. A key step was to select the appropriate double angle formula. Also, realising the denominator and substituting θ values from (ii) were not well done.

Qn 4 (continued)

(b) (i) $\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP}$

$= \underset{\sim}{a} + \underset{\sim}{p}$

Well done by most students.

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{p} = |\underset{\sim}{a}||\underset{\sim}{p}| \cos \phi$

$\underset{\sim}{a} \cdot \underset{\sim}{p} = r^2 \cos \phi$ since $|\underset{\sim}{a}| = |\underset{\sim}{p}| =$ radius

(1)

$\overrightarrow{BA} \cdot \overrightarrow{BP} = |\overrightarrow{BA}||\overrightarrow{BP}| \cos \theta$

$\overrightarrow{BA} \cdot \overrightarrow{BP} = 2r\,|\underset{\sim}{a} + \underset{\sim}{p}| \cos \theta$ (2)

The vast majority of students made minimal progress. Writing down $\underset{\sim}{a} \cdot \underset{\sim}{p}$ and $\overrightarrow{BA} \cdot \overrightarrow{BP}$ was a necessary first step.

Also $\overrightarrow{BA} \cdot \overrightarrow{BP} = 2\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{p})$

$= 2\underset{\sim}{a} \cdot \underset{\sim}{a} + 2\underset{\sim}{a} \cdot \underset{\sim}{p}$

using (1) and (2)

$\therefore 2r\,|\underset{\sim}{a} + \underset{\sim}{p}| \cos \theta = 2|\underset{\sim}{a}|^2 + 2(r^2 \cos \phi)$ (*)

$2r\,|\underset{\sim}{a} + \underset{\sim}{p}| \cos \theta = 2r^2(1 + \cos \phi)$

Making a connection between $\cos \theta$ and $\cos \phi$ (*) was crucial.

$|\underset{\sim}{a} + \underset{\sim}{p}| \cos \theta = r(1 + \cos \phi)$

$|\underset{\sim}{a} + \underset{\sim}{p}|^2 \cos^2 \theta = r^2 (1 + \cos \phi)^2$

No marks were awarded for proofs that did not use the dot product.

$(\underset{\sim}{a} + \underset{\sim}{p}) \cdot (\underset{\sim}{a} + \underset{\sim}{p}) \cos^2 \theta = r^2 (1 + \cos \phi)^2$

$(\underset{\sim}{a} \cdot \underset{\sim}{a} + 2\underset{\sim}{a} \cdot \underset{\sim}{p} + \underset{\sim}{p} \cdot \underset{\sim}{p}) \cos^2 \theta = r^2 (1 + \cos \phi)^2$

$(|\underset{\sim}{a}|^2 + 2r^2 \cos \phi + |\underset{\sim}{p}|^2) \cos^2 \theta = r^2 (1 + \cos \phi)^2$ using (1)

$2r^2 (1 + \cos \phi) \cos^2 \theta = r^2 (1 + \cos \phi)^2$ since $|\underset{\sim}{a}| = |\underset{\sim}{p}| = r$

$2 \cos^2 \theta = 1 + \cos \phi$

$2 \cos^2 \theta - 1 = \cos \phi$

$\cos 2\theta = \cos \phi$

$\therefore 2\theta = \phi$