SYDNEY GRAMMAR SCHOOL

CANDIDATE NUMBER

2019 Term IV Alternative Assessment

Form VI Mathematics Extension 2

Wednesday 27th November 2019

General Instructions

- Working time — 1 hour 15 minutes
- Attempt all questions
- Write using black pen
- Calculators approved by NESA may be used
- A loose reference sheet is provided separate to this paper
- Pencils may be used for graphs and annotating diagrams

Total Marks: 43

Section I (10 marks) Question 1
- Relevant mathematical reasoning and calculations are required
- Answer the questions in this paper in the spaces provided

Section II (14 marks) Question 2–3
- Relevant mathematical reasoning and calculations are required
- Answer the questions in this paper in the spaces provided

Section III (19 marks) Question 4–5
- Relevant mathematical reasoning and calculations are required
- Answer the questions in this paper in the spaces provided

Collection

- Write your candidate number on this page and on the start of the separate sections
- Place everything inside this question booklet

Checklist
- Reference sheet
- Candidature: 82 pupils

Writer: WJM

Section I

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Record your answers in the space provided in this paper.

QUESTION ONE (10 marks) **Marks**

(a) (i) Write down the **converse** of the following statement: **1**

If $n \geq 100$, then $\sqrt{n} \geq 10$.

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(ii) Write down the **contrapositive** of the following statement: **1**

If $n \geq 0$, then $\sqrt{n}$ is a real number.

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(b) Write a true mathematical implication in which the converse is **not** true. **2**

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QUESTION ONE (Continued)

(c) (i) Prove that $\sqrt{7}$ is irrational. **3**

(ii) Consider the following statement: **1**

If I add two irrational numbers, then the result will be irrational.

Prove that this statement is false by providing a counterexample.

QUESTION ONE (Continued)

(d) The right-hand side of the following statement, which has been underlined, contains an error. **2**

$$p \text{ is a multiple of } q \Rightarrow \underline{\exists\ m \in \mathbf{Z} \text{ such that } q = pm}$$

Circle the error in the underlined section of the statement above, and give a brief explanation of why it is an error.

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End of Section I

The paper continues in the next section

CANDIDATE NUMBER

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
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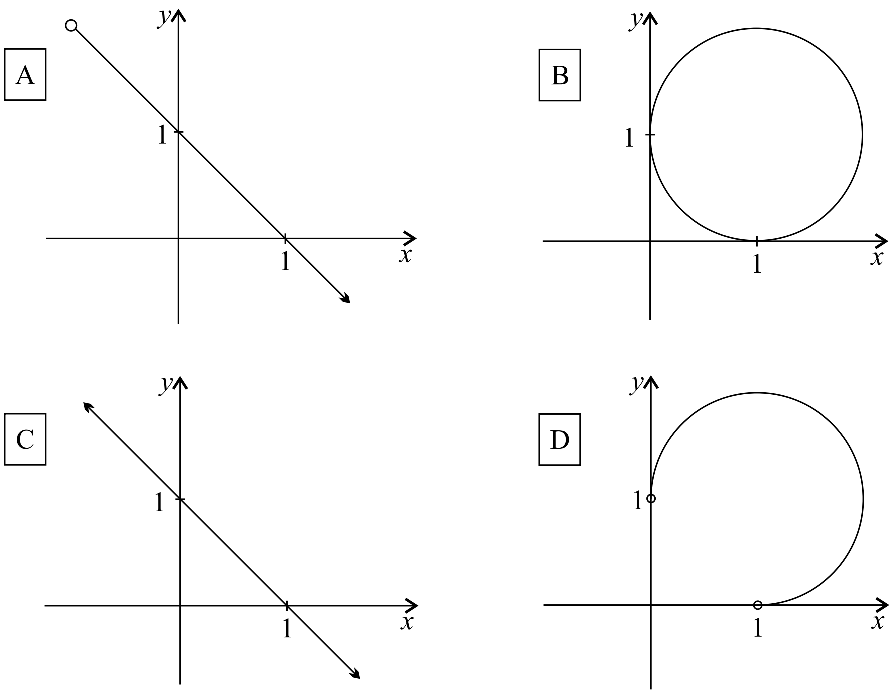
Record your answers in the space provided in this paper.

QUESTION TWO (4 marks) **Marks**

4

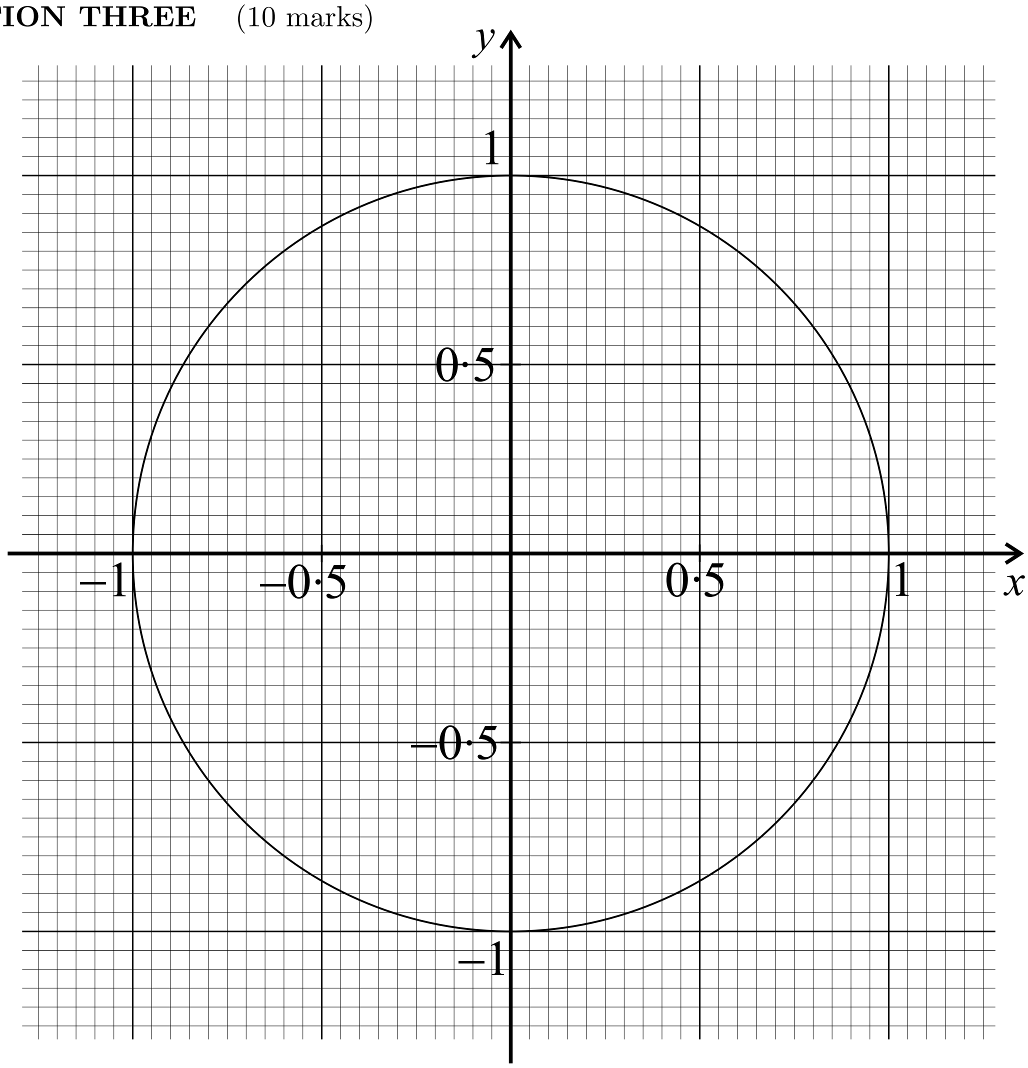


A student was tasked with sketching four graphs of a complex number z given four different restrictions. The student did not label their answers, and they did not complete the questions in any particular order. The four diagrams above represent the student's responses, labelled $\boxed{A}$, $\boxed{B}$, $\boxed{C}$ and $\boxed{D}$.

Complete the table below by matching each restriction with the diagram that best represents the graph of the complex number.

Restriction	Diagram
$\lvert z-1-i\rvert = 1$	
$\lvert z\rvert = \lvert z-1-i\rvert$	
$\arg(z+1-2i) = -\dfrac{\pi}{4}$	
$\arg(z-1) - \arg(z-i) = \dfrac{\pi}{4}$	

QUESTION THREE (10 marks) **Marks**



The diagram above shows the circle $x^2 + y^2 = 1$ in the complex plane.

(a) On the diagram above, plot a point Z_1 representing a complex number z such that **2**

$$\frac{1}{2} < |z| < 1 \quad \text{and} \quad 0 < \arg(z) < \frac{\pi}{2}.$$

(b) Referring to the complex number z that you have plotted in part (a), for the complex number z^2, calculate:

(i) the modulus, **1**

....

(ii) the argument. **1**

....

(c) Hence plot the point Z_2, representing the complex number z^2, on your diagram. **1**

QUESTION THREE (Continued)

(d) Plot the points Z_3 and Z_4, representing the complex numbers z^3 and z^4 on your diagram, clearly labelling each. **2**

(e) What can be said about the moduli of successive powers of z? **1**

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(f) Let the point Z_0 represent the number 1 on the Argand diagram, and let Z_n represent the complex number z^n for all $n \geq 0$. Giving an explanation for your answer, predict the limiting value of the vector sum **2**

$$\overrightarrow{Z_0Z_1} + \overrightarrow{Z_1Z_2} + \overrightarrow{Z_2Z_3} + \overrightarrow{Z_3Z_4} + \ldots .$$

Give your answer as a complex number.

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End of Section II

The paper continues in the next section

Extra writing space

If you use this space, clearly indicate which question you are answering.

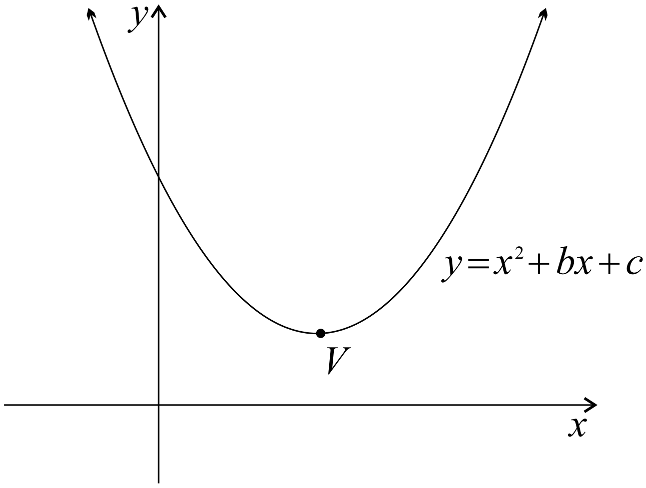
CANDIDATE NUMBER

Section III

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Record your answers in the space provided in this paper.

QUESTION FOUR (11 marks) **Marks**

(a)



The diagram above shows the parabola $y = x^2 + bx + c$, where b and c are real. The point V represents the vertex of the parabola.

(i) Explain why the equation $x^2 + bx + c = 0$ has no real roots. **1**

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(ii) Let one of the roots of $x^2 + bx + c = 0$ be $p + iq$, where p and q are real and $q \neq 0$. Write down the other root, giving a reason. **1**

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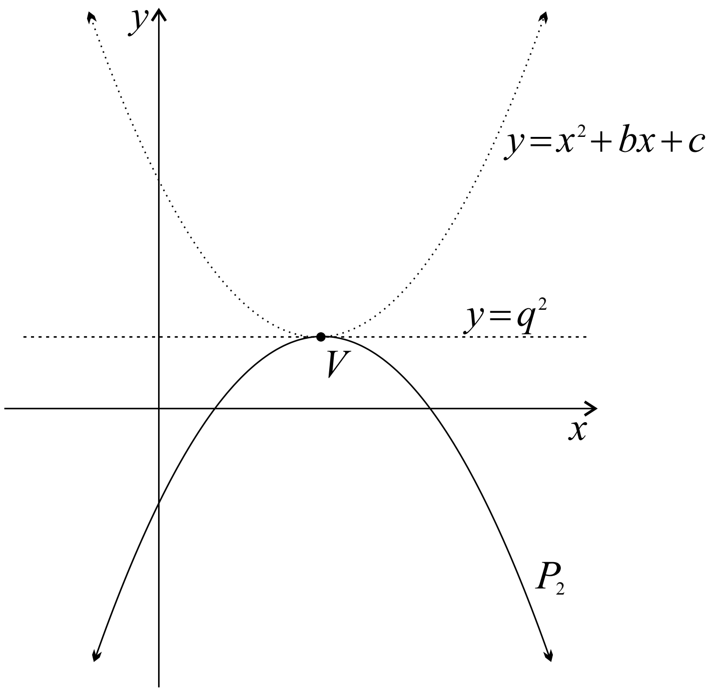
QUESTION FOUR (Continued)

(iii) Show that the coordinates of V are $(p,\, q^2)$. **2**

QUESTION FOUR (Continued)

(b)



The diagram above shows a new parabola P_2, created by reflecting the parabola in part (a) about the line $y = q^2$.

A curve can be reflected about a line $y = k$ by replacing y with $2k - y$. That is, if the curve $y = f(x)$ is reflected about the line $y = k$, the new curve will have equation $2k - y = f(x)$.

(i) Find the equation of P_2. Give your answer using only the variables x and y, and constants p and q. **2**

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(ii) Find the x-intercepts of P_2. **2**

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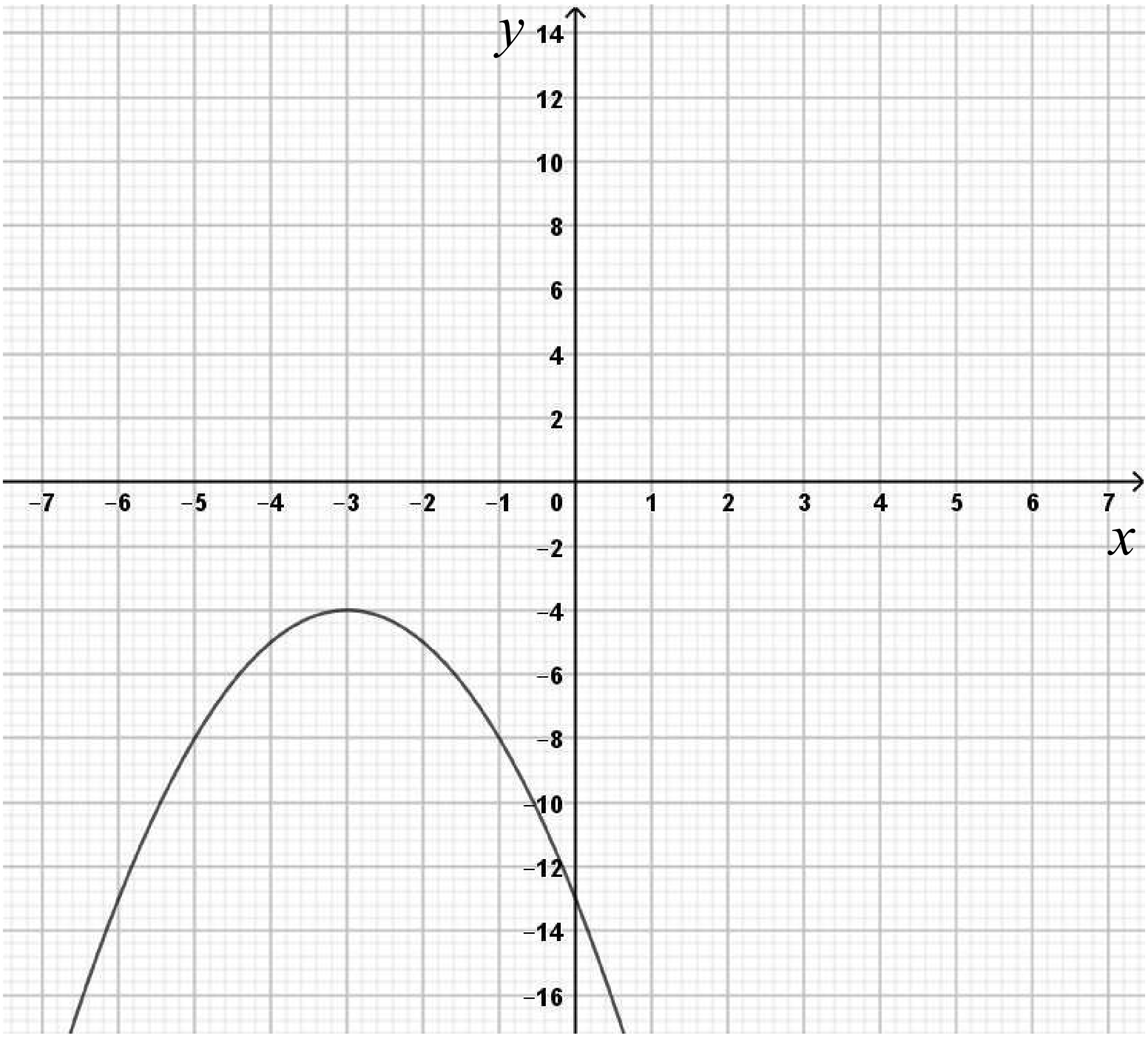
QUESTION FOUR (Continued)

(c)



The diagram above is a screen shot of the parabola $y = h(x)$ drawn using graphing software. Using parts (a) and (b), add an appropriate reflection to the diagram and write down the roots of the equation $h(x) = 0$ in the space provided below. $\boxed{3}$

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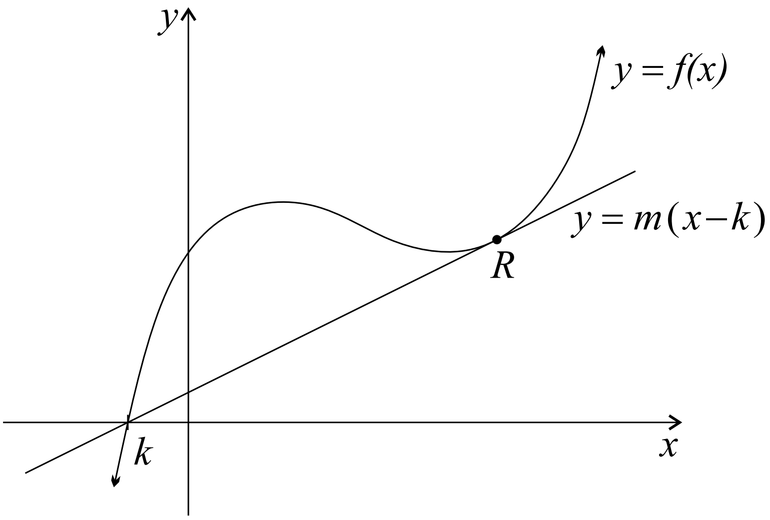
QUESTION FIVE (8 marks) **Marks**

(a)



The diagram above shows the monic cubic polynomial $f(x) = (x - k)(x^2 + bx + c)$, which has zeroes k, $p + iq$, and $p - iq$. The line $y = m(x - k)$ is drawn such that it intersects $y = f(x)$ at $(k, 0)$ and is tangent to $y = f(x)$ at R.

(i) Write down the equation of $f(x)$ with variable x and constants k, p, and q. You do not need to expand your expression. **1**

Hint: You may wish to refer to your answers to Question Four (a).

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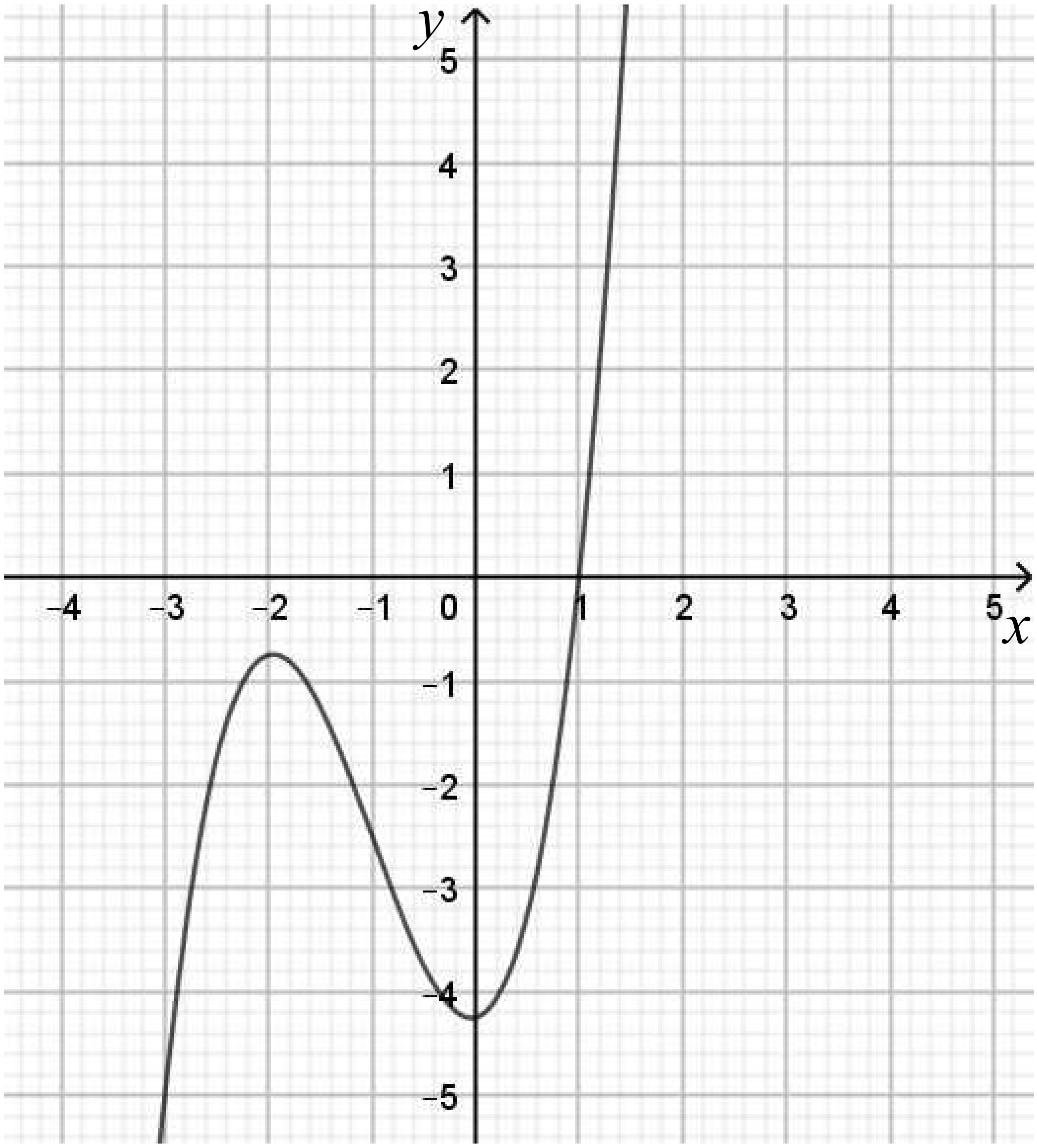
QUESTION FIVE (Continued)

(ii) Show that $m = q^2$. **3**

(iii) Find the x-coordinate of R. **1**

QUESTION FIVE (Continued)

(b)



The diagram above is a screen shot of a monic cubic drawn using graphing software. Using part (a), add any appropriate information to the diagram above and write down the zeroes of the cubic polynomial in the space provided below. **3**

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SYDNEY GRAMMAR SCHOOL

SOLUTIONS

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QUESTION ONE (10 marks) **Marks**

(a) (i) Write down the **converse** of the following statement: **1**

If $n \geq 100$, then $\sqrt{n} \geq 10$.

If $\sqrt{n} \geqslant 10$, $n \geqslant 100$

(ii) Write down the **contrapositive** of the following statement: **1**

If $n \geq 0$, then $\sqrt{n}$ is a real number.

If $\sqrt{n}$ is not a real number, then $n < 0$.

(b) Write a true mathematical implication in which the converse is **not** true. **2**

If a number is divisible by 6, then it is divisible by 2.

QUESTION ONE (Continued)

(c) (i) Prove that $\sqrt{7}$ is irrational. **3**

By way of contradiction, assume that $\sqrt{7}$ is irrational. i.e. $\exists\, m, n \in \mathbb{Z}$ such that $\sqrt{7} = \frac{m}{n}$, and m and n have no common factors.

Then $7 = \frac{m^2}{n^2}$

$7n^2 = m^2$

Since 7 is a factor of the LHS, 7 is a factor of m^2. This means that 7 is a factor of m. (*)

$\therefore \exists\, k \in \mathbb{Z}$ such that $m = 7k$.

$\therefore 7n^2 = (7k)^2$

$7n^2 = 49k^2$

$n^2 = 7k^2$

By the same logic as (*), 7 is a factor of n. This is a contradiction, as m and n have no common factors other than 1.

$\therefore \sqrt{7}$ is irrational.

(ii) Consider the following statement: **1**

If I add two irrational numbers, then the result will be irrational.

Prove that this statement is false by providing a counterexample.

Note that $-\sqrt{7}$ is irrational, so $1-\sqrt{7}$ is irrational.

$$\underbrace{\sqrt{7}}_{\text{irrational}} + \underbrace{(1-\sqrt{7})}_{\text{irrational}} = \underbrace{1}_{\text{rational}}$$

QUESTION ONE (Continued)

(d) The right-hand side of the following statement, which has been underlined, contains an error. **2**

p is a multiple of $q \Rightarrow \underline{\exists\, m \in \mathbf{Z} \text{ such that } q = pm}$

Circle the error in the underlined section of the statement above, and give a brief explanation of why it is an error.

$q = pm$ means that q is a multiple of p. The statement would be correct if it said "$p = qm$."

End of Section I

The paper continues in the next section

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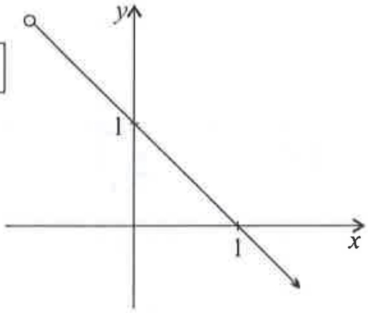
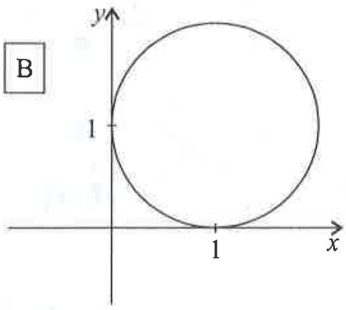
QUESTION TWO (4 marks)

Marks

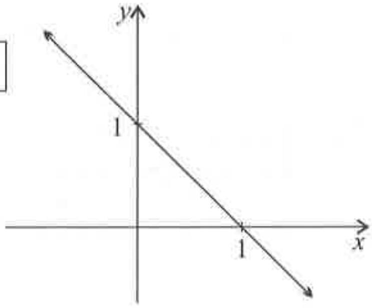
A



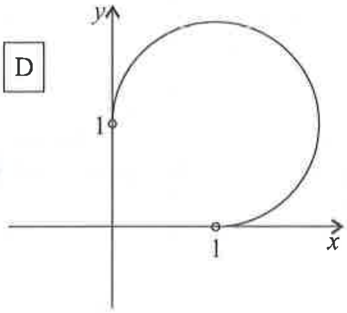
B



C



D



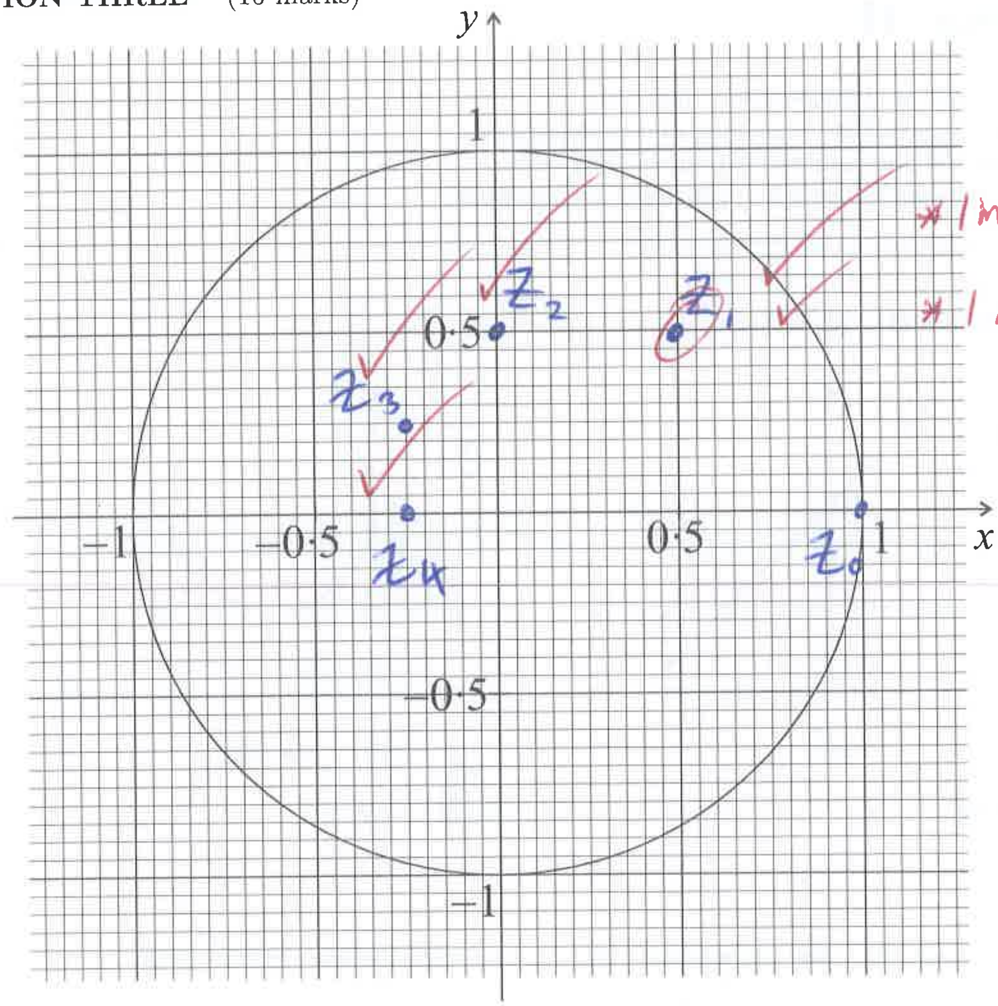
A student was tasked with sketching four graphs of a complex number z given four different restrictions. The student did not label their answers, and they did not complete the questions in any particular order. The four diagrams above represent the student's responses, labelled $\boxed{A}$, $\boxed{B}$, $\boxed{C}$ and $\boxed{D}$.

Complete the table below by matching each restriction with the diagram that best represents the graph of the complex number. **4**

Restriction	Diagram
$\lvert z-1-i\rvert = 1$	B
$\lvert z\rvert = \lvert z-1-i\rvert$	C
$\arg(z+1-2i) = -\dfrac{\pi}{4}$	A
$\arg(z-1) - \arg(z-i) = \dfrac{\pi}{4}$	D

QUESTION THREE (10 marks)

Marks



* 1 mark: correct modulus

* 1 mark: correct argument.

The diagram above shows the circle $x^2 + y^2 = 1$ in the complex plane.

(a) On the diagram above, plot a point Z_1 representing a complex number z such that **2**

$$\frac{1}{2} < |z| < 1 \quad \text{and} \quad 0 < \arg(z) < \frac{\pi}{2}.$$

(b) Referring to the complex number z that you have plotted in part (a), for the complex number z^2, calculate:

(i) The modulus, **1**

$|z| = \sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2} = \frac{\sqrt{2}}{2}$ $\therefore$ $|z^2| = \frac{1}{2}$ ✓

(ii) the argument (in degrees). **1**

$\arg(z) = 45°$, $\therefore$ $\arg(z^2) = 90°$ ✓

(c) Hence plot the point Z_2, representing the complex number z^2, on your diagram. **1**

QUESTION THREE (Continued)

(d) Plot the points Z_3 and Z_4, representing the complex numbers z^3 and z^4 on your diagram, clearly labelling each. **2**

(e) What can be said about the moduli of successive powers of z? **1**

The modulus always gets smaller.
i.e. $|z^{n+1}| < |z^n|$.

(f) Let the point Z_0 represent the number 1 on the Argand diagram, and let Z_n represent the complex number z^n for all $n \geq 0$. Giving an explanation for your answer, predict the limiting behaviour of ~~S, where~~ the vector sum **2**

$$\cancel{S =} \overrightarrow{Z_0Z_1} + \overrightarrow{Z_1Z_2} + \overrightarrow{Z_2Z_3} + \overrightarrow{Z_3Z_4} + \ldots .$$

Give your answer as a complex number

Note that $\lim_{n \to \infty} |z^n| = 0$.

The points $Z_0, Z_1, Z_2, \ldots$ spiral about the origin, getting closer to the origin.

$\therefore$ If O is the origin, then
$\overrightarrow{Z_0Z_1} + \overrightarrow{Z_1Z_2} + \overrightarrow{Z_2Z_3} + \ldots = \overrightarrow{Z_0O}$
$= -1$

End of Section II

The paper continues in the next section

CANDIDATE NUMBER

Section III

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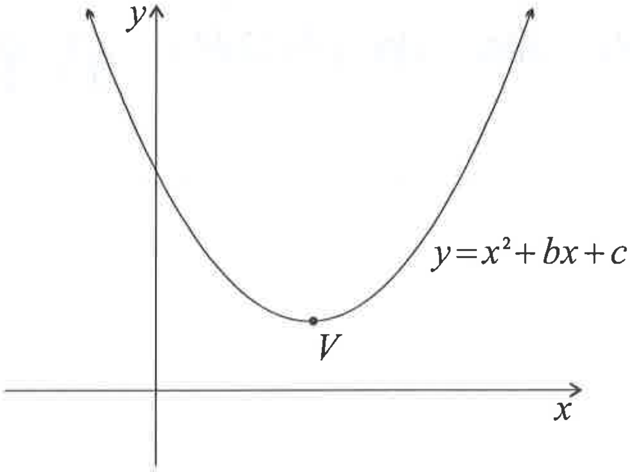
QUESTION FOUR (11 marks) **Marks**

(a)



The diagram above shows the parabola $y = x^2 + bx + c$, where b and c are real. The point V represents the vertex of the parabola.

(i) Explain why the equation $x^2 + bx + c = 0$ has no real roots. **1**

The parabola $y = x^2+bx+c$ doesn't cut the x-axis, so there are no real numbers x that can be substituted so that $y=0$.

(ii) Let one of the roots of $x^2+bx+c = 0$ be $p+iq$, where p and q are real and $q \neq 0$. Write down the other root, giving a reason. **1**

As the coefficients of the quadratic are real, any roots occur in conjugate pairs. $\therefore$ the other root is $p-iq$.

QUESTION FOUR (Continued)

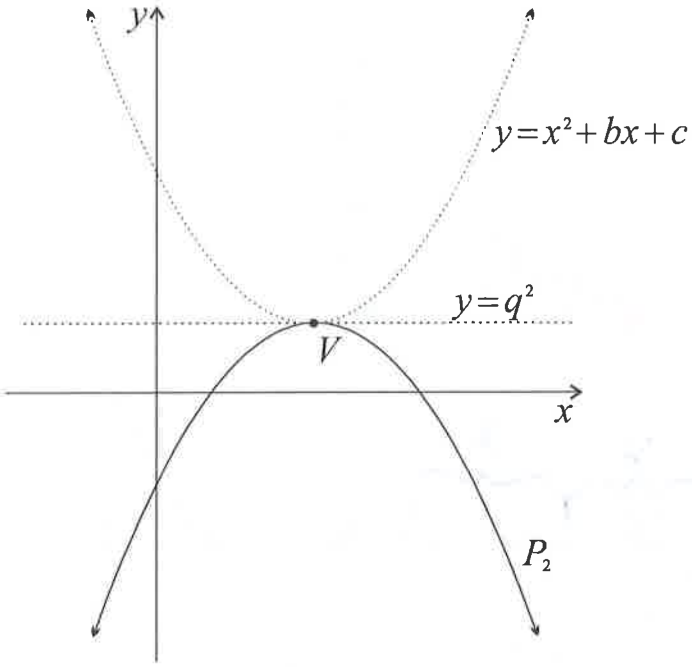
(iii) Show that the coordinates of V are (p, q^2). **2**

The equation of the parabola can be written as:
$$y = (x-(p+iq))(x-(p-iq))$$
$$= x^2 - 2px + p^2 + q^2$$
$$= (x-p)^2 + q^2$$

$\therefore$ The vertex has coordinates (p, q^2).

QUESTION FOUR (Continued)

(b)



The diagram above shows a new parabola P_2, created by reflecting the parabola in (a) about the line $y = q^2$.

A curve can be reflected about a line $y = k$ by replacing y with $2k - y$. That is, if the curve $y = f(x)$ is reflected about the line $y = k$, the new curve will have equation $2k - y = f(x)$.

(i) Find the equation of P_2. Give your answer using only the variables x and y, and constants p and q. **2**

$2q^2 - y = (x-p)^2 + q^2$

$y = -(x-p)^2 + q^2$

(ii) Find the x-intercepts of P_2. **2**

when $y=0$: $0 = -(x-p)^2 + q^2$

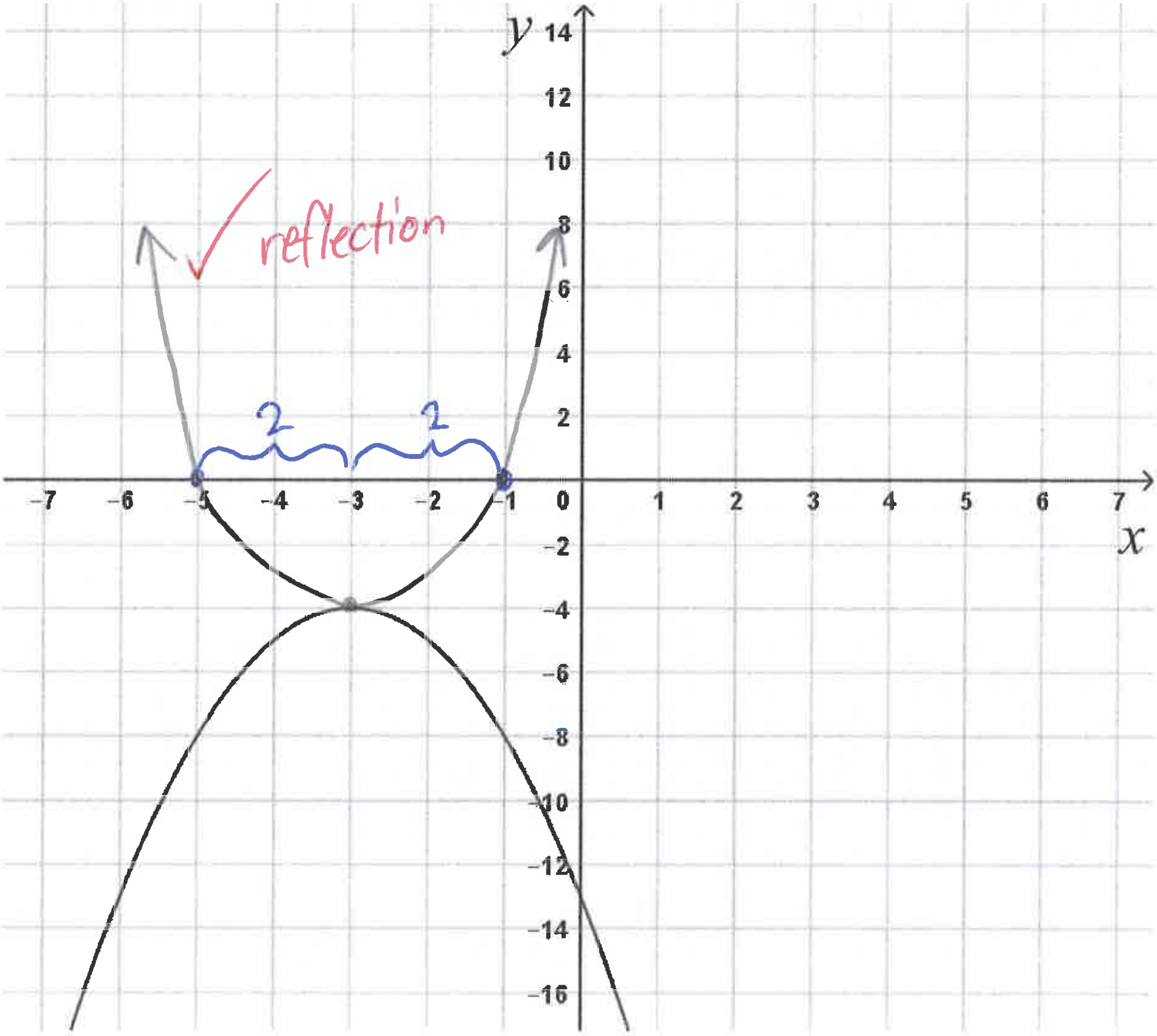
$(x-p)^2 = q^2$

$x - p = \pm q$

$x = p \pm q$

QUESTION FOUR (Continued)

(c)



The diagram above is a screen shot of the parabola $y = h(x)$ drawn using graphing software. Using parts (a) and (b), add an appropriate reflection to the diagram and write down the roots of the equation $h(x) = 0$ in the space provided below. **3**

The roots of the reflection are -3 ± 2

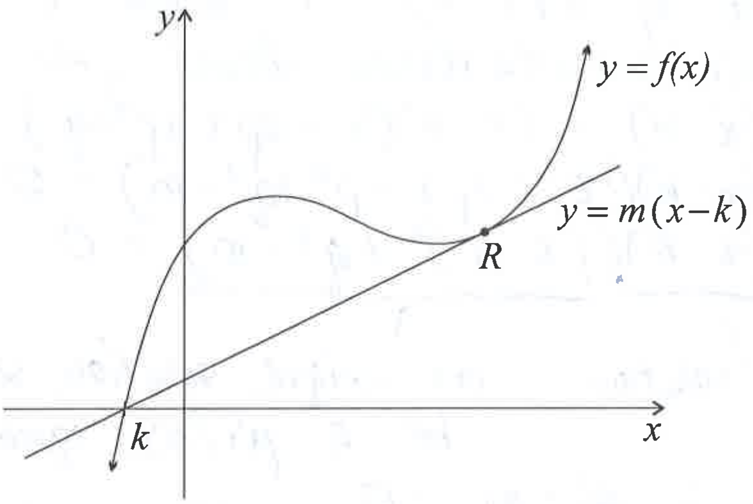
$\therefore$ The roots of $h(x) = 0$ are $-3 \pm 2i$

real part

imaginary part.

QUESTION FIVE (8 marks) **Marks**

(a)



The diagram above shows the monic cubic polynomial $f(x) = (x-k)(x^2+bx+c)$, which has zeroes k, $p+iq$, and $p-iq$. The line $y = m(x-k)$ is drawn such that it intersects $y = f(x)$ at $(k, 0)$ and is tangent to $y = f(x)$ at R.

(i) Write down the equation of $f(x)$ with variable x and constants k, p, and q. You do not need to expand your expression. **1**

Hint: You may wish to refer to your answers to Question Four (a).

From 4(a)(iii), the quadratic (monic) with roots $p \pm iq$ is $x^2 - 2px + p^2 + q^2$

$\therefore f(x) = (x-k)(x^2 - 2px + p^2 + q^2)$

QUESTION FIVE (Continued)

(ii) Show that $m = q^2$. **3**

The line $y = m(x-k)$ is tangent to $y = f(x)$, so solving simultaneously should yield 2 unique solutions

$$m(x-k) = (x-k)(x^2 - 2px + p^2 + q^2)$$

$$(x-k)(x^2 - 2px + p^2 + q^2 - m) = 0$$

$$\underbrace{(x-k)}_{\text{one solution}}\underbrace{((x-p)^2 + q^2 - m)}_{\text{one unique solution, so must be a perfect square.}} = 0$$

$$\therefore q^2 - m = 0$$

$$m = q^2.$$

Any correct method correctly applying the ideas of a double-root.

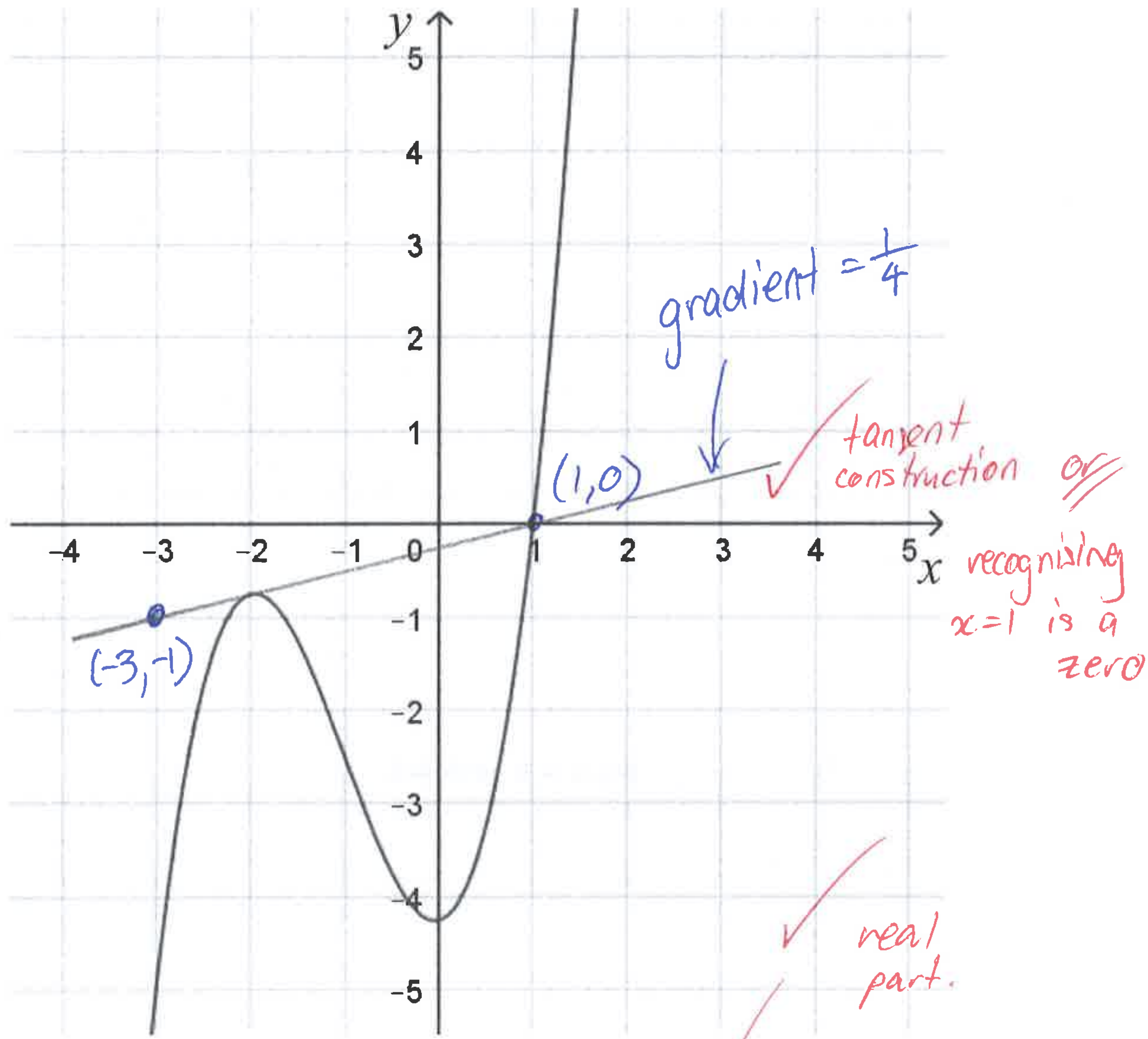
(iii) Find the x-coordinate of R. **1**

The x-coord. of R is the unique solution of the eqn. in (ii) that is not k.

i.e. $x = p$.

QUESTION FIVE (Continued)

(b)



The diagram above is a screen shot of a monic cubic drawn using graphing software. Using part (a), add any appropriate information to the diagram above and write down the zeroes of the cubic polynomial in the space provided below. **3**

From (a), the complex zeroes are $p \pm iq$, where p is the x-coord. of R and q is the square root of the gradient of the tangent $(m = q^2)$.

$\therefore$ The zeroes are $1,\ -2 \pm i\sqrt{\frac{1}{4}}$

$\Rightarrow \underbrace{1,\quad -2 + \frac{i}{2},\quad -2 - \frac{i}{2}}_{}$

all 3 zeroes ✓

END OF PAPER