Sydney Technical High School

Name: ....

Maths Class: ....

Year 12
Mathematics Extension 2

HSC Course

Assessment 3

June, 2024

Time allowed: 120 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided.

Section I Multiple Choice
Questions 1-6
6 Marks

Section II Questions 7-12
56 Marks

This Page

Intentionally

Left Blank

Section I

6 marks
Attempt Questions 1-6
Allow about 10 minutes for this section
Use the multiple-choice answer sheet for Questions 1-6

---

1. Which of the following is equivalent to $\int xsec^2(x^2)\,dx$?

A. $\frac{1}{2}\tan(x^2)+C$

B. $2\tan(x^2)+C$

C. $\frac{1}{6}\sec^3(x^2)+C$

D. $\frac{1}{3}\sec^3(x^2)+C$

2. What is the angle between the vectors $\underset{\sim}{a}=\begin{bmatrix}-2\\1\\1\end{bmatrix}$ and $\underset{\sim}{b}=\begin{bmatrix}1\\-2\\1\end{bmatrix}$

A. $30°$

B. $60°$

C. $120°$

D. $150°$

3. With a suitable substitution $\int_0^{\frac{\pi}{3}} \sin^3 x \cos^5 x \; dx$ can be expressed as

A. $\int_{0.5}^{1} (u^7 - u^5) \, du$

B. $-\int_0^{\frac{\sqrt{3}}{2}} (u^3 - u^5 + u^7) \, du$

C. $\int_{\frac{\pi}{3}}^{0} (u^7 - u^5) \, du$

D. $\int_{0.5}^{1} (u^5 - u^7) \, du$

4. In 3D space, the cartesian equation $(x-2)^2 + (z-5)^2 = 9 \;\; y \in [-4,4]$, geometrically describes:

A. A sphere radius = 3 centred at $(2,0,5)$ and truncated by the planes $y = -4$ and $y = 4$

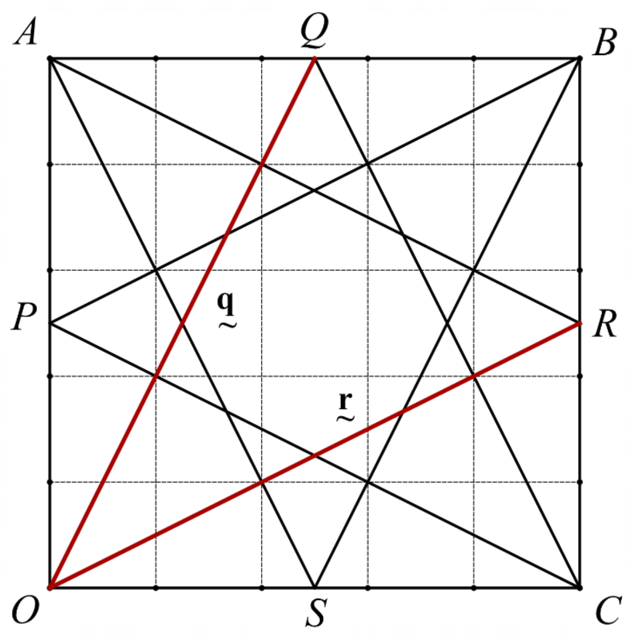
B. A cylinder with a base in the xz plane of radius 3, centred at $(2,0,5)$ and height of 8

C. A spiral curve tracing the cylinder described in B.

D. A horizontal paraboloid with a vertex at $(2,0,5)$ and truncated by the plane $y = 4$

5. The sides of a square are subdivided into 5 equal parts forming 25 smaller squares.

The corners of the large square are connected to the midpoints of the opposite side, $P, Q, R,$ and S as shown in the diagram below.



$\underset{\sim}{q} = \overrightarrow{OQ}$, and $\underset{\sim}{r} = \overrightarrow{OR}$. Which of the following represents $\overrightarrow{OB}$?

(**Hint:** let the horizontal and vertical sides of the smaller squares represent 1 unit of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ respectively)

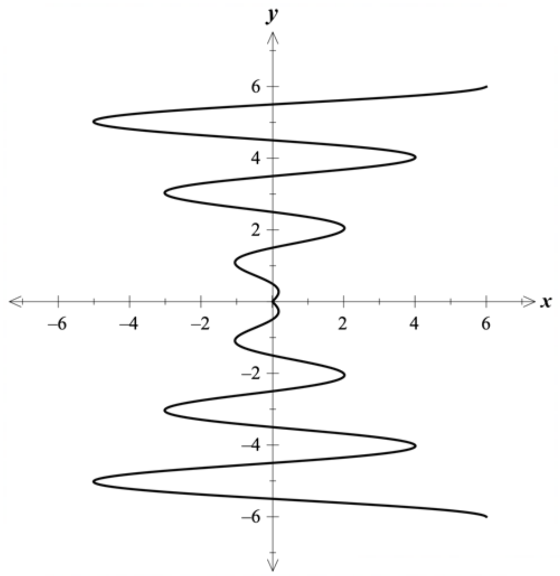
A. $\dfrac{2\underset{\sim}{q} + 2\underset{\sim}{r}}{5}$

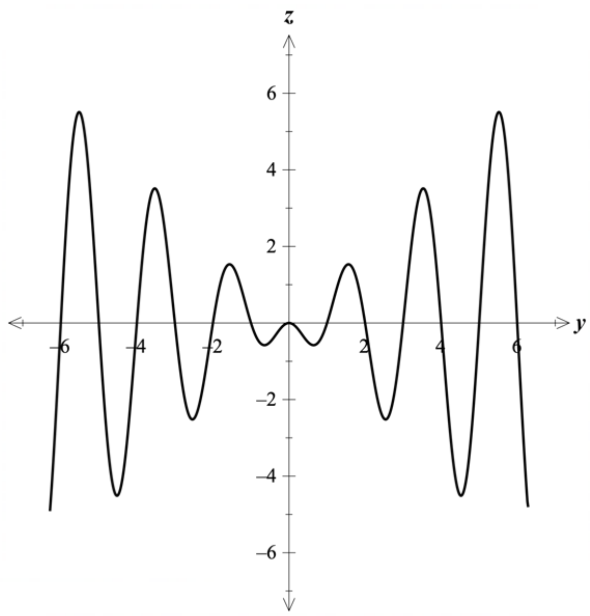
B. $\dfrac{2\underset{\sim}{q} + 2\underset{\sim}{r}}{3}$

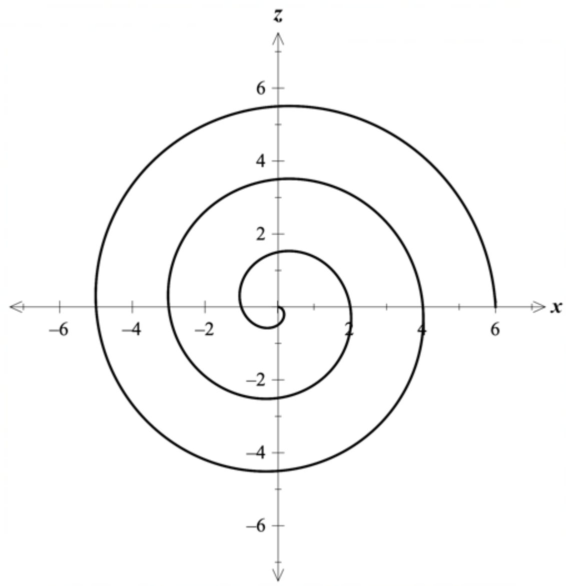
C. $\dfrac{3\underset{\sim}{q} + 3\underset{\sim}{r}}{5}$

D. $\dfrac{3\underset{\sim}{q} + 2\underset{\sim}{r}}{5}$

6. These 3 diagrams represent the projections of a 3D curve onto the xy , yz, and xz planes respectively.







Which of the following could be a parametric equation for this $3D$ curve ?

A. $(\,|t|\cos\pi t\,, t\,, -t\sin\pi t), -6 \le t \le 6$

B. $(\,t\cos\pi t\,, t\,, -t\sin\pi t), -6 \le t \le 6$

C. $(\,t\cos\pi t\,, t\sin\pi t,\ t), -6 \le t \le 6$

D. $(t\cos\pi t\,, t\,, -|t|\sin\pi t), -6 \le t \le 6$

Section II

Attempt Questions 7-12 (56 marks)
Allow about 110 minutes for this section

- Answer each question on a new page in the answer booklet
- In questions 7-12, your responses **must include** relevant mathematical reasoning and/or calculations.

Question 7 (9 marks) Start a new page **Marks**

a) i) Find the values of A and B so that $\dfrac{2x+1}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$ **2**

ii) Hence find the exact value of $\displaystyle\int_4^5 \frac{2x+1}{(x-2)(x-3)}\, dx$ **2**

b) By considering the dot product of two 3 dimensional vectors or otherwise, show that for any $a_k, b_k \in \mathbb{R}$

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$ **2**

c) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x + \cos x}$ using the substitution $t = \tan\dfrac{x}{2}$ **3**

End of Question 7 please turn over

Question 8 (10 marks) Start a new page **Marks**

a) The points A and B have position vectors $\underset{\sim}{a} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$ respectively

i) Find $\overrightarrow{AB}$ **1**

ii) Find the unit vector $\underset{\sim}{\hat{a}}$ **1**

iii) Find the vector equation, $\underset{\sim}{v}$, of the line passing through the point $C\ (11, 7, 2)$ and parallel to $\underset{\sim}{a}$ **2**

iv) It is known than $\underset{\sim}{v}$ and the line through AB intersect (do not prove this).
Show that this point of intersection is $(5, 9, 6)$ **2**

v) Show that $\underset{\sim}{v}$ and AB are not perpendicular **1**

b) Find the exact value of $\int_1^2 x(x-1)^{\frac{3}{2}}\,dx$ **3**

End of Question 8 please turn over

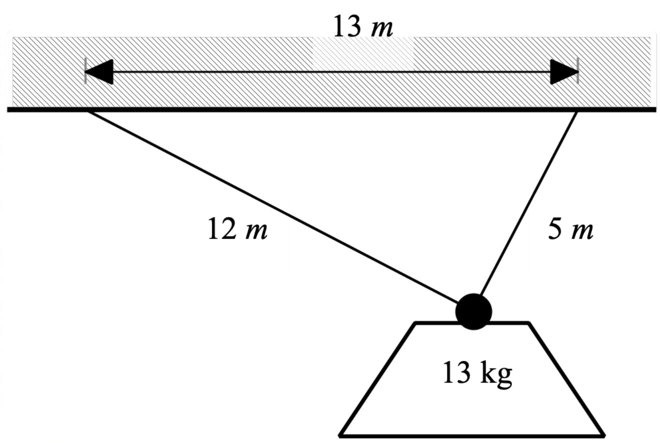
Question 9 (9 marks) Start a new page **Marks**

a) State the Cartesian equation of a sphere with a radius of 4 centred on $P\ (1, 2, 3)$ **1**

b) By using integration by parts or otherwise find $\int (3x^2 + 1)\arctan(x)\,dx$ **3**

c) Prove that $9x^2y^2 + 4y^2(x+1) \geq 16xy^2 \quad x, y \in \mathbb{R}$ **2**

d) A 13 kg weight is held in equilibrium by a 5 metre rope and a 12 metre rope that are attached 13 meters apart to the horizontal ceiling as shown below. Taking acceleration due to gravity to be $10\ \mathrm{m/s^2}$, find the exact tension in the ropes in Newtons. **3**



End of Question 9 please turn over

Question 10 (10 marks) Start a new page **Marks**

a) The point C divides the interval AB between $A(5,-2,-8)$ and $B\ (2,\ 10,-5)$ in the ratio $2:1$ (**Hint:** A representative 2D diagram should help you with this question)

i) Show that the position vector $\overrightarrow{OC} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}$ **2**

ii) Find the projection of $\overrightarrow{OC}$ onto the vector $\overrightarrow{OD} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ **1**

iii) Find the exact area of ΔOCD **3**

b) Use the substitution $x = 2\tan\theta$ to find $\displaystyle\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\ dx$ **4**

End of Question 10 please turn over

Question 11 (9 marks) Start a new page **Marks**

a) i) By using integration by parts, show that

$$\int_1^k \ln x \, dx = k \ln k - k + 1$$ **2**

ii) For the curve $y = \ln x$, use the trapezoidal rule on the intervals with integer endpoints $1,2,3, \dots, k$ to show that

$$\int_1^k \ln x \, dx \approx \frac{1}{2} \ln k + \ln[(k-1)!]$$ **2**

iii) Hence and assuming that $\ln x$ is concave down throughout its domain deduce that

$$k! < e\sqrt{k}\left(\frac{k}{e}\right)^k$$ **2**

b) i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **1**

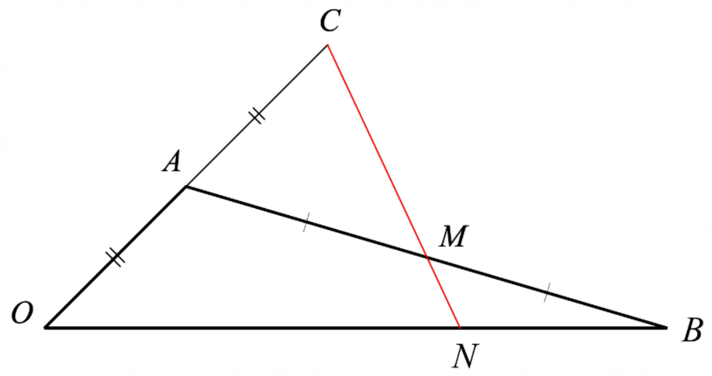
ii) Hence find the exact value of $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} \, dx$ **2**

End of Question 11 please turn over

Question 12 (9 marks) Start a new page **Marks**

a) The figure below shows the triangle OAB. A point C is produced such that $\overrightarrow{OC} = 2\overrightarrow{OA}$.

M is the midpoint of AB. A straight line through C and M intersects OB at the point N



Given that $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$

i) Find $\overrightarrow{CM}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ **2**

ii) Hence show that $\overrightarrow{ON} = \left(2 - \frac{3}{2}\lambda\right)\underset{\sim}{a} + \frac{1}{2}\lambda\underset{\sim}{b}$ **2**

iii) Show that $ON: NB = 2: 1$ **2**

b) A number is divisible by 11 if and only if the difference between the sum of its digits in odd positions, and sum of its digits in even positions, is divisible by 11. For example, 869 is divisible by 11 as $8 + 9 - 6 = 11$ which is clearly divisible by 11.

By writing a 3 digit integer N as $100a + 10b + c$ where a, b, and c are its digits and $a \neq 0$, prove that N is divisible by 11 if and only if $a - b + c$ is divisible by 11.

(you can assume that 0 and negative multiples of 11 are divisible by 11) **3**

End of Exam

2024-Y12-EX2-ASS3-SOLS MC (P01)

Multiple Choice

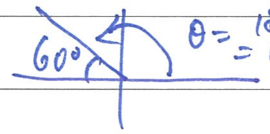
1	2	3	4	5	6
A	C	D	B	B	A

1. $\int x\sec^2(x^2)\,dx = \frac{1}{2}\int 2x\sec^2(x^2)\,dx = \frac{1}{2}\tan(x^2) + C$

$\therefore$ A

2. $\cos\theta = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \dfrac{-2-2+1}{\sqrt{2^2+1+1}\times\sqrt{2^2+1+1}} = -1/2$

$\cos\theta = -1/2 \quad \therefore$ in Q2



$60°$

$\theta = 180-60 = 120°$

C

3. $\int_0^{\pi/3}\sin^3\cos^5 dx = \int_0^{\pi/3}\sin^2 x\cdot\cos^5 x\cdot\sin x\,dx$

$= \int_0^{\pi/3}(1-\cos^2 x)(\cos^5 x)\cdot\sin x\,dx$

$u = \cos x$

$du = -\sin x$

$x=0 \Rightarrow u = 1$

$x = \pi/3 \quad u = 0.5$

$= -\int_1^{0.5}(1-u^2)(u^5)\,du$

$= \int_{0.5}^{1}(u^5 - u^7)\,du \quad \therefore$ D

4. point moves in a circle in (x+z) plane.
and straight line along y

$\therefore$ a cylinder centre (2,0,5) radius 3
with base in x/z plane ... $\therefore$ B

(Q5) $\overset{\uparrow j}{\underset{\sim}{}}\rightarrow \underset{\sim}{i}$ let each small square side represent 1 unit in $\underset{\sim}{i}, \underset{\sim}{j}$ direction.

Then we need $\overrightarrow{OB} = (5\underset{\sim}{i} + 5\underset{\sim}{j})$

what we have is $\overrightarrow{OQ}$ and $\overrightarrow{OR}$

$\overrightarrow{OQ} = \frac{5}{2}\underset{\sim}{i} + 5\underset{\sim}{j}$ $\overrightarrow{OR} = 5\underset{\sim}{i} + \frac{5}{2}\underset{\sim}{j}$

$\alpha\overrightarrow{OQ} + \gamma\overrightarrow{OR} = \overrightarrow{OB}$ (any point of Cartesian Plane can be represented as a linear combination of 2 non-parallel vectors in the plane....)

$\alpha\begin{pmatrix} 5/2 \\ 5 \end{pmatrix} + \gamma\begin{pmatrix} 5 \\ 5/2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$5/2\,\alpha + 5\gamma = 5 \Rightarrow 5\alpha + 10\gamma = 10$ — ①

$5\alpha + 5/2\,\gamma = 5 \Rightarrow 10\alpha + 5\alpha = 10$ — ②

Solving... $\alpha = \gamma = 2/3$ ∴ B

Note: you could just eyeball it noting $\overrightarrow{OQ} + \overrightarrow{OR} = \frac{15}{2}\underset{\sim}{i} + \frac{15}{2}\underset{\sim}{j}$ so the direction is right so $\overrightarrow{OB}$ is just a scalar multiple of the sum, the use ratios....

(6) Note a 'circular spiral' in x-z plane ∴ the sin/cos pairs must be in the x, z positions so eliminate $\cancel{C}$
also means $y = t$

now examining on the y-z projection.

z component is even wrt $y = t$

i.e $z = f(t)$ must be an even function.

∴ eliminate $\cancel{B}$ as $z = t\cos\pi t$ which is odd (odd×even = odd)

eliminate $\cancel{D}$ as again $z = t\cos\pi t$ which is odd.

must be A which is even $= |t|\cos\pi t$

note: There are many approaches you don't have to use odd/even, usually you just follow the curves near transition points to see if the sign is correct and it is moving in the right direction.

2024-EX2-TASK3-SOLUTIONS

(Q7)

a) (i) $$\frac{2x+1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$= \frac{A(x-3) + B(x-2)}{(x-2)(x-3)}$$

$$2x+1 = A(x-3) + B(x-2)$$

$$2\times3+1 = A\times0 + B(3-2)$$ sub in $x=3$

$$\therefore \underline{B = 7}$$

$$2\times2+1 = A(2-3) + B(2-2)$$ Sub in $x=2$

$$\therefore \underline{A = -5}$$

(ii) $$\int_4^5 \frac{2x+1}{(x-2)(x-3)}\,dx = \int_4^5 \left(\frac{-5}{x-2} + \frac{7}{x-3}\right) dx.$$

$$= \Big[7\ln|x-3| - 5\ln|x-2|\Big]_4^5$$

$$= 7\ln 2 - 5\ln 3 - (7\cancel{\ln 1} - 5\ln 2)$$

$$= 12\ln 2 - 5\ln 3$$

$$= \ln\left(\frac{4096}{243}\right)$$

(b) let $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Consider the two forms of scalar product.

P02

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$$

$$a_1b_1 + a_2b_2 + a_3b_3 = \sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2} \cdot \cos\theta$$

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)\cos^2\theta$$

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$ as $\underline{0 \leq \cos^2\theta \leq 1}$

✓

(c) $$\int_0^{\pi/2} \frac{dx}{1 + \sin x + \cos x}$$

$$= \int_{t=0}^{t=1} \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\, dt$$

$$= \int_0^1 \frac{1}{\frac{1+t^2 + 2t + 1 - t^2}{\cancel{1+t^2}}} \times \frac{2}{\cancel{1+t^2}}\, dt$$

$$= \int_0^1 \frac{2\,dt}{1+\cancel{t^2}+2t+1-\cancel{t^2}} = \int_0^1 \frac{2\,dt}{2+2t}$$

$$= \int_0^1 \frac{dt}{1+t} = \Big[\ln(1+t)\Big]_0^1$$

$$= \underline{\ln 2}$$

$t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(1+t^2)$

$\therefore dx = \frac{2\,dt}{1+t^2}$

$x = 0 \quad t = 0$

$x = \pi/2 \quad t = 1$

(Q8) (a)

(i) $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix}1\\5\\4\end{pmatrix} - \begin{pmatrix}-3\\1\\2\end{pmatrix} = \begin{pmatrix}4\\4\\2\end{pmatrix}$

(ii) $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \dfrac{1}{\sqrt{(-3)^2+1^2+2^2}}\begin{pmatrix}-3\\1\\2\end{pmatrix}$

$\therefore \hat{\underset{\sim}{a}} = \dfrac{1}{\sqrt{14}}\begin{pmatrix}-3\\1\\2\end{pmatrix}$

(iii) $\underset{\sim}{v} = \begin{pmatrix}11\\7\\2\end{pmatrix} + \lambda\begin{pmatrix}-3\\1\\2\end{pmatrix}$

direction vector.

position vector of a point on line.

(iv) line $\overrightarrow{AB} = \begin{pmatrix}-3\\1\\2\end{pmatrix} + \mu\begin{pmatrix}4\\4\\2\end{pmatrix}$

as it is known they intersect our check does not need to be exhaustive, can either sub in (5, 9, 6) into equations of both lines to verify, or (better solution) solve simultaneously to find λ or μ then use this to confirm point.

$$\begin{pmatrix}11\\7\\2\end{pmatrix} + \lambda\begin{pmatrix}-3\\1\\2\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix} + \mu\begin{pmatrix}4\\4\\2\end{pmatrix}$$

$11 - 3\lambda = -3 + 4\mu$

$7 + \lambda = 1 + 4\mu$

$\therefore 14 - 3\lambda = 6 + \lambda$

$4\lambda = 8$

$\therefore \lambda = 2$

sub into either eqn.

Point $= \begin{pmatrix}11\\7\\2\end{pmatrix} + 2\begin{pmatrix}-3\\1\\2\end{pmatrix} = \begin{pmatrix}5\\9\\6\end{pmatrix}$

verified.

$\left(\mu = \dfrac{7+2-1}{4} = 2\right)$

(a)(i) to test for right angle test the scaler product of the direction vectors of lines.

direction vector of $\underset{\sim}{v} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ direction vector of $\vec{AB} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$

$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} = -12+4+4 \neq 0 \therefore$ not perpendicular.

(b) $$\int_1^2 x(x-1)^{3/2}\,dx$$

$u = x-1 \therefore x = u+1$

$du = dx$

$x=1 \quad u=0$

$x=2 \quad u=1$

$$= \int_{u=0}^{u=1} (u+1)\cdot u^{3/2}\,du$$

$$= \int_0^1 (u^{5/2}+u^{3/2})\,du = \left[\frac{2}{7}u^{7/2} + \frac{2}{5}u^{5/2}\right]_0^1$$

$$= \frac{2}{7} + \frac{2}{5}$$

$$= \frac{24}{35}$$

(9) (a) $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$.

(b) $$\int (3x^2+1)\arctan x\,dx$$

$u = \arctan x$

$u' = \dfrac{1}{x^2+1}$

$v' = 3x^2+1$

$v = x^3+x$

$$= uv - \int vu'\,dx$$

$$= \arctan x\,(x^3+x) - \int \frac{x^3-x}{x^2+1}\,dx$$

$$= \arctan x\,(x^3+x) - \int x\,dx$$

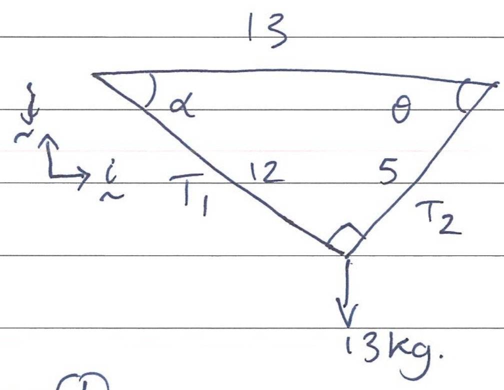
$$= \arctan x\,(x^3+x) - \frac{x^2}{2} + C$$

POS

(9a) (c) $\boxed{\begin{array}{l} \text{RTP} \quad 9x^2y^2+4y^2(x+1) \geq 16xy^2 \\ \text{i.e} \quad 9x^2y^2+4y^2(x+1)-16xy^2 \geq 0 \end{array}}$

$$\begin{aligned} LHS &= 9x^2y^2+4y^2(x+1)-16xy^2 \\ &= y^2(9x^2+4x+4-16x) \\ &= y^2(9x^2-12x+4) \\ &= y^2(3x-2)^2 \\ &\geq 0 \quad \text{as both factors are squares.} \end{aligned}$$

$\therefore \quad 9x^2y^2+4y^2(x+1) \geq 16xy^2$



note we have a right triangle (Pythagorean triad)

(d) horizontally

$T_2\cos\theta\,\underset{\sim}{i} - T_1\cos\alpha\,\underset{\sim}{i} = 0$

$T_1\cos\alpha = T_2\cos\theta$

$\therefore \quad T_2 = \dfrac{\cos\alpha}{\cos\theta}T_1 \quad$ — (1)

vertically

$T_1\sin\alpha\,\underset{\sim}{j} + T_2\sin\theta\,\underset{\sim}{j} - 13\times10\,\underset{\sim}{j} = 0.$

$T_1\sin\alpha + T_2\sin\theta = 130 \quad$ — (2).

$T_1\sin\alpha + \left(\dfrac{\cos\alpha}{\cos\theta}T_1\right)\sin\theta = 130.$

$T_1(\sin\alpha + \cos\alpha\tan\theta) = 130.$

$T_1\left(\dfrac{5}{13} + \dfrac{12}{3}\cdot\dfrac{12}{5}\right) = 130.$

$$T_1\left(\frac{5\times5+12\times12}{13\times5}\right)=130.$$

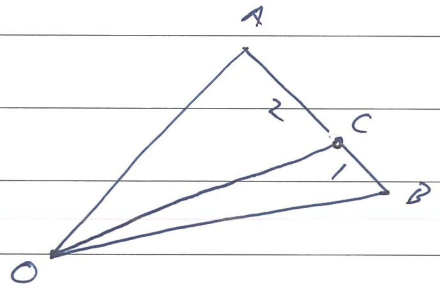
$$T_1=\frac{130\times13\times5}{25+144}=\frac{130\times5\times13}{169}=\underline{50\,N}$$

$$T_2=\frac{\cos\alpha}{\cos\theta}T_1=\frac{12}{13}\div\frac{5}{13}\cdot 50\,N$$

$$=\frac{12}{5}\times50=\underline{120\,N}$$

(Q10)

(a)



$$\vec{OA}=\begin{pmatrix}5\\-2\\-8\end{pmatrix}\qquad \vec{OB}=\begin{pmatrix}2\\10\\-5\end{pmatrix}$$

method 1 using ratio divide formula.

$$\vec{OC}=\frac{1}{2+1}\begin{pmatrix}5\times1+2\times2\\-2\times1+10\times2\\-8\times1+-5\times2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}9\\18\\-18\end{pmatrix}=\begin{pmatrix}3\\6\\-6\end{pmatrix}\checkmark$$

<u>method 2</u> vector methods.

$$\vec{OC}=\vec{OA}+\frac{2}{3}\vec{AB}$$

$$=\begin{pmatrix}5\\-2\\-8\end{pmatrix}+\frac{2}{3}\begin{pmatrix}2-5\\10--2\\-5--8\end{pmatrix}=\begin{pmatrix}5\\-2\\-8\end{pmatrix}+\frac{2}{3}\begin{pmatrix}-3\\12\\3\end{pmatrix}$$

$$=\begin{pmatrix}5\\-2\\-8\end{pmatrix}+\begin{pmatrix}-2\\8\\2\end{pmatrix}=\begin{pmatrix}3\\6\\-6\end{pmatrix}\checkmark$$

(10) (a) ii

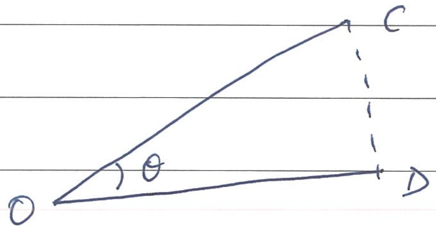
$\vec{OD} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$ $\vec{OC} = \begin{pmatrix}3\\6\\-6\end{pmatrix}$

$$\text{Proj}_{\vec{OD}}\vec{OC} = \frac{\vec{OC}\cdot\vec{OD}}{|\vec{OD}|^2}\vec{OD}$$

$$= \frac{(3\times2 + 6\times1 + -6\times-1)}{(2^2+1^2+1^2)}\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$= \frac{6+6+6}{6}\begin{pmatrix}2\\1\\-1\end{pmatrix} = 3\begin{pmatrix}2\\1\\-1\end{pmatrix} = \underline{\begin{pmatrix}6\\3\\-3\end{pmatrix}}$$

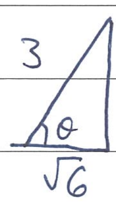
(iii)



area $= \frac{1}{2}|\vec{OC}||\vec{OD}|\sin\theta$

need angle. $\cos\theta = \frac{\vec{OC}\cdot\vec{OD}}{|\vec{OC}||\vec{OD}|} = \frac{18}{\sqrt{81}\sqrt{6}}$

$\cos\theta = \frac{18}{9\sqrt{6}} = \frac{2}{\sqrt{6}} = \underline{\underline{\frac{\sqrt{6}}{3}}}$



$\sin\theta = \frac{\sqrt{9-6}}{3} = \frac{\sqrt{3}}{3}$

$\therefore A = \frac{1}{2}|\vec{OC}||\vec{OD}|\cdot\sin\theta = \frac{1}{2}\cdot9\cdot\sqrt{6}\cdot\frac{\sqrt{3}}{3}$

$= \underline{\frac{9\sqrt{2}}{2}\ u^2}$

(Q10)b

$x = 2\tan\theta$

$\frac{dx}{d\theta} = 2\sec^2\theta$

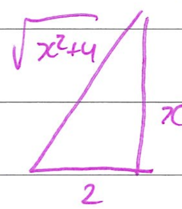
$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{(4+x^2)^{3/2}}\,dx = \int \frac{2\sec^2\theta}{(4+4\tan^2\theta)^{3/2}}\,d\theta$$

$$= \frac{2}{8}\int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta = \frac{1}{4}\int \frac{1}{\sec\theta}$$

$\tan\theta = \frac{x}{2}$

$\sin\theta = \frac{x}{\sqrt{x^2+4}}$



$$= \frac{1}{4}\int \cos\theta = \frac{1}{4}\sin\theta + C$$

(undo substitution) $\left(\sin\theta = \frac{x}{\sqrt{x^2+4}}\right)$

$$= \underline{\frac{x}{4\sqrt{x^2+4}} + C}$$

(Q11)

(a) i

$u = \ln x \quad u' = 1/x$

$v' = 1 \quad v = x$

$$\int_1^K \ln x\,dx = uv - \int_1^K u'v\,dx$$

$$= \left[x\ln x\right]_1^K - \int_1^K \frac{x}{x}\,dx = (K\ln K - \ln 1) - [x]_1^K$$

$$= K\ln K - (K-1) = K\ln K - K + 1 \checkmark$$

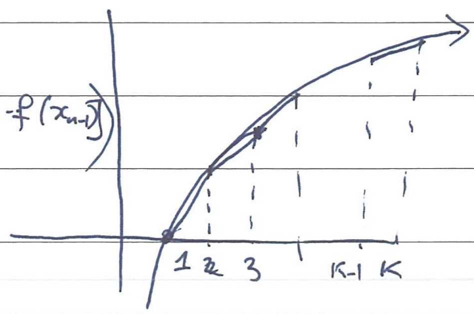
a(ii)

$$\int_1^K \ln x\,dx \simeq \frac{b-a}{2n}\left(f(a)+f(b) + 2\left[f(x_1)+\cdots f(x_{n-1})\right]\right)$$

in this case $f(x) = \ln x$

$K-1$ intervals $\therefore n = K-1$

$b = K \quad a = 1$



$\therefore \int_1^K \ln x\,dx \simeq \frac{K-1}{2(K-1)}\left(\ln 1 + \ln K + 2(\ln 2 + \ln 3 + \cdots \ln(K-1)\right)$

$= \frac{1}{2}(\ln K + 2(\ln 2 + \ln 3 + \cdots \ln(K-1))$

$= \frac{1}{2}\ln K + \ln(2 \times 3 \times \cdots \cdots (K-1))$

$= \frac{1}{2}\ln K + \ln((K-1)!)$ ✓ as required.

(iii) as $\ln x$ concave down area of trapezia < exact area.

i.e $\frac{1}{2}\ln K + \ln((K-1)!) < K\ln K - K + 1$

$\ln K + \ln((K-1)!) < K\ln K - K + \frac{1}{2}\ln K + 1$

$\ln(K(K-1)!) < \ln K^K - \ln e^K + \ln\sqrt{K} + \ln e$

$\ln(K!) < \ln\left(\frac{K^K}{e^K}\cdot\sqrt{K}\cdot e\right)$

$\ln(K!) < \ln\left(e\sqrt{K}\left(\frac{K}{e}\right)^K\right)$

$K! < e\sqrt{K}\left(\frac{K}{e}\right)^K$ as required.

(b) (i) RTP $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$RHS = \int_0^a f(a-x)\,dx = -\int_{u=a}^{u=0} f(u)\,du$

$u = a - x$
$du = -dx$
$dx = -du$.

$= \int_0^a f(u)\,du$

$x=0 \quad u=a$
$x=a \quad u=0$

x (u/just placeholder)

$= \int_0^a f(x)\,dx = LHS$

(11) (b)

(ii) $$I = \int_0^{\pi/2} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}}\,dx$$

$$= \int_0^{\pi/2} \frac{e^{\sin(\pi/2 - x)}}{e^{\sin(\pi/2 - x)} + e^{\cos(\pi/2 - x)}}\,dx$$

$$= \int_0^{\pi/2} \frac{e^{\cos x}}{e^{\cos x} + e^{\sin x}}\,dx$$ (complementary identity).

$$2I = \int_0^{\pi/2} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}}\,dx + \int_0^{\pi/2} \frac{e^{\cos x}}{e^{\sin x} + e^{\cos x}}\,dx.$$

$$= \int_0^{\pi/2} \left(\frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} + \frac{e^{\cos x}}{e^{\sin x} + e^{\cos x}} \right) dx$$

$$= \int_0^{\pi/2} \frac{e^{\sin x} + e^{\cos x}}{e^{\sin x} + e^{\cos x}}\,dx$$

$$= \int_0^{\pi/2} 1\,dx = \left[x \right]_0^{\pi/2}$$

$$2I = \pi/2$$

$$\therefore I = \pi/4$$

(Q12) (a)

(i) $\vec{CM} = \vec{CM} + \vec{AM}$

$= -\vec{OA} + \frac{1}{2}(\vec{AB})$

$= -\underset{\sim}{a} + \frac{1}{2}(\vec{OB} - \vec{OA})$

$= -\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

$\vec{CM} = -\frac{3}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

(ii) $\vec{ON} = \vec{OC} + \vec{CN}$

$= \vec{OC} + \lambda\vec{CM}$

$= 2\underset{\sim}{a} + \lambda(-3/2\,\underset{\sim}{a} + 1/2\,\underset{\sim}{b})$

$= (2 - \frac{3}{2}\lambda)\underset{\sim}{a} + \frac{1}{2}\lambda\underset{\sim}{b}$ ✓ as required.

(iii) we note $\vec{ON}$ has no $\underset{\sim}{a}$ component

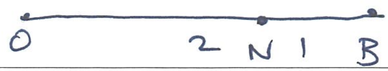
$\therefore (2 - \frac{3}{2}\lambda)\underset{\sim}{a} = \underset{\sim}{0}$ i.e $2 - \frac{3}{2}\lambda = 0$

$2 - \frac{3}{2}\lambda = 0$ $\quad \frac{3}{2}\lambda = 2$ $\therefore$ $\underline{\lambda = \frac{4}{3}}$

Sub back into equation for $\vec{ON}$

$\vec{ON} = 0\underset{\sim}{a} + \frac{1}{2} \cdot \frac{4}{3}\underset{\sim}{b}$

$\vec{ON} = \frac{2}{3}\vec{OB}$



$\therefore$ N divides OB in the ratio 2 : 1

ON : NB = 2 : 1

Q12) b)

$$N = 100a + 10b + c$$

RTP $N = 11M$ iff $a - b + c = 11K$ $N, K, M \in \mathbb{Z}$

$\Rightarrow$ $a - b + c = 11K$

$$100a + 10b + c = 99a + a + 11b - b + c$$
$$= 99a + 11b + a - b + c$$
$$= 11(9a + b) + 11K$$ which is divisible by 11

$\Leftarrow$ $N = 11M$

$$100a + 10b + c = 11M.$$

$$99a + 11b + a - b + c = 11M$$

$$a - b + c = 11M - 99a - 11b$$
$$= 11(M - 9a + b).$$

which is divisible by 11

$\therefore$ N is divisible by 11 iff $a-b+c$ is divisible by 11

(i.e N is divisible by 11 iff the difference between the sum of its digits in odd positions, and sum of its digits in even digits is divisible by 11 (for a 3 digit number)

Note can do for 4 digits etc... but for general case need modulo arithmetic which is beyond the syllabus...)

— The end —