TRINITY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

YEAR 12 2008 HALF YEARLY EXAMINATION

EXTENSION II MATHEMATICS

STUDENT NUMBER: ______________________________

ASSESMENT TASK 3

Time Allowed – *3 hours (Plus 5 minutes Reading Time)*
Total marks – 120

WEIGHTING 30% towards final result

Outcomes referred to: E1, E2, E6, E8 and E9 (refer booklet)

INSTRUCTIONS:

1. Attempt **ALL** questions.
2. Show all necessary working.
3. Put your Student Number **ONLY** on each writing sheet.
4. Begin **each** question on a **new page**.
5. Marks are shown next to each question.
6. Additional writing sheets may be obtained from the supervisor upon request.
7. Only approved Board of Studies calculators are permitted.
8. A Standard Integrals sheet is provided on a separate sheet.

Question 1: *Use a SEPARATE Writing Booklet.*

Total Marks 15

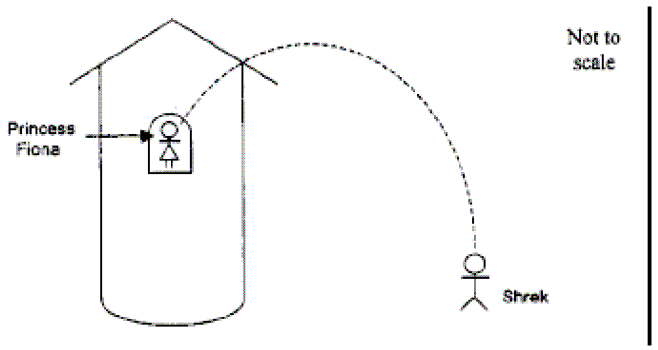
(a) (i) A particle is moving along the x-axis so that at any time $t \geq 0$ its velocity is v and its acceleration is $\frac{dv}{dt}$. Prove that $\frac{dv}{dt} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$ **2**

(ii) A particle is moving along the x-axis in simple harmonic motion with its velocity given by the equation $v^2 = 4x - 2x^2$. Find

(α) Its amplitude. **2**

(β) Its period. **2**

(b)



Princess Fiona is locked up in a tower, 80m above the ground.
To gain the attention of Shrek, Princess Fiona throws a lentil at an angle of elevation of θ and an initial velocity of $50ms^{-1}$.

(i) Derive the equations for the horizontal and vertical displacements of the lentil t seconds after it is thrown. (Use $g = 10\,ms^{-2}$.) **4**

(ii) Shrek is 300m from the base of the tower when he is hit by the lentil. Find the values of the initial angle of projection θ, correct to the nearest degree, if Shrek is 2m tall. **5**

Question 2: *Use a SEPARATE Writing Booklet.*

Total Marks 15

(a) It is given that $\left(2+\cos x\right)\left(2-\cos y\right)=3$, where $0<x<\pi$ and $0<y<\pi$.

(i) Show that $\cos y=\dfrac{1+2\cos x}{2+\cos x}$ and $\sin y=\dfrac{\sqrt{3}\sin x}{2+\cos x}$. **2**

(ii) Hence show that $\dfrac{dy}{dx}=\dfrac{\sqrt{3}}{2+\cos x}$ **3**

(b) The function $f(x)$ is given by $f(x)=a+\dfrac{b\sin x}{x}$, $x\neq 0$ and $f(0)=0$,

where *a* and *b* are non-zero real numbers.

(i) Show that $f(x)$ is an even function. **1**

(ii) Find the general solution of the equation $f(x)=a$. **2**

(iii) If $\lim\limits_{x\to\infty} f(x)=1$ and $f(x)$ is continuous at $x=0$,

find the values of *a* and *b*. **2**

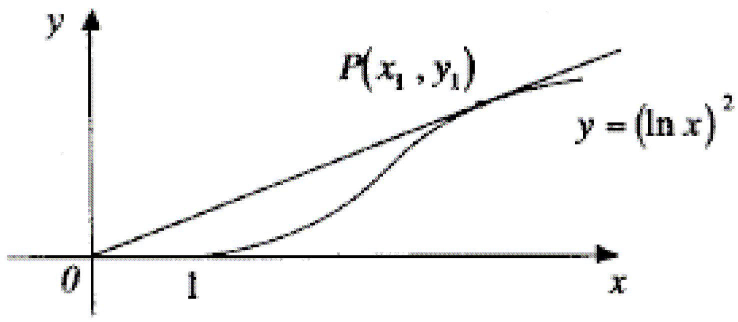
(c) Find $\displaystyle\int \frac{1}{1-x^2}\,dx$. **2**

(d) Use the substitution $u=e^x-1$ to find $\displaystyle\int \frac{e^{2x}}{\left(e^x-1\right)^2}\,dx$. **3**

Question 3: *Use a SEPARATE Writing Booklet.*

Total Marks 15

(a)



The diagram shows the graph of the function $f(x) = (\ln x)^2$.

$P(x_1,\ y_1)$ is a point on the curve such that the tangent to the curve at P passes through the origin O.

(i) By considering the gradient of the line OP in two different ways, show that P is the point $(e^2,\ 4)$. **2**

(ii) Find the set of values of the real number k such that the equation $f(x) = kx$ has two distinct real roots. **1**

(iii) Use integration by parts to show that

$$\int (\ln x)^2\ dx = x(\ln x)^2 - 2x\ln x + 2x + c\ .$$

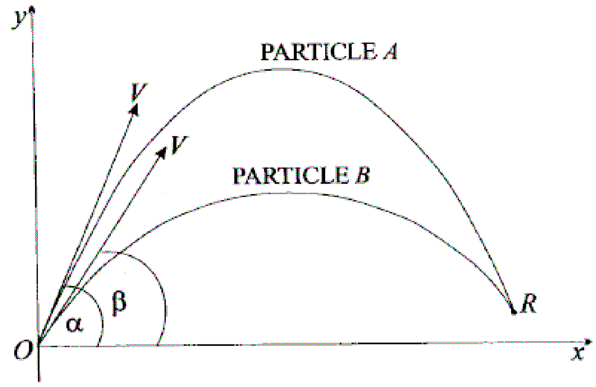
Hence find the exact area of the region bounded by the curve $y = f(x)$, the x axis and the line OP. **4**

(iv) Find the equation of the inverse function $f^{-1}(x)$. **1**

(v) Find the equation of the tangent to the curve $y = f^{-1}(x)$ that passes through the origin. **1**

Question 3 continued next page

(b) *(continued from previous page)*



The diagram above shows two particles A and B projected from the origin. Particle A is projected with an initial velocity of $V\,ms^{-1}$ at an angle α. Particle B is projected T seconds later with the same initial velocity $V\,ms^{-1}$ but at an angle of β. The particles collide at a point R.

(i) You may assume that the equations of the paths of A and B are:

For A: $$y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha$$

For B: $$y = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta$$

Show that the *x*-coordinate of the point R of collision is

$$x = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}.$$ **3**

(ii) You may assume that the equation of the horizontal displacement of A is $x = VT\cos\alpha$.

(α) Write down the equation for the horizontal displacement of B (Remember that B is projected T seconds after A). **1**

(β) Show that the difference T in the times of projection is

$$T = \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)}.$$ **2**

<u>Question 4:</u> ***Use a SEPARATE Writing Booklet.***

Total Marks 15

(a) Use the table of standard integrals to find the area enclosed between the curve $y = \dfrac{1}{\sqrt{4 - x^2}}$, the x-axis and the ordinates at $x = -1$ and $x = 1$. **2**

(b) Find the volume of the solid of revolution formed when the area enclosed by the curve $y = \dfrac{1}{\sqrt{4 - x^2}}$ and the x-axis between the ordinates at $x = -1$ and $x = 1$ is rotated around the x-axis. **3**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{2 - 2\cos 2x}\ dx$. **2**

(d) Evaluate $\displaystyle\int_0^1 x\sqrt{1 - x}\ dx$ **3**

(e) (i) If $\mathbf{I}_n = \displaystyle\int_1^e x(\ln x)^n\ dx$; $n = 0, 1, 2, 3, \ldots\ldots$ **5**

Show that $\mathbf{I}_n = \dfrac{e^2}{2} - \dfrac{n}{2}\mathbf{I}_{n-1}$; $n = 0, 1, 2, 3, \ldots..$

(ii) Hence evaluate $\displaystyle\int_1^e x(\ln x)^3\ dx$

Question 5: ***Use a SEPARATE Writing Booklet.***

Total Marks 15

(a) Two complex numbers are given by

$$z = 3 - 4i \text{ and } w = 2 - 2i.$$ **6**

(i) Find the values of the product $\bar{z}w$.

(ii) Find the two square roots of *z*.

(iii) Express *w* in modulus-argument form and hence find the value of w^4.

(b) The locus of a point P which moves in the complex plane is represented by the equation $\left|z-(3+4i)\right| = 5$. **4**

(i) Sketch the locus of the point P.

(ii) Find the maximum value of the modulus of *z* and write down the value of *arg z* when P is in the position of maximum modulus.

(iii) Find the modulus of *z* when $\arg z = \tan^{-1}\left(\frac{1}{2}\right)$.

(c) Using the complex number $z = 1 - i$:
find the real numbers *a* and b such that $az = 1 - \frac{b}{z}$ **2**

(d) In an Argand diagram, the point P represents the complex number *z*, the point Q represents the complex number *w* and the point R represents the complex number $z - w$ with O being the origin.
It is given that $z - w = iz$. **3**

(i) Describe the geometric properties of the triangle POR, providing full reasoning for your answer.

(ii) Find the size of the angle QPR.

Question 6: Use a SEPARATE Writing Booklet.

Total Marks 15

(a) Consider the curve defined by $2x^2 + xy - y^2 = 0$. At the point $(2, 4)$ on the curve, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **4**

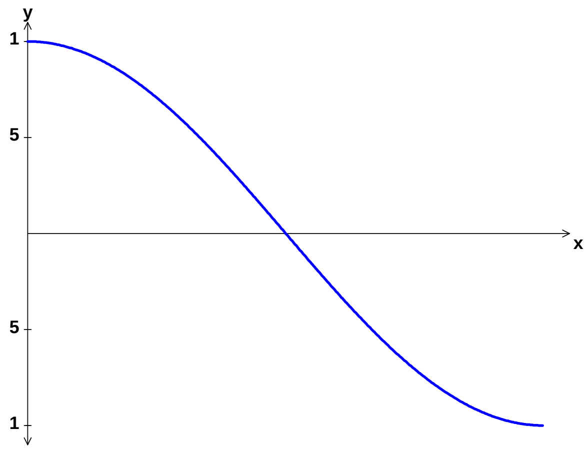
(b) Consider the function $f(x) = \frac{x}{1-x^2}$.

(i) Show that the function is increasing for all values of x in its domain. **1**

(ii) Sketch the graph of $y = f(x)$ showing the intercepts on the axes and the equations of any asymptotes **2**

(iii) Find the values of k such that the equation $\frac{x}{1-x^2} = kx$ has three distinct real roots. **2**

(c)



The graph of $f(x) = \cos x$ is shown above in the interval $0 \le x \le \pi$.

Provide separate half-page sketches of the graphs of the following:

(i) $y = [f(x)]^2$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \frac{1}{|f(x)|}$ **2**

Question 7: *Use a SEPARATE Writing Booklet.*

Total Marks 15

(a) Solve the equation $|z|^2 + 2i\,\overline{z} = 4i + 7$, expressing any answers in the form $z = a + ib$ where a and b are real. **3**

(b) A, B and C are the angles of a triangle.

Show that $(\cos A + i\sin A)(\cos B + i\sin B) + (\cos C - i\sin C) = 0$ **3**

(c) Using $z = \cos\theta + i\sin\theta$

(i) Express $1 + z$ in modulus-argument form.

Hence show that $(1+z)^4 = 16\cos^4\left(\frac{\theta}{2}\right)(\cos 2\theta + i\sin 2\theta)$. **3**

(ii) Use the Binomial Theorem expansion of $(1+z)^4$ to show that

$$1 + 4\cos\theta + 6\cos 2\theta + 4\cos 3\theta + \cos 4\theta = 16\cos^4\left(\frac{\theta}{2}\right)\cos 2\theta,$$

and find a corresponding expression for

$4\sin\theta + 6\sin 2\theta + 4\sin 3\theta + \sin 4\theta$. **3**

(iii) Hence show that

$$\frac{4\sin\theta + 6\sin 2\theta + 4\sin 3\theta + \sin 4\theta}{1 + 4\cos\theta + 6\cos 2\theta + 4\cos 3\theta + \cos 4\theta} = \tan 2\theta,$$

and

$$\frac{4\sin\theta + 4\sin 3\theta + \sin 4\theta}{1 + 4\cos\theta + 4\cos 3\theta + \cos 4\theta} = \tan 2\theta.$$ **3**

Question 8: *Use a SEPARATE Writing Booklet.*

Total Marks 15

(a) (i) Use the substitution $x = 10\sqrt{2}\sin\theta$ to show that

$$\int_{-10}^{10} \sqrt{200 - x^2}\ dx = 100 + 50\pi ,$$

then use a geometrical argument to verify this result. **4**

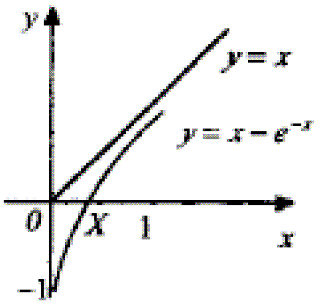
(ii) A mould from a model railway tunnel is made by rotating the region bounded by the curve $y = \sqrt{200 - x^2}$ and the x-axis between the lines $x = -10$ and $x = 10$ through $180°$ about the line $x = 100$ (where all measurements are in cm).
Use the method of cylindrical shells to show that the volume Vcm^3 of the tunnel is given by

$$V = \pi \int_{-10}^{10} (100 - x)\sqrt{200 - x^2}\ dx .$$

Hence find the volume of the tunnel in m^3 correct to 2 significant figures. **4**

Question 8 continued on next page

(b) *(Continued from previous page)*



The diagram shows the graph of the curve $y = x - e^{-x}$, $x \geq 0$.

This curve makes an intercept X on the x-axis, where $0 < X < 1$.

The region bounded by the curve and the line $y = x$ between $x = 0$ and $x = X$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by

$$V = 2\pi \int_0^X x e^{-x} \, dx .$$ **3**

(ii) Hence show that $V = 2\pi\left(1 - X - X^2\right)$. **4**

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Student Number: ______

EXTENSION 2 MATHEMATICS - HALF-YEARLY 2008

SOLUTIONS - PR

Question 1:

(a) (i)
$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} \quad ✓$$
$$= \frac{dv}{dx}.(v)$$
$$= \frac{dv}{dx}.\frac{d(\frac{1}{2}v^2)}{dv} = \frac{d(\frac{1}{2}v^2)}{dx} \quad ✓$$

(ii) $v^2 = 4x - 2x^2$

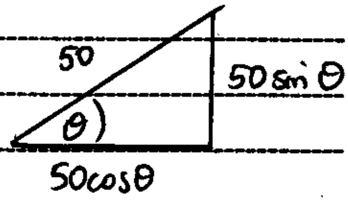
(α) $v = 0 \quad \therefore \quad 0 = 2x(2-x)$ ✓

$\therefore \quad x = 0, \; x = 2$

$\therefore$ The amplitude is $\frac{0+2}{2} = 1$ ✓

(β)
$$v^2 = 4x - 2x^2$$
$$= 2(2x - x^2)$$
$$\therefore \frac{1}{2}v^2 = 2x - x^2$$
$$a = \ddot{x} = 2 - 2x \qquad (a = -n^2x) \quad ✓$$
$$= 2(1-x)$$
$$= -2(x-1)$$
$$\therefore \quad n^2 = 2$$
$$\therefore \quad n = \sqrt{2}$$

Period: $T = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$ ✓



(b) (i) $\ddot{x} = 0$

$\dot{x} = c$

when $t=0$, $\dot{x} = 50\cos\theta$ $\therefore c = 50\cos\theta$

$\therefore \dot{x} = 50\cos\theta$

$x = \int \dot{x}\,dt = \int 50\cos\theta\,dt = 50t\cos\theta + c$

when $t=0$, $x=0$ $\therefore c=0$

$\therefore x = 50t\cos\theta$

$\ddot{y} = -10$

$\dot{y} = \int -10\,dt = -10t + c$

when $t=0$, $\dot{y} = 50\sin\theta$ $\therefore c = 50\sin\theta$

$\dot{y} = -10t + 50\sin\theta$

$y = \int(-10t + 50\sin\theta)\,dt = -5t^2 + 50t\sin\theta + c$

when $t=0$, $y = 80$ $\therefore c = 80$

$\therefore y = -5t^2 + 50t\sin\theta + 80$

(ii) when $x = 300$, $y = 2$.

$\therefore 300 = 50t\cos\theta$

$t = 6/\cos\theta$

Sub into horizontal equation: $2 = -5\left(6/\cos\theta\right)^2 + 50\left(6/\cos\theta\right)\sin\theta + 80$

$\therefore 0 = -180\sec^2\theta + 300\tan\theta + 78$

$0 = -180(\tan^2\theta + 1) + 300\tan\theta + 78$

$= -180\tan^2\theta + 300\tan\theta - 102$

$0 = 180\tan^2\theta - 300\tan\theta + 102$

$$\therefore \tan\theta = \frac{300 \pm \sqrt{300^2 - 4\times180\times102}}{2\times180}$$

$\doteqdot 1.19$ or 0.475

$\therefore \theta = 49°58'$ or $25°27'$

$\therefore$ Initial angle of projection could be either 25 or 50 degrees.

Question 2:

(a) $(2+\cos x)(2-\cos y) = 3$

$$2-\cos y = \frac{3}{2+\cos x}$$

$$\therefore \cos y = 2 - \frac{3}{2+\cos x} \quad ✓$$

$$\cos y = \frac{4+2\cos x - 3}{2+\cos x}$$

$$= \frac{1+2\cos x}{2+\cos x}$$

$\cos^2 y + \sin^2 y = 1$

$$\therefore \sin^2 y = 1 - \left(\frac{1+2\cos x}{2+\cos x}\right)^2$$

$$= \frac{(2+\cos x)^2 - (1+2\cos x)^2}{(2+\cos x)^2} \quad ✓$$

$$= \frac{3 - 3\cos^2 x}{(2+\cos x)^2}$$

$$= \frac{3(1-\cos^2 x)}{(2+\cos x)^2}$$

$$\sin^2 y = \frac{3\sin^2 x}{(2+\cos x)^2}$$

$$\therefore \sin y = \frac{\sqrt{3}\sin x}{2+\cos x}$$

(ii) $(2+\cos x)(2-\cos y) = 3$

Product rule using implicit differentiation.

$u = 2+\cos x \quad \therefore u' = -\sin x$ ✓

$v = 2-\cos y \quad \therefore v' = \sin y(dy/dx)$

$\therefore -\sin x(2-\cos y) + (2+\cos x)\sin y(dy/dx) = 0$

$$\therefore \frac{dy}{dx} = \frac{\sin x(2-\cos y)}{\sin y(2+\cos x)} \quad ✓$$

Substituting for sin y : $\frac{\sin y}{\sin x} = \frac{\sqrt{3}}{2+\cos x}$

$\therefore \; dy/dx = \left(\frac{2+\cos x}{\sqrt{3}}\right)\left(\frac{2-\cos y}{2+\cos y}\right)$

but $(2+\cos x)(2-\cos y) = 3$ ✓

$\therefore \; dy/dx = \frac{3}{\sqrt{3}(2+\cos y)}$

$= \frac{\sqrt{3}}{2+\cos y}$

(b)(i) Even function if $f(-x) = f(x)$

$\therefore \; f(-x) = a + \frac{b\sin(-x)}{-x}$ ✓

$= a + \frac{-b\sin x}{-x}$

$= a + \frac{b\sin x}{x}$

$= f(x) \quad \therefore \; f(x)$ is even

(ii) $f(x) = a$

$\therefore \quad a = a + \frac{b\sin x}{x}$

$\therefore \quad 0 = \frac{b\sin x}{x} \qquad x \neq 0$

$\therefore \; \cancel{b}\sin x = 0$ ✓

$\therefore \; x = n\pi, \quad n = \pm 1, \pm 2, \pm 3, \ldots$ ✓

(iii) $\left|\frac{b\sin x}{x}\right| \leq \left|\frac{b}{x}\right| \rightarrow$ as $x \rightarrow \infty$

$\therefore \lim_{x\to\infty} f(x) = a \quad \therefore \; a = 1$ ✓

$\lim_{x\to 0} f(x) = a + b\lim_{x\to 0}\frac{\sin x}{x} = a+b \quad \therefore \; b = -1$ ✓

(c) $\int \frac{1}{1-x^2}\,dx = \int \frac{1}{(1+x)(1-x)}\,dx$

Partial fractions: $\frac{a}{(1+x)} + \frac{b}{(1-x)} \equiv \frac{1}{(1+x)(1-x)}$

$\therefore\ a(1-x) + b(1+x) \equiv 1$

$\therefore\ a + b + x(b-a) \equiv 1$

$\therefore\ a+b = 1$ and $b-a=0$

$\therefore\ a=b$ and $2b=1$ $b = 1/2$

$\therefore \int \frac{1}{1-x^2}\,dx = \int \left(\frac{1/2}{1+x} + \frac{1/2}{1-x}\right) dx$ ✓

$= \frac{1}{2}\left(\ln|1+x| - \ln|1-x|\right) + c$

$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$ ✓

(d) $\int \frac{e^{2x}}{(e^x-1)^2}\,dx$ Let $u = e^x - 1 \Rightarrow e^x = u+1$

$\frac{du}{dx} = e^x \Rightarrow du = e^x dx$

$= \int \frac{e^x}{(e^x-1)^2}\, e^x\,dx$

$= \int \frac{u+1}{u^2}\,du$ ✓

$= \int \left(\frac{u}{u^2} + \frac{1}{u^2}\right) du$

$= \int \left(\frac{1}{u} + \frac{1}{u^2}\right) du$

$= \ln|u| - \frac{1}{u} + c$ ✓

$= \ln|e^x - 1| - \frac{1}{e^x-1} + c$ ✓

Question 3:

(a) (i) Gradient OP $= y_1/x_1 = \dfrac{(\ln x_1)^2}{x_1}$

$y = (\ln x)^2 \therefore \dfrac{dy}{dx} = \dfrac{2\ln x}{x}$ @ $x = x_1$

$\therefore dy/dx = \dfrac{2\ln x_1}{x_1}$

Equate gradients $\therefore \dfrac{(\ln x_1)^2}{x_1} = \dfrac{2\ln x_1}{x_1}$ ✓

$\therefore (\ln x_1)^2 - 2\ln x_1 = 0$

$\ln x_1(\ln x_1 - 2) = 0$ ✓

$\therefore \ln x_1 = 0$ (Not possible) or $\ln x_1 = 2$

$\ln x_1 = 2$

$\therefore x_1 = e^2 \quad y_1 = (\ln e^2)^2$

$= 2^2$

$= 4$

$\therefore P(e^2, 4)$

(ii) $f(x) = kx$ has two distinct real roots if the line $y = kx$ cuts the curve in two points

$\therefore 0 < k <$ gradient OP

$\therefore 0 < k < 4e^{-2}$ ✓

(iii) $\int (\ln x)^2 dx = \int 1 \times (\ln x)^2 dx$ $\quad u = x \quad v = (\ln x)^2$

$u' = 1 \quad v' = \dfrac{2\ln x}{x}$

$= x(\ln x)^2 - \int x \cdot \dfrac{2\ln x}{x} dx$ ✓

$= x(\ln x)^2 - \int 2\ln x\, dx$ $\quad u = x \quad v = \ln x$

$= x(\ln x)^2 - 2\left[x\ln x - \int x \cdot \tfrac{1}{x} dx\right]$ ✓ $\quad u' = 1 \quad v' = 1/x$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

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$$A = \tfrac{1}{2}e^2 \cdot 4 - \int_1^{e^2} (\ln x)^2\, dx$$

$$= 2e^2 - \left[x(\ln x)^2 - 2x\ln x + 2x \right]_1^{e^2} \quad ✓$$

$$= 2e^2 - \left(e^2(\ln e^2)^2 - 2e^2 \ln e^2 + 2e^2 - (0 - 0 + 2) \right)$$

$$= 2e^2 - (4e^2 - 4e^2 + 2e^2 - 2)$$

$$= 2 \quad ✓$$

Area is 2 square units

(iv) $y = (\ln x)^2 \qquad x \geq 1$

$\sqrt{y} = \ln x$

$e^{\sqrt{y}} = x$

$\therefore f^{-1}(x) = e^{\sqrt{x}}$ ✓

(v) The required tangent is the reflection of OP in the line $y = x$. It passes through $(4, e^2)$ and has equation

$$y = \tfrac{1}{4}e^2 x \quad ✓$$

(b) (i) Equate equations for A and B

$$\frac{-gx^2}{2V^2}\sec^2\alpha + x\tan\alpha = \frac{-gx^2}{2V^2}\sec^2\beta + x\tan\beta \quad ✓$$

$$\therefore \frac{gx^2}{2V^2}(\sec^2\alpha - \sec^2\beta) = x(\tan\alpha - \tan\beta)$$

$$\frac{gx}{2V^2}(\tan^2\alpha - \tan^2\beta) = (\tan\alpha - \tan\beta) \qquad x \neq 0 \quad ✓$$

$$\therefore \frac{gx}{2V^2} = \frac{\tan\alpha - \tan\beta}{\tan^2\alpha - \tan^2\beta}$$

$\therefore \quad x = \frac{2V^2}{g} \times \left(\frac{1}{(\tan\alpha + \tan\beta)} \right)$ $\qquad \tan\alpha \neq \tan\beta$

$= \frac{2V^2}{g} \left(\frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta} \right)$

$= \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}$ ✓

(ii) (α) $\quad x = VT\cos\alpha$

New equation: $\quad x = V(t - T)\cos\beta$ ✓

(β) when A is at R:

$Vt\cos\alpha = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}$

$\therefore \quad t = \frac{2V\cos\beta}{g\sin(\alpha+\beta)}$ ✓

When B is at R:

$V(t-T)\cos\beta = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}$

$t - T = \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}$

$\therefore \quad T = t - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}$

$= \frac{2V\cos\beta}{g\sin(\alpha+\beta)} - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}$ ✓

$= \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)}$

Question 4 :

(a) $A = \int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx$ Even Function

$= 2\int_{0}^{1} \frac{1}{\sqrt{4-x^2}}\,dx$

$= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$ ✓

$= 2\left(\sin^{-1}(1/2) - \sin^{-1}(0)\right)$

$= 2\,(\pi/6)$

$= \pi/3$ units^2 ✓

(b) $V = \pi\int_{-1}^{1}\left(\frac{1}{\sqrt{4-x^2}}\right)^2 dx$ ✓

$= \pi\int_{-1}^{1} \frac{1}{4-x^2}\,dx$

$= \pi\int_{-1}^{1}\left(\frac{a}{2-x}\right)+\left(\frac{b}{2+x}\right)dx$

$a(2+x)+b(2-x) \equiv 1$

$ax - bx + 2a + 2b \equiv 1$

$= \frac{\pi}{4}\int_{-1}^{1}\left(\frac{1}{2-x}+\frac{1}{2+x}\right)dx$ ✓

$\therefore a-b=0$ & $2a+2b=1$

$\therefore a=b$ $a+b=1/2$

$\therefore a = b = 1/4$

$= \frac{\pi}{4}\left[\ln\left|\frac{2+x}{2-x}\right|\right]_{-1}^{1}$

$= \frac{\pi}{4}\left(\ln 3 - \ln(1/3)\right)$

$= \frac{2\ln 3\,\pi}{4} = \frac{\pi\ln 3}{2}$ units^3 $(= 1.73)$ ✓

(c) $\int_0^{\pi/2} \sqrt{2-2\cos 2x}\, dx = \int_0^{\pi/2} \sqrt{2}\sqrt{1-\cos 2x}\, dx$

$= \int_0^{\pi/2} \sqrt{2}\sqrt{2\sin^2 x}\, dx$ ✓

$= \sqrt{2}\int_0^{\pi/2} \sqrt{2}\sin x\, dx$

$= 2\int_0^{\pi/2} \sin x\, dx$

$= 2\left[-\cos x\right]_0^{\pi/2} dx$

$= 2(0 - -1)$

$= 2$ ✓

(d) $\int_0^1 x\sqrt{1-x}\, dx$ Let $u = 1-x \quad \therefore x = 1-u$

$x=1 \quad u=0 \quad \frac{du}{dx} = -1$

$= \int_1^0 (1-u)\, u^{1/2} \cdot -du$ ✓ $\quad x=0 \quad u=1$

$dx = -du$

$= -\int_1^0 (u^{1/2} - u^{3/2})\, du$

$= \int_0^1 (u^{1/2} - u^{3/2})\, du$

$= \left[\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_0^1$ ✓

$= \left(\frac{2}{3} - \frac{2}{5}\right) - (0)$

$= \frac{4}{15}$ ✓

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(e) (i) $I_n = \int_1^e x(\ln x)^n \, dx \qquad n = 0, 1, 2, 3 \ldots$

$= \left[\frac{x^2}{2}(\ln x)^n\right]_1^e - \int_1^e \frac{x^2}{2} \cdot \frac{n}{x}(\ln x)^{n-1} dx \qquad u = x^2/2 \quad v = (\ln x)^n$ ✓

$u' = x \qquad v' = \frac{n}{x}(\ln x)^{n-1}$

$= \left(\frac{e^2}{2} - 0\right) - \frac{n}{2}\int_1^e x(\ln x)^{n-1} dx$ ✓

$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$

(ii) $\int_1^e x(\ln x)^3 \, dx$ implies I_3

$\therefore I_3 = \frac{e^2}{2} - \frac{3}{2} I_2$

$= \frac{e^2}{2} - \frac{3}{2}\left[\frac{e^2}{2} - \frac{2}{2} I_1\right]$

$= \frac{e^2}{2} - \frac{3}{2}\left[\frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{1}{2} I_0\right)\right]$ ✓

$= \frac{e^2}{2} - \frac{3}{4} I_0$

$= \frac{e^2}{2} - \frac{3}{4}\left(\frac{e^2}{2} - \frac{1}{2}\right) \qquad I_0 = \int_1^e x(\ln x)^0 dx$

$= \frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8} \qquad = \int_1^e x \, dx$

$= \frac{e^2}{8} + \frac{3}{8}$ ✓ $\qquad = \frac{e^2}{2} - \frac{1}{2}$ ✓

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Question 5 :

(a) (i) $\bar{z}w = (3+4i)(2-2i)$

$= 6+8i-6i-8i^2$

$= 14+2i$ ✓

(ii) $(a+ib)^2 = 3-4i$ $\qquad a, b \in R$

$a^2-b^2+2abi = 3-4i$

$\therefore a^2-b^2=3$ and $ab=-2$ ✓

$\therefore a = {}^{-2}/_{b}$

$\therefore \left({}^{-2}/_{b}\right)^2 - b^2 = 3$

$4 - b^4 - 3b^2 = 0$

$b^4+3b^2-4 = 0$

$(b^2+4)(b^2-1) = 0$

$\therefore b = \pm 1 \qquad b = \sqrt{-2}$ (Not applicable)

$\therefore b=+1 \quad a=-2$

$b=-1 \quad a=2$

$\therefore$ Roots are $-2+i$ and $2-i$ ✓

(iii) $w = 2-2i$

$|w| = \sqrt{4+4}$ $\qquad \tan^{-1}\left(\frac{-2}{2}\right) = -\frac{\pi}{4}$

$= 2\sqrt{2}$

$\therefore w = 2\sqrt{2}\,\text{cis}\left(-{}^{\pi}/_{4}\right)$ ✓

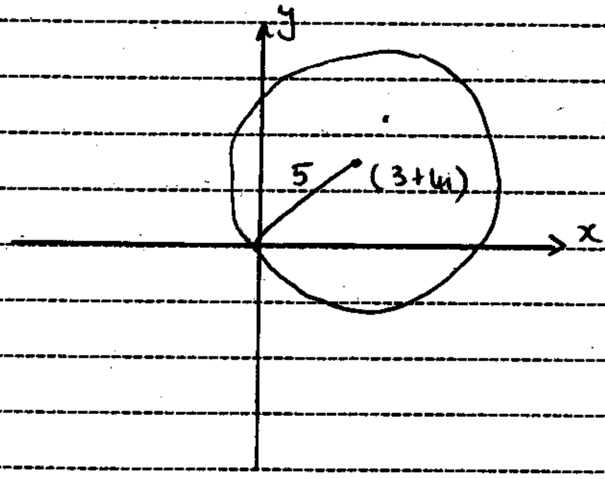
$\therefore w^4 = (2\sqrt{2})^4\,\text{cis}(4)\left(-\frac{\pi}{4}\right)$ ✓

$= 64\,\text{cis}(-\pi)$

$= 64\,\text{cis}\,\pi$

$= -64$ ✓

(b) (i) $|z-(3+4i)|=5$



✓

(ii) maximum value of $|z| = 10$; diameter of the circle.

$z = 6+8i$

$\arg z = \tan^{-1}(8/6)$ ✓

$= \tan^{-1}(4/3)$

(iii) if $\arg z = \tan^{-1}(\frac{1}{2})$

$\therefore$ $y = \frac{1}{2}x$ is a line

Need to find when this line intersects the circle

$(x-3)^2 + (y-4)^2 = 25$

sub for $y = x/2$

$(x-3)^2 + (x/2 - 4)^2 = 25$ ✓

$x^2 - 6x + 9 + \frac{x^2}{4} - 4x + 16 = 25$

$\frac{5x^2}{4} - 10x = 0$

$x(\frac{5x}{4} - 10) = 0$

$\therefore x = 0$ or $x = 8$

$y = 0$ $y = 4$

$\therefore$ Intersect at $(0,0)$ and $(8,4)$

$\therefore$ $z = 8+4i$

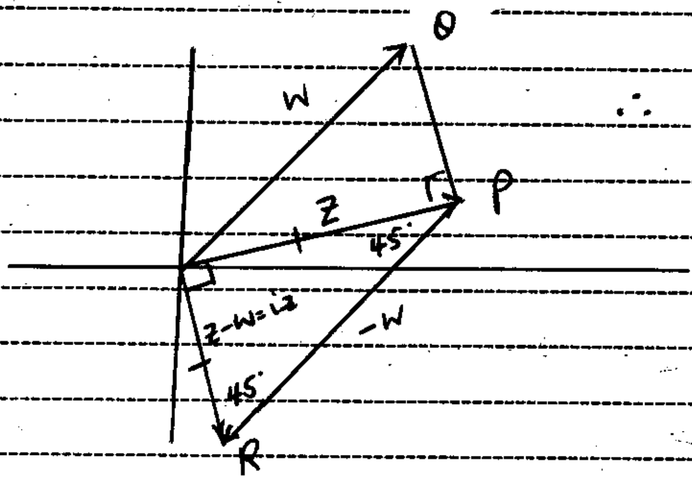
$|z| = \sqrt{80}$

$= 4\sqrt{5}$ ✓

(c) (i) POR is an isosceles right angled triangle. ✓

$|z| = |iz|$ ∴ OP = OR ∴ isosceles ✓
OP is rotated through 90° anticlockwise to get OR

(ii)



∴ $\hat{QPR} = 135°$ ✓

$\left(= \frac{3\pi^c}{4}\right)$

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Question 6:

(a) $2x^2 + xy - y^2 = 0$

$\therefore \quad 4x + x\,dy/dx + y - 2y\,dy/dx = 0$

$4x + y = dy/dx\,(2y - x)$

$\therefore \quad dy/dx = \frac{(4x+y)}{(2y-x)}$ ✓

dy/dx @ $(2,4)$ $\quad \therefore \frac{dy}{dx} = \frac{4(2)+4}{2(4)-2} = \frac{12}{6} = 2$ ✓

$\frac{d^2y}{dx^2} = \frac{(2y-x)(4+dy/dx) - (4x+y)(2\,dy/dx - 1)}{(2y-x)^2}$ ✓

at $(2,4)$

$\frac{d^2y}{dx^2} = \frac{6(6) - 12(3)}{36} = 0$ ✓

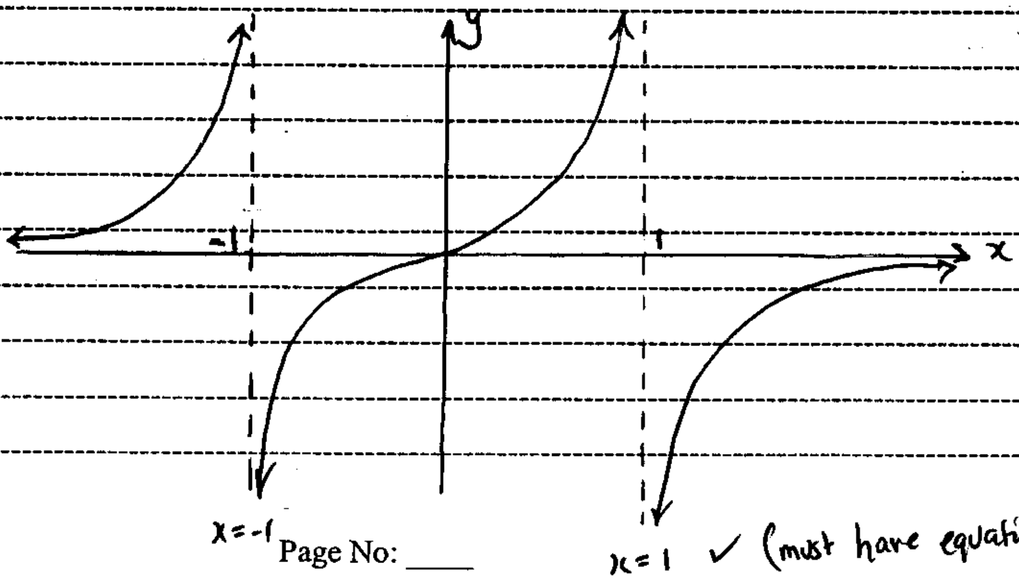
(b) (i) $f(x) = \frac{x}{1-x^2}$

$f'(x) = \frac{(1-x^2)(1) - (x)(-2x)}{(1-x^2)^2}$

$f'(x) = \frac{x^2+1}{(1-x^2)^2}$ ✓

$f'(x)$ is always positive, so function always increasing.

(ii)



✓

✓ (must have equations)

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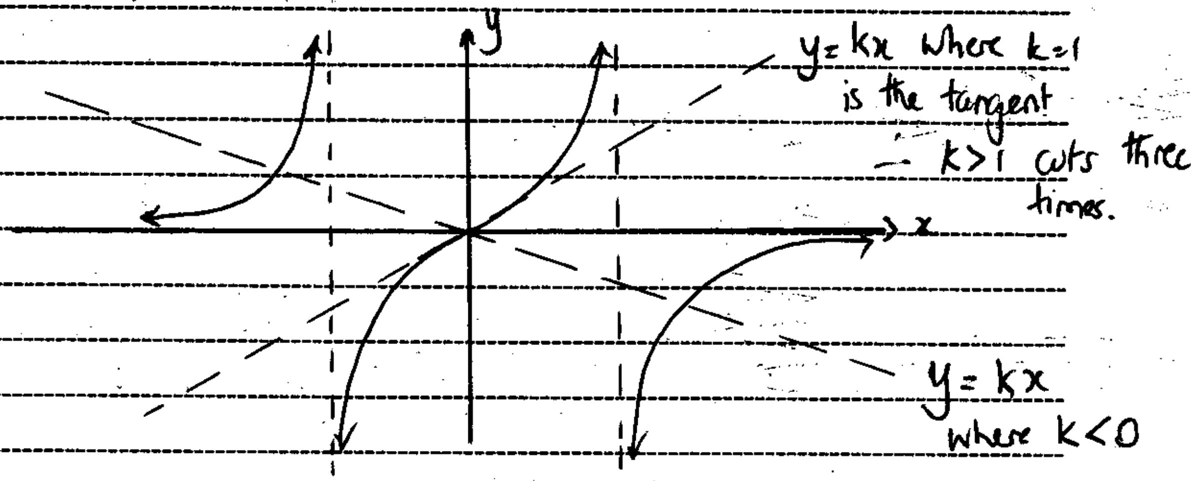
(iii) $\dfrac{x}{1-x^2} = kx$ has 3 distinct roots

$\therefore$ line $y = kx$ intersects $y = \dfrac{x}{1-x^2}$

three times $\Rightarrow$ 3 distinct roots

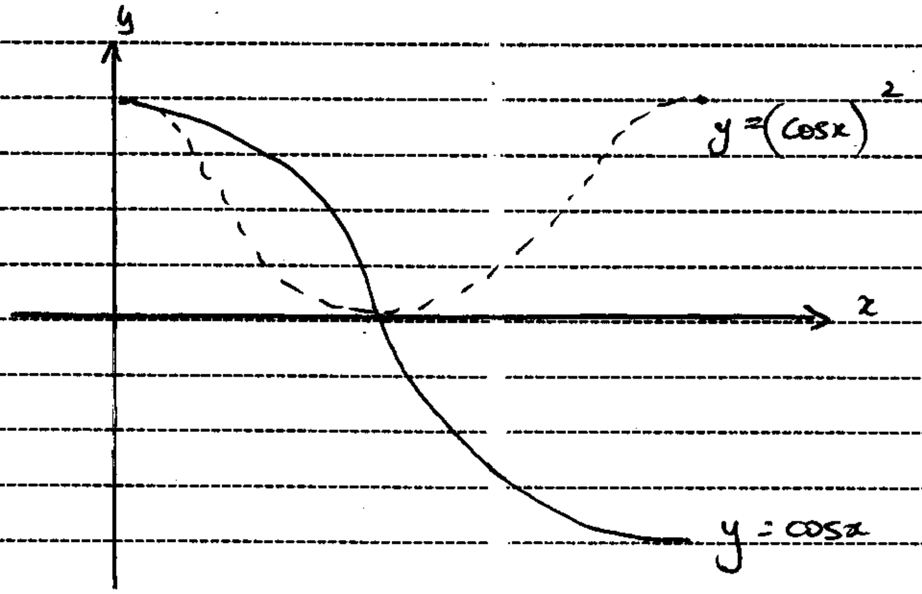
From the sketch below we can see:

Either $k < 0$ or $k > 1$ as 3 roots ✓✓



(c)

(i)



✓ shape

✓ intercepts

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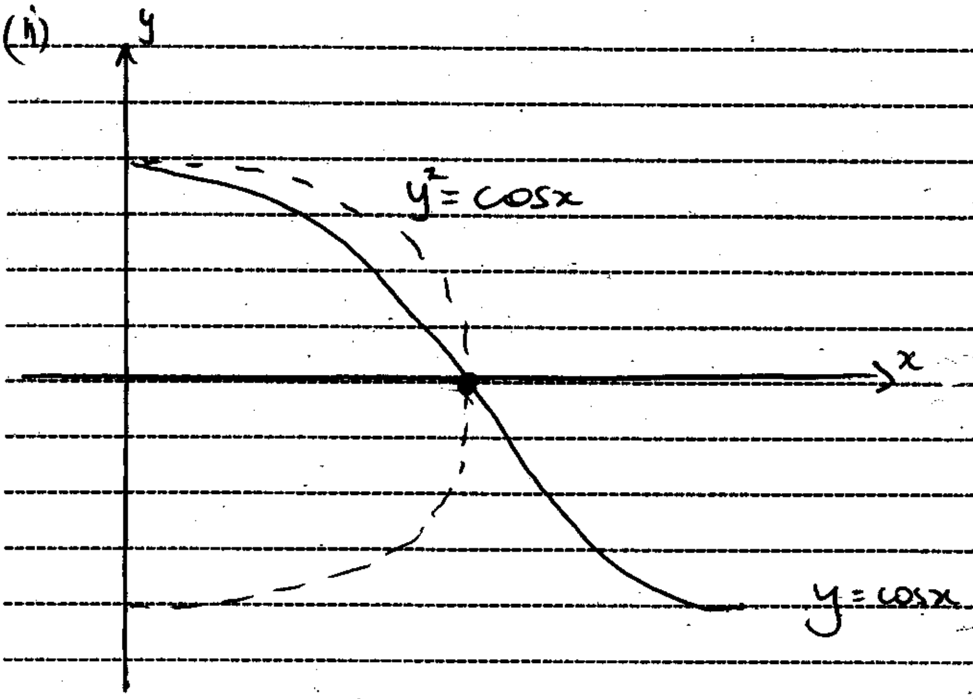
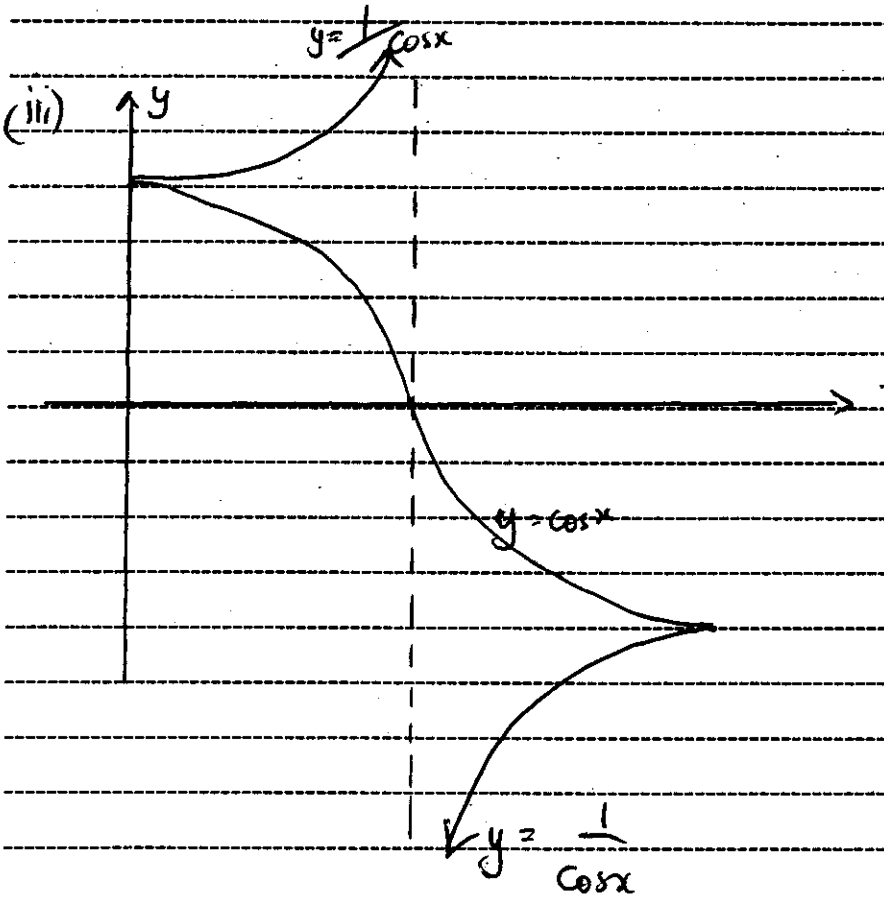
(ii)



✓ Shape

✓ intercepts

(iii)



✓ shape

✓ asymptote

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Question 7:

(a) $|z|^2 + 2i\bar{z} = 4i + 7$ $\qquad z = a + ib$

$a^2 + b^2 + 2i(a - ib) = 4i + 7$ ✓

$2ai + a^2 + b^2 + 2b = 4i + 7$

Equate real and imaginery parts

$\therefore\ 2a = 4 \qquad a^2 + b^2 + 2b = 7$ ✓

$a = 2 \qquad 4 + b^2 + 2b = 7$

$b^2 + 2b - 3 = 0$

$(b+3)(b-1) = 0$

$b = -3 \quad b = 1$

$\therefore$ The two solutions are : $z = 2 + i$ or $z = 2 - 3i$ ✓

(b) $(\cos A + i\sin A)(\cos B + i\sin B) + (\cos C - i\sin C) = 0$

$(\cos A + i\sin A)(\cos B + i\sin B) = \cos A\cos B - \sin A\sin B$ ✓
$+ i(\sin A\cos B + \cos A\sin B)$
$= \cos(A+B) + i\sin(A+B)$ ✓

Now: $A + B = \pi - C$ [angle sum of triangle]

$\therefore \cos(A+B) = \cos(\pi - C)$

$\cos(A+B) = -\cos C$ ✓

and $\sin(A+B) = \sin(\pi - C)$

$\sin(A+B) = \sin C$

$= -\cos C + i\sin C$

$= -(\cos C - i\sin C)$

$\therefore (\cos A + i\sin A)(\cos B + i\sin B) + (\cos C - i\sin C)$
$= -(\cos C - i\sin C) + (\cos C - i\sin C) = 0$

(c) (i) $1+z = 1+\cos\theta + i\sin\theta$

$= 2\cos^2(\theta/2) + i\,(2\sin(\theta/2)\cos(\theta/2))$ ✓

$= 2\cos(\theta/2)\left[\cos(\theta/2) + i\sin(\theta/2)\right]$ ✓

Use De Moivre's Theorem:

$$(1+z)^4 = 2^4\cos^4\left(\frac{\theta}{2}\right)\left(\cos(4)\left(\frac{\theta}{2}\right) + i\sin(4)\left(\frac{\theta}{2}\right)\right)$$ ✓

$$= 16\cos^4\left(\frac{\theta}{2}\right)\left(\cos 2\theta + i\sin 2\theta\right)$$

(ii) $(1+z)^4 = 1 + 4z + {}^4C_2 z^2 + {}^4C_1 z^3 + z^4$

$= 1 + 4z + 6z^2 + 4z^3 + z^4$

$z = \cos\theta + i\sin\theta$ $\quad z^2 = \cos 2\theta + i\sin 2\theta$ $\quad z^3 = \cos 3\theta + i\sin 3\theta$

$z^4 = \cos 4\theta + i\sin 4\theta$

✓

$\therefore (1+z)^4 = 1 + 4(\cos\theta + i\sin\theta) + 6(\cos 2\theta + i\sin 2\theta)$
$+ 4(\cos 3\theta + i\sin 3\theta) + (\cos 4\theta + i\sin 4\theta)$

$\therefore$ Equate real and imaginery components

✓

$$\therefore 16\cos^4\left(\frac{\theta}{2}\right)\cos 2\theta = 1 + 4\cos\theta + 6\cos 2\theta + 4\cos 3\theta + \cos 4\theta$$

✓

$$\therefore 16\cos^4\left(\frac{\theta}{2}\right)\sin 2\theta = 4\sin\theta + 6\sin 2\theta + 4\sin 3\theta + \sin 4\theta$$

(iii) $$\frac{4\sin\theta + 6\sin 2\theta + 4\sin 3\theta + \sin 4\theta}{1 + 4\cos\theta + 6\cos 2\theta + 4\cos 3\theta + \cos 4\theta} = \frac{16\cancel{\cos^4(\theta/2)}\sin 2\theta}{16\cancel{\cos^4(\theta/2)}\cos 2\theta} = \tan 2\theta$$ ✓

$$\frac{4\sin\theta + 4\sin 3\theta + \sin 4\theta}{1 + 4\cos\theta + 4\cos 3\theta + \cos 4\theta} = \frac{\left[16\cos^4(\theta/2) - 6\right]\sin 2\theta}{\left[16\cos^4(\theta/2) - 6\right]\cos 2\theta} = \tan 2\theta$$ ✓ ✓

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Question 8

(a) (i) $x = 10\sqrt{2}\sin\theta$ $\qquad x = 10 \quad \theta = \pi/4$

$\therefore \dfrac{dx}{d\theta} = 10\sqrt{2}\cos\theta$ $\qquad x = -10 \quad \theta = -\pi/4$

$$\therefore \int_{-10}^{10} \sqrt{200 - x^2}\, dx = \int_{-\pi/4}^{\pi/4} \sqrt{200 - 200\sin^2\theta} \,.\, 10\sqrt{2}\cos\theta \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \sqrt{200}\cos\theta \,.\, 10\sqrt{2}\cos\theta \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} 200\cos^2\theta \, d\theta$$

$$= 200 \int_{-\pi/4}^{\pi/4} \cos^2\theta \, d\theta \quad ✓$$

$$= 100 \int_{-\pi/4}^{\pi/4} (1 + \cos 2\theta) \, d\theta$$

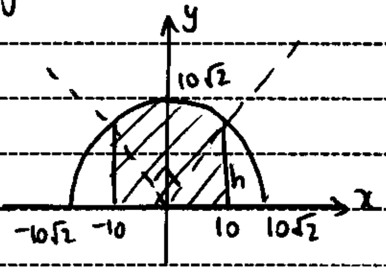
$$= 100 \left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/4}^{\pi/4} \quad ✓$$

$$= 100 \left[\left(\frac{\pi}{4} + \frac{1}{2}(1)\right) - \left(\frac{-\pi}{4} - \frac{1}{2}\right)\right]$$

$$= 100\left(\frac{\pi}{2} + 1\right)$$

$$= 100 + 50\pi \quad ✓$$

Diagram:

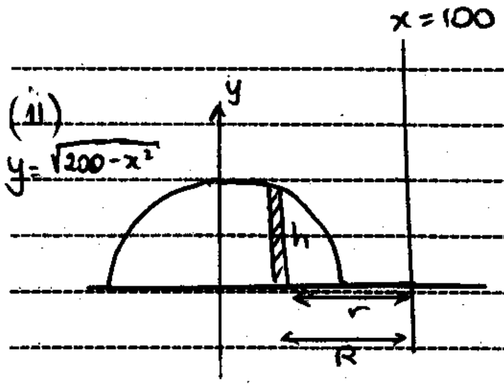


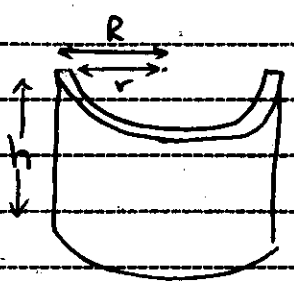
Geometrically integral gives shaded area comprising $\frac{1}{4}$ circle ($r = 10\sqrt{2}$) and two right triangles ($h = 10$). ✓

Shaded area $= \frac{1}{4}\pi(10\sqrt{2})^2 + 2 \times \frac{1}{2} \times 10 \times 10$

$= 50\pi + 100$

(ii)





$h = \sqrt{200 - x^2}$

$r = 100 - x$

$R = 100 - x + \delta x$

$$\delta V = \tfrac{1}{2}\pi\left(R^2 - r^2\right)h$$

$$= \tfrac{1}{2}\pi(R+r)(R-r)h$$

$R - r = \delta x$

$$= \tfrac{1}{2}\pi\left(200 - 2x + \delta x\right)\delta x\, h$$

$$= \tfrac{1}{2}\pi\left(2(100-x) + \delta x\right)\delta x\, h$$

Ignoring terms in $(\delta x)^2$

$$\delta V = \pi(100 - x)\,h\,\delta x \quad ✓$$

$$= \pi(100-x)\sqrt{200-x^2}\;\delta x$$

$$\therefore\ V = \pi \lim_{\delta x \to 0} \sum_{-10}^{10} (100-x)\sqrt{200-x^2}\;\delta x \quad ✓$$

$$= \pi \int_{-10}^{10} (100-x)\sqrt{200-x^2}\,dx$$

$$V = 100\pi \int_{-10}^{10} \sqrt{200-x^2}\,dx - \pi \int_{-10}^{10} x\sqrt{200-x^2}\,dx \quad ✓$$

$x\sqrt{200-x^2}$ is an odd function, therefore integral is zero

$$\therefore\ V = 100\pi\left[100 + 50\pi\right] + 0$$

$$= 5000\pi(2+\pi) \times 10^{-6}\ m^3$$

$$= 0.081\ m^3 \quad ✓$$

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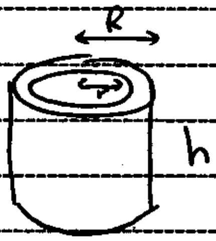
(b)

(i)



$R = x + \delta x$

$r = x$

$h = x - (x - e^{-x})$

$= e^{-x}$

$\delta V = \pi(R^2 - r^2)h$

$= \pi(R-r)(R+r)h$

$= \pi(x+\delta x - x)(x + \delta x + x)e^{-x}$

$= \pi(\delta x)(2x + \delta x)e^{-x}$

Ignoring second order terms

$\therefore \delta V = 2\pi e^{-x} \cdot x\,\delta x$

$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{X} 2\pi x e^{-x}\,\delta x$

$V = 2\pi \int_0^X x e^{-x}\,dx$

(ii) $\int_0^X x e^{-x}\,dx = \left[-x e^{-x}\right]_0^X + \int_0^X e^{-x}\,dx$

$= -Xe^{-X} - \left[e^{-x}\right]_0^X$

$= -Xe^{-X} - (e^{-X} - 1)$

$= 1 - e^{-X} - Xe^{-X}$

But $0 = X - e^{-X} \implies e^{-X} = X$

$\therefore V = 2\pi(1 - X - X^2)$