Sydney Girls High School

MATHEMATICS EXTENSION 2

Year 12

Assessment Task 3

2022

Reading time :	5 minutes
Writing time :	60 minutes
Total marks :	50
Topics:	Complex numbers II, Vectors, Integration

General Instructions:

- There are four (4) questions.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale, unless indicated otherwise.
- Board-approved calculators may be used.
- Write using a black or blue pen.
- White-out or correction tape is NOT to be used.
- A Mathematics reference sheet is provided.

Student Name:__________________________ **Teacher Name:**______________

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Question 1 **13 marks**

(a) Find $\int \frac{1}{x^2+4x+8}\,dx$. **2**

(b) Find $\int \frac{x}{x-1}\,dx$. **2**

(c) (i) Express $-1+i\sqrt{3}$ in mod-arg form. **2**

(ii) Hence, find $\left(-1+i\sqrt{3}\right)^6$ in the form $a+ib$, where a and b are real. **1**

(d) Use Euler's formula to simplify $e^{3i\theta}-e^{-3i\theta}$. **2**

(e) Consider the points $P(-1,\ 0,\ 3)$, $Q(2,\ 4,\ -2)$ and $R(-7,\ -8,\ 13)$.

(i) Find the distance PQ. **1**

(ii) Find the midpoint, M, of the interval PQ. **1**

(iii) Show that $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel. **2**

End of Question 1

Question 2 **12 marks**

(a) Find $\displaystyle\int \frac{e^{2x}}{e^{x}-1}\,dx$. **2**

(b) (i) Find integers A and B such that **2**

$$\frac{7x}{(x+3)(2x-1)} = \frac{A}{x+3}+\frac{B}{2x-1}.$$

(ii) Hence, find $\displaystyle\int \frac{7x}{(x+3)(2x-1)}\,dx$. **1**

(c) Suppose that $-1-i=e^{x+iy}$, where x and y are real and $-\pi < y \le \pi$. **2**
Find the exact values of x and y.

(d) Find the value of p, given that $p\underset{\sim}{i}-\underset{\sim}{k}$ is perpendicular to $(p+2)\underset{\sim}{i}-3\underset{\sim}{j}-\underset{\sim}{k}$. **2**

(e) Consider the points $A(-3, 4, 1)$ and $B(7, 5, 2)$.

(i) Write a vector equation of the line AB. **1**

(ii) Determine whether $C(17, 6, 5)$ lies on the line AB. **2**

End of Question 2

Question 3 **12 marks**

(a) Find $\displaystyle\int \frac{x}{\sqrt{10x - x^2}}\, dx$. **3**

(b) Find $\displaystyle\int \sin^{-1} x \; dx$. **2**

(c) Find $\displaystyle\int \tan^3 x \; dx$. **2**

(d) Suppose that $z \neq -1$ is a cube root of -1. **2**

Show that $\dfrac{1}{iz-1} - \dfrac{1}{iz+1} = 2z^2$.

(e) (i) The line ℓ has the following vector equation: **2**

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}.$$

Find the point on the line ℓ which is closest to the origin.

(ii) Using part (i), or otherwise, determine the value of a such that **1**

the line ℓ is a tangent to the sphere with equation $x^2 + y^2 + z^2 = a^2$.

End of Question 3

Question 4 **13 marks**

(a) Let α be the complex fifth root of unity with smallest positive argument.

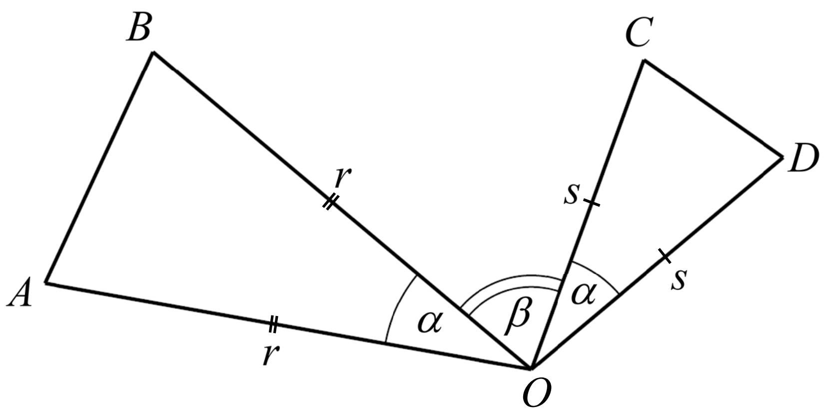
Let $u = \alpha + \alpha^4$ and $v = \alpha^2 + \alpha^3$.

(i) Show that $u + v = -1$. **1**

(ii) Given $u > v$ and the property $(u-v)^2 = (u+v)^2 - 4uv$, **2**
show that $u - v = \sqrt{5}$.

(iii) Deduce that $\cos\dfrac{2\pi}{5} = \dfrac{1}{4}\left(\sqrt{5}-1\right)$. **2**

(b) In the diagram below, ΔAOB and ΔCOD are isosceles with $OA = OB = r$ and $OC = OD = s$. Furthermore, $\angle AOB = \angle COD = \alpha$ and $\angle BOC = \beta$.
Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$ and $\overrightarrow{OD} = \underset{\sim}{d}$.



(i) Use the property $\left|\underset{\sim}{v}\right|^2 = \underset{\sim}{v}\cdot\underset{\sim}{v}$ to find expressions for $\left|\overrightarrow{AC}\right|$ and $\left|\overrightarrow{BD}\right|$ **2**
in terms of r, s, α and β.

(ii) Hence, by considering $\overrightarrow{AC}\cdot\overrightarrow{BD}$, show that the lines AC and BD **2**
meet at an angle of α.

Question 4 continues on the next page

Question 4 (continued)

(c) (i) The integral I_n is defined by **3**

$$I_n = \int_0^{2\pi} (1+\cos\theta)^n \, d\theta, \text{ for all integers } n \geq 0.$$

Show that $I_n = \dfrac{2n-1}{n} I_{n-1}$ for $n \geq 1$.

(ii) Hence, evaluate $\displaystyle\int_0^{2\pi} (1+\cos\theta)^3 \, d\theta$. **1**

End of paper

SOLUTIONS - Ext. 2 Task 3, 2022

Question 1

(a) $$\int \frac{1}{x^2+4x+8}\,dx = \int \frac{dx}{(x+2)^2+4} \leftarrow (1)$$

• Well done.

$$\therefore I = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C \leftarrow (1)$$

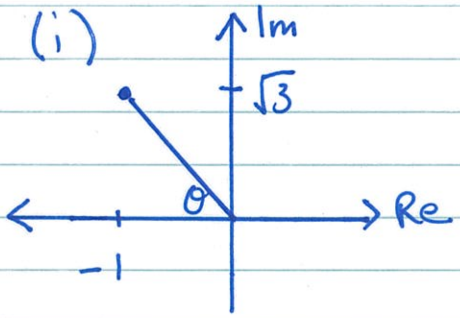
(b) $$\int \frac{x\,dx}{x-1} = \int\left(\frac{x-1}{x-1} + \frac{1}{x-1}\right)dx$$

• Well done.

$$= \int\left(1 + \frac{1}{x-1}\right)dx \leftarrow (1)$$

$$\therefore I = x + \ln|x-1| + C \leftarrow (1)$$

(c) (i)



$$\tan\theta = \frac{\sqrt{3}}{1}$$

$$\theta = \frac{\pi}{3}$$

$$\arg(-1+\sqrt{3}i) = \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3} \leftarrow (1) \text{ OR}$$

$$|-1+\sqrt{3}i| = \sqrt{(-1)^2+(\sqrt{3})^2}$$

$$= \sqrt{4}$$

$$= 2$$

• Many students did not get the correct argumet. Draw a diagram to fix this error.

$$\therefore -1+\sqrt{3}i = 2\operatorname{cis}\left(\frac{2\pi}{3}\right) \leftarrow (1)$$

(ii) $$(-1+\sqrt{3}i)^6 = \left(2\operatorname{cis}\frac{2\pi}{3}\right)^6$$

$$= 2^6\operatorname{cis}\left(6\times\frac{2\pi}{3}\right)$$

• Well done.

$$= 64\operatorname{cis}4\pi$$

$$= 64+0i \text{ (or } 64) \leftarrow (1)$$

(d) $e^{3\theta i} - e^{-3\theta i} = (\cos 3\theta + i\sin 3\theta)$
$- (\cos(-3\theta) + i\sin(-3\theta))$ ← (1)

• Well done.

$= \cos 3\theta + i\sin 3\theta - (\cos 3\theta - i\sin 3\theta)$

$= \cos 3\theta + i\sin 3\theta - \cos 3\theta + i\sin 3\theta$

$= 2i\sin 3\theta$ ← (1)

(e)(i) $PQ = \sqrt{(2+1)^2 + (4-0)^2 + (-2-3)^2}$

$= \sqrt{9 + 16 + 25}$

• Well done.

$= \sqrt{50}$ ← OR (1)

$\therefore PQ = 5\sqrt{2}$ units

(ii) $M = \left(\frac{-1+2}{2}, \frac{0+4}{2}, \frac{3-2}{2}\right)$

$\therefore M = \left(\frac{1}{2}, 2, \frac{1}{2}\right)$ ← (1)

• Mostly well done but a number of students got it wrong.

(iii) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$

$= \begin{pmatrix} -7 \\ -8 \\ 13 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -6 \\ -8 \\ 10 \end{pmatrix}$ ← (1)

$\overrightarrow{PR} = -2\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$

$= -2\overrightarrow{PQ}$

Since $\overrightarrow{PR} = \lambda\overrightarrow{PQ}$, (1)

$\overrightarrow{PR} \parallel \overrightarrow{PQ}$.

• Students needed to idetify the correct relationship between $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ to get the 2nd mark.

Question 2

(a) (2 marks)

✓ [1] for substantial progress

✓ [1] for correct final answer

$$\begin{aligned} u &= e^x \\ du &= e^x dx \\ &= u dx \\ dx &= \frac{du}{u} \end{aligned}$$

$$\begin{aligned} \int \frac{e^{2x}}{e^x - 1}\,dx &= \int \frac{u^2}{u-1}\,\frac{du}{u} \\ &= \int \frac{u}{u-1}\,du \\ &= \int \frac{u-1}{u-1} + \frac{1}{u-1}\,du \\ &= u + \ln|u-1| + C \\ &= e^x + \ln|e^x - 1| + C \end{aligned}$$

Alternatively,

$$\begin{aligned} \int \frac{e^{2x}}{e^x - 1}\,dx &= \int \frac{e^x(e^x-1)+e^x}{e^x-1}\,dx \\ &= \int e^x + \frac{e^x}{e^x-1}\,dx \\ &= e^x + \ln|e^x - 1| + C \end{aligned}$$

Comment:

- Some students who used the substitution $u = e^x - 1$ rearranged it incorrectly to $e^x = u - 1$.

(b) (i) (2 marks)

✓ [1] for correct value of A

✓ [1] for correct value of B

$$7x = A(2x-1) + B(x+3)$$

Let $x = \frac{1}{2}$:

$$\frac{7}{2} = \frac{7}{2}B \qquad \therefore B = 1$$

Let $x = -3$:

$$-21 = -7A \qquad \therefore A = 3$$

(ii) (1 mark)

✓ [1] for correct final answer

$$\begin{aligned} &\int \frac{7x}{(x+3)(2x-1)}\,dx \\ &= \int \frac{3}{x+3} + \frac{1}{2x-1}\,dx \\ &= 3\ln|x+3| + \frac{1}{2}\ln|2x-1| + C \end{aligned}$$

(c) (2 marks)

✓ [1] for correct value of x

✓ [1] for correct value of y

$$\begin{aligned} -1 - i &= e^{x+iy} \\ \sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)} &= e^x e^{iy} \\ \therefore e^x = \sqrt{2},\ y &= -\frac{3\pi}{4} \\ \therefore x = \ln\sqrt{2},\ y &= -\frac{3\pi}{4} \end{aligned}$$

Comment:

- Some students did not recognise that $e^{x+iy} = e^x e^{iy}$.
- Some students expressed $-1 - i = \sqrt{2}e^{i\frac{5\pi}{4}}$, but did not realise that the condition $-\pi < y \leq \pi$ was provided.

(d) (2 marks)

- ✓ [1] for obtaining $p(p+2)+1=0$
- ✓ [1] for correct value of p

$$\begin{aligned}(p\underset{\sim}{i}-\underset{\sim}{k})\cdot\left((p+2)\underset{\sim}{i}-3\underset{\sim}{j}-\underset{\sim}{k}\right)&=0\\ p(p+2)+1&=0\\ p^2+2p+1&=0\\ (p+1)^2&=0\\ \therefore p&=-1\end{aligned}$$

Comment:

- Some students did not realise the coefficient of $\underset{\sim}{j}$ in the vector $p\underset{\sim}{i}-\underset{\sim}{k}$ is 0.

(e) i. (1 mark)

- ✓ [1] for correct final answer (other answers possible)

$$\begin{aligned}\underset{\sim}{d}&=\overrightarrow{AB}\\ &=\begin{pmatrix}10\\1\\1\end{pmatrix}\\ \therefore\underset{\sim}{r}&=\begin{pmatrix}-3\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}10\\1\\1\end{pmatrix}\end{aligned}$$

Comment:

- Some students did not know that the vector equation of a line is given by $\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$, where $\underset{\sim}{d}=\overrightarrow{AB}$; some students simply wrote $\underset{\sim}{r}=\lambda\underset{\sim}{d}$.

ii. (2 marks)

- ✓ [1] for obtaining 3 simultaneous equations
- ✓ [1] for correct conclusion

$$\begin{pmatrix}17\\6\\5\end{pmatrix}=\begin{pmatrix}-3\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}10\\1\\1\end{pmatrix}$$

$$\begin{aligned}17=-3+10\lambda &\implies \lambda=2\\ 6=4+\lambda &\implies \lambda=2\\ 5=1+\lambda &\implies \lambda=4\end{aligned}$$

$\therefore C$ does NOT lie on the line.

Extension 2 : Task 3

$\boxed{Q3}$ a) $\int \frac{x}{\sqrt{10x - x^2}}\,dx$

$$= -\frac{1}{2}\int \frac{2(5-x)}{\sqrt{10x-x^2}}\,dx + \int \frac{5\,dx}{\sqrt{25-(x-5)^2}} \quad (*)$$

$$= -\frac{1}{2}\times 2\sqrt{10x-x^2} + 5\sin^{-1}\left(\frac{x-5}{5}\right) + c$$

$$= -\sqrt{10x-x^2} + 5\sin^{-1}\left(\frac{x-5}{5}\right) + c'$$

AW 2 marks for having correct integrand from step (*) and and either has $-\sqrt{10x-x^2}$ OR $5\sin^{-1}\left(\frac{x-5}{5}\right)$ in the final answer.

b) $\int \sin^{-1}x\,dx$

Let $u = \sin^{-1}x$ $\quad v' = 1$

$u' = \frac{1}{\sqrt{1-x^2}}$ $\quad v = x$

$$I = uv - \int u'v\,dx$$

$$= x\sin^{-1}x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$= x\sin^{-1}x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1}x + \frac{1}{2}\times 2\sqrt{1-x^2} + c$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + c$$

Note: $\sin^{-1}x$ is an inverse Trig fn $\sin^{-1}x \neq \frac{1}{\sin x} = \operatorname{cosec}x$ as some students did.

$\boxed{Q3}$

c) $\int \tan^3 x\,dx = \int \tan x \cdot \tan^2 x\,dx$

$= \int \tan x\,(\sec^2 x - 1)\,dx$

$= \int \tan x \cdot \sec^2 x\,dx - \int \tan x\,dx$

Let $u = \tan x$

$\frac{du}{dx} = \sec^2 x \quad \therefore du = \sec^2 x\,dx$

$=$

$= \int u\,du - \int \frac{\sin x}{\cos x}\,dx$

$= \frac{u^2}{2} + \int \frac{-\sin x}{\cos x}\,dx$

$= \frac{\tan^2 x}{2} + \ln|\cos x| + C$

d) $z^3 = -1 \quad \therefore z^3 + 1 = 0$

$(z+1)(z^2 - z + 1) = 0$

But $z \neq -1 \therefore z^2 - z + 1 = 0$

$1 + z^2 = z$ (*)

$LHS = \frac{iz + 1 - iz + 1}{-z^2 - 1} = \frac{2}{-z^2 - 1} = \frac{-2}{z}$ from (*)

$\frac{-2}{z} \times \frac{z^2}{z^2} = \frac{-2z^2}{z^3} = \frac{-2z^2}{-1} = 2z^2$

$= RHS$

Q3

e) $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

p is a variable point on l

$P(2+3\lambda, 2+\lambda, -1+4\lambda)$

O is the origin: $O(0, 0, 0)$

$$\overrightarrow{OP} = \begin{bmatrix} 2+3\lambda \\ 2+\lambda \\ -1+4\lambda \end{bmatrix} \quad (*)$$

$$|\overrightarrow{OP}|^2 = (2+3\lambda)^2 + (2+\lambda)^2 + (4\lambda-1)^2$$

$$|\overrightarrow{OP}| = \sqrt{9\lambda^2 + 12\lambda + 4 + \lambda^2 + 4\lambda + 4 + 16\lambda^2 - 8\lambda + 1}$$

$$|\overrightarrow{OP}| = \sqrt{26\lambda^2 + 8\lambda + 9}$$

$$= \sqrt{26\left(\lambda^2 + \frac{8}{26}\lambda + \frac{9}{26}\right)}$$

$$= \sqrt{26\left[\left(\lambda + \frac{2}{13}\right)^2 + \frac{109}{338}\right]}$$

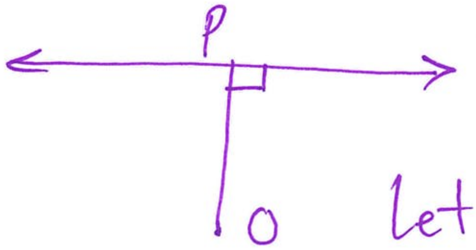
$|\overrightarrow{OP}|$ is minimum $\therefore$ $\lambda + \frac{2}{13} = 0 \therefore \lambda = \frac{-2}{13}$

OR point $P \begin{bmatrix} 20/13 \\ 24/13 \\ -21/13 \end{bmatrix}$ subs $\lambda = \frac{-2}{13}$ into $(*)$

ii) Thus the value of a = radius of the sphere = $|\overrightarrow{OP}|$ when $\lambda = -\frac{2}{3}$

$$a = \sqrt{\frac{109}{13}} \text{ units.}$$

$\boxed{Q3/e}$ Alternate Method:



$\underset{\sim}{r} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

Let P be the point on $\underset{\sim}{r}$ and $\vec{OP} \perp \underset{\sim}{r}$. ($O$ is the origin).

$$\vec{OP} = \begin{bmatrix} 2 + 3\lambda \\ 2 + \lambda \\ -1 + 4\lambda \end{bmatrix}$$

$$\vec{OP} \perp \underset{\sim}{r} : \begin{bmatrix} 2 + 3\lambda \\ 2 + \lambda \\ -1 + 4\lambda \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = 0$$

$$6 + 9\lambda + 2 + \lambda - 6 + 16\lambda = 0$$

$$26\lambda + 4 = 0$$

$$\lambda = \frac{-2}{13}$$

Thus $P\left(\frac{20}{13}, \frac{24}{13}, \frac{-21}{13}\right)$

$$a = |\vec{OP}| = \sqrt{\left(\frac{20}{13}\right)^2 + \left(\frac{24}{13}\right)^2 + \left(\frac{-21}{13}\right)^2}$$

$$= \sqrt{\frac{109}{13}}$$

Question 4(a)(i)

<table>
<tr><th>· Provides correct proof.</th><th>1 mark</th></tr>
<tr><td>

The roots of $z^5 - 1 = 0$ are $1, \alpha, \alpha^2, \alpha^3$ and α^4.

The sum of the roots of this polynomial is $-\dfrac{b}{a} = 0$.

$$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$$

$$\alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1 \quad (🐸)$$

$$\begin{aligned} u + v &= \alpha + \alpha^4 + \alpha^2 + \alpha^3 \\ &= -1 \checkmark \qquad \text{(in view of (🐸))} \end{aligned}$$

</td><td>This part was done fairly well. It is important for students to recognise that if α is the root with smallest positive argument, then the roots are $1, \alpha, \alpha^2, \alpha^3$ and α^4. Some students successfully used an alternative approach using the factorisation $\alpha^5 - 1 = (\alpha - 1)(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4)$.</td></tr>
</table>

Question 4(a)(ii)

<table>
<tr><th>· Provides correct proof.</th><th>2 marks</th></tr>
<tr><th>· Substitutes previous answer and expands.</th><th>1 mark</th></tr>
<tr><td>

$$\begin{aligned} (u - v)^2 &= (u + v)^2 - 4uv \\ &= (-1)^2 - 4(\alpha + \alpha^4)(\alpha^2 + \alpha^3) \\ &= 1 - 4(\alpha^3 + \alpha^4 + \alpha^6 + \alpha^7) \checkmark \\ &= 1 - 4(\alpha^3 + \alpha^4 + \alpha + \alpha^2) \qquad \text{(since } \alpha^5 = 1\text{)} \\ &= 1 - 4(-1) \qquad \text{(in view of (🐸))} \\ &= 5 \end{aligned}$$

$$u - v = \pm 5$$

But $u > v$ so $u - v = \sqrt{5}$. ✓

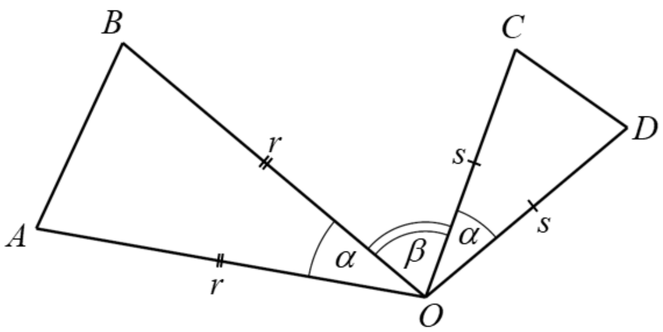
</td><td>Some students converted every power of α to expressions in cos and sin. Note that working directly with cos and sin complicates matters and obscures the important aspects of the question. However, even though their solution involved more work, such students were often able to obtain full marks in this part anyway.</td></tr>
</table>

Question 4(a)(iii)

· Provides correct proof.	2 marks
· Solves for u or equivalent merit.	1 mark
Add the results from part (i) and (ii) together: $$u+v+u-v=-1+\sqrt{5}$$ $$2u=\sqrt{5}-1$$ $$u=\frac{\sqrt{5}-1}{2}$$ Now, $$u=\alpha+\alpha^4$$ $$=\alpha+\overline{\alpha}$$ $$=2\,\mathrm{Re}(\alpha)$$ $$=2\cos\frac{2\pi}{5}.$$ Therefore, $$2\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{2}$$ $$\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}.$$	As always, students are strongly encouraged to look for ways of using the previous part of a question in order to do the next part. The word 'deduce' in the question statement is a clue that the previous part(s) should be used. An alternative approach used the sum and product of roots to obtain a quadratic equation in $\cos\frac{2\pi}{5}$ which could then be solved using the quadratic formula. Students were generally less successful using this approach, since it involved more work and created more chances for error.

Question 4(b)(i)

· Provides correct solution.	2 marks
· Uses the given property and expands.	1 mark



Using the given property $|\underset{\sim}{v}|^2 = \underset{\sim}{v} \cdot \underset{\sim}{v}$:

$$\left|\overrightarrow{AC}\right|^2 = \overrightarrow{AC} \cdot \overrightarrow{AC}$$

$$= (\underset{\sim}{c} - \underset{\sim}{a}) \cdot (\underset{\sim}{c} - \underset{\sim}{a})$$

$$= \underset{\sim}{c} \cdot \underset{\sim}{c} - 2\,\underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{a} \cdot \underset{\sim}{a} \checkmark$$

$$= s^2 - 2rs\cos(\alpha + \beta) + r^2 \qquad \text{(from the diagram)}$$

Similarly:

$$\left|\overrightarrow{BD}\right|^2 = \overrightarrow{BD} \cdot \overrightarrow{BD}$$

$$= (\underset{\sim}{d} - \underset{\sim}{b}) \cdot (\underset{\sim}{d} - \underset{\sim}{b})$$

$$= \underset{\sim}{d} \cdot \underset{\sim}{d} - 2\,\underset{\sim}{b} \cdot \underset{\sim}{d} + \underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$= s^2 - 2rs\cos(\alpha + \beta) + r^2 \qquad \text{(from the diagram)}$$

Therefore:

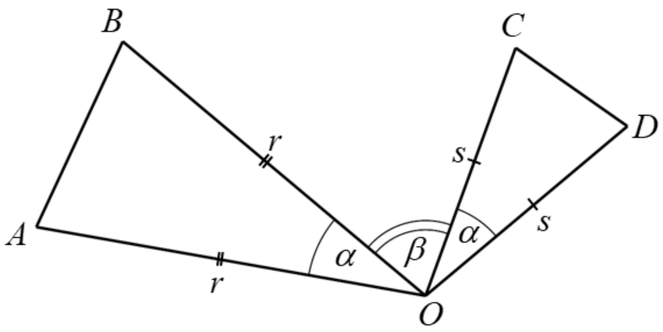
$$\left|\overrightarrow{AC}\right| = \sqrt{s^2 - 2rs\cos(\alpha + \beta) + r^2}$$

$$\left|\overrightarrow{BD}\right| = \sqrt{s^2 - 2rs\cos(\alpha + \beta) + r^2} \checkmark$$

No marks were awarded for using the cosine rule, since the question specifically asked for students to use the property $|\underset{\sim}{v}|^2 = \underset{\sim}{v} \cdot \underset{\sim}{v}$. If instead the question had asked 'using the property **or otherwise**', then the cosine rule would have been accepted. The point of this question was to assess students knowledge of the dot product, in particular the distributive law and the geometrical definition.

Question 4(b)(ii)

· Provides correct proof.	2 marks
· Uses (b)(i) or the sums to product formula appropriately.	1 mark



We have $\overrightarrow{AC}\cdot\overrightarrow{BD} = \left|\overrightarrow{AC}\right|\left|\overrightarrow{BD}\right|\cos\theta$, where θ is the angle between AC and BD. Also:

$$\begin{aligned}
\overrightarrow{AC}\cdot\overrightarrow{BD} &= (\underset{\sim}{c}-\underset{\sim}{a})\cdot(\underset{\sim}{d}-\underset{\sim}{b})\\
&= \underset{\sim}{c}\cdot\underset{\sim}{d}-\underset{\sim}{c}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{d}+\underset{\sim}{a}\cdot\underset{\sim}{b}\\
&= s^2\cos\alpha - rs\cos\beta - rs\cos(2\alpha+\beta)+r^2\cos\alpha\\
&= s^2\cos\alpha - rs\left[\cos\beta+\cos(2\alpha+\beta)\right]+r^2\cos\alpha\\
&= s^2\cos\alpha - rs\left[2\cos(\alpha+\beta)\cos\alpha\right]+r^2\cos\alpha \checkmark
\end{aligned}$$

(using $\frac{1}{2}[\cos(A-B)+\cos(A+B)] = \cos A\cos B$)

$$\begin{aligned}
&= \left(s^2-2rs\cos(\alpha+\beta)+r^2\right)\cos\alpha\\
&= \left|\overrightarrow{AC}\right|\left|\overrightarrow{BD}\right|\cos\alpha \quad \text{(in view of part (i))}
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|\overrightarrow{AC}\right|\left|\overrightarrow{BD}\right|\cos\theta &= \left|\overrightarrow{AC}\right|\left|\overrightarrow{BD}\right|\cos\alpha\\
\cos\theta &= \cos\alpha\\
\theta &= \alpha \checkmark
\end{aligned}$$

(Note: $\cos x$ is one-to-one for $x\in[0,\pi]$.)

This part was challenging. Again, students are encouraged to find ways of using the previous parts of a question in the following parts. The first key insight was using the previous part to find $\left|\overrightarrow{AC}\right|\left|\overrightarrow{BD}\right| = s^2-2rs\cos(\alpha+\beta)+r^2$. The second insight was using the sums to products formula to find $\cos\beta+\cos(2\alpha+\beta) = 2\cos(\alpha+\beta)\cos\alpha$.

Question 4(c)(i)

· Provides correct proof.	3 marks
· Uses integration by parts appropriately and simplifies.	2 marks
· Splits off a factor of $(1+\cos\theta)$ and expands.	1 mark

$$
\begin{aligned}
I_n &= \int_0^{2\pi} (1+\cos\theta)(1+\cos\theta)^{n-1}\,d\theta \\
&= \int_0^{2\pi} (1+\cos\theta)^{n-1}\,d\theta + \int_0^{2\pi} \cos\theta(1+\cos\theta)^{n-1}\,d\theta \\
I_n &= I_{n-1} + \int_0^{2\pi} \underbrace{\cos\theta}_{v'}\ \underbrace{(1+\cos\theta)^{n-1}}_{u}\,d\theta \quad \text{(🐢)} \\
&= I_{n-1} + \Big[uv\Big]_0^{2\pi} - \int_0^{2\pi} vu'\,d\theta \\
&= I_{n-1} + \cancel{\Big[(1+\cos\theta)^{n-1}\sin\theta\Big]_0^{2\pi}}^{\,=0} \\
&\qquad - (n-1)\int_0^{2\pi} \sin\theta\,(1+\cos\theta)^{n-2} \times -\sin\theta\,d\theta \\
&= I_{n-1} + (n-1)\int_0^{2\pi} \sin^2\theta\,(1+\cos\theta)^{n-2}\,d\theta \\
&= I_{n-1} + (n-1)\int_0^{2\pi} (1-\cos^2\theta)(1+\cos\theta)^{n-2}\,d\theta \\
&= I_{n-1} + (n-1)\int_0^{2\pi} (1-\cos\theta)(1+\cos\theta)(1+\cos\theta)^{n-2}\,d\theta \\
&= I_{n-1} + (n-1)\int_0^{2\pi} (1-\cos\theta)(1+\cos\theta)^{n-1}\,d\theta \\
&= I_{n-1} + (n-1)\Big(I_{n-1} - \underbrace{\int_0^{2\pi} \cos\theta\,(1+\cos\theta)^{n-1}\,d\theta}_{=I_n - I_{n-1}\text{ from (🐢)}}\Big) \\
&= I_{n-1} + (n-1)\Big(I_{n-1} - \big(I_n - I_{n-1}\big)\Big) \\
I_n &= I_{n-1} + (n-1)\big(2I_{n-1} - I_n\big) \\
nI_n &= (2n-1)I_{n-1} \\
I_n &= \frac{2n-1}{n} I_{n-1}
\end{aligned}
$$

This question was challenging and had many places where students could lose marks. In order to improve at reduction formula questions, students need to quickly decide whether their intended strategy is going to work before committing time and energy working it out. For example, a common strategy was to use integration by parts from the outset, which lead to a term involving $\theta + \sin\theta$ in the integrand. Students who tried this approach received zero marks, because there is no possible way to get rid of the θ and return to an expression for I_n. Students should realise early on that they need to abandon their plans and try something else. Some students who changed their approach ended up with partial marks for their solution.

Question 4(c)(ii)

· Provides correct solution.	1 mark
$I_0 = \int_0^{2\pi} d\theta$ $= 2\pi$ $I_1 = I_0$ $= 2\pi$ $I_2 = \frac{3}{2} I_1$ $= 3\pi$ $I_3 = \frac{5}{3} I_2$ $= 5\pi$ ✓	This part was fairly well done. Students were able to obtain a mark here without answering (c)(i).