Sydney Girls High School

YEAR 12 MATHEMATICS EXTENSION 2

2024 HSC Task 3

Reading time: 5 minutes

Writing time: 75 minutes

Total marks: 55

Section I – 5 marks

- Attempt Questions 1-5
- Allow about 5-10 minutes for this section

Section II – 50 marks

- Attempt Question 6-9
- Allow about 65-70 minutes for this section

Topics: Complex Numbers 2, Vectors, Integration

General Instructions:

- There are nine (9) questions which are not all of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Diagrams are NOT to scale, unless indicated otherwise.
- NESA approved calculators may be used.
- Write using a black or blue pen.
- White-out or correction tape is NOT to be used.
- A correct answer without working will be awarded a maximum of 1 mark.

Student Name:________________________ **Teacher:**____________________________

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Section I

5 marks
Attempt Questions 1-5
Allow about 5-10 minutes for this section

Use the multiple-choice answer sheet for Questions 1-5.

Questions **Marks**

1 A sphere with equation $(x-a)^2+(y-2a)^2+(z-2a)^2=81$ passes through the origin. **1**

Which of the following is a possible value of a?

A. $a=0$

B. $a=3$

C. $a=6$

D. $a=9$

2 What are the cube roots of $-2-2i$? **1**

A. $\sqrt{2}\text{ cis}\left(-\frac{\pi}{4}\right)$, $\sqrt{2}\text{ cis}\left(-\frac{5\pi}{12}\right)$, $\sqrt{2}\text{ cis}\left(\frac{11\pi}{12}\right)$

B. $\sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right)$, $\sqrt{2}\text{ cis}\left(-\frac{5\pi}{12}\right)$, $\sqrt{2}\text{ cis}\left(-\frac{11\pi}{12}\right)$

C. $\sqrt{2}\text{ cis}\left(-\frac{\pi}{4}\right)$, $\sqrt{2}\text{ cis}\left(\frac{5\pi}{12}\right)$, $\sqrt{2}\text{ cis}\left(-\frac{11\pi}{12}\right)$

D. None of the above

3 Suppose that a and b are non-zero constants. **1**

Which of the following could be a unit vector?

A. $\frac{1}{a}\underset{\sim}{i}+b\underset{\sim}{j}+\underset{\sim}{k}$

B. $a^2b^2\underset{\sim}{i}+2ab\underset{\sim}{j}+2\underset{\sim}{k}$

C. $a\underset{\sim}{i}+\frac{1}{b}\underset{\sim}{j}+\frac{1}{a}\underset{\sim}{k}$

D. $a^2\underset{\sim}{i}+2ab\underset{\sim}{j}+2b^2\underset{\sim}{k}$

4 The integral **1**

$$\int_1^2 \frac{\sqrt{x^2-1}-2x^2\sqrt{3}}{x}\,dx,$$

is evaluated using the trigonometric substitution $x=\sec\theta$.

Which of the following is the correct value of the integral?

A. $-2\sqrt{3}-\dfrac{\pi}{3}$

B. $-4\sqrt{3}-\dfrac{\pi}{3}$

C. $-4\sqrt{3}+\dfrac{\pi}{3}$

D. $-2\sqrt{3}+\dfrac{\pi}{3}$

5 Let $P(x)$ and $Q(x)$ be real-coefficient polynomials without any common factors. Suppose that **1**

$$\int \frac{P(x)}{Q(x)}\,dx = (x-3)^2 + \ln\Big|(x-a_1)(x-a_2)(x-a_3)\cdots(x-a_{2024})\Big| + C,$$

where a_1, a_2, a_3, $\ldots$, a_{2024} are constants.

What is the degree of $P(x)$?

A. 2023

B. 2024

C. 2025

D. 2026

Examination continues overleaf...

Section II

50 marks
Attempt Questions 6-9
Allow about 65-70 minutes for this section

Answer each question on the writing paper supplied. Start each question on a NEW page. Extra writing paper are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (13 marks) Start on a NEW page **Marks**

(a) Use integration by parts to find **3**

$$\int \tan^{-1} x \, dx.$$

(b) (i) Find real numbers A, B and C such that **3**

$$\frac{5x^2 - 5x + 14}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+4}.$$

(ii) Hence find **2**

$$\int \frac{5x^2 - 5x + 14}{(x-2)(x^2+4)} \, dx.$$

Examination continues overleaf...

(c) The lines ℓ_1 and ℓ_2 have vector equations

$$\underset{\sim}{r}_1 = \begin{pmatrix} -9 \\ 0 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ and } \underset{\sim}{r}_2 = \begin{pmatrix} 3 \\ 1 \\ 17 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$

respectively where $\lambda,\ \mu \in \mathbb{R}$.

The point A with coordinates $(5, 7, 3)$ lies on ℓ_1. The point B is the reflection of the point A in the line ℓ_2.

(i) Find the point of intersection P of the lines ℓ_1 and ℓ_2. **2**

(ii) Show that ℓ_1 and ℓ_2 are perpendicular to each other. **1**

(iii) Find the coordinates of the point B. **2**

End of Question 6

Question 7 (12 marks) Start on a NEW page **Marks**

(a) By using the substitution $t = \tan x$, show that **3**

$$\int \operatorname{cosec} 2x \, dx = \frac{1}{2} \ln \left| \tan x \right| + C.$$

(b) Find the least positive integer value of n for which $(-1+i)^n$ is real. **3**

(c) Let $z = a + ib$, where $a > 0$ and $b > 0$, be represented by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

(i) Find the vector representation for $\dfrac{1}{z}$. **2**

(ii) Let the angle between the two vectors represented by z and $\dfrac{1}{z}$ be θ. Show that **2**

$$\theta = \cos^{-1}\left(\frac{a^2 - b^2}{a^2 + b^2}\right).$$

(iii) Hence show that **2**

$$\cos^{-1}\left(\frac{a^2 - b^2}{a^2 + b^2}\right) = 2\tan^{-1}\left(\frac{b}{a}\right).$$

End of Question 7

Question 8 (13 marks) Start on a NEW page **Marks**

(a) Let $\omega = e^{i\frac{3\pi}{5}}$.

(i) Show that $w^5 = -1$, and hence show that **3**

$$\left(\omega + \omega^{-1}\right)^2 - \left(\omega + \omega^{-1}\right) - 1 = 0.$$

(ii) Show that **2**

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right).$$

(iii) Hence find the exact value of $\cos\dfrac{3\pi}{5}$. **2**

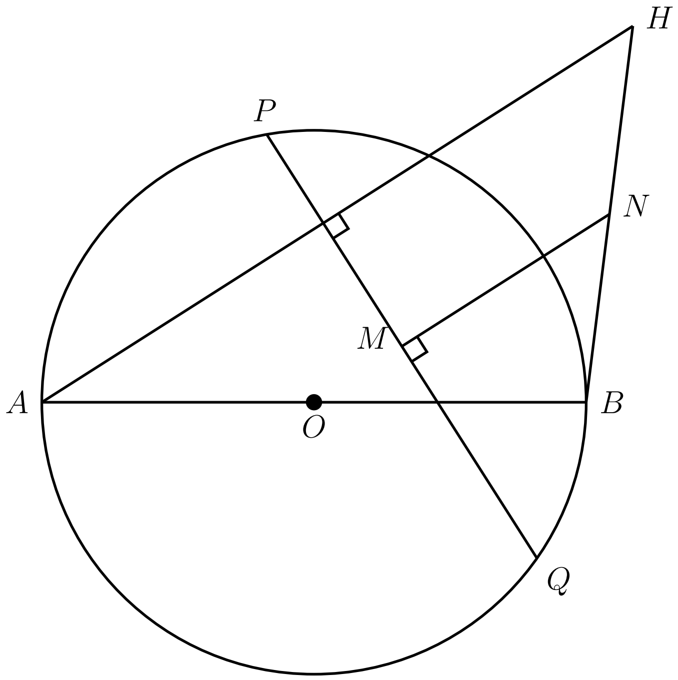
Examination continues overleaf...

(b) In the diagram below, AB is a diameter of a circle with centre O and PQ is a chord of the circle that is ***not*** perpendicular to AB.

The perpendicular from A to PQ is produced to the point H outside the circle.

M is the midpoint of PQ and N is the point on BH such that MN is perpendicular to PQ.

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OP} = \underset{\sim}{p}$, $\overrightarrow{OQ} = \underset{\sim}{q}$ and $\overrightarrow{AH} = \underset{\sim}{h}$.



(i) By writing $\overrightarrow{OM}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$, or otherwise, show that the points O, M and N are collinear. **2**

(ii) If $\overrightarrow{BN} = \lambda\overrightarrow{BH}$ for some scalar λ, express $\overrightarrow{ON}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{h}$ and λ. **1**

(iii) Hence, or otherwise, show that N is the midpoint of BH. **3**

End of Question 8

Question 9 (12 marks) Start on a NEW page **Marks**

(a) Let

$$I_n = \int_1^2 \frac{(x-1)^{\frac{n}{2}}}{x}\,dx,$$

where $n \geq -1$ is an integer.

(i) Show that $I_{-1} = \dfrac{\pi}{2}$. (Note: The subscript of I is -1, not 1.) **3**

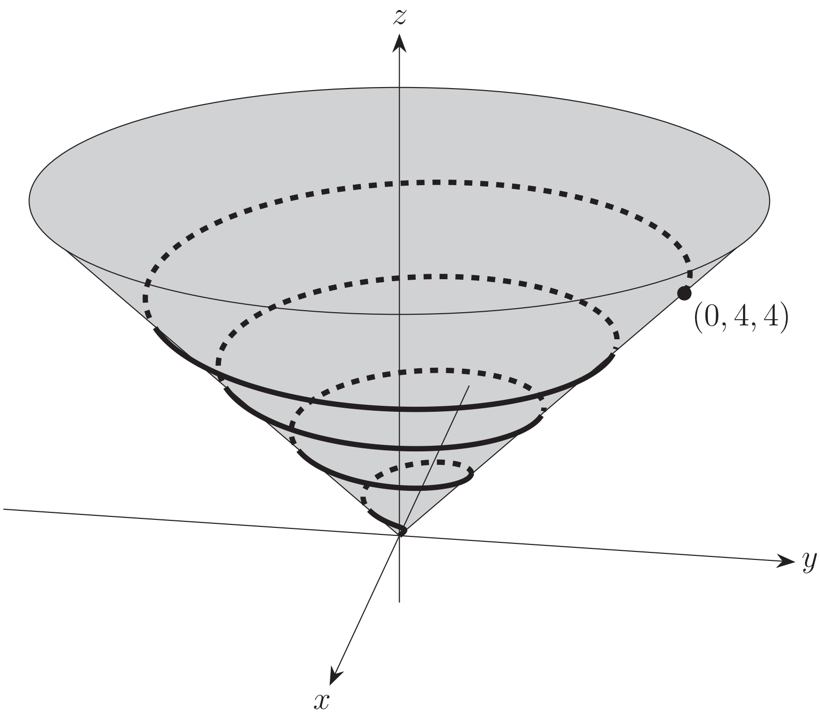
(ii) Show that $I_n = \dfrac{2}{n} - I_{n-2}$, for $n \geq 1$. **3**

(iii) Hence find I_5. **1**

Examination continues overleaf...

(b) The diagram below shows a cone with apex at the origin, and the equation of the surface of the cone is given by $z = \sqrt{x^2 + y^2}$. **3**

A particle is initially at the origin and moves along the curve $\mathscr{C}$ on the surface of the cone, ending at the point $(0, 4, 4)$, as shown.



By considering the graph of $y = f(x)g(x)$ for $x \geq 0$, where $f(x) = x$ and $g(x) = \sin(2\pi x)$, give a possible set of parametric equations that describe the curve $\mathscr{C}$.

(c) Find **2**

$$\int \left(1 - 2\sin^2\left(\frac{x}{2}\right)\right) \sec^3(\sin x)\tan^5(\sin x)\, dx.$$

End of paper

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2024 Year 12 Mathematics Extension 2 Task 3 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	C
3	D
4	A
5	C

Question 1 (1 mark)

Sub $(0,0,0)$ into the given equation of the sphere:

$$\begin{aligned}(0-a)^2+(0-2a)^2+(0-2a)^2&=81\\ a^2+4a^2+4a^2&=81\\ 9a^2&=81\\ a^2&=9 \quad \therefore a=\pm 3 \quad \therefore \textbf{B}\end{aligned}$$

Question 2 (1 mark)

$$\begin{aligned}z^3&=-2-2i\\ &=2\sqrt{2}\text{ cis}\left(-\frac{3\pi}{4}+2k\pi\right)\\ &=2\sqrt{2}\text{ cis}\left(\frac{-3+8k}{4}\pi\right)\\ \therefore z&=2^{\frac{1}{3}}2^{\frac{1}{6}}\text{ cis}\left(\frac{-3+8k}{12}\pi\right),\quad k=0,\pm 1\\ &=\sqrt{2}\text{ cis}\left(-\frac{\pi}{4}\right),\ \sqrt{2}\text{ cis}\left(\frac{5\pi}{12}\right),\ \sqrt{2}\text{ cis}\left(-\frac{11\pi}{12}\right)\quad \therefore \textbf{C}\end{aligned}$$

Question 3 (1 mark)

For (A),

$$\begin{aligned}|\underset{\sim}{u}|^2&=\frac{1}{a^2}+b^2+1\\ &>1\end{aligned}$$

$\therefore |\underset{\sim}{u}|>1 \quad \therefore$ can't be (A)

For (B),

$$\begin{aligned}|\underset{\sim}{u}|^2&=a^4b^4+4a^2b^2+4\\ &>4\end{aligned}$$

$\therefore |\underset{\sim}{u}|>2 \quad \therefore$ can't be (B)

For (C),

$$\begin{aligned}|\underset{\sim}{u}|^2&=a^2+\frac{1}{b^2}+\frac{1}{a^2}\\ &\geq 2+\frac{1}{b^2}\\ &>2\end{aligned}$$

$\therefore |\underset{\sim}{u}|>\sqrt{2} \quad \therefore$ can't be (C)

$\therefore$ **D**

Question 4 (1 mark)

$$x = \sec\theta \qquad x = 1,\ \cos\theta = 1 \quad \therefore \theta = 0$$

$$dx = \sec\theta\tan\theta\, d\theta \qquad x = 2,\ \cos\theta = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{3}$$

$$\begin{aligned}
\int_1^2 \frac{\sqrt{x^2-1} - 2x^2\sqrt{3}}{x}\, dx &= \int_0^{\frac{\pi}{3}} \frac{\sqrt{\sec^2\theta - 1} - 2\sec^2\theta\sqrt{3}}{\sec\theta} \sec\theta\tan\theta\, d\theta \\
&= \int_0^{\frac{\pi}{3}} \left(\tan\theta - 2\sqrt{3}\sec^2\theta\right)\tan\theta\, d\theta \\
&= \int_0^{\frac{\pi}{3}} \tan^2\theta - 2\sqrt{3}\sec^2\theta\tan\theta\, d\theta \\
&= \int_0^{\frac{\pi}{3}} \sec^2\theta - 1 - 2\sqrt{3}\sec^2\theta\tan\theta\, d\theta \\
&= \left[\tan\theta - \theta - \cancel{2}\sqrt{3}\,\frac{\tan^2\theta}{\cancel{2}}\right]_0^{\frac{\pi}{3}} \\
&= \sqrt{3} - \frac{\pi}{3} - \sqrt{3}(\sqrt{3})^2 \\
&= -2\sqrt{3} - \frac{\pi}{3} \qquad \therefore \text{(A)} \ \checkmark
\end{aligned}$$

Question 5 (1 mark)

Analysing the RHS, it can be deduced that

$$\int \frac{P(x)}{Q(x)}\, dx = \int 2(x-3) + \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{2024}}\, dx$$

$\therefore$ $P(x)$ must have degree 2025 and $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_{2024})$, so that after polynomial division,

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= 2(x-3) + \frac{R(x)}{(x-a_1)(x-a_2)\cdots(x-a_{2024})} \\
&= 2(x-3) + \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{2024}} \qquad \therefore \text{(C)} \ \checkmark
\end{aligned}$$

Question 6 (a) (3 marks)

$$\left.\begin{aligned} \text{Let } u &= \tan^{-1} x \qquad & v' &= 1 \\ u' &= \frac{1}{1+x^2} \qquad & v &= x \end{aligned}\right\} \checkmark$$

$$\begin{aligned} \therefore \int \tan^{-1} x \, dx &= x \tan^{-1} x - \int \frac{x}{1+x^2} \, dx \quad \checkmark \\ &= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C \quad \checkmark \end{aligned}$$

Marking Scheme
✓ [1] for obtaining the correct expressions for u, v', u' and v, or equivalent merit
✓ [1] for correctly applying the integration by parts formula
✓ [1] for correct final answer

Marker's Comments:

- Generally well done.
- Some students were not able to find $\int \frac{x}{1+x^2} \, dx$ correctly.

Question 6 (b)(i) (3 marks)

$$5x^2 - 5x + 14 = A(x^2+4) + (Bx+C)(x-2)$$

Sub $x = 2$

$$24 = 8A \qquad \therefore A = 3 \quad \checkmark$$

Sub $x = 0$

$$14 = 4A - 2C$$
$$14 = 12 - 2C$$
$$2C = -2 \qquad \therefore C = -1 \quad \checkmark$$

Equate coefficients of x^2

$$5 = A + B \qquad \therefore B = 2 \quad \checkmark$$

Marking Scheme
✓ [1] for correct value of A, or equivalent merit
✓ [1] for correct value of C, or equivalent merit
✓ [1] for correct value of B, or equivalent merit

Marker's Comments:

- Generally well done.
- Some students made careless errors when finding A, B and C. Students are advised to label their equations and to be careful when solving simultaneous equations.

Question 6 (b)(ii) (2 marks)

$$\int \frac{5x^2-5x+14}{(x-2)(x^2+4)}\,dx = \int \frac{3}{x-2}+\frac{2x-1}{x^2+4}\,dx$$

$$= 3\ln|x-2| + \int \frac{2x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx \quad \checkmark$$

$$= 3\ln|x-2| + \ln\left(x^2+4\right) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \quad \checkmark$$

Marking Scheme
✓ [1] for correct progress
✓ [1] for obtaining correct primitive

Marker's Comments:

- Generally well done.
- Some students were not able to find $\displaystyle\int \frac{2x}{x^2+4}\,dx$ correctly;

  e.g. some students had incorrectly written that $\displaystyle\int \frac{2x}{x^2+4}\,dx = x\tan^{-1}\left(\frac{x}{2}\right)$.

Question 6 (c)(i) (2 marks)

Solving $\underset{\sim}{\mathrm{r}_1}$ and $\underset{\sim}{\mathrm{r}_2}$ simultaneously:

$$\begin{aligned} -9+2\lambda &= 3+3\mu \implies 2\lambda-3\mu=12 && (1)\\ \lambda &= 1-\mu && (2)\\ 10-\lambda &= 17+5\mu && (3) \end{aligned}$$

$(2) \to (1)$

$$\begin{aligned} 2-2\mu-3\mu &= 12\\ -5\mu &= 10 \qquad \therefore \mu=-2 \ \checkmark \end{aligned}$$

$\therefore P(-3,3,7)$ ✓

Marking Scheme		
✓	[1]	for obtaining the correct value of μ at the point of intersection, or equivalent merit
✓	[1]	for obtaining the correct coordinates of P

Marker's Comments:

- Generally well done, apart from some careless errors made.
- Majority of the students gave their answer as $P=\begin{pmatrix}-3\\3\\7\end{pmatrix}$, which was not penalised this time. However, students are encouraged to use proper notation; i.e. giving P as $P(-3,3,7)$.

Question 6 (c)(ii) (1 mark)

$$\underset{\sim}{d}_1 \cdot \underset{\sim}{d}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$
$$= 6 - 1 - 5$$
$$= 0$$

$\therefore \ell_1 \perp \ell_2$ ✓

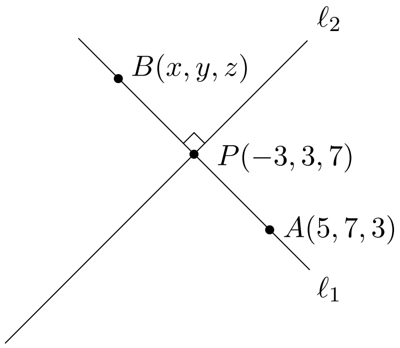
Marking Scheme

✓ [1] for correctly showing that $\ell_1 \perp \ell_2$

Marker's Comments:

- Generally well done.
- Some students had attempted to show that $\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = 0$, which is an incorrect approach.

Question 6 (c)(iii) (2 marks)



P is the midpoint of A and B ✓

$$\therefore \frac{x+5}{2} = -3, \qquad \frac{y+7}{2} = 3, \qquad \frac{z+3}{2} = 7$$

$$\therefore x = -11,\ y = -1,\ z = 11$$

$$\therefore B(-11, -1, 11)$$ ✓

Marking Scheme

✓	[1]	for recognising that P is the midpoint of A and B, or equivalent merit
✓	[1]	for correct coordinates of B

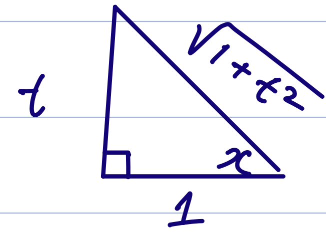
Marker's Comments:

- Generally not done well. Students who recognised that P is the midpoint of A and B were successful in determining the coordinates of B.
- Students who also recognised that $\overrightarrow{AP} = \overrightarrow{PB}$ were also successful. Thus students who found $\overrightarrow{AP}$ (or $\overrightarrow{PA}$) correctly were awarded 1 mark for correct progress, and 2 marks if they correctly deduced the coordinates of B thereafter.

$\boxed{Q7}$ $t = \tan x$

$$\frac{dt}{dx} = \sec^2 x$$

$$dx = \frac{dt}{\sec^2 x} = \cos^2 x \, dt = \frac{dt}{1+t^2}$$



$$\int \operatorname{cosec} 2x \, dx = \int \frac{1}{\sin 2x} dx$$

$$= \int \frac{1+t^2}{2t} \cdot \frac{dt}{1+t^2}$$

$$= \frac{1}{2} \int \frac{dt}{t}$$

$$= \frac{1}{2} \ln|t| = \frac{1}{2} \ln|\tan(x)| + c$$

1 mark for express : $dx = \dfrac{dt}{1+t^2}$

2 marks for having integrand $\displaystyle\int \frac{1+t^2}{2t} \cdot \frac{dt}{1+t^2}$

3 marks for correct working and solution.

b) $$(-1+i)^n = \left(\sqrt{2} \operatorname{cis} \frac{3\pi}{4}\right)^n$$

$$= (\sqrt{2})^n \left[\cos \frac{3\pi n}{4} + i \sin \frac{3\pi n}{4}\right]$$

$(-1+i)^n$ is real $\therefore$ $\sin \dfrac{3\pi n}{4} = 0$

$$\frac{3\pi}{4} n = k\pi \ , \ k = 0, 1, 2 \ldots$$

$$3n = 4k$$
$$n = \frac{4k}{3}$$
$$n = 4 \quad \text{when} \quad k = 3$$

$n = 4$ is the least positive integer for which $(-1+i)^n$ is real.

1 mark for expressing $(-1+i) = cis\frac{3\pi}{4}$

2 marks for showing $\sin\frac{3\pi}{4}n = 0$

3 marks for correct working and answer.

c) i) $z = a + ib$

$$\frac{1}{z} = \frac{1}{a+ib} \times \frac{(a-ib)}{(a-ib)}$$

$$= \frac{a-ib}{a^2-b^2}$$

$$= \begin{pmatrix} \frac{a}{a^2-b^2} \\ \frac{-b}{a^2-b^2} \end{pmatrix}$$

Award zero for express vector presentation of $\frac{1}{z} = \begin{pmatrix} a \\ -b \end{pmatrix}$

1 mark for multiply by the conjugate

2 marks for correct answer.

c/ii) $$\cos\theta = \frac{z \cdot \frac{1}{z}}{|z|\left|\frac{1}{z}\right|}$$

$$\cos\theta = \frac{\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} \frac{a}{a^2+b^2} \\ \frac{-b}{a^2+b^2} \end{pmatrix}}{\sqrt{a^2+b^2} \cdot \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{b}{a^2+b^2}\right)^2}}$$

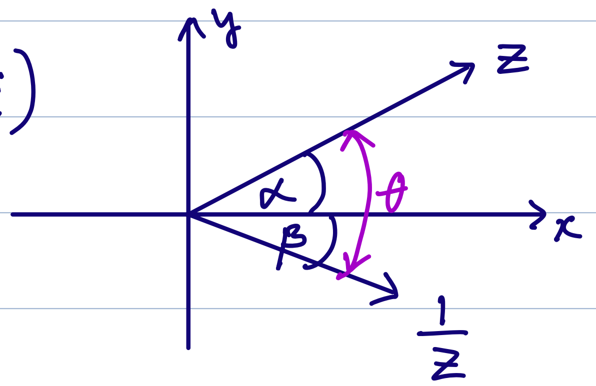
$$\cos\theta = \frac{\frac{a^2-b^2}{a^2+b^2}}{\frac{a^2+b^2}{a^2+b^2}} = \frac{a^2-b^2}{a^2+b^2}$$

$$\theta = \cos^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right)$$

1 mark for attempt to use the scalar product rule from the correct vectors from part i

2 marks for correct prove.

c/iii)



$$\tan\alpha = \frac{b}{a}$$

$$\alpha = \tan^{-1}\frac{b}{a}$$

$$\tan|\beta| = \left|\frac{\frac{-b}{a^2+b^2}}{\frac{a}{a^2+b^2}}\right|$$

$$\beta = \tan^{-1}\frac{b}{a}$$

$$\theta = \alpha + \beta = 2\tan^{-1}\frac{b}{a}$$

OR $\cos^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$ from ii

1 mark for recognising the $|Arg z| = \left|Arg \frac{1}{z}\right|$
/

2 marks for correct prove.

Question 8 (a)(i) (3 marks)

$$\begin{aligned}\omega^5 &= \left(e^{i\frac{3\pi}{5}}\right)^5 \\ &= e^{i3\pi} \\ &= \cos(3\pi) + i\sin(3\pi) \\ &= -1 \quad \checkmark\end{aligned}$$

$$\begin{aligned}&(\omega+\omega^{-1})^2 - (\omega+\omega^{-1}) - 1 \\ &= \omega^2 + 2 + \omega^{-2} - \omega - \omega^{-1} - 1 \\ &= \omega^2 + 2 + \omega^{-2}(-\omega^5) - \omega - \omega^{-1}(-\omega^5) - 1, \text{ since } -\omega^5 = 1 \quad \checkmark \\ &= 1 - \omega + \omega^2 - \omega^3 + \omega^4 \\ &= 0, \quad \text{since } \omega^5 + 1 = 0\end{aligned}$$

$$\therefore (\omega+1)(1-\omega+\omega^2-\omega^3+\omega^4) = 0$$

$$\therefore 1-\omega+\omega^2-\omega^3+\omega^4 = 0, \text{ since } \omega \neq -1 \quad \checkmark$$ ($\omega \neq -1$ needed to be written)

Marking Scheme		
✓	[1]	for correctly showing that $\omega^5 = -1$
✓	[1]	for correct progress and multiplying ω^{-2} and ω^{-1} by $-\omega^5$
✓	[1]	for correctly showing that $(\omega+\omega^{-1})^2 - (\omega+\omega^{-1}) - 1 = 0$, with justification

Marker's Comments:

- Generally not done well. Students who had their solutions set out like the one shown above were usually successful.
- Some students were also successful in showing the required result by first showing that $\omega^5 = -1$, then showing that $1-\omega+\omega^2-\omega^3+\omega^4 = 0$, then divided both sides by ω^2, followed by using algebraic techniques.
- Most students had difficulty in attempting to show that $(\omega+\omega^{-1})^2 - (\omega+\omega^{-1}) - 1 = 0$, and some students were also not able to show that $\omega^5 = -1$ or that $1-\omega+\omega^2-\omega^3+\omega^4 = 0$.
- Some students also incorrectly factorised $\omega^5 + 1$ as $(\omega+1)(1+\omega+\omega^2+\omega^3+\omega^4)$.
- Some students had also found the complex fifth roots of -1, but did not make clear progress thereafter to show the required result.

Question 8 (a)(ii) (2 marks)

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \quad \checkmark \end{aligned}$$

$$\therefore e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\therefore \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \checkmark$$

Marking Scheme
✓ [1] for correct progress
✓ [1] for correctly showing the result

Marker's Comments:

- Generally well done.

Question 8 (a)(iii) (2 marks)

Let $\theta = \dfrac{3\pi}{5}$

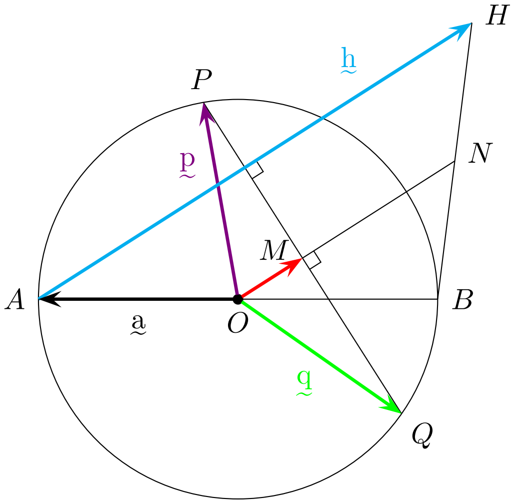
$$\cos\frac{3\pi}{5} = \frac{1}{2}\left(e^{i\frac{3\pi}{5}} + e^{-i\frac{3\pi}{5}}\right), \text{ from (ii)}$$
$$= \frac{1}{2}(\omega + \omega^{-1})$$
$$\therefore \omega + \omega^{-1} = 2\cos\frac{3\pi}{5}$$
$$\left(2\cos\frac{3\pi}{5}\right)^2 - 2\cos\frac{3\pi}{5} - 1 = 0, \text{ from (i)}$$
$$4\cos^2\frac{3\pi}{5} - 2\cos\frac{3\pi}{5} - 1 = 0 \quad \checkmark$$
$$\therefore \cos\frac{3\pi}{5} = \frac{2 \pm \sqrt{4+16}}{8}$$
$$= \frac{2 \pm 2\sqrt{5}}{8}$$
$$= \frac{1 \pm \sqrt{5}}{4}$$
$$\therefore \cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}, \text{ since } \cos\frac{3\pi}{5} < 0 \quad \checkmark$$

Marking Scheme
✓ [1] for correct progress
✓ [1] for correct value of $\cos\frac{3\pi}{5}$

Marker's Comments:

- Generally well done, however some students had incorrectly deduced that $\cos\frac{3\pi}{5} = \frac{1+\sqrt{5}}{4}$, as they thought that $\frac{3\pi}{5}$ was acute (i.e. in the first quadrant, and hence $\cos\frac{3\pi}{5} > 0$).

Question 8 (b)(i) (2 marks)



$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP} + \overrightarrow{PM}\\ &= \underset{\sim}{p} + \frac{1}{2}\overrightarrow{PQ}\\ &= \underset{\sim}{p} + \frac{1}{2}(\underset{\sim}{q} - \underset{\sim}{p})\\ &= \frac{1}{2}(\underset{\sim}{p} + \underset{\sim}{q}) \quad \checkmark\end{aligned}$$

$$\begin{aligned}\overrightarrow{OM} \cdot \overrightarrow{PQ} &= \frac{1}{2}(\underset{\sim}{p} + \underset{\sim}{q}) \cdot (\underset{\sim}{q} - \underset{\sim}{p})\\ &= \frac{1}{2}(|\underset{\sim}{q}|^2 - |\underset{\sim}{p}|^2)\\ &= \frac{1}{2}(r^2 - r^2), \text{ where } r \text{ is the radius}\\ &= 0\end{aligned}$$

$\therefore \angle OMQ = 90^\circ$

$\therefore O$, M and N are collinear (straight angle) ✓

Marking Scheme

✓ [1] for correct expression of $\overrightarrow{OM}$, or equivalent merit

✓ [1] for correctly proving that O, M and N are collinear (giving the reason 'straight angle' was not required to obtain the mark)

Marker's Comments:

- Most students had correctly found the expression of $\overrightarrow{OM}$, but were not able to use this fact to deduce that O, M and N are collinear. Students who correctly showed that $\overrightarrow{OM} \cdot \overrightarrow{PQ} = 0$ were awarded the second mark.

Question 8 (b)(ii) (1 mark)

$$
\begin{aligned}
\overrightarrow{ON} &= \overrightarrow{OB} + \overrightarrow{BN} \\
&= -\underset{\sim}{a} + \lambda\overrightarrow{BH} \\
&= -\underset{\sim}{a} + \lambda(2\underset{\sim}{a} + \underset{\sim}{h}) \\
&= (2\lambda - 1)\underset{\sim}{a} + \lambda\underset{\sim}{h} \quad \checkmark
\end{aligned}
$$

Marking Scheme

✓ [1] for correct expression of $\overrightarrow{ON}$

Marker's Comments:

- Mixed responses. The main error made by students was deducing that $\overrightarrow{ON} = (\lambda - 1)\underset{\sim}{a} + \lambda\underset{\sim}{h}$.

Question 8 (b)(iii) (3 marks)

$$\overrightarrow{ON} \cdot \overrightarrow{PQ} = 0$$

$$\Big((2\lambda - 1)\underset{\sim}{a} + \lambda\underset{\sim}{h}\Big) \cdot (\underset{\sim}{q} - \underset{\sim}{p}) = 0$$

$$(2\lambda - 1)\underset{\sim}{a} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) + \lambda\underset{\sim}{h} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) = 0$$

$$\therefore (2\lambda - 1)\underset{\sim}{a} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) = 0 \text{ (since } AH \perp PQ) \quad \checkmark$$

$$\therefore 2\lambda - 1 = 0, \text{ as } PQ \text{ is } \textbf{\textit{not}} \text{ perpendicular to } AB, \text{ so } \underset{\sim}{a} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) \neq 0 \quad \checkmark$$

$$\lambda = \frac{1}{2}$$

$$\therefore \overrightarrow{BN} = \frac{1}{2}\overrightarrow{BH} \quad \therefore N \text{ is the midpoint of } BH \quad \checkmark$$

Marking Scheme	
✓ [1]	for showing that $(2\lambda - 1)\underset{\sim}{a} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) = 0$ (giving the reason 'since $AH \perp PQ$' was not required to obtain the mark)
✓ [1]	for deducing that $2\lambda - 1 = 0$, WITH proper justification
✓ [1]	for correctly concluding that N is the midpoint of BH

Marker's Comments:

- This part proved difficult for majority of the students. Students are advised to learn the techniques shown in the above suggested solutions, and to reattempt this question until mastery.
- Students who noted that $ON \parallel AH$ ***with reasoning***, and subsequently wrote

$$\overrightarrow{ON} = \mu\overrightarrow{AH}$$

$$(2\lambda - 1)\underset{\sim}{a} + \lambda\underset{\sim}{h} = \mu\underset{\sim}{h}$$

$$\therefore 2\lambda - 1 = 0, \quad \lambda = \mu \quad \text{(since } \underset{\sim}{a} \text{ and } \underset{\sim}{h} \text{ are not parallel)}$$

$$\therefore \lambda = \frac{1}{2}$$

$$\therefore \overrightarrow{BN} = \frac{1}{2}\overrightarrow{BH} \quad \therefore N \text{ is the midpoint of } BH$$

  were also successful in showing the required result.

  This is a more elegant alternative solution.

- Some students had also deduced the result by using the fact that $\triangle ONB$ and $\triangle AHB$ are similar. To be awarded full marks from this approach, full working and reasoning needed to be given. Furthermore, students who used this approach are encouraged to reattempt this question using vector methods, as the use of Euclidean Geometry (e.g. similar triangles) may not work in other questions.

Question 9(a)(i)	Marks
· Provides correct solution.	3 marks
· Finds integral in terms of u correctly.	2 marks
· Identifies the substitution $u = \sqrt{x+1}$ or $x = \sec^2\theta$.	1 mark

$$I_{-1} = \int_1^2 \frac{dx}{x\sqrt{x-1}}$$

Use the substitution $u = \sqrt{x-1}$. ✓

$$\begin{aligned} u &= \sqrt{x-1} & x = 1 &\implies u = 0 \\ du &= \frac{dx}{2\sqrt{x-1}} & x = 2 &\implies u = 1 \\ du &= \frac{dx}{2u} & u^2 &= x - 1 \\ 2u\,du &= dx & x &= u^2 + 1 \end{aligned}$$

Therefore

$$\begin{aligned} I_{-1} &= \int_0^1 \frac{2u}{(u^2+1)u}\,du \checkmark \\ &= 2\int_0^1 \frac{1}{u^2+1}\,du \\ &= 2\left[\tan^{-1}(u)\right]_0^1 \\ &= 2\left(\frac{\pi}{4} - 0\right) \\ &= \frac{\pi}{2} \checkmark \end{aligned}$$

Comments

· This question was challenging.

· Successful students used one of the substitutions $u = \sqrt{x-1}$, $u^2 = x - 1$ or $x = \sec^2\theta$.

· Integration by parts was not successful.

· Some errors occurred when finding limits for θ using $x = \sec^2\theta$.

· Sometimes, letting u be the square root appearing in the integrand is a good strategy.

Question 9(a)(ii)	Marks
· Provides correct solution.	3 marks
· Expands and simplifies.	2 marks
· Splits off a factor of $(x-1)$.	1 mark

$$
\begin{aligned}
I_n &= \int_1^2 \frac{(x-1)^{\frac{n}{2}}}{x}\,dx \\
&= \int_1^2 \frac{(x-1)^{\frac{n}{2}-1}(x-1)}{x}\,dx \\
&= \int_1^2 \frac{x(x-1)^{\frac{n}{2}-1}}{x}\,dx - \int_1^2 \frac{(x-1)^{\frac{n}{2}-1}}{x}\,dx \\
&= \int_1^2 (x-1)^{\frac{n}{2}-1}\,dx - \int_1^2 \frac{(x-1)^{\frac{n-2}{2}}}{x}\,dx \\
&= \left[\frac{2}{n}(x-1)^{\frac{n}{2}}\right]_1^2 - I_{n-2} \\
&= \frac{2}{n} - I_{n-2}
\end{aligned}
$$

Comments

· This question was also challenging.

· Again, integration by parts was not successful.

· Fractional powers caused several students to make errors when using the index laws.

Question 9(a)(iii)	**Marks**
· Provides correct solution.	1 mark

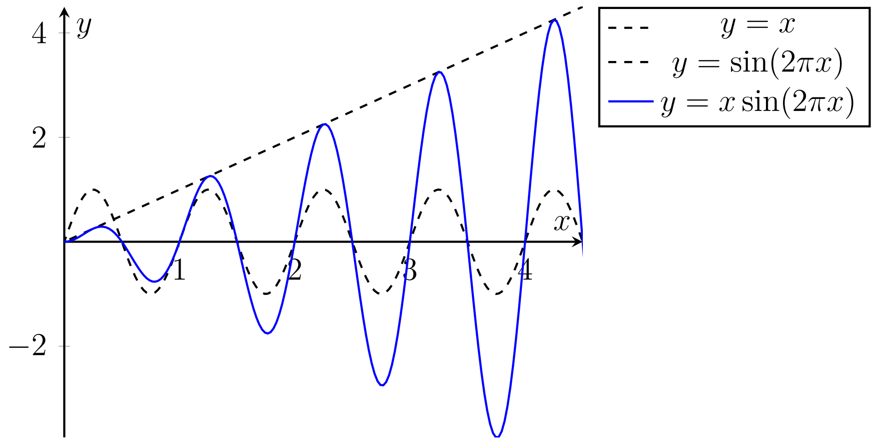
$$
\begin{aligned}
I_1 &= \frac{2}{1} - I_{-1} \\
&= 2 - \frac{\pi}{2} \\
I_3 &= \frac{2}{3} - I_1 \\
&= \frac{2}{3} - \left(2 - \frac{\pi}{2}\right) \\
&= \frac{\pi}{2} - \frac{4}{3} \\
I_5 &= \frac{2}{5} - I_3 \\
&= \frac{2}{5} - \left(\frac{\pi}{2} - \frac{4}{3}\right) \\
&= \frac{26}{15} - \frac{\pi}{2} \checkmark
\end{aligned}
$$

Comments

- Quite well done.
- This part can be done without having attempted (a)(i) or (a)(ii).
- You are allowed to use the result of (i) without having proved it.
- Students are strongly encouraged to attempt this question for a relatively easy mark.

Question 9(b)	Marks
· Provides correct solution.	3 marks
· Finds $y = \pm t\cos(2\pi t)$ if $z = t$.	2 marks
· Finds $x = z\sin(2\pi z)$.	1 mark

Following the hint, let's consider the sketch of $y = f(x)g(x)$ for $x \geq 0$ where $f(x) = x$ and $g(x) = \sin x$. We use the method of multiplication of y-values from the Year 11 Extension 1 course (assumed knowledge for Year 12 Extension 2).



We realise that this sketch is how the conical spiral would appear in the original 3D diagram when it is viewed side-on from the positive $y-$axis, rotating 90° clockwise so the $x-$axis is vertical and the $z-$axis is horizontal. Thus, the curve $x = z\sin(2\pi z)$ is the projection of the conical spiral on to the $xz-$plane.

Let $z = t$ so that $x = t\sin(2\pi t)$.

Substitute these into the equation of the cone, $z = \sqrt{x^2 + y^2}$:

$$\begin{aligned} t &= \sqrt{(t\sin(2\pi t))^2 + y^2} \\ t^2 &= t^2\sin^2(2\pi t) + y^2 \\ y^2 &= t^2 - t^2\sin^2(2\pi t) \\ y^2 &= t^2(1 - \sin^2(2\pi t)) \\ y^2 &= t^2\cos^2(2\pi t) \\ y &= \pm t\cos(2\pi t) \end{aligned}$$

However, the spiral passes through $(0,\, 4,\, 4)$. Therefore, when $y = 4$, $z = t = 4$.

$$\begin{aligned} 4 &= \pm 4\cos(2\pi \times 4) \\ 4 &= \pm 4 \times 1 \\ 4 &= \pm 4 \end{aligned}$$

Thus, we must take the positive square root. Hence, the final answer is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t\sin(2\pi t) \\ t\cos(2\pi t) \\ t \end{pmatrix}, \text{ for } 0 \leq t \leq 4.$$

Comments

· This question was very challenging.

· Writing down $y = x\sin(2\pi x)$ and manipulating it was not awarded any marks.

· Students were required to connect the sketch $y = x\sin(2\pi x)$ to the given diagram.

· Varied answers for x, y, z in terms of t seemed consistent with random guessing.

· The domain restriction $0 \leq t \leq 4$ was an important detail required for full marks.

Question 9(c)	Marks
· Provides correct solution.	2 marks
· Finds integral in terms of $u = \sin x$ and uses $\tan^2 u = \sec^2 u - 1$.	1 mark

Let the required integral be I. First simplify using $1 - 2\sin^2\left(\frac{x}{2}\right) = \cos x$.
The required integral becomes:

$$I = \int \sec^3(\sin x)\tan^5(\sin x)\cos x\,dx.$$

Now let $u = \sin x$ so that $du = \cos x\,dx$.

$$\begin{aligned}
\therefore I &= \int \sec^3 u\,\tan^5 u\,du \\
&= \int \sec^3 u\,(\tan^2 u)^2 \tan u\,du \\
&= \int \sec^3 u\,(\sec^2 u - 1)^2 \tan u\,du \checkmark \\
&= \int \sec^3 u\,(\sec^4 u - 2\sec^2 u + 1)\tan u\,du \\
&= \int (\sec^7 u - 2\sec^5 u + \sec^3 u)\tan u\,du \\
&= \int (\sec^6 u\,\sec u\,\tan u - 2\sec^4 u\,\sec u\,\tan u + \sec^2 u\,\sec u\,\tan u)\,du \\
&= \frac{1}{7}\sec^7 u - \frac{2}{5}\sec^5 u + \frac{1}{3}\sec^3 u + C \\
&= \frac{1}{7}\sec^7(\sin x) - \frac{2}{5}\sec^5(\sin x) + \frac{1}{3}\sec^3(\sin x) + C \checkmark
\end{aligned}$$

Comments

· Quite well done.
· The given integral is "a sheep in wolf's clothing".
· It simplifies enormously after using the double angle result and $u = \sin x$.
· However, more progress was needed to obtain the first mark.
· Students needed to apply the Pythagorean identity to get $\tan^4 u = (\sec^2 u - 1)^2$.