Sydney Technical High School

Name: ....

Maths Class: ....

Year 12
Mathematics Extension 2

HSC Course

Assessment 3

May, 2021

Time allowed: 120 minutes

General Instructions:

- Marks for each question are indicated on the question.
- NESA approved calculators may be used
- In Questions 8-10, show all relevant mathematical reasoning and/or calculations
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using <u>black</u> pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided.

Total marks – 67

Section I - 7 Marks
- Attempt Question 1-7 on the sheet provided
- Allow about 10 minutes for this section

Section II – 60 Marks
- Attempt Questions 8-10
- Allow about 1 hour and 50 minutes for this section

Section I

7 Marks
Attempt Questions 1-7
Allow about 10 minutes for this section

Use the multiple choice answer sheet for Questions 1-7.

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1. Which of the following is equivalent to $\sqrt{i^3}$?

(A) $e^{\frac{i\pi}{4}}$

(B) $e^{\frac{i\pi}{2}}$

(C) $e^{\frac{3i\pi}{4}}$

(D) $e^{-\frac{i\pi}{2}}$

2. If $\underset{\sim}{v}$ is the vector from the origin to the point $(2, 3, 2)$, what is $\underset{\sim}{\hat{v}}$?

(A) 7

(B) $\frac{1}{7}(2\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k})$

(C) $\frac{1}{\sqrt{17}}(2\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k})$

(D) $\sqrt{17}$

3. Let $\underset{\sim}{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. Which of the following is a correct expression for $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}$?

(A) $\frac{1}{2}\underset{\sim}{i} - \underset{\sim}{j} + \frac{3}{2}\underset{\sim}{k}$

(B) $\frac{3}{2}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} + \underset{\sim}{k}$

(C) $\frac{1}{2}\sqrt{14}$

(D) $-2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$

4. A sphere has centre $\underset{\sim}{c} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and passes through the point with position vector $\underset{\sim}{v} = 4\underset{\sim}{i} + 4\underset{\sim}{j} + 4\underset{\sim}{k}$. Its Cartesian equation is:

(A) $(x-2)^2 + (y-2)^2 + (z-2)^2 = 2$

(B) $(x-2)^2 + (y-2)^2 + (z-2)^2 = 4$

(C) $(x-2)^2 + (y-2)^2 + (z-2)^2 = 2\sqrt{3}$

(D) $(x-2)^2 + (y-2)^2 + (z-2)^2 = 12$

5. Consider the statement $y = 2x^2 + 1$. Under what conditions is this statement true?

(A) $\forall x, \forall y \in \mathbb{R}$

(B) $\forall y, \exists x \in \mathbb{R}$

(C) $\forall x, \exists y \in \mathbb{R}$

(D) $\forall x, \forall y \geq 1 \in \mathbb{R}$

6. Which of the following statements is NOT true?

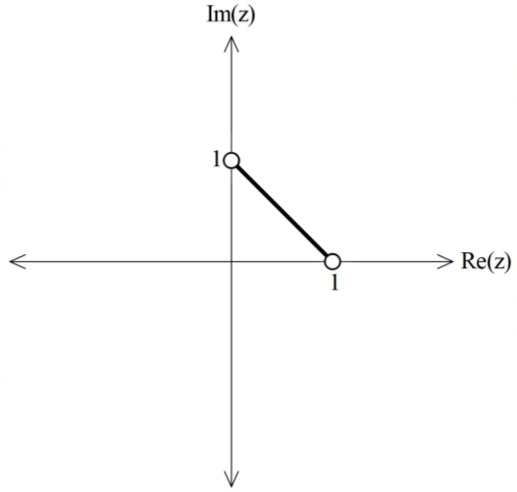
(A) If an integer is divisible by 54, then it is divisible by both 9 and 6.

(B) If an integer is divisible by both 9 and 6, then it is divisible by 54.

(C) If two statements are equivalent, then each is a necessary condition for the other to be true.

(D) If two statements are equivalent, then each is a sufficient condition for the other to be true.

7. The locus of z is displayed on the Argand diagram below.



Which of the following is the correct equation for the locus of z?

(A) $\arg\left(\frac{z-i}{z-1}\right) = 0$

(B) $\arg\left(\frac{z-i}{z-1}\right) = \pi$

(C) $\arg\left(\frac{z+i}{z+1}\right) = 0$

(D) $\arg\left(\frac{z+i}{z+1}\right) = \pi$

Section II

Total marks – 60
Attempt Question 8-10
Allow about 1 hour and 50 minutes for this section

<u>Begin each question on a NEW page</u>

In Questions 8-10, your responses should include relevant mathematical reasoning and/or calculations.

Question 8 (20 marks) Begin a NEW page.

a) For $z = 1 + \sqrt{3}i$.

i. Find $|z|$ and $\arg(z)$ **2**

ii. Express z in polar form. **1**

iii. Find the smallest positive value of $n \in \mathbb{Z}$ such that $z^n \in \mathbb{R}$. **1**

b) Express w in the form $a + ib$ if $\frac{w}{1+w} = 1 + i$. **2**

c) Prove that $\log_5 7$ is irrational. **2**

d) A sphere has vector equation $\left|\underset{\sim}{v} - \left(\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}\right)\right| = 14$. **2**
Determine whether $P(2, 3, 4)$ lies inside, outside, or on the surface of the sphere.

e) Let $P(z) = z^4 - 4z^3 - 3z^2 + 50z - 52$. **3**
Solve $P(z) = 0$ if $z = 3 - 2i$ is a root of the polynomial.

f) For the quadratic equation $z^2 - 3z + (3 + i) = 0$

i. Find the square roots of $-3 - 4i$. **2**

ii. Hence, or otherwise, solve the quadratic equation under $\mathbb{C}$. **2**

g) The line l_1 passes the points $(a, -2, 4)$ and $(5, 3, -2)$, while l_2 passes $(3, 8, -2)$ and $(2, -2, a)$, where $a \in \mathbb{R}$. Find all possible values of a if $l_1 \perp l_2$. **3**

End of Question 8

Question 9 (20 marks) Begin a NEW page.

a) A line passes through the points $A(1,3,-2)$ and $B(2,-1,5)$.

i. Show that the vector equation of the line AB is given by: **1**

$$\underset{\sim}{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$$

ii. Determine if the point $C(3,-5,12)$ lies on the line. **2**

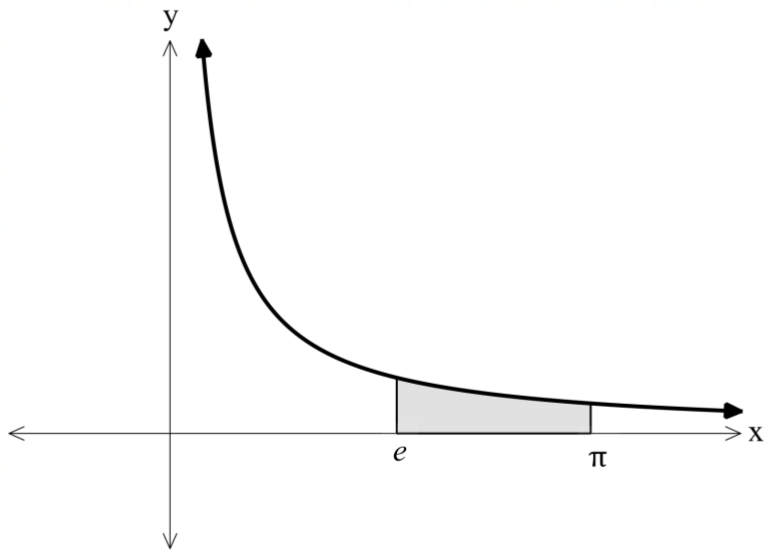
iii. Consider a line with vector equation:

$$\underset{\sim}{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

α) Explain why $\underset{\sim}{v}$ and $\underset{\sim}{u}$ are not parallel. **1**

β) Determine whether $\underset{\sim}{v}$ and $\underset{\sim}{u}$ intersect or are skew. **3**

b) The diagram below shows the area under $y = \frac{1}{x}$ from $x = e$ to $x = \pi$.



i. Use a rectangular upper bound of the area to show that $\ln\left(\frac{\pi}{e}\right) < \frac{\pi}{e} - 1$ **2**

ii. Hence show that $e^{\pi} > \pi^{e}$ **1**

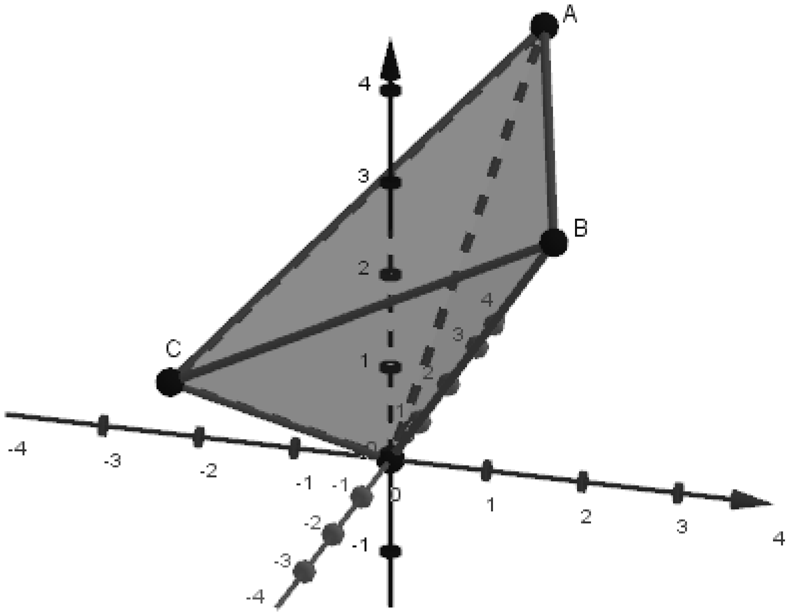
c) For $\forall a, b \in \mathbb{R}$, consider the statement

$$a^3 + ab^2 \geq a^2b + b^3 \Longrightarrow a \geq b$$

i. State the contrapositive of the statement. **1**

ii. Hence, prove the statement. **2**

d) A triangular pyramid has vertices $O(0\ ,0,0)$, $A(1,\ 2,\ 4)$, $B(2,-1,3)$, and $C(-2,\ -1,1)$, as shown below.



i. Find the lengths of OA and OB. **2**

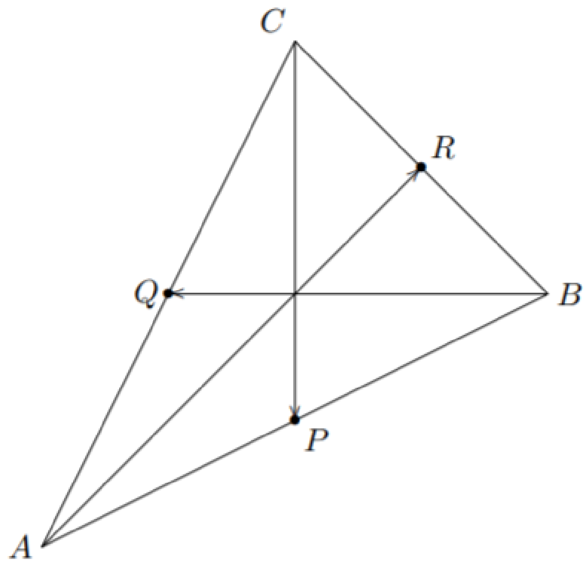
ii. Find the size of $\angle AOB$, to the nearest degree. **2**

iii. Given that OC is perpendicular to the face AOB, find the volume of the triangular pyramid $OABC$, to two decimal places. **3**

End of Question 9

Question 10 (20 marks) Begin a NEW page.

a) The median of a triangle is a line from the vertex to the midpoint of the opposite side. **3**
Using the diagram below, prove that the sum of the medians of a triangle is the zero vector.



b) Considering $z = \cos\theta + i\sin\theta$

i. Use de Moivre's theorem with z^5 to show that **3**

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

ii. Hence, solve **2**

$$16x^5 - 20x^3 + 5x = \frac{1}{2}$$

Leave your answers to two decimal places.

c) A line, which passes through $(3, 2, 5)$, has vector equation **3**

$$\underset{\sim}{v} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

Find the perpendicular distance from $Q(5, 3, 10)$ to $\underset{\sim}{v}$.
(Hint: Draw a diagram)

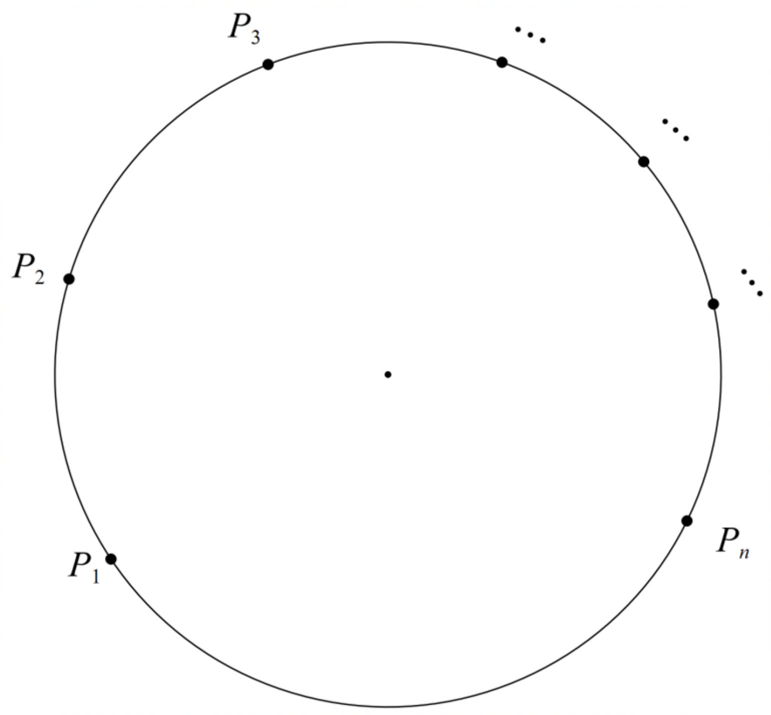
d) For real numbers $a, b, c, d > 0$, show that

i. $\frac{a+b}{2} \geq \sqrt{ab}$ **1**

ii. $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$ **2**

iii. $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$ **2**

e) n points, labelled $P_1, P_2, P_3, \ldots, P_n$ are placed around a circle such that no three points are collinear, as shown on the diagram below. Every set of three points forms a triangle. **4**



Use mathematical induction to show that, the number of distinct triangles formed is:

$$\frac{n(n-1)(n-2)}{6}, \text{where } n \geq 3$$

End of Examination

X2 T2 Solutions

1. $i = e^{\frac{\pi i}{2}}$

$i^{\frac{3}{2}} = e^{\frac{3\pi i}{2}}$

C.

2. $|\underset{\sim}{v}| = \sqrt{2^2+3^2+2^2} = \sqrt{17}$

$\therefore \hat{\underset{\sim}{v}} = \frac{1}{\sqrt{17}}(2\underset{\sim}{i}+3\underset{\sim}{j}+2\underset{\sim}{k})$

C.

3. $$\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}} \times \underset{\sim}{b} = \frac{3-2+6}{1+4+9} \times \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{3}{2} \end{bmatrix}$$

A.

4. $r^2 = (4-2)^2 + (4-2)^2 + (4-2)^2$

$= 12$

$r = 2\sqrt{3}$

D.

5. "For all x, exists a y": C.

6. B. Counterexample of 18.

7. $\arg(z-i) - \arg(z-1) = \pi$ B.

$\arg(z-i) = \pi + \arg(z-1)$

8. a) $z = 1+\sqrt{3}\,i$

i) $|z| = \sqrt{1^2+(\sqrt{3})^2}$

$= 2$

$\arg(z) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \frac{\pi}{3}$

ii) $2\operatorname{cis}\frac{\pi}{3}$

iii) $z^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$

Real at $\sin\frac{n\pi}{3} = 0$

$\therefore n = 3$

b) $\frac{w}{1+w} = 1+i$

$w = (1+i) + w(1+i)$

$-iw = 1+i$

$w = \frac{1+i}{-i} \times \frac{i}{i}$

$= \frac{i+i^2}{-1\times-1}$

$= -1+i$

c) Assume $\log_5 7$ is rational

$\therefore \log_5 7 = \frac{p}{q} \quad p, q \in \mathbb{Z},\ q \neq 0$

HCF of $p, q = 1$

$5^{\frac{p}{q}} = 7$

$5^p = 7^q$

As 5^p is divisible by 5, 7^q is divisible by 5.

This a contradiction as 7^q is not divisible by 5

$\therefore \log_5 7$ is irrational.

d) Distance from $(1,3,1)$ to $P(2,3,4)$

$= \sqrt{(2-1)^2 + (3-3)^2 + (4-1)^2}$

$= \sqrt{10}$

< 14

$\therefore P$ is inside the sphere.

e) $3-2i$ is a root $\Rightarrow 3+2i$ is a root

(real coefficients)

Let roots be $\alpha, \beta, 3-2i, 3+2i$

Sum of roots: $3-\cancel{2i}+3+\cancel{2i}+\alpha+\beta = -(-4)$

$\alpha+\beta = -2$

$\alpha = -2-\beta \quad —(1)$

Product of roots: $(3-2i)(3+2i)\alpha\beta = -52$

$13\alpha\beta = -52$

$\alpha\beta = -4$ — (2)

Sub (1) into (2) $\beta(-2-\beta) = -4$

$-2\beta - \beta^2 = -4$

$\beta^2 + 2\beta - 4 = 0$

$\beta = \dfrac{-2 \pm \sqrt{2^2 + 4 \times 1 \times 4}}{2}$

$= -1 \pm \sqrt{5}$

$\therefore z = 3+2i, 3-2i, -1+\sqrt{5}, -1-\sqrt{5}$

f) i) $-3-4i = (a+ib)^2$

$= a^2 - b^2 + 2abi$

$\therefore a^2 - b^2 = -3$ $\quad 2ab = -4$

$ab = -2$

By inspection $a=1, b=-2$ or $a=-1, b=2$

$\therefore$ Square roots are $1-2i$, $1+2i$

ii) $z = \dfrac{3 \pm \sqrt{9 - 4 \times 1 (3+i)}}{2}$

$= \dfrac{3 \pm \sqrt{-3-4i}}{2}$

$= \dfrac{3 \pm (1-2i)}{2}$ From i)

$= 2-i, 1+i$

g) Direction vectors:

$$l_1 = \begin{bmatrix} 5-a \\ 3+2 \\ -2-4 \end{bmatrix} = \begin{bmatrix} 5-a \\ 5 \\ -6 \end{bmatrix}$$

$$l_2 = \begin{bmatrix} 2-3 \\ -2-8 \\ a+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -10 \\ a+2 \end{bmatrix}$$

as $l_1 \perp l_2$,

$$\begin{bmatrix} 5-a \\ 5 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -10 \\ a+2 \end{bmatrix} = 0$$

$$-(5-a) - 5\times10 - 6(a+2) = 0$$

$$-5+a - 50 - 6a - 12 = 0$$

$$-5a = 67$$

$$a = \frac{-67}{5}$$

9. a) i) $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} 2-1 \\ -1-3 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$

$\therefore$ Vector equation $\underset{\sim}{r} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$

ii) Test $\begin{bmatrix} 3 \\ -5 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$

$\lambda \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 14 \end{bmatrix}$

As $\lambda = 2$ solves this, $(3, -5, 12)$ lies on the line.

iii) α) The direction vectors

$\begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ are not parallel

(Cannot express one as the scalar multiple of the other).

β) If $\begin{bmatrix}1\\2\\-1\end{bmatrix} + \mu\begin{bmatrix}-1\\3\\1\end{bmatrix} = \begin{bmatrix}1\\3\\-2\end{bmatrix} + \lambda\begin{bmatrix}1\\-4\\7\end{bmatrix}$

Then

$1-\mu = 1+\lambda$ $\quad$ $2+3\mu = 3-4\lambda$ —(2) $\quad$ $-1+\mu = -2+7\lambda$ —(3)

$\lambda = -\mu$ —(1)

Sub (1) into (2): $2+3\mu = 3-4\times(-\mu)$

$3\mu = 1+4\mu$

$-\mu = 1$

$\mu = -1$

$\therefore \lambda = 1$

Test in (3)

$LHS = -1+(-1)$ $\quad$ $RHS = -2+7\times 1$

$= -2$ $\quad$ $= 5$

$LHS \neq RHS$

$\therefore$ No value of λ and μ works

$\therefore$ $\underset{\sim}{r}$ and $\underset{\sim}{\mu}$ are skew.

b) i) Upper rectangle gives

$$\int_e^{\pi} \frac{1}{x}\,dx < \frac{1}{e} \times (\pi - e)$$

$$\Big[\ln|x|\Big]_e^{\pi} < \frac{\pi}{e} - 1$$

$$\ln\pi - \ln e < \frac{\pi}{e} - 1$$

$$\ln\left(\frac{\pi}{e}\right) < \frac{\pi}{e} - 1$$

ii) $\therefore \quad e^{\frac{\pi}{e} - 1} > \frac{\pi}{e} \qquad \times e$

$$e^{\frac{\pi}{e}} > \pi$$

$$e^{\pi} > \pi^{e}$$

c) i) Contrapositive $a < b \Rightarrow a^3 + ab^2 < a^2b + b^3$

ii) If $a < b$, then

$$a(a^2+b^2) < b(a^2+b^2) \qquad \text{as } a^2+b^2 \geq 0$$

$$a^3 + ab^2 < a^2b + b^3$$

$\therefore \quad a < b \Rightarrow a^3 + ab^2 < a^2b + b^3$

$\therefore$ Contrapositive holds.

$$a^3 + ab^2 \geq a^2b + b^3 \Rightarrow a \geq b$$

d) i) $|OA| = \sqrt{1^2+2^2+4^2} = \sqrt{21}$ $\quad |OB| = \sqrt{2^2+(-1)^2+3^2} = \sqrt{14}$

ii) $\vec{OA} \cdot \vec{OB} = 1\times 2 - 2\times 1 + 4\times 3 = 12$

$\therefore\ 12 = \sqrt{21} \times \sqrt{14} \times \cos\theta$

$\cos\theta = \dfrac{12}{\sqrt{21} \times \sqrt{14}}$

$\therefore \angle AOB = 46^\circ$ (n. d.)

iii) $V = \frac{1}{3} \times \text{Area } \Delta AOB \times |OC|$

$= \frac{1}{3} \times \frac{1}{2} \times \sqrt{14} \times \sqrt{21} \times \sin 46^\circ \times \sqrt{2^2+1^2+1^2}$

$= 5.04$ units^3

(Answer is exactly 5 if $\angle AOB$ isn't rounded)

10. a) Sum of diagonals:

$\vec{AR}+\vec{BQ}+\vec{CP} = (\vec{AB}+\vec{BR}) + (\vec{BA}+\vec{AQ}) + (\vec{CA}+\vec{AP})$

using medians $= \cancel{\vec{AB}} + \frac{1}{2}\vec{BC} - \cancel{\vec{AB}} + \frac{1}{2}\vec{AC} - \vec{AC} + \frac{1}{2}\vec{AB}$

$= \frac{1}{2}\vec{BC} - \frac{1}{2}\vec{AC} + \frac{1}{2}\vec{AB}$

$= \frac{1}{2}(\vec{BC} + \vec{CA} + \vec{AB})$

$= \frac{1}{2}(\vec{AB} + \vec{BC} + \vec{CA})$

$= \frac{1}{2}(\vec{AA})$

$= \frac{1}{2} \times \underset{\sim}{0}$

$= \underset{\sim}{0}$

b) i) $z^5 = (\cos\theta + i\sin\theta)^5$

$\cos 5\theta + i\sin 5\theta = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$

$-10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating imaginary components: $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5(1-2\sin^2\theta + \sin^4\theta)\sin\theta - (10-10\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

ii) Let $x = \sin\theta$; $\therefore \sin 5\theta = \frac{1}{2}$

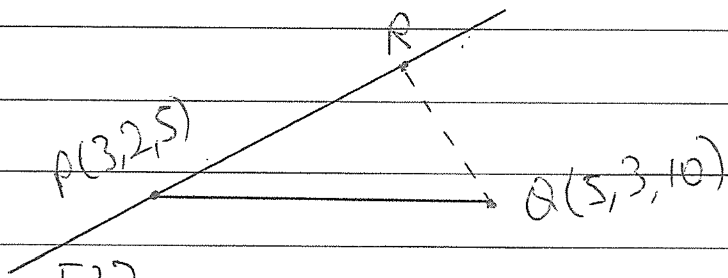
$5\theta = \frac{\pi}{6}, \frac{\pi}{6}+2\pi, \frac{\pi}{6}+4\pi, \frac{\pi}{6}+6\pi, \frac{\pi}{6}+8\pi$

$\theta = \frac{\pi}{30}, \frac{13\pi}{30}, \frac{5\pi}{6}, \frac{37\pi}{30}, \frac{49\pi}{30}$

$\therefore x = \sin\frac{\pi}{30}, \sin\frac{13\pi}{30}, \sin\frac{5\pi}{6}, \sin\frac{37\pi}{30}, \sin\frac{49\pi}{30}$

$= 0.10, 0.98, 0.5, -0.67, -0.91$

c)



$$\overrightarrow{PQ} = \begin{bmatrix} 5-3 \\ 3-2 \\ 10-5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

$$\overrightarrow{PR} = \text{proj } \overrightarrow{PQ} \text{ onto } \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$= \frac{2\times -1 + 1\times 2 + 5\times -2}{1^2+2^2+2^2} \times \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$= \frac{-10}{9}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$\therefore \overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ}$$

$$= -\frac{10}{9}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8}{9} \\ -\frac{29}{9} \\ -\frac{25}{9} \end{bmatrix}$$

$$\therefore \text{Perpendicular distance} = \sqrt{\left(\frac{8}{9}\right)^2 + \left(\frac{29}{9}\right)^2 + \left(\frac{25}{9}\right)^2}$$

$$= \sqrt{\frac{170}{9}}$$

$$= \frac{\sqrt{170}}{3}$$

d) i) $(\sqrt{a}-\sqrt{b})^2 \geq 0$

$a - 2\sqrt{ab} + b \geq 0$

$a+b \geq 2\sqrt{ab}$

$\frac{a+b}{2} \geq \sqrt{ab}$

ii) From i)

$\frac{a+b}{2} + \frac{b+c}{2} \geq \sqrt{ab} + \sqrt{cd}$

$\frac{a+b+c+d}{2} \geq \sqrt{ab} + \sqrt{cd}$

Using i) with $a = \sqrt{ab}$, $b = \sqrt{cd}$

$\frac{\sqrt{ab}+\sqrt{cd}}{2} \geq \sqrt{\sqrt{ab} \times \sqrt{cd}}$

$\sqrt{ab} + \sqrt{cd} \geq 2\sqrt[4]{abcd}$

$\therefore \frac{a+b+c+d}{2} \geq 2\sqrt[4]{abcd}$

$\frac{a+b+cd}{4} \geq \sqrt[4]{abcd}$

iii) In ii), let $a = \frac{a}{b}$, $b = \frac{b}{c}$, $c = \frac{c}{d}$, $d = \frac{d}{a}$

$$\therefore \frac{\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}\right)}{4} \geq \sqrt[4]{\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{a}}$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4 \times \sqrt[4]{1}$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$$

e) Test: $n=3$. $\frac{n(n-1)(n-2)}{6} = \frac{3\times(3-1)\times(3-2)}{6}$

$= 1$

As only one triangle is formed by three points, statement is true for $n=3$

Assume true for $n=k$, i.e. $\frac{k(k-1)(k-2)}{6}$ triangles are formed by k points.

Hence prove true for $n=k+1$, aim to show $(k+1)$ points form: $\frac{(k+1)(k)(k-1)}{6}$ triangles.

When a $(k+1)^{th}$ point is added, it forms a triangle with any selection of 2 points, so ${}^kC_2$ new triangles are formed.

$\therefore$ Total number of triangles: $\frac{k(k-1)(k-2)}{6} + {}^kC_2$ by assumption

$$= \frac{k(k-1)(k-2)}{6} + \frac{\cancel{k!}\,(k)(k-1)}{2!\,\cancel{(k-2)!}}$$

$$= \frac{k(k-1)(k-2) + 3k(k-1)}{6}$$

$$= \frac{k(k-1)\left[(k-2)+(3)\right]}{6}$$

$$= \frac{(k+1)(k)(k-1)}{6}$$

$\therefore$ Statement is true for $n=k+1$ if it is true for $n=k$.

As it is true for $n=3$, it is also true for $n=3+1=4, 5, 6, \ldots$

Hence, by mathematical induction, it is true for all positive integers, n.

Extension 2 Assessment 3 Markers Comments'

Question 8

c) Once you arrived at 5^p=7^q you had to say why this is not possible since neither 7 nor 5 to any power can be divisible by the other.

d) If you worked out the distance to be root 10 it had to be compared to the radius which is 14, or if you used 10 it had to be compared to 196.

e) This question could have been done using sums and products of the roots of the quartic and solving simultaneously or by inspection, finding the other quadratic factor and solving it. Either method required utmost care.

f) Finding square roots is fiddly but if an error was made, carry through was paid for part (ii)

g) Direction vectors required (in the right directions, some candidates reversed them), next dot product =0 and then solving carefully for a.

Question 9

a)

i.+ii. Well done, just some arithmetic mistakes.

iii.

α. Direction Vectors/Vector Lines/Position Vectors same/different means nothing for the lines being parallel. You need to tell me that the direction vectors were not scalar multiples of each other, OR direction vectors were not parallel.

β. Relatively well done, most were able to find λ and μ which were not solutions to all three equations

b)

i. Literally says use rectangular upper bound of the area, so, don't use the lower bound, don't type it into your calculator. None of these are answering the question. Also, you needed to show where $\ln\frac{\pi}{e}$ was from; sometimes, it appeared, just, like magic.

ii. Please be careful when doing the algebra, here many students lost the mark due to algebraic mistakes – like flipping the inequality randomly, or even dropping -1, adding a +1.

c)

i. Some forgot to that contrapositive meant that you had to flip the statements/reverse the implication arrow. Others forgot to exclude the equality for the negation or the statements. Simply stating "not" with the implication was incorrect.

ii. Many ways to do this, but you needed to PROVE and explain yourself. The key for most methods was recognising and stating that "$a^2 + b^2 > 0$ and NOT 0, and/or $a - b < 0$ and NOT 0.

d)

i.+ii. Relatively well done, except for some arithmetic mistakes.

iii. If you aren't going to use the facts given in the question, then you need to be very sure that your new found "properties" are real properties, not just made up. It tells you that OC is perpendicular to the FACE of AOB, not perpendicular to OA, or perpendicular to OB (it was the perpendicular height)

We got a variety of triangle bases, a variety of formulas for a pyramid (IT'S ON THE FORMULA SHEET) and a variety of heights. Some made assumptions of where right angles lived, sometimes incorrectly, sometimes correctly. Please take a look at the solution.

Question 10

a)

There were many successful methods of completing this question. The most elegant used vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{b} - \underset{\sim}{a}$ as vectors for side lengths of the triangle. Every method needed to use the fact that medians of the triangle cut each side in half.
Common errors included using vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$ without any clear ideas as to what the vectors were, incorrectly writing that a side length was $\frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$, and assuming that a sum added to $\underset{\sim}{0}$ without any clear reason as to why.

b)

Part i) was generally well done, though some students got lost attempting to apply $z^n - z^{-n}$.
Part ii) had a nasty trick to it. Many students accessed a mark by finding some solutions to the equation. However, the total of five solutions were needed for full marks. The difficult part was that the first five ordered solutions to $\sin 5\theta = \frac{1}{2}$ give repeated solutions in the polynomial. This led students to believe that there were double roots.

c)

There were several opportunities for students to be lost in this question. Most commonly, students used $\begin{bmatrix} 5 \\ 3 \\ 10 \end{bmatrix}$ as a vector (i.e. the position vector of Q). A simple diagram shows that the vector from the origin is not useful.

If students realised this, they typically made good progress, either applying to dot product or using $\left| proj_{\underset{\sim}{a}}\underset{\sim}{b} - \underset{\sim}{b} \right|$. Mistakes after this point were typically clerical errors.

d)

Part i) was fairly well done. Those who were stuck at this point generally started by assuming the result that they want to prove.

Parts ii) and particularly iii) caused more difficulty. Students are advised to build familiarity with these types of questions, being aware that they should aim to use previous parts. This was especially efficient in using part ii) to get to iii) through a simple substitution. Students should also indicate when they have used a previous part, or the reasoning behind any step that is not just a simple manipulation.

e)

Potentially due to its position at the end of the examination paper, this question caused a lot of difficulty. First, many students seemed unaware that adding the extra triangles caused by P_{k+1} was the process needed. Instead, they seemed to make up new mathematical equations that needed to be solved, such as claiming that it was equivalent to a convenient series, or claiming that the total number of triangles was $3k$.
Those who did realise that you needed to add the extra triangles struggled to see that the number was $\binom{k}{2}$ or $\frac{k(k-1)}{2}$. Instead, answers such as 1 and k were used, leading to further mistakes in making it fit the answer.