Sydney Girls High School

MATHEMATICS EXTENSION 2

Year 12

Assessment Task 1

2023

Reading time :	5 minutes
Writing time :	60 minutes
Total marks :	50
Topics:	Complex Numbers (partial topic), Proof (partial topics)

General Instructions:

- There are four (4) questions which **are not** of equal value.
- Attempt all questions.
- Start each question on a NEW page.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- A correct answer without working may be awarded a maximum of 1 mark.
- Write on one side of the paper only.
- Diagrams are NOT to scale, unless indicated otherwise.
- Board-approved calculators may be used.
- Write using a black or blue pen.
- White-out or correction tape is NOT to be used.
- A Mathematics reference sheet is provided.

Student Name:______________________________ **Teacher Name:**______________________

Question 1 (13 marks)

a) Write the negation of: $\exists$ a real number $x \in \mathbb{R}$ such that $x^2 - 25 = 0$. 1

b) Given $z = 1 - 2i$ and $w = 1 + i$. Evaluate:

i) $z + w$ 1

ii) $\dfrac{z}{\overline{w}}$ 2

c) Prove by counterexample that the statement $x^2 + y^2 < 2xy$ for all $x, y \in \mathbb{R}$ is false. 2

d) Convert $\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$ into cartesian form. 2

e) Prove by contrapositive that: if $3n + 2$ is even, then n is even, given $n \in \mathbb{Z}^+$. 2

f) Factorise $z^2 + 6iz + 7$. Hence solve $z^2 + 6iz + 7 = 0$. 3

End of Question 1

Question 2 (12 marks) Begin writing on a NEW page.

a) Convert $\dfrac{1}{1+i}$ into modulus argument form. 2

b) Prove that the sum of any four consecutive numbers is always even. 2

c) Given: $z = 2-3i$ is a zero of $p(z) = z^3 + z^2 - 7z + 65$, find the other two zeros. 2

d) Find the square roots of the complex number $7+24i$. 3

e) Given: $a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + a^{m-3}b^2 + ... + b^{m-1})$. Prove $3^{2n} - 1$ is divisible by 4 without using mathematical induction. 3

End of Question 2

Question 3 (13 marks) Begin writing on a NEW page.

a) On the Argand diagram, let A represent $z_1 = 1 + i\sqrt{3}$, let B represent $z_2 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and let C represent $z_3 = z_1 + z_2$. Find the exact coordinates of point C. 1

b) Prove the following statement is false: 2
"There is a Pythagorean Triad where all numbers are odd."

c) Sketch the following regions, showing all important features. You do not need to include axis intercepts.

i) $|z+i| \leq |z-2|$ 2

ii) $\frac{-\pi}{4} < Arg(z+1-i\sqrt{5}) \leq \frac{\pi}{4}$. 2

d) Let A and B be points in the complex plane representing $2i$ and $6+4i$ respectively. Suppose AB is one side of a square $ABCD$. Find all the possible complex numbers representing C and D. 3

e) Use proof by contradiction to show that $\frac{1}{\sqrt{2}}$ is irrational. 3

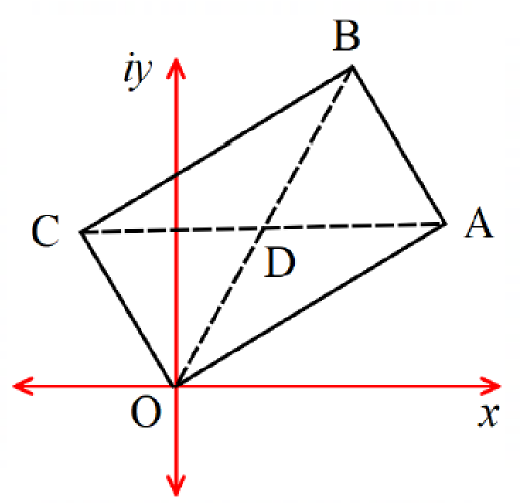
End of Question 3

Question 4 (12 marks) Begin writing on a NEW page.

a) Prove that a three-digit number is divisible by 9 if and only if the sum of its digits is divisible by 9. 3

b) In the Argand diagram, $OABC$ is a rectangle, where $OA = 2OC$. 2
The vertex A corresponds to the complex number w.
The diagonals OB and AC intersect at D.

Find the complex number that represents D in terms of w.



c) Prove that $4^n + 111(2^n) + 2024$ is composite for all $n \in \mathbb{Z}^+$. 2

d) z_1, z_2 and z_2 are three complex numbers which satisfy $z_1 + z_2 + z_3 = 0$. 2

i) Given that the arguments of z_1, z_2 and z_3 are α, β and γ respectively, and their moduli are 1, k and $2-k$ respectively, where $0 < k < 2$.

Express z_1, z_2 and z_3 in modulus-argument form.

ii) Prove that: $k^2 + (2-k)^2 + 2k(2-k)\cos(\beta - \gamma) = 1$. 3

End of Task.

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Q1

e) prove by contrapositive:

If n is Not even (odd) then $3n+2$ is Not even (odd).

Let $n = 2k-1$, $k \in \mathbb{Z}^+$

$$3n+2 = 3(2k-1)+2$$
$$= 6k-1$$
$$= 2(3k)-1$$
$$= 2p-1, \quad p \in \mathbb{Z}^+$$

This is odd.

* One mark will be deducted for letting $n = 2k+1$, $k \in \mathbb{Z}^+$, as this will leave out 1 as an odd number.

f) $z^2 + 6iz + 7$

$(z+7i)(z-i)$ ✓

$$z^2 + 6iz + 7 = 0$$
$$(z+7i)(z-i) = 0$$
$$\begin{cases} z = -7i \checkmark \\ z = i \checkmark \end{cases}$$

Students must factorise $z^2+6iz+7$ into $(z+7i)(z-i)$ then solve for z to get 3 Marks.

$\boxed{Q1}$

a) $\forall x \in R, \; x^2 - 25 \neq 0$

b) $z = 1 - 2i \quad , \quad w = 1 + i$

i) $z + w = 1 - 2i + 1 + i$
$= 2 - i$

ii) $\dfrac{z}{\overline{w}} = \dfrac{1-2i}{1-i} \times \dfrac{(1+i)}{(1+i)}$

$= \dfrac{1 + i - 2i + 2}{1+1} = \dfrac{3}{2} - \dfrac{1}{2}i$

c) $x^2 + y^2 < 2xy \quad , \quad x, y \in R$

Counterexample: For $x = 1 \; , \; y = 1$

$LHS = 1^2 + 1^2 = 2$

$RHS = 2(1)(1) = 2$

$LHS = RHS = 2 \quad \therefore \; LHS \not< RHS$

$\therefore x^2 + y^2 < 2xy$ is false.

OR any logical values of x, y that disproves the original statement.

d) $\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= 1 + i$

Marks | **Question 2**

2 (a)

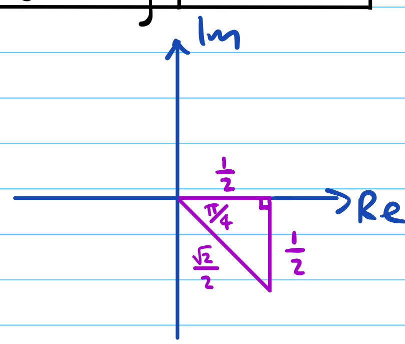
Criteria	Marks
• Provides correct solution	2
• Finds $\frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i$ OR • Finds modulus or argument correctly	1

$$\frac{1}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{1-i}{1^2+1^2}$$

$$= \frac{1}{2} - \frac{1}{2}i \checkmark$$

$$= \frac{1}{\sqrt{2}} \text{cis}\left(-\frac{\pi}{4}\right) \checkmark$$



$$\left|\frac{1}{2} - \frac{1}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}$$

Comments:

- Students who sketched the location of $\frac{1}{2} - \frac{1}{2}i$ were more successful than those using inverse tan with no sketch.

Marks

Question 2 (continued)

2 (b)

Criteria	Marks
• Provides correct solution	2
• Calculates $n+(n+1)+(n+2)+(n+3)=4n+6$ or equivalent merit	1

Let the four consecutive numbers be $n, n+1, n+2, n+3$.

$\therefore$ Their sum is

$$n+n+1+n+2+n+3 = 4n+6$$
$$= 2(2n+3)$$

which is even, as required.

Comments:

- Some students split this question into cases depending on whether the first number was odd or even. While this is often a useful strategy, it was not necessary here.
- Some students wrote $2n, 2n+1, 2n+2, 2n+3$, which incorrectly assumes the first number is even.

Marks	Question 2 (continued)
2	(c)

Criteria	Marks
• Provides correct solution	2
• Applies the conjugate root theorem OR • Considers the sum or product of roots	1

Since all of the coefficients are real, the conjugate root theorem applies.

Hence, $\overline{2-3i} = 2+3i$ is also a root.

Let the remaining root be α.

Therefore, the sum of roots is

$$2+\cancel{3i}+2\cancel{-3i}+\alpha = -\frac{b}{a}$$

$$4+\alpha = -1$$

$$\alpha = -5$$

$\therefore$ The other two roots are $2+3i$ and -5.

Comments:

- This question was very well done. The majority of students applied the conjugate root theorem and sum of roots to efficiently find all roots.

Marks

3

Question 2 (continued)

(d)

Criteria	Marks
• Provides correct solution	3
• Attempts to solve $x^2-y^2=7$ and $2xy=24$ OR • Finds both square roots without working	2
• Sets up $(x+iy)^2 = 7+24i$ OR • Finds one square root without working	1

We want $x+iy$ such that

$$(x+iy)^2 = 7+24i$$

$$x^2-y^2+2xyi = 7+24i$$

Equate real and imaginary parts:

$$x^2-y^2 = 7$$

$$2xy = 24$$

$$xy = 12$$

By inspection, $x = \pm 4$, $y = \pm 3$

$\therefore$ The square roots of $7+24i$ are $4+3i$ and $-4-3i$

Comments:

- Quite well done. Some working out was required for full marks.

Marks **Question 2 (continued)**

(e)

3

Criteria	Marks
• Provides correct solution	3
• Progress proving $3^{2n-1}+3^{2n-2}+\dots+3+1$ is even OR • Writes $9^n-1=(9-1)(9^{n-1}+9^{n-2}+\dots+1)$ OR • Progress proving $(3^n-1)(3^n+1)$ divisible by 4 OR • Writes $3^{2n}-(-1)^{2n}=(3+1)(3^{2n-1}+3^{2n-2}(-1)+\dots+(-1)^{2n-1})$	2
• Writes $3^{2n}-1=(3-1)(3^{2n-1}+3^{2n-2}+\dots+3+1)$ OR • Writes $3^{2n}-1=9^n-1$ OR • Writes $3^{2n}-1=(3^n-1)(3^n+1)$ OR • Writes $3^{2n}-1=3^{2n}-(-1)^{2n}$	1

$$\begin{aligned}3^{2n}-1 &= 9^n-1 \checkmark\\ &= (9-1)(9^{n-1}+9^{n-2}+\dots+9+1) \checkmark\\ &= 8(9^{n-1}+9^{n-2}+\dots+9+1)\\ &= 4(2\times9^{n-1}+2\times9^{n-2}+\dots+2)\end{aligned}$$

which is divisible by 4. ✓

Note: This method proves the stronger statement that $3^{2n}-1$ is divisible by 8.

Comments:

- As you can see from the marking guidelines above, this was an interesting question with many viable approaches.
- The most common difficulty involved starting with

$$3^{2n} - 1 = (3-1)(3^{2n-1} + 3^{2n-2} + \dots + 3 + 1)$$

and then trying to show that

$$3^{2n-1} + 3^{2n-2} + \dots + 3 + 1 \quad \text{is even.}$$

Students who attempted to argue that powers of 3 are odd, and the sum of many odd numbers is even, were close and awarded 2 marks. The crucial detail required for full marks is that the sum consists of $2n$ terms, and $2n$ is even. So the sum of an <u>even number</u> of odd terms is even, as required.

Question 3 (13 marks)

a) On the Argand diagram, let A represents $z_1 = 1 + i\sqrt{3}$, let B represents $z_2 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and let C represents $z_3 = z_1 + z_2$. Find the exact coordinates of point C. 1

$z_1 = 1 + i\sqrt{3}$

$z_2 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -1 + i\sqrt{3}$

Students need to read questions carefully!

$z_3 = z_1 + z_2$

$= 1 + i\sqrt{3} - 1 + i\sqrt{3}$

$\therefore z_3 = 2\sqrt{3}\,i$

hence $C = (0, 2\sqrt{3})$ ✓

One mark was awarded for the correct coordinates.

b) Prove the following statement is false:
"There is a Pythagorean Triad where all numbers are odd." 2

It is sufficient to prove that a Pythagorean Triad does not exist where all numbers are odd.

Assume all numbers are odd, prove by contradiction:

Let the triad of odd numbers be a, b, c where $a=2p+1$, $b=2q+1$, $c=2r+1$, $p,q,r \in \mathbb{Z}^+$

$$a^2+b^2=c^2$$

$$\begin{aligned} LHS &= (2p+1)^2+(2q+1)^2 \\ &= 4p^2+4p+1+4q^2+4q+1 \\ &= 4(p^2+p+q^2+q)+2 \\ &= 2(2p^2+2p+2q^2+2q+1) \\ &= 2m, \quad m \in \mathbb{Z}^+ \end{aligned}$$

$\therefore$ LHS is even

Students had to make a clear statement for one mark

$$\begin{aligned} RHS &= (2r+1)^2 \\ &= 4r^2+4r+1 \\ &= 4(r^2+r)+1 \\ &= 4n+1, \quad n \in \mathbb{Z}^+ \end{aligned}$$

$\therefore$ RHS is odd

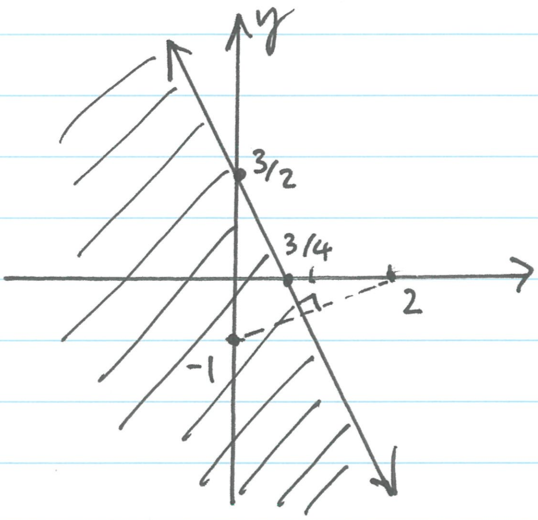
Since LHS is even and RHS is odd
This is a contradiction
That is: $a^2+b^2=c^2$ does not exist where all numbers are odd.
Hence statement is false.

A single counterexample was not awarded any marks.

c) Sketch the following regions: (show all working).

i) $|z+i| \le |z-2|$ $\Rightarrow |z-(0-i)| \le |z-(2+0i)|$ 2

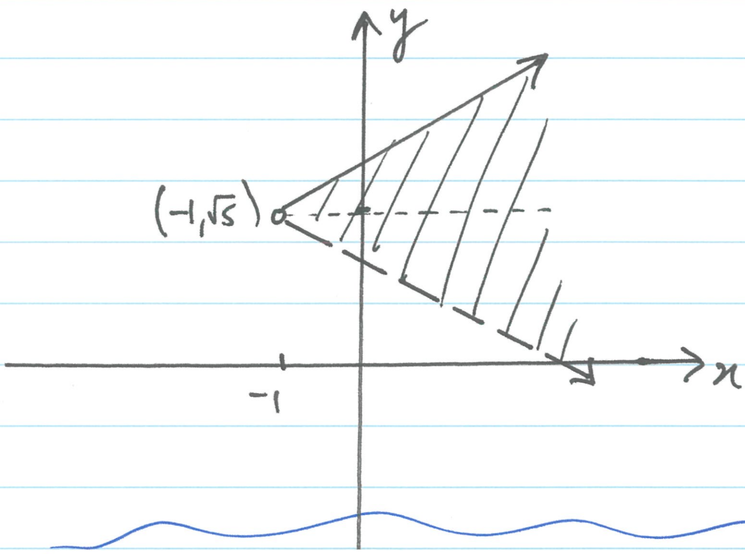
$\Rightarrow$ perp. bisector of interval joining $(0,-1)$ and $(2,0)$.



perp. solid line ✓
region below line ✓

ii) $\frac{-\pi}{4} < Arg(z+1-i\sqrt{5}) \le \frac{\pi}{4}$. 2

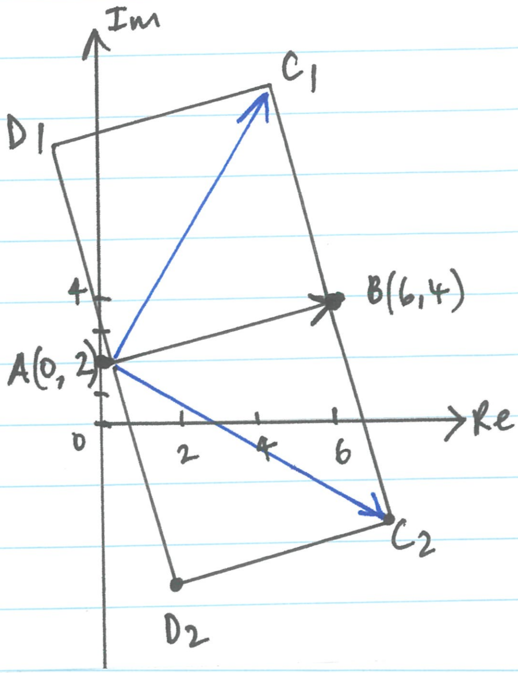
$-\frac{\pi}{4} < Arg\left(z-(-1+\sqrt{5}\,i)\right) \le \frac{\pi}{4}$.



region between rays ✓
open circle at point $(-1,\sqrt{5})$.
solid / dotted rays. ✓

*(Care must be taken to determine the points of origin in part i) and ii))

d) Let A and B be points representing $2i$ and $4i+6$ respectively.
Suppose AB is one side of a square $ABCD$. Find all the possible complex numbers representing C and D. 3



$\overrightarrow{AB} = 6+4i-2i$
$= 6+2i$

$\overrightarrow{AD_1}$ anticlockwise rotation by i

$\therefore \overrightarrow{AD_1} = i \times \overrightarrow{AB}$
$= i(6+2i)$
$= -2+6i$

point $D_1 = \overrightarrow{OA} + \overrightarrow{AD_1}$
$= 0+2i-2+6i$
$\therefore D_1 = -2+8i = (-2,8)$

$\overrightarrow{AC_1} = \overrightarrow{AD_1} + \overrightarrow{AB}$ (diagonal)
$= -2+6i+6+2i$
$= 4+8i$

point $C_1 = \overrightarrow{OA} + \overrightarrow{AC_1}$
$= 0+2i+4+8i$
$\therefore C_1 = 4+10i = (4,10)$

Marks were awarded for position vectors or points

A labelled diagram helped in marking student responses.

$\overrightarrow{AD_2}$ clockwise rotation by i

$\therefore \overrightarrow{AD_2} = -i \times \overrightarrow{AB}$
$= -i(6+2i)$
$= 2-6i$

point $D_2 = \overrightarrow{OA} + \overrightarrow{AD_2}$
$= 0+2i+2-6i$
$\therefore D_2 = 2-4i = (2,-4)$

$\overrightarrow{AC_2} = \overrightarrow{AD_2} + \overrightarrow{AB}$ (diagonal)
$= 2-6i+6+2i$
$= 8-4i$

point $C_2 = \overrightarrow{OA} + \overrightarrow{AC_2}$
$= 0+2i+8-4i$
$\therefore C_2 = 8-2i = (8,-2)$

e) Use proof by contradiction to show that $\frac{1}{\sqrt{2}}$ is irrational. 3

Assume $\frac{1}{\sqrt{2}}$ is rational. That is:

$\frac{1}{\sqrt{2}} = \frac{p}{q}$ where $p, q \in Z^+$, $q \neq 0$, p, q have no common factors.

$\therefore \frac{1}{2} = \frac{p^2}{q^2}$

Students received one mark up until this point

$q^2 = 2p^2 \quad \ldots (1)$

$\therefore q^2$ is even $\Rightarrow q$ is even

Let $q = 2k$ sub into (1) $k \in Z^+$

$(2k)^2 = 2p^2$

$4k^2 = 2p^2$

$p^2 = 2k^2$

$\therefore p^2$ is even $\Rightarrow p$ is even

p, q have a common factor of 2.

Contradiction!

hence $\frac{1}{\sqrt{2}}$ is irrational.

*A clear statement of proof and showing that p and q have a 'common factor' was required for 3 marks.

Question 4

(a) Let abc be a three-digit number.

Suppose that $a+b+c = 9k,\ k \in \mathbb{Z}$

$$
\begin{aligned}
abc &= 100a + 10b + c \\
&= 100a + 10b + 9k - a - b \\
&= 99a + 9b + 9k \\
&= 9(\underbrace{11a + b + k}_{\in \mathbb{Z} \text{ since } a,b,k \in \mathbb{Z}})
\end{aligned}
$$

$\therefore$ abc is div. by 9. ✓✓ (1 M for correct progress*)

Now suppose that $100a + 10b + c = 9k,\ k \in \mathbb{Z}$

$$
\begin{aligned}
a+b+c &= a + b + 9k - 100a - 10b \\
&= 9k - 99a - 9b \\
&= 9(\underbrace{k - 11a - b}_{\in \mathbb{Z} \text{ since } a,b,k \in \mathbb{Z}})
\end{aligned}
$$

$\therefore$ $a+b+c$ is div by 9. ✓

$\therefore$ Proposition is proven.

Comment

- Some students did not know that the three-digit number abc could be expressed as $100a + 10b + c$.

- Some students also did not prove the converse.

* Correct progress : Writing $a+b+c = 9k$

or

$100a + 10b + c = 9k$

(b) $\vec{OC} = \vec{OA} \times i \times \frac{1}{2}$

$= w \times \frac{1}{2} i$

$= \frac{1}{2} w i$ ✓

$\therefore$ D represents $\dfrac{w + \frac{1}{2} w i}{2}$

$= \dfrac{2w + wi}{4}$

$= \dfrac{w(2+i)}{4}$ ✓

Comment

- Some students incorrectly wrote

$\vec{OC} = 2wi$ or $\vec{OC} = \frac{w}{2}$

- Some students also thought that D can be found by finding $\frac{1}{2}\vec{AC}$.

(c) $4^n + 111(2^n) + 2024$

$= (2^2)^n + 111(2^n) + 2024$

$= (2^n)^2 + 111(2^n) + 2024$

$= (2^n + 23)(2^n + 88)$ ✓

This is needed to show that the expression has factors other than 1 and itself, and hence it is composite.

Each factor is an integer > 1 since $n \in \mathbb{Z}^+$. ✓

$\therefore 4^n + 111(2^n) + 2024$ is composite.

Method 2

$4^n + 111(2^n) + 2024$

$= 2^{2n} + 111(2^n) + 2024$

$= 2\underbrace{\left(2^{2n-1} + 111(2^{n-1}) + 1012\right)}_{\in \mathbb{Z} \text{ with value } > 1}$ ✓

$\therefore 4^n + 111(2^n) + 2024$ is an even number that is greater than 2, so it is composite. ✓

required (students who stated that since it is even, it must be composite. Not true - counterexample: 2)

Comment

To achieve full marks, students had to have a careful justification of why the expression is composite.

See the above annotations.

(d)

(i) $z_1 = \text{cis}\alpha$

$z_2 = k\,\text{cis}\beta$

$z_3 = (2-k)\,\text{cis}\gamma$

(ii) $z_1 + z_2 + z_3 = 0$

$z_2 + z_3 = -z_1$

$\therefore |z_2 + z_3| = |z_1|$

$\therefore \left| k\cos\beta + (2-k)\cos\gamma + i\left(k\sin\beta + (2-k)\sin\gamma\right)\right| = 1$

$\left(k\cos\beta + (2-k)\cos\gamma\right)^2 + \left(k\sin\beta + (2-k)\sin\gamma\right)^2 = 1$

$k^2(\cos^2\beta + \sin^2\beta) + 2k(2-k)\left(\cos\beta\cos\gamma + \sin\beta\sin\gamma\right)$
$+ (2-k)^2\left(\cos^2\gamma + \sin^2\gamma\right) = 1$

$\therefore k^2 + (2-k)^2 + 2k(2-k)\cos(\beta - \gamma) = 1$

Method 2

$z_1 + z_2 + z_3 = 0$

$\text{cis}\,\alpha + k\,\text{cis}\,\beta + (2-k)\,\text{cis}\,\gamma = 0$

Equating Re and Im parts:

$$\left.\begin{aligned} \cos\alpha + k\cos\beta + (2-k)\cos\gamma &= 0 \\ \sin\alpha + k\sin\beta + (2-k)\sin\gamma &= 0 \end{aligned}\right\} \checkmark$$

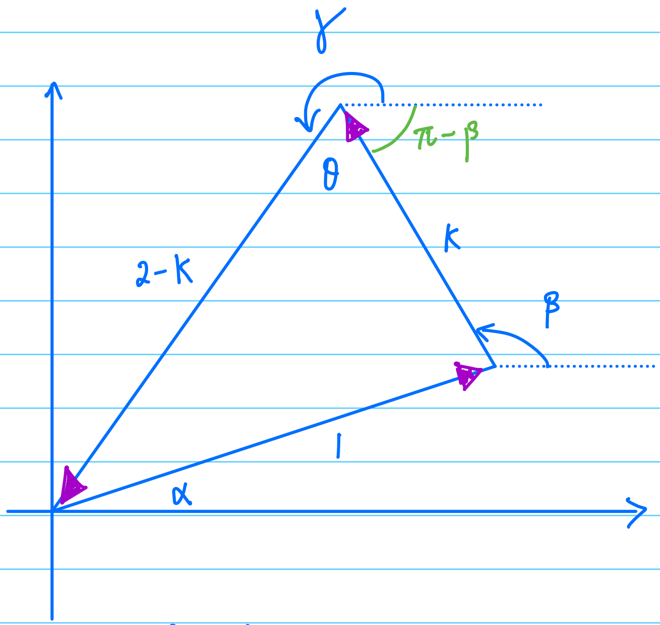
$$\therefore \begin{cases} k\cos\beta + (2-k)\cos\gamma = -\cos\alpha & (1) \\ k\sin\beta + (2-k)\sin\gamma = -\sin\alpha & (2) \end{cases}$$

$(1)^2 + (2)^2$:

$$k^2(\cos^2\beta + \sin^2\beta) + 2k(2-k)(\cos\beta\cos\gamma + \sin\beta\sin\gamma)$$
$$+ (2-k)^2(\cos^2\gamma + \sin^2\gamma) = \cos^2\alpha + \sin^2\alpha$$

$$\therefore k^2 + (2-k)^2 + 2k(2-k)\cos(\beta-\gamma) = 1 \quad \checkmark\checkmark$$

Method 3



$$\theta = 2\pi - (\pi - \beta) - \gamma$$

$$= \pi + \beta - \gamma$$

$$\therefore \; 1^2 = k^2 + (2-k)^2 - 2k(2-k)\cos(\pi + \beta - \gamma)$$

$$1 = k^2 + (2-k)^2 - 2k(2-k)(-\cos(\beta - \gamma)) \quad (\cos(\pi + A) = -\cos A)$$

$$\therefore \; k^2 + (2-k)^2 + 2k(2-k)\cos(\beta - \gamma) = 1$$

Comment

Not well done.

Students who resorted to the cosine rule needed a careful diagram. Incorrect diagrams or using the cosine rule incorrectly were penalised.