
CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2017 Half-Yearly Examination

FORM VI

MATHEMATICS EXTENSION 2

Wednesday 22nd February 2017

General Instructions

- Writing time — 1 hour 30 minutes
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 55 Marks

- All questions may be attempted.

Section I – 7 Marks

- Questions 1–7 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 48 Marks

- Questions 8–11 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eight.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 74 boys

Examiner

LRP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Given that $z = x + iy$, which expression is equal to $|z|$?

(A) $x + y$

(B) $z\overline{z}$

(C) $x^2 + y^2$

(D) $\sqrt{z\overline{z}}$

QUESTION TWO

Given that $z = i\sqrt{3} - 1$, which of the following represents the complex number z?

(A) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(B) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(C) $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

(D) $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

QUESTION THREE

Which expression is equal to $\displaystyle\int \frac{1}{\sqrt{6x - x^2}}\,dx$?

(A) $\sin^{-1}\left(\dfrac{x+3}{3}\right) + C$

(B) $-\sin^{-1}\left(\dfrac{x+3}{3}\right) + C$

(C) $\sin^{-1}\left(\dfrac{x-3}{3}\right) + C$

(D) $\sin^{-1}\left(\dfrac{3-x}{3}\right) + C$

Examination continues next page ...

QUESTION FOUR

Which of the following is a primitive of $\cos^3 x \sin x$?

(A) $4\cos^4 x - 3\cos^2 x$

(B) $\dfrac{\cos^4 x}{4}$

(C) $\dfrac{\sin^4 x}{4} - \dfrac{\sin^2 x}{2}$

(D) $-\dfrac{\cos^4 x}{4}$

QUESTION FIVE

Given $x^2y - y^2 = 1$, which of the following is the correct result for $\dfrac{dy}{dx}$?

(A) $\dfrac{1 - 2xy}{x^2 - 2y}$

(B) $\dfrac{2xy}{x^2 + 2y}$

(C) $\dfrac{2xy}{x^2 - 2y}$

(D) $\dfrac{2xy}{2y - x^2}$

QUESTION SIX

Which expression is equal to $\displaystyle\int \frac{dx}{e^x + 1}$?

(A) $-\dfrac{e^x}{(e^x + 1)^2} + C$

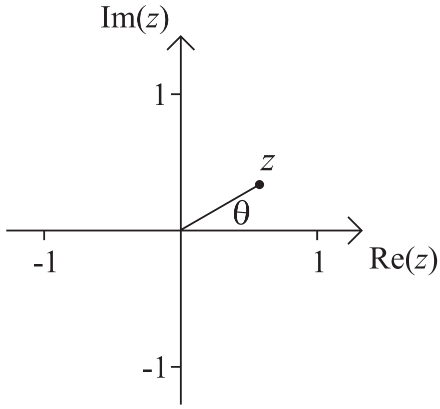
(B) $x - \log_e(e^x + 1) + C$

(C) $x - \dfrac{1}{e^x} + C$

(D) $\dfrac{\log_e(e^x + 1)}{e^x} + C$

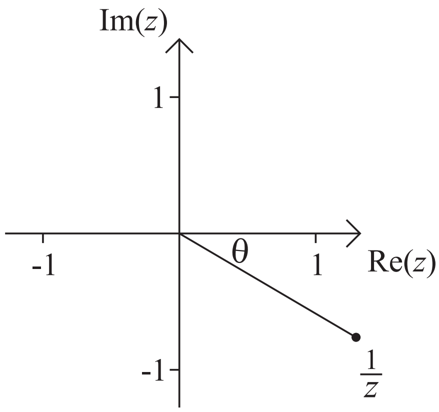
Examination continues overleaf ...

QUESTION SEVEN

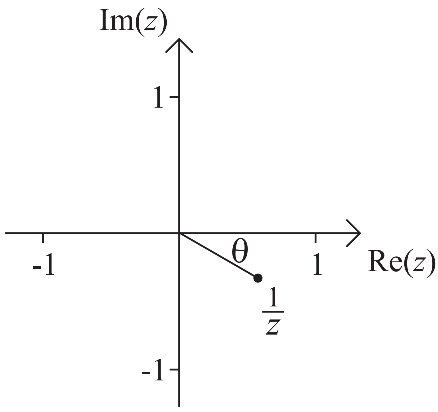


The Argand diagram above shows the complex number z. Which of the following diagrams best shows the complex number $\dfrac{1}{z}$?

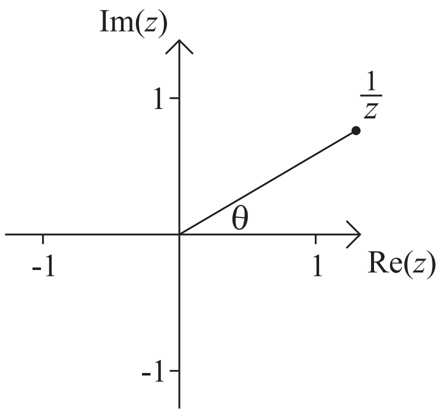
(A)



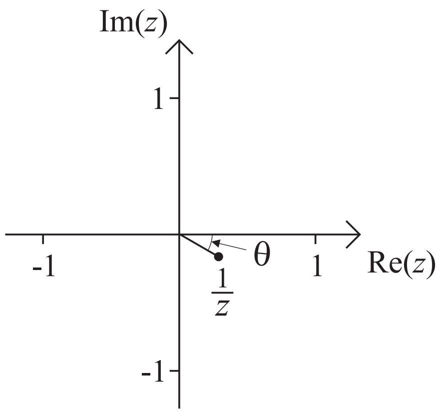
(B)



(C)



(D)



End of Section I

Examination continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) Let $w = -1 + i$ and $z = 4 + 3i$.

(i) Find $2w - \overline{z}$. **1**

(ii) Find $\operatorname{Im}\left(\dfrac{z}{i}\right)$. **1**

(iii) Express $\dfrac{w}{z}$ in the form $a + ib$, where a and b are real numbers. **2**

(b) (i) Find the two square roots of $5 - 12i$. **2**

(ii) Hence, or otherwise, solve $z^2 - (1 - 4i)\,z - (5 - i) = 0$. **2**

(c) Let $z = \cos\theta + i\sin\theta$, where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. **2**
Plot points representing the positions of z, $2z$ and z^2 on an Argand diagram.

(d) Sketch the locus defined by the intersection of $0 \leq \arg(z + 2) \leq \dfrac{\pi}{6}$ and $\operatorname{Re}(z) \geq 0$. **2**

Examination continues overleaf ...

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Use the compound–angle formulae to show that **1**

$$\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}.$$

(ii) Hence evaluate $\displaystyle\int_0^{\pi} \cos 3x \sin 2x \, dx$. **2**

(b) Find:

(i) $\displaystyle\int \frac{6x^3 + 3x^2 - 2}{2x+1} \, dx$ **2**

(ii) $\displaystyle\int \sin^3 x \cos^2 x \, dx$. **2**

(c) Evaluate $\displaystyle\int_1^{e} \frac{(\log_e x)^2}{x} \, dx$. **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{2 - \cos x} \, dx$. **3**

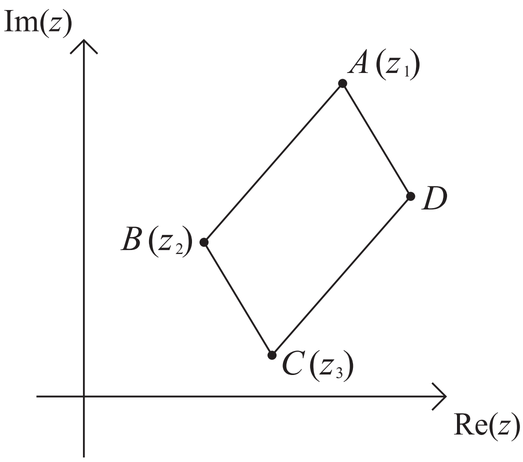
QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Find real numbers A, B, C and D such that **2**

$$\frac{x}{x^4-1} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}.$$

(ii) Hence find $\displaystyle\int \frac{x}{x^4-1}\,dx$. **2**

(b) **1**



In the Argand diagram above, $ABCD$ is a parallelogram. The points A, B and C represent the complex numbers z_1, z_2 and z_3 respectively. Find the complex number represented by the vector $\overrightarrow{BD}$ in terms of z_1, z_2 and z_3.

(c) A complex number z satisfies $|z+4-3i| = 2$.

(i) Sketch the locus of z in the complex plane. **1**

(ii) Find the maximum value of $|z|$. **1**

(iii) Determine the values of k for which the simultaneous equations **1**

$$|z+4-3i| = 2 \quad \text{and} \quad |z+4| = k$$

have exactly one solution for z.

(d) (i) Use a suitable substitution to show that $\displaystyle\int e^{\sqrt{x}}\,dx = \int 2ue^u\,du$. **1**

(ii) Hence use integration by parts to evaluate $\displaystyle\int_0^4 e^{\sqrt{x}}\,dx$. **3**

QUESTION ELEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) A complex number z satisfies $|z-2| = 2$ and $0 < \arg(z) < \frac{\pi}{2}$.

(i) Use a diagram to show that $\arg(z-2) = 2\arg(z)$. **1**

(ii) Show that $\arg\left(z^2 - 2z\right) = \dfrac{3\arg(z-2)}{2}$. **1**

(b) Let $f(x) = \tan^{-1}\left(x + \dfrac{1}{x}\right)$, where $x > 0$.

(i) Find the coordinates of the minimum stationary point. You do not have to justify that it is a minimum turning point. **2**

(ii) Find the equation of the horizontal asymptote. **1**

(iii) Find $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0^+} f'(x)$. **2**

(iv) Sketch the graph of the function. **1**

(c) Let $z = x + iy$ be a complex number, where x and y are real numbers. Suppose that z satisfies

$$z\overline{z} + (1-2i)\,z + (1+2i)\,\overline{z} \leq 3.$$

(i) Find the equation of the locus of z and sketch this locus on an Argand diagram. **2**

(ii) Find the maximum and minimum values of $x + y$. **2**

HINT: Consider the line $x + y = k$.

End of Section II

END OF EXAMINATION

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

EXTENSION 2

Half Yearly 2017 - SOLUTIONS

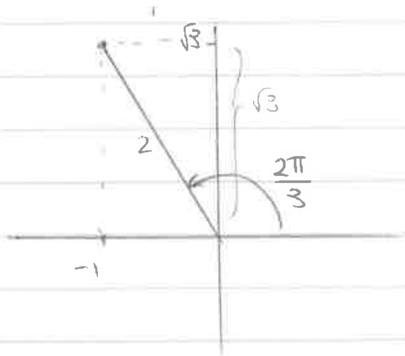
Q1. D
Q2. C
Q3. C
Q4. D
Q5. D
Q6. B
Q7. A

Working:

Q1. $$z\bar{z} = (x+iy)(x-iy)$$
$$= x^2 + y^2$$
$$= |z|^2$$

$$\therefore |z| = \sqrt{z\bar{z}}$$ D

Q2.



$$\arg(z) = \pi - \frac{\pi}{3}$$
$$= \frac{2\pi}{3}$$ C

Q3. $-(x^2-6x) = 9-(x^2-6x+9)$
$= 3^2-(x-3)^2$

$\therefore \int \frac{1}{\sqrt{6x-x^2}}\,dx = \int \frac{1}{\sqrt{3^2-(x-3)^2}}\,dx$

$= \sin^{-1}\left(\frac{x-3}{3}\right) + C$ C

Q4. $\int \cos^3 x \sin x \, dx$

$= -\int \cos^3 x(-\sin x)\,dx$

$= -\frac{\cos^4 x}{4} + C$ D

Q5. $2xy + x^2\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2y-x^2) = 2xy$

$\therefore \frac{dy}{dx} = \frac{2xy}{2y-x^2}$ D

Q6. $\int \frac{dx}{e^x+1} = \int \frac{e^x+1-e^x}{e^x+1}\,dx$

$= \int \left(1 - \frac{e^x}{e^x+1}\right)dx$

$= x - \ln(e^x+1) + C$ B

Q7. $\frac{1}{z} = 1\text{cis}0 \div r\,\text{cis}\,\theta$
$= \frac{1}{r}\text{cis}(-\theta)$

$r \doteqdot \frac{2}{3} \quad \therefore \frac{1}{r} \doteqdot \frac{3}{2}$ A

QUESTION 8

a) $\omega = -1 + i$ and $z = 4 + 3i$

i) $2\omega - \bar{z} = 2(-1+i) - (4-3i)$
$= -6 + 5i$ ✓

ii) $\text{Im}\left(\frac{z}{i}\right) = \text{Im}\left(\frac{4+3i}{i} \times \frac{-i}{-i}\right)$

$= \text{Im}(3 - 4i)$
$= -4$ ✓

iii) $\frac{\omega}{z} = \frac{-1+i}{4+3i} \times \frac{4-3i}{4-3i}$ ✓

$= \frac{-4 + 3i + 4i + 3}{16 + 9}$

$= -\frac{1}{25} + \frac{7}{25}i$ ✓

b) i) Let $(x+iy)^2 = 5 - 12i$

$\therefore \quad x^2 + y^2 = 5$
$\quad\quad 2xy = -12$ ✓
$\quad\quad xy = -6$

By inspection $x = 3$, $y = -2$ or $x = -3$, $y = 2$

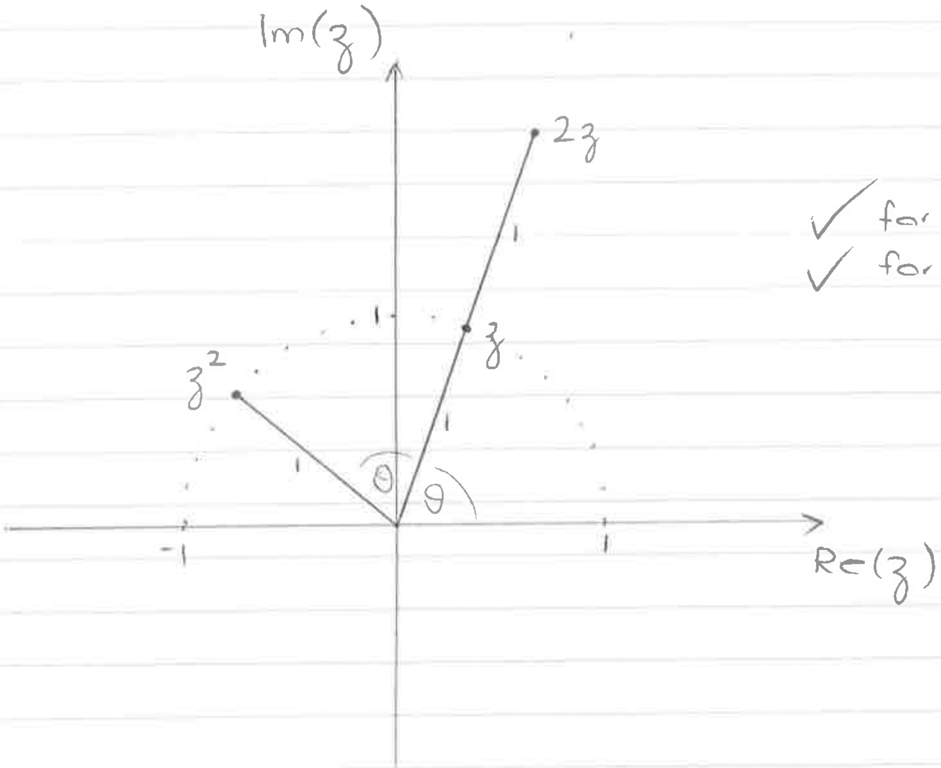
$\therefore$ the square roots are $3 - 2i$ or $-3 + 2i$ ✓

ii) $\Delta = (-(1-4i))^2 - 4 \times 1 \times -(5-i)$
$= 1 - 8i - 16 + 20 - 4i$
$= 5 - 12i$ ✓

$z = \frac{1 - 4i + 3 - 2i}{2}$ or $\frac{1 - 4i - 3 + 2i}{2}$

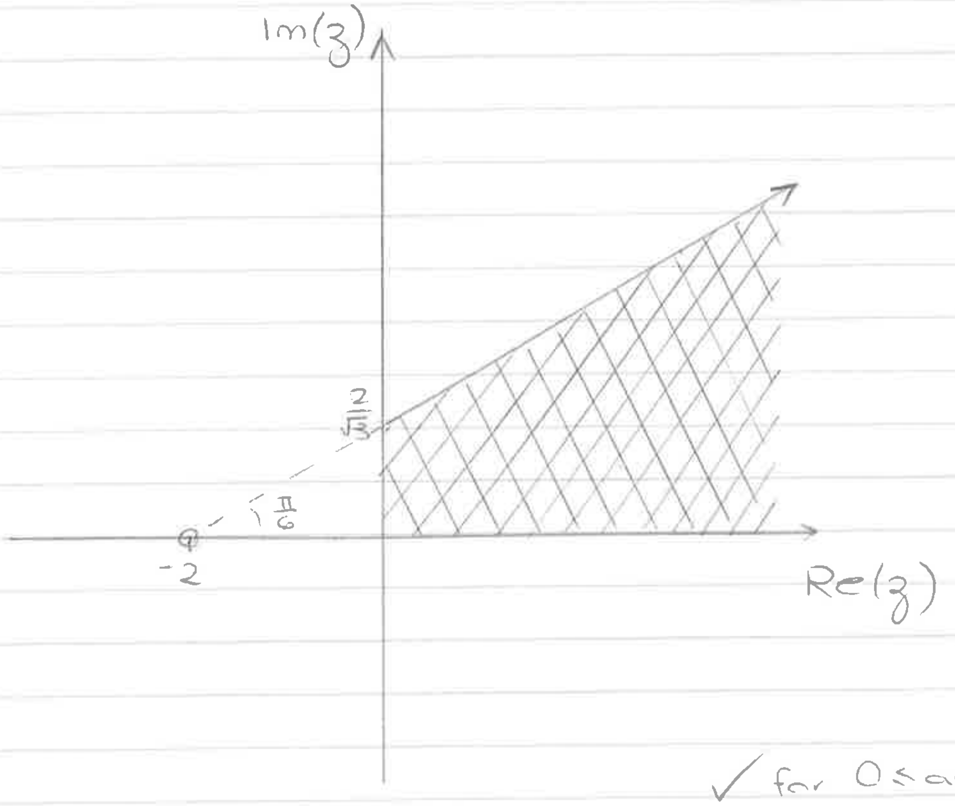
$= 2 - 3i$ $\quad -1 - i$ ✓

c)



✓ for two correct
✓ for all correct

d)



✓ for $0 \le \arg(z+2) \le \frac{\pi}{6}$
✓ for intersection with $\mathrm{Re}(z) \ge 0$

QUESTION 9:

a) i) RHS $= \dfrac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$

$= \dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta-(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{2}$

$= \dfrac{2\cos\alpha\sin\beta}{2}$ ✓ show that...

$= \cos\alpha\sin\beta$

$=$ LHS

ii) $\displaystyle\int_0^{\pi}\cos 3x\sin 2x\,dx = \int_0^{\pi}\frac{\sin(3x+2x)-\sin(3x-2x)}{2}\,dx$

$\displaystyle = \frac{1}{2}\int_0^{\pi}(\sin 5x-\sin x)\,dx$ ✓

$\displaystyle = \frac{1}{2}\left[-\frac{\cos 5x}{5}+\cos x\right]_0^{\pi}$

$\displaystyle = \frac{1}{2}\left(-\frac{(-1)}{5}+(-1)-\left(-\frac{1}{5}+1\right)\right)$

$= -\dfrac{4}{5}$ ✓

b) i) $\displaystyle\int\frac{6x^3+3x^2-2}{2x+1}\,dx = \int\frac{3x^2(2x+1)-2}{2x+1}\,dx$ ✓

$\displaystyle = \int\left(3x^2-\frac{2}{2x+1}\right)dx$

$= x^3 - \ln|2x+1| + C$ ✓

b) ii) $\int \sin^3 x \cos^2 x \, dx$

$= \int \sin x (1-\cos^2 x)\cos^2 x \, dx$

$= \int (\cos^2 x - \cos^4 x)\sin x \, dx$ ✓

Let $u = \cos x$

$du = -\sin x \, dx$

$= -\int (u^2 - u^4) \, du$

$= -\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$

$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$ ✓

c) $\int_1^e \frac{(\log_e x)^2}{x} \, dx$

Let $u = \log_e x$

$du = \frac{1}{x} dx$ ✓

x	1	e
u	0	1

$= \int_0^1 u^2 \, du$

$= \left[\frac{u^3}{3}\right]_0^1$

$= \frac{1}{3}$ ✓

d) $\int_0^{\frac{\pi}{3}} \frac{1}{2-\cos x} \, dx$

Let $t = \tan\frac{x}{2}$

$x = 2\tan^{-1} t$

$dx = \frac{2}{1+t^2} dt$ ✓

$\cos x = \frac{1-t^2}{1+t^2}$

x	0	$\frac{\pi}{3}$
t	0	$\frac{1}{\sqrt{3}}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{2 - \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2} \, dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{2+2t^2-1+t^2} \, dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{1+3t^2} \, dt$ ✓

$= \frac{2}{3} \int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{(\frac{1}{\sqrt{3}})^2 + t^2}$

$= \frac{2}{3} \left[\sqrt{3} \tan^{-1}(\sqrt{3}t)\right]_0^{\frac{1}{\sqrt{3}}}$

$= \frac{2\sqrt{3}}{3}\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \frac{2\sqrt{3}}{3} \times \frac{\pi}{4} = \frac{\pi\sqrt{3}}{6}$ ✓

QUESTION 10:

a) i) $\frac{x}{(x^2+1)(x+1)(x-1)} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

$x = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x+1)(x-1)$

Sub $x=1$: $1 = 4A$
$\therefore A = \frac{1}{4}$

Sub $x=-1$: $-1 = -4B$
$\therefore B = \frac{1}{4}$ ✓

Sub $x=0$: $0 = \frac{1}{4} - \frac{1}{4} - D$
$\therefore D = 0$

Coeff of x^3: $0 = \frac{1}{4} + \frac{1}{4} + C$
$\therefore C = -\frac{1}{2}$ ✓

ii) $\int \frac{x}{x^4-1}\,dx = \int \left(\frac{1}{4(x-1)} + \frac{1}{4(x+1)} - \frac{x}{2(x^2+1)} \right) dx$

$= \frac{1}{4}\int \left(\frac{1}{x-1} + \frac{1}{x+1} - \frac{2x}{x^2+1} \right) dx$ ✓

$= \frac{1}{4}\left(\ln|x-1| + \ln|x+1| - \ln(x^2+1) \right) + C$ ✓

$= \frac{1}{4}\ln\left| \frac{(x-1)(x+1)}{x^2+1} \right| + C$

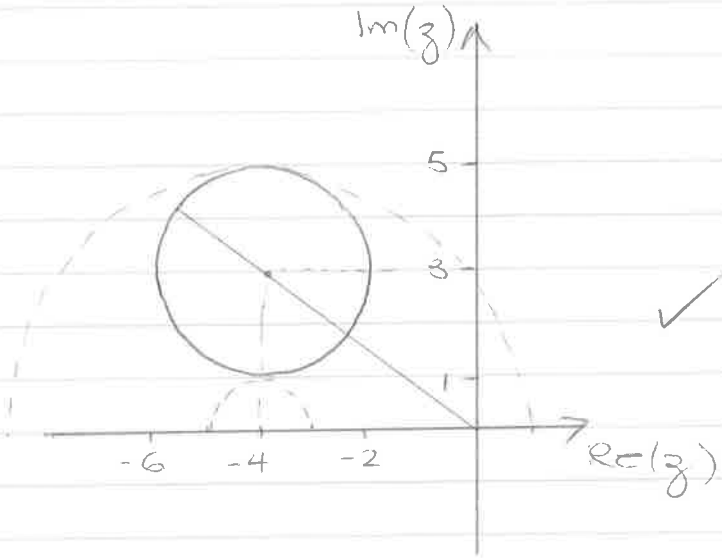
b) $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$= \overrightarrow{BC} + \overrightarrow{BA}$ ($\overrightarrow{CD} = \overrightarrow{BA}$ since ABCD is a parallelogram)

$= z_3 - z_2 + z_1 - z_2$

$= z_1 - 2z_2 + z_3$ ✓

c) $\left| z - (-4+3i) \right| = 2$



ii) $|z|_{max} = \sqrt{4^2+3^2} + 2$
$= 7$

iii) $\left| z - (-4) \right| = k$

$k = 1$ or 5

d) i) Let $u = \sqrt{x}$
$\therefore x = u^2$
$\frac{dx}{d} = 2u$
$dx = 2u \, du$

$$\int e^{\sqrt{x}} dx = \int e^u \times 2u \, du$$
$$= \int 2ue^u \, du$$

d) ii) $\int_0^4 e^{\sqrt{x}}\,dx$ $\qquad u = \sqrt{x}$

x	0	4
u	0	2

$= \int_0^2 2ue^u\,du$

$= 2\int_0^2 u \frac{d}{du}(e^u)\,du$ ✓

$= 2\left\{ [ue^u]_0^2 - \int_0^2 1 \times e^u\,du \right\}$ ✓

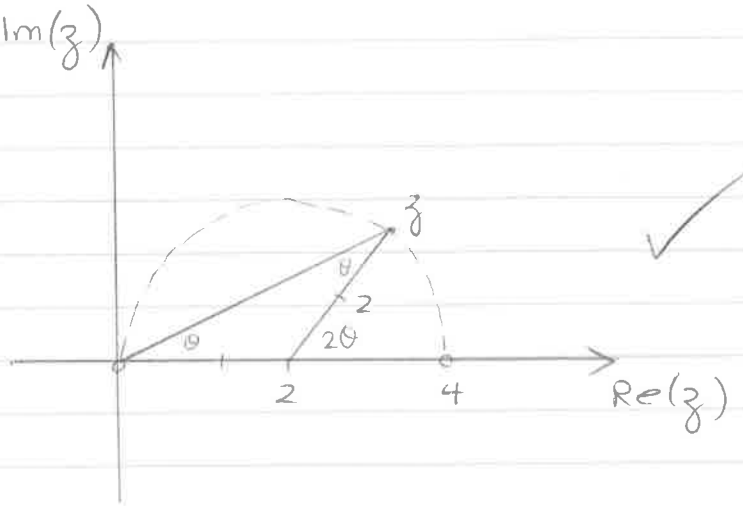
$= 2\left(2e^2 - 0 - [e^u]_0^2\right)$

$= 2\left(2e^2 - (e^2 - e^0)\right)$

$= 2\left(e^2 + 1\right)$ ✓

QUESTION 11:

a) i)



$\arg z = \theta$

$\arg(z-2) = 2\theta$

$= 2\arg z$

angles at centre and circumference on the same arc.

or// external angle of isosceles triangle ...

ii) $\arg(z^2-2z) = \arg(z(z-2))$

$= \arg z + \arg(z-2)$

$= \dfrac{\arg(z-2)}{2} + \arg(z-2)$

$= \dfrac{3\arg(z-2)}{2}$

b) i) $f(x) = \tan^{-1}(x+\frac{1}{x})$

$f'(x) = \dfrac{1}{1+(x+\frac{1}{x})^2} \times (1-\frac{1}{x^2})$

$= 0$ when $1-\frac{1}{x^2} = 0$

$\therefore x = \pm 1$

BUT $x > 0$

$\therefore x = 1$

$y = \tan^{-1} 2$

$\therefore$ stationary point at $(1, \tan^{-1} 2)$

ii) $y = \lim_{x \to \infty} \tan^{-1}\left(x + \frac{1}{x}\right)$

$= \lim_{x \to \infty} \tan^{-1} x$

$= \frac{\pi}{2}$

$\therefore$ horizontal asymptote at $y = \frac{\pi}{2}$.

iii) $\lim_{x \to 0^+} \tan^{-1}\left(x + \frac{1}{x}\right) = \lim_{x \to \infty} \tan^{-1}\left(x + \frac{1}{x}\right)$

$\therefore \lim_{x \to 0} f(x) = \frac{\pi}{2}$

$$f'(x) = \frac{1}{1 + \left(x + \frac{1}{x}\right)^2} \times \left(1 - \frac{1}{x^2}\right)$$

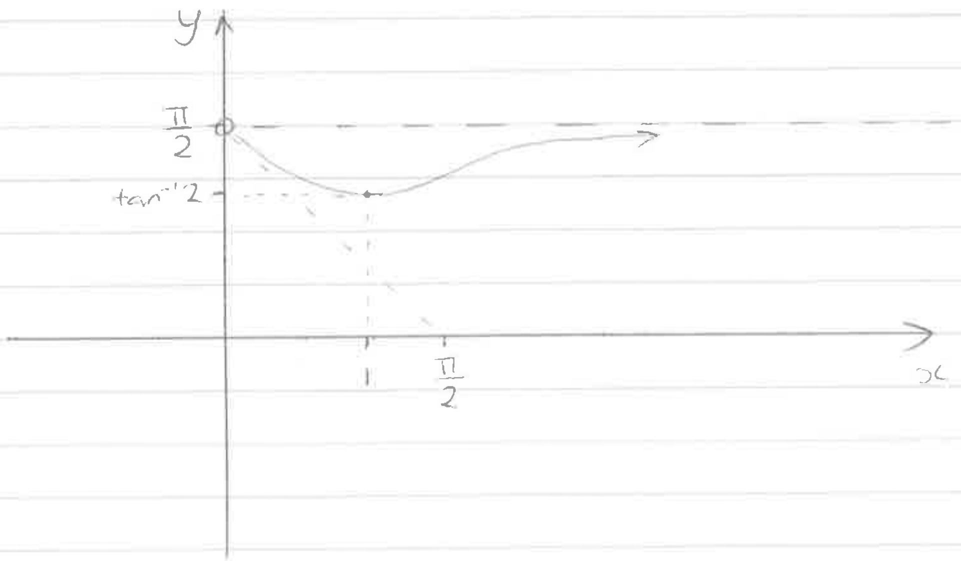
$$= \frac{1}{1 + x^2 + 2 + \frac{1}{x^2}} \times \frac{x^2 - 1}{x^2}$$

$$= \frac{x^2 - 1}{x^4 + 3x^2 + 1} \qquad x \neq 0$$

$\therefore \lim_{x \to 0} f'(x) = \frac{-1}{1}$

$= -1$

iv)



c) i) $x^2 + y^2 + (1-2i)(x+iy) + (1+2i)(x-iy) \leq 3$

$$x^2 + y^2 + x + \cancel{iy} - \cancel{2ix} + 2y + x - \cancel{iy} + \cancel{2ix} + 2y \leq 3$$

$$x^2 + 2x + 1 + y^2 + 4y + 4 \leq 3 + 1 + 4$$

$$(x+1)^2 + (y+2)^2 \leq 8$$

OR//

$$(z + (1+2i))(\overline{z + (1+2i)}) \leq 3 + |1+2i|^2$$

$$|z + 1 + 2i|^2 \leq 8$$

$$|z - (-1-2i)| \leq 2\sqrt{2}$$

OR//

$$z\bar{z} + (1-2i)z + \overline{(1-2i)z} \leq 3$$

$$z\bar{z} + 2\,\mathrm{Re}((1-2i)z) \leq 3$$

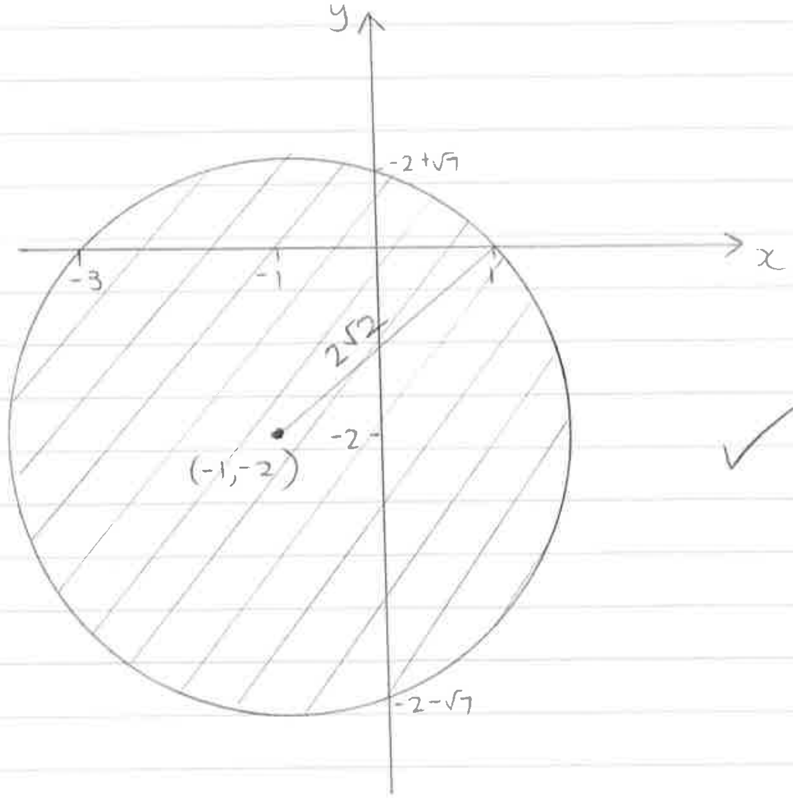
$$x^2 + y^2 + 2\,\mathrm{Re}((1-2i)(x+iy)) \leq 3$$

$$x^2 + y^2 + 2(x + 2y) \leq 3$$

$$x^2 + 2x + 1 + y^2 + 4y + 4 \leq 3 + 1 + 4$$

$$(x+1)^2 + (y+2)^2 \leq 8$$

c) i) cont...

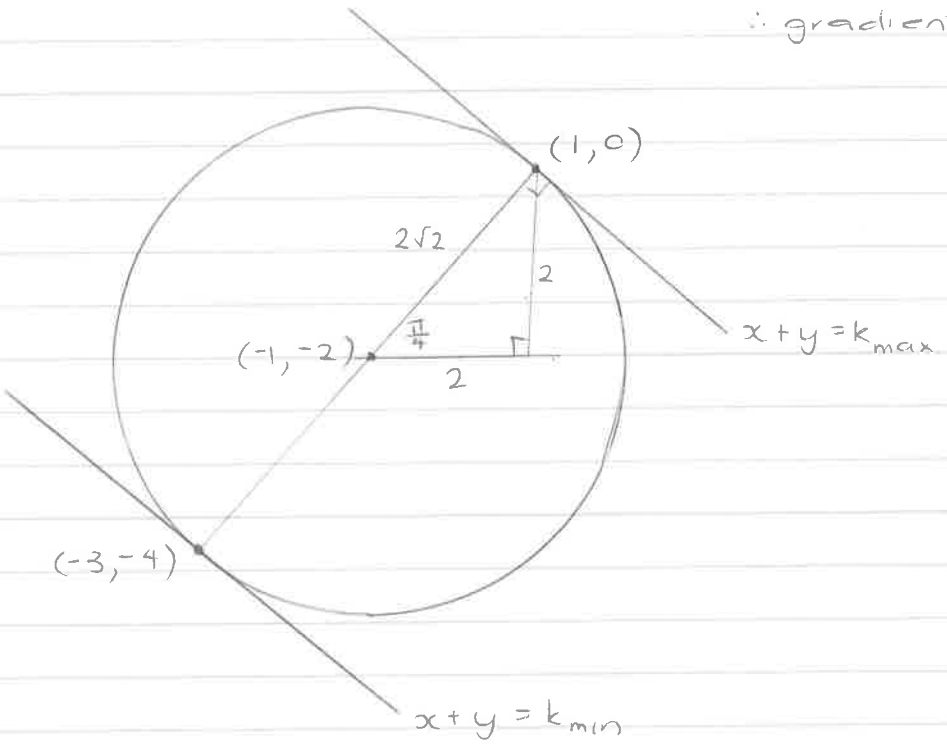


(intercept values not req'd for mark)

ii) the max & min values will occur when the line is tangent to the circle.

gradient of $x+y=k$ is -1

$\therefore$ gradient of perpendicular is 1.



$(1,0)$ satisfies equation $\therefore (x+y)_{max} = 1+0$
$= 1$

$(-3,-4)$ satisfies equation $\therefore (x+y)_{min} = -3-4$
$= -7$