Trinity Grammar School

Mathematics Department

2013

HALF-YEARLY EXAMINATION

HSC ASSESSMENT TASK 3

Year 12

Mathematics Extension 2

Board of Studies Student Number
(Year 12 only)

Class Teacher:

……………………………………………………………………

Name:
(Year 11 only)

……………………………………………………………………

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Only approved calculators for this course are allowed in this task
- A table of Standard Integrals (if required) in this course is supplied
- Show all necessary working
- Write your Board of Studies Student Number (Year 12 HSC) or Name (Year 11) and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating **N/A** and your Student Number or Name.

- **HSC Assessment Weighting:** 30%

Total marks – 100

Section I

10 marks

- Attemtpt all Questions
- Allow about **15** minutes for this section

Section II

90 marks

- Attempt all Questions
- Allow about **2** hours **45** minutes for this section

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Section I **10 marks**

- Shade the correct response on the multiple-choice answer sheet provided
- Each question is worth 1 mark

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1 Which of the following complex numbers has a modulus of 1?

(A) $1 + i$

(B) $\dfrac{1+i}{1-i}$

(C) $(1+i)^2$

(D) $\sqrt{1+i}$

2 The locus defined by $\arg\left(\dfrac{z}{z+1}\right) = \dfrac{5\pi}{11}$ is a

(A) circle

(B) minor arc of a circle

(C) semi-circle

(D) major arc of a circle

3 If S and R represent the two complex numbers z_1 and z_2 respectively such that $ki(z_2 - z_1) = -z_1$ where k is a non-zero real number, then ΔOSR is

(A) right-angled at S

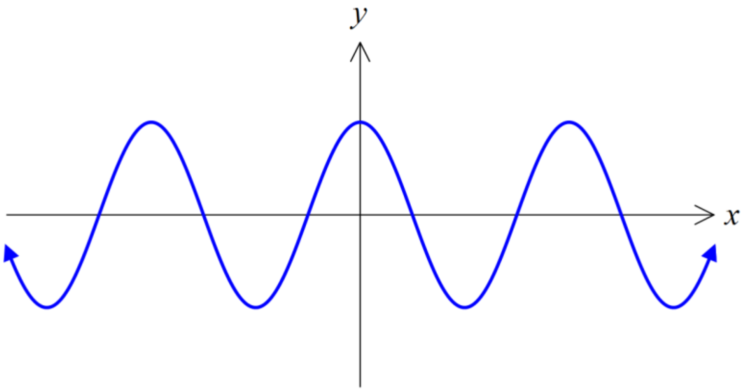
(B) isosceles with $OS = OR$

(C) isosceles with $OS = SR$

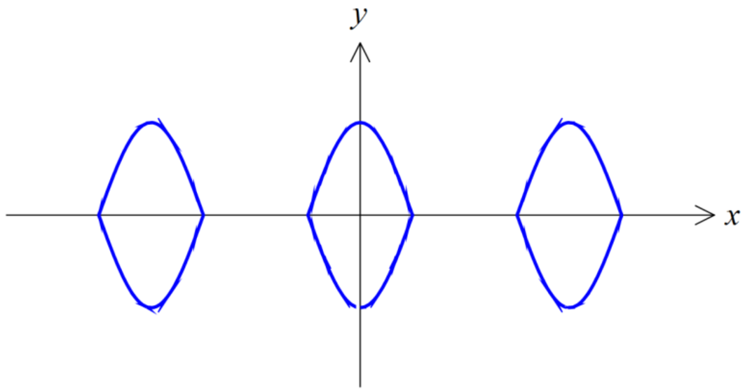
(D) right-angled at O

4 Which of the following graphs best illustrates $|y| = \cos x$?

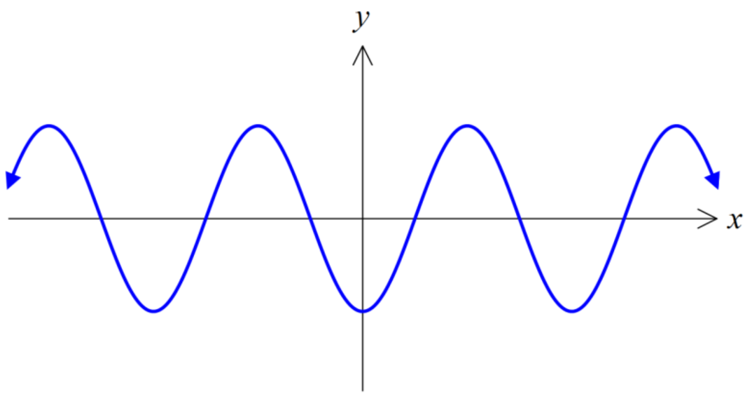
(A)



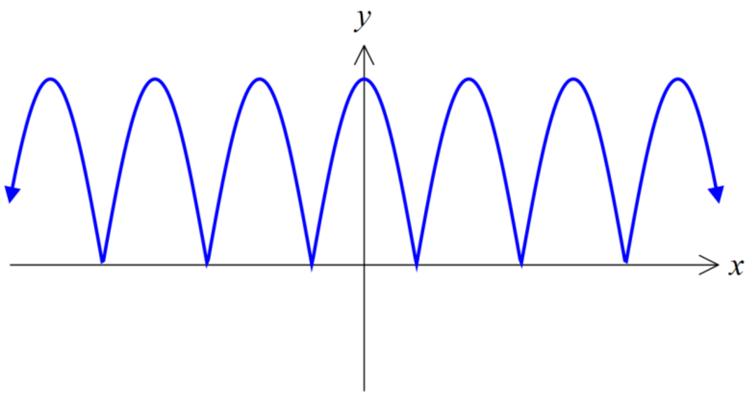
(B)



(C)



(D)



5 The value of $\lim_{x \to 0}\left(\frac{\sqrt{a^2+x} - a}{x+x^2}\right)$ given $a > 0$ is

(A) 0

(B) $\frac{1}{2a}$

(C) $\frac{1}{a}$

(D) undefined

6 A curve is defined by the equation $2x^3 - y^2 = 7$.
The slope of the curve at the point where $y = -3$ is

(A) –4

(B) –2

(C) 2

(D) 4

7 Let $I_n = \int \frac{(\ln x)^n}{x^2}\, dx$. For $n \geq 1$

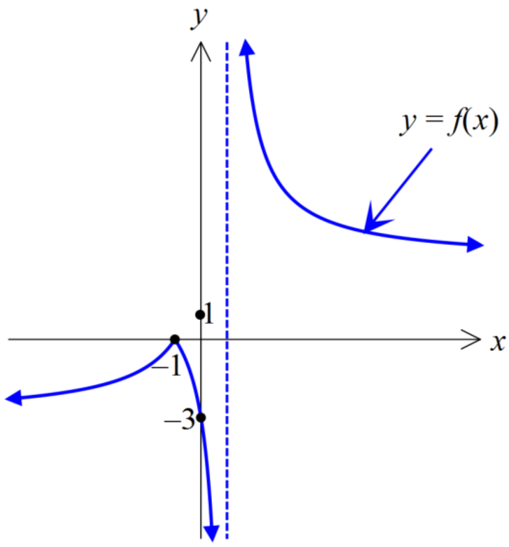
(A) $I_n = nI_{n-1} + \frac{1}{x}\ln x^n$

(B) $I_n = nI_{n-1} + \frac{1}{x}(\ln x)^n$

(C) $I_n = nI_{n-1} - \frac{1}{x}\ln x^n$

(D) $I_n = nI_{n-1} - \frac{1}{x}(\ln x)^n$

8 Part of the graph of $f(x) = \dfrac{|3x+3|}{x-1}$ is shown below.



Not to scale

Let $g(x) = x+1$. The solution to the inequality $f(x) \geq g(x)$ is

(A) $x < -2$ or $1 < x \leq 4$

(B) $x \leq -2$ or $1 \leq x < 4$

(C) $x \leq -2$ or $1 < x \leq 4$

(D) $x < -2$ or $1 < x < 4$

9 The value of $\int_0^1 x(x^2+1)^5 \; dx$ is

(A) $\dfrac{21}{4}$

(B) $\dfrac{16}{3}$

(C) $\dfrac{63}{6}$

(D) $\dfrac{32}{3}$

10 $P(z)$ is a polynomial in z of degree 4 with real coefficients.

Which of the following statements must be false?

(A) $P(z) = 0$ has no real roots

(B) $P(z) = 0$ has four real roots

(C) $P(z) = 0$ has one (repeated) real root and two non-real roots

(D) $P(z) = 0$ has only one real root and three non-real roots

End of Section I
Section II commences on the next page

Section II **90 marks**

- Begin each question in a SEPARATE writing booklet or answer sheet
- Show all relevant mathematical reasoning and/or calculations

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Question 11 **(15 marks)** Use a SEPARATE writing booklet

(a) Use the table of standard integrals or otherwise, to find $\int \frac{dx}{\sqrt{9x^2-4}}$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 y \cos^3 y \quad dy$. **3**

(c) (i) Using division, express $\frac{x^3}{1+x}$ in the form $q(x)+\frac{r}{d(x)}$, where d and q are polynomials in x and r is a constant. **2**

(ii) Hence, evaluate $\int_0^1 \frac{x}{1+\sqrt{x}}\, dx$, correct to two decimal places. **3**
Initially, use the substitution $u=\sqrt{x}$.

(d) (i) Find real numbers A and B such that $\frac{1}{(2x-1)(x+2)} \equiv \frac{A}{2x-1}+\frac{B}{x+2}$. **2**

(ii) Hence or otherwise, find $\int \frac{5}{3\sin x - 4\cos x}\, dx.$ **3**

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Given the complex number $z = 7 - 3i$, find

(i) $|z - \overline{z}|$ **1**

(ii) $\arg(z - \overline{z})$ **1**

(b) (i) Express $\omega = \dfrac{\sqrt{2}}{1-i}$ in modulus-argument form. **2**

(ii) Use the principle of mathematical induction to prove de Moivre's theorem, which states that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where n is a positive integer. **3**

(iii) Hence, using de Moivre's theorem, find ω^5 in the form $x + yi$. **2**

(c) Sketch the region in the Argand plane satisfying the following inequalities simultaneously. **3**

$$\begin{cases} \dfrac{\pi}{6} \le \arg z \le \dfrac{5\pi}{6} \\ 1 \le \operatorname{Im}(z) \le 2 \\ |z| \le 4 \end{cases}$$

(d) (i) Find the values of h and k such that $3 + 2x - x^2 = k - (x-h)^2$. **1**

(ii) Hence, show that $\displaystyle\int_0^1 \frac{dx}{\sqrt{3 + 2x - x^2}} = \frac{\pi}{6}$. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Let $f(x) = x^2 - 4x$.

On **separate** $\frac{1}{3}$ page number planes, sketch and show the important features of the graph of

(i) $y = f(x)$ **1**

(ii) $y = |\, f(x) \,|$ **1**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **2**

(b) Sketch **and** describe the locus of the complex number $z = x + yi$ such that **3**

$$2|\, z - z_1 \,| = |\, z - z_2 \,|,$$

where $z_1 = \dfrac{7}{2} + 2i$ and $z_2 = -1 - i$.

(c) It is known that $ax^4 + 4bx + c = 0$ has a double root at $x = \alpha$.

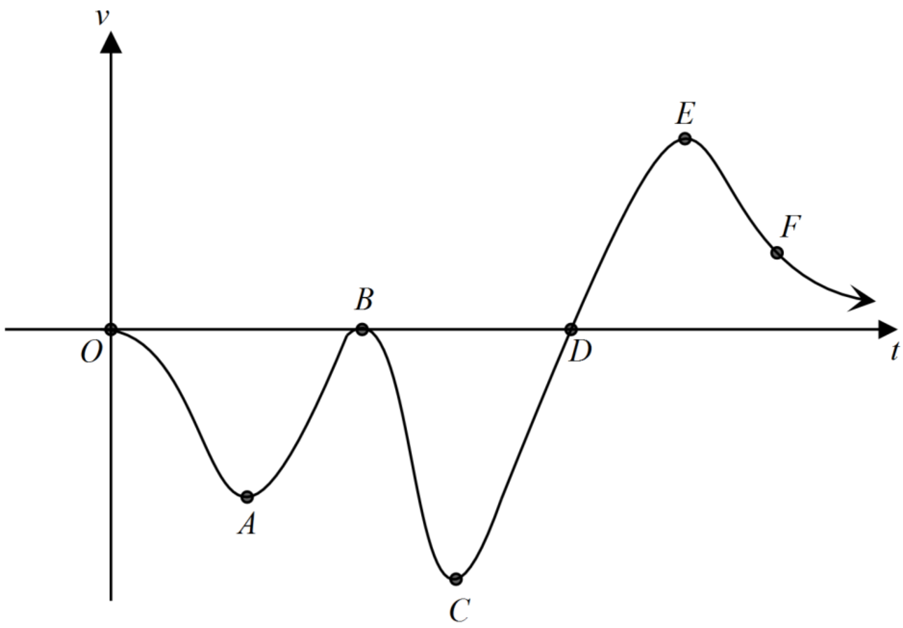
(i) Show that $\alpha^3 = -\dfrac{b}{a}$. **1**

(ii) Hence or otherwise, solve the equation, $27x^4 - 32x + 16 = 0$, given that it has a double root. **3**

Question 13 continues on the next page

Question 13 continued

(d) The diagram below shows the velocity v (in m/s) of a particle at any time $t \geq 0$.



On a suitable number plane, sketch a possible graph of the displacement x (in metres) of the particle at any time $t \geq 0$, given the particle is initially at rest, 3 m to the right of O. **2**

Question 14 commences on the next page

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) The curves $y = x^2 - 4$ and $y = \dfrac{1}{x}$ intersect at the points $P(\alpha, y_1)$, $Q(\beta, y_2)$ and $R(\gamma, y_3)$.

(i) Explain why α, β and γ are the roots of $x^3 - 4x - 1 = 0$. **1**

(ii) Hence, form a cubic polynomial equation whose roots are α^2, β^2 and γ^2. **2**

(iii) Hence, state a cubic polynomial equation whose roots are $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}$ and $\dfrac{1}{\gamma^2}$. **1**

(iv) Hence or otherwise, prove that $OP^2 + OQ^2 + OR^2 = 24$ where O is the origin. **2**

(b) The point P with coordinates (p, q) lies on the graph $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, $a > 0$.

The tangent to the curve at P cuts the x-axis at $(m, 0)$ and the y-axis at $(0, n)$.

(i) Use implicit differentiation to show $\dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$. **2**

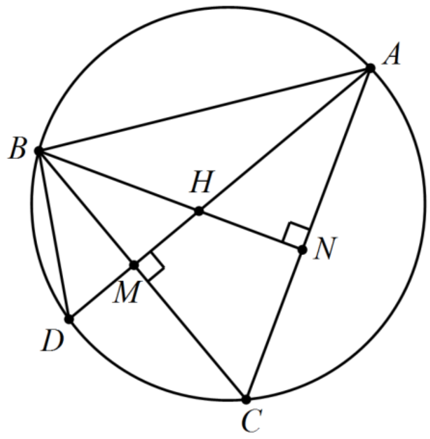
(ii) Find, in terms of p and/or q the value of m and of n. **2**

(iii) Prove that $m + n = a$. **2**

Question 14 continues on the next page

Question 14 continued

(c)



Not to scale

Copy this diagram into your writing booklet.

The points A, B and C lie on the circumference of the circle shown above. **3**

The altitudes BN and AM of triangle ABC meet at H. The interval AM is extended to meet the circle at D.

Prove (with reasons) that $HM = MD$.

Question 15 commences on the next page

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) (i) Show that $z = -1 + \sqrt{3}i$ is a root of the equation $z^4 - 4z^2 - 16z - 16 = 0$. **2**

(ii) Hence, find all the other roots of $z^4 - 4z^2 - 16z - 16 = 0$. **3**

(b) A helicopter H is moving vertically upwards with a speed of 10 m/s. The helicopter is h metres directly above a point O which is situated on level ground. After some time t (in seconds), the helicopter is observed from a point P, which is also on level ground, at a distance of 40 m from O.

Let $\angle HPO = \theta$ radians and $PH = x$ metres.

When h = 30 metres,

(i) find the rate of change of $\angle HPO$; **2**

(ii) show that the rate of change of PH is 6 m/s. **3**

(c) For integers $n \geq 0$, let $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$.

(i) Show, using integration by parts, that $I_{n+1} = \frac{1}{2n}\left[\left(\frac{1}{2}\right)^n + (2n-1)I_n\right]$. **3**

(ii) Hence, evaluate I_5. **2**

Question 16 commences on the next page

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are integers.

Suppose α is an integer root of the equation $p(x) = 0$.

(i) Prove that α must be a factor of the constant e in the expression $p(x) = ax^4 + bx^3 + cx^2 + dx + e$. **2**

(ii) Prove that the polynomial $q(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$ has no integer roots. **2**

(b) (i) Prove that $\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$, where v (or velocity) is a function of x (or displacement) and t is time in seconds. **2**

(ii) A particle moves in SHM on a horizontal line and its acceleration is given by $\ddot{x} = 36 - 9x$, where x is the particle's displacement (in metres) after t seconds.

(α) Find the particle's period **and** state its centre of motion. **2**

(β) If the particle is initially at rest at $x = 6$ m, find, using (b)(i) its amplitude. **2**

(c) (i) Prove, using the substitution $u = a - x$ that **2**

$$\int_0^a f(x) \; dx = \int_0^a f(a-x) \; dx.$$

(ii) Hence, evaluate $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$. **3**

End of Section II

End of Examination

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \ , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

2013 HSC MATHEMATICS EXT 2

SUGGESTED SOLUTIONS : TASK 3

SECTION I: MULTIPLE CHOICE.

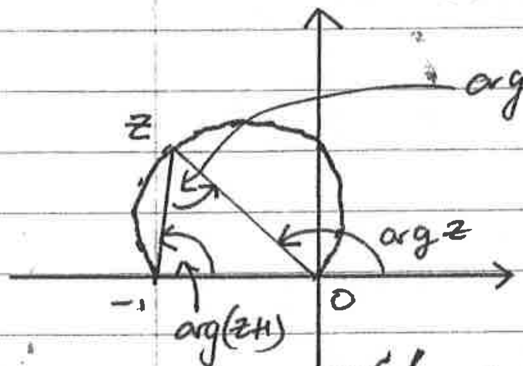
1) $\boxed{B}$ $\because \left|\dfrac{1+i}{1-i}\right| = \dfrac{\sqrt{1^2+1^2}}{\sqrt{1^2+(-1)^2}}$

$= \dfrac{\sqrt{2}}{\sqrt{2}}$

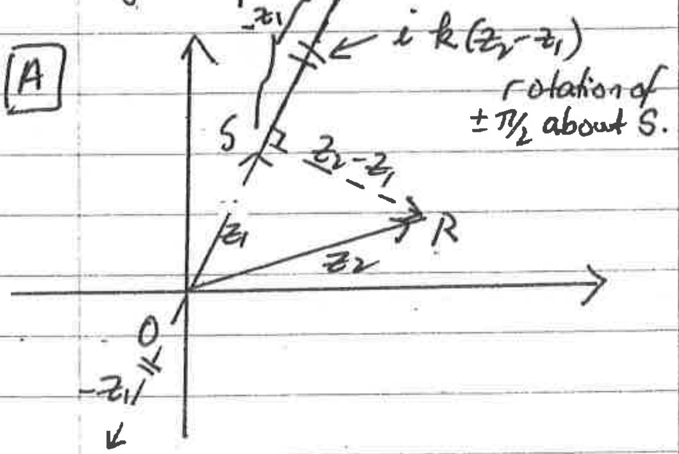
$= 1$

2) $\boxed{D}$ $\arg\left(\frac{z}{z+1}\right) = \arg z - \arg(z+1)$

$\doteq 5\pi/11$



3) $\boxed{A}$



4) $\boxed{B}$

5) $\boxed{B}$ $\displaystyle\lim_{x\to 0} \frac{\left(\sqrt{a^2+x}-a\right)\left(\sqrt{a^2+x}+a\right)}{(x+x^2)\left(\sqrt{a^2+x}+a\right)}$

$= \displaystyle\lim_{x\to 0} \frac{a^2+x-a^2}{x(x+1)\left(\sqrt{a^2+x}+a\right)}$

$= \displaystyle\lim_{x\to 0} \frac{1}{(x+1)\left(\sqrt{a^2+x}+a\right)}$

$= \dfrac{1}{\sqrt{a^2}+a}$

$= \dfrac{1}{2a}$.

6) $\boxed{A}$ $2x^3 - y^2 = 7$

$\therefore 6x^2 - 2y\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2}{y}$

at $y=-3$, $2x^3 = 7+9$

$\therefore x = 2$

hence $\dfrac{dy}{dx} = \dfrac{3\times 2^2}{-3}$

$= 12/-3$

$= -4$.

7) $\boxed{D}$ let $u = (\ln x)^n$ $\quad dv = \dfrac{1}{x^2}$

$du = n\left(\frac{1}{x}\right)(\ln x)^{n-1}$ $\quad v = -\dfrac{1}{x}$

$\therefore I_n = uv - \int v\, du$

$= -\dfrac{1}{x}(\ln x)^n + n\displaystyle\int \frac{1}{x^2}(\ln x)^{n-1}\,dx$

$= n I_{n-1} - \dfrac{1}{x}(\ln x)^n + c$.

8) $\boxed{C}$ Critical points when

$$x+1 = \frac{3x+3}{x-1}$$
$$x^2-1 = 3x+3$$
$$(x-4)(x+1)=0$$
$$\therefore\ x = 4 \text{ or } x=-1$$

expected.

Also: $x+1 = \dfrac{-3x-3}{x-1}$

i.e. $x^2-1 = -3x-3$

$$(x+2)(x+1)=0$$
$$x=-2 \text{ or } x=-1$$

expected

From the graph the required solution to $f > g$ is

$\{x: x \le -2 \text{ or } 1 < x \le 4\}$.

9) $\boxed{A}$ $\displaystyle\int_0^1 x(x^2+1)^5\,dx$

$$= \frac{1}{2}\int_0^1 2x(x^2+1)^5\,dx$$
$$= \frac{1}{2}\cdot\frac{1}{6}(x^2+1)^6\Big]_0^1$$
$$= \frac{1}{12}(x^2+1)^6\Big|_0^1$$
$$= \frac{1}{12}[64-1]$$
$$= 63/12$$
$$= 21/4.$$

10) $\boxed{D}$ by the Fundamental Theorem of Algebra and other associated laws;

A polynomial $p(z)=0$ of degree 4 can have non-real roots where in this case (real coefficients) you could have 2 conjugate pairs of unreal roots. $\therefore$ A is a possibility. B is possible because all roots can be real (distinct or equal). C can be true from B. However, in $\boxed{D}$ not possible – conjugate root theorem plus Fundamental Root theorem.

1. B
2. D
3. A
4. B
5. B
6. A
7. D
8. C
9. A
10. D

SECTION II:

11) a) From the table $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx$

$$= \ln\left(x+\sqrt{x^2-a^2}\right)+C$$

$$\therefore \int \frac{dx}{\sqrt{9x^2-4}} = \int \frac{dx}{\sqrt{9(x^2-4/9)}}$$
$$= \frac{1}{3}\int \frac{dx}{\sqrt{x^2-(\frac{2}{3})^2}}$$
$$= \frac{1}{3}\ln\left(x+\sqrt{x^2-\frac{4}{9}}\right)+C.$$

[will accept $\frac{1}{3}\ln\left[3x+\sqrt{9x^2-4}\right]+C$]

b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 y(\cos^2 y)\cos y\,dy$

$$= \int_0^{\frac{\pi}{2}} \sin^2 y(1-\sin^2 y)\cos y\,dy$$
$$= \int_0^{\frac{\pi}{2}} \sin^2 y\cos y - \sin^4 y\cos y\,dy$$
$$= \left[\frac{\sin^3 y}{3} - \frac{\sin^5 y}{5}\right]_0^{\pi/2}$$
$$= \left(\frac{\sin^3\pi/2}{3} - \frac{\sin^5\pi/2}{5}\right)-(0-0)$$
$$= \frac{1}{3}-\frac{1}{5}$$
$$= 2/15.$$

c) i)

$$\begin{array}{r} x^2 - x + 1 \\ 1+x\,\overline{)\,x^3\qquad\qquad} \\ \underline{x^3+x^2} \\ -x^2 \\ \underline{-x^2-x} \\ x \\ \underline{x+1} \\ -1 \end{array}$$

$$\therefore \frac{x^3}{x+1} = (x^2-x+1) - \frac{1}{1+x}$$

ii) $u=\sqrt{x}$ $\therefore x=u^2$

$$\frac{dx}{du}=2u$$
$$dx = 2u\,du$$

when $x=0$, $u=0$
$x=1$, $u=1$

$$\therefore \int_0^1 \frac{x}{\sqrt{x}+1}\,dx = \int_0^1 \frac{u^2\cdot 2u\,du}{1+u}$$
$$= 2\int_0^1 \frac{u^3}{1+u}\,du$$
$$= 2\int_0^1 u^2-u+1-\frac{1}{1+u}\,du$$

from (i)

$$= 2\left[\frac{u^3}{3}-\frac{u^2}{2}+u-\ln(1+u)\right]_0^1$$
$$= 2\left[\frac{1}{3}-\frac{1}{2}+1-\ln 2-0+0-0+\ln 1\right]$$
$$= 2\left(\frac{1}{3}-\frac{1}{2}+1-\ln 2\right)$$
$$= 5/3 - 2\ln 2 \quad \left[\text{or } \frac{5}{3}-\ln 4\right]$$
$$\doteq 0.28 \ (2\text{ d.p.}).$$

d) i) $A(x+2)+B(2x-1) \equiv 1$

let $x=-2$, $B(-4-1)=1$

$$B = -1/5$$

$x=\frac{1}{2}$, $A(2.5)=1$

$$A = 2/5$$

1 d) ii) let $t = \tan \frac{x}{2}$

$\therefore \cos x = \frac{1-t^2}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{dx}{dt} = \frac{2}{1+t^2}$ [since $\sec^2 x = 1+\tan^2 x$]

$\therefore 5\int \frac{dx}{3\sin x - 4\cos x}$

$= 5\int \frac{dx}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,dt$

$= 5\int \frac{1}{2t^2+3t-2}\,dt$

$= 5\int \frac{1}{(2t-1)(t+2)}\,dt$

$= 5\int \frac{2/5}{2t-1} - \frac{1/5}{t+2}\,dt$ from (i)

$= \int \frac{2}{2t-1} - \frac{1}{t+2}\,dt$

$= \ln|2t-1| - \ln|t+2| + c$

$= \ln\left|\frac{2\tan\frac{x}{2}-1}{\tan\frac{x}{2}+2}\right| + c.$

12) a) $z = 7-3i$

i) $z - \bar{z} = 7-3i-7-3i$

$= -6i$

$\therefore |z-\bar{z}| = |-6i|$

$= \sqrt{(-6)^2}$

$= 6.$

ii) $\arg(z-\bar{z}) = \arg(-6i)$

$= -\pi/2$ (will accept $3\pi/2$)

b) $|w| = \frac{\sqrt{2}}{\sqrt{1^2+(-1)^2}} = \frac{\sqrt{2}}{\sqrt{2}}$

$= 1$

$\arg(w) = \arg\left(\frac{\sqrt{2}\operatorname{cis}0}{\sqrt{2}\operatorname{cis}(-\pi/4)}\right)$

$= 0 - (-\pi/4)$

$= \pi/4$

$\therefore w = \operatorname{cis}\pi/4.$

ii) Step 1: prove true for $n=1$.

LHS $= \cos\theta + i\sin\theta$

RHS $= \cos(1\times\theta) + i\sin(1\times\theta)$

$= \cos\theta + i\sin\theta$

$=$ LHS.

Step 2: Assume the statement is true for $n=k$ (k is a positive integer and $1 \le k \le n$).

i.e. $(\operatorname{cis}\theta)^k = \operatorname{cis}(k\theta)$ — ①.

Prove the statement is true for $n=k+1$. That is RTP

$(\operatorname{cis}\theta)^{k+1} = \operatorname{cis}(k+1)\theta$ — ②

In the LHS of ②

$(\operatorname{cis}\theta)^{k+1} = (\operatorname{cis}\theta)\cdot(\operatorname{cis}\theta)^k$

$= (\operatorname{cis}\theta)[\operatorname{cis}(k\theta)]$ from ①

$= [\cos\theta + i\sin\theta][\cos k\theta + i\sin k\theta]$

$= \cos\theta\cos k\theta + i\cos\theta\sin k\theta + i\sin\theta\cos k\theta - \sin\theta\sin k\theta$

$= \cos(\theta+k\theta) + i\sin(\theta+k\theta)$

$= \cos\theta(k+1) + i\sin\theta(k+1)$

$= \operatorname{cis}(k+1)\theta$

$=$ RHS.

Step 3: The statement is true for $n=1$. Whenever it is true for $n=k$, it is also true for $n=k+1$ and for all positive integer values of $n \ge 1$.

iii) $w^5 = (\operatorname{cis}\pi/4)^5$

$= \operatorname{cis}5\pi/4$ from (ii)

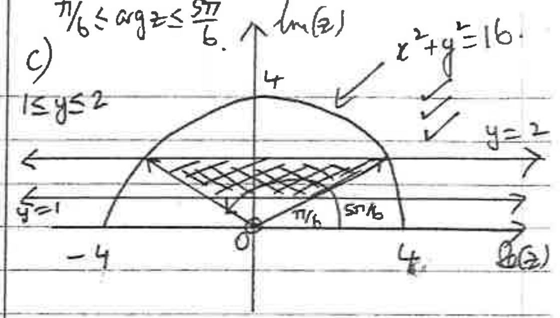
$= \operatorname{cis}(\pi+\pi/4)$

$= -\cos\pi/4 - i\sin\pi/4$

$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$ ($x+yi$ form).

c) $\pi/6 \le \arg z \le \frac{5\pi}{6}$

$1 \le y \le 2$



d) i) by completing the square

$-(-3-2x+x^2)$

$= -\left(x^2-2x+\left(-\frac{2}{2}\right)^2 - \left(-\frac{2}{2}\right)^2 - 3\right)$

$= -\left(x^2-2x+(-1)^2-4\right)$

$= -[(x-1)^2-4]$

$= 4-(x-1)^2$

$\therefore k=4, h=1.$

ii) $\int_0^1 \frac{dx}{\sqrt{3+2x-x^2}}$

$= \int_0^1 \frac{1}{\sqrt{4-(x-1)^2}}$

$= \left[\sin^{-1}\left(\frac{x-1}{2}\right)\right]_0^1$

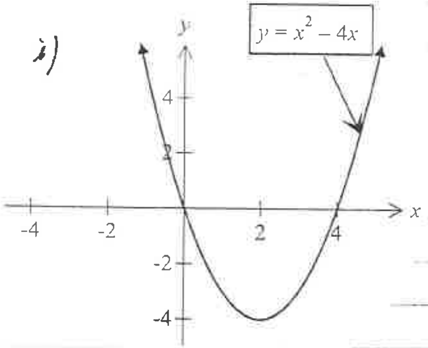
$= \sin^{-1}(0) - (\sin^{-1}-\frac{1}{2})$

$= \sin^{-1}\frac{1}{2}$

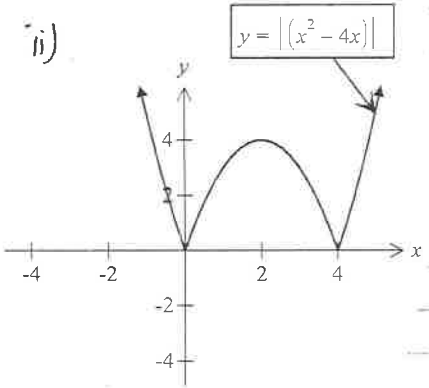
$= \pi/6.$

13 (a)

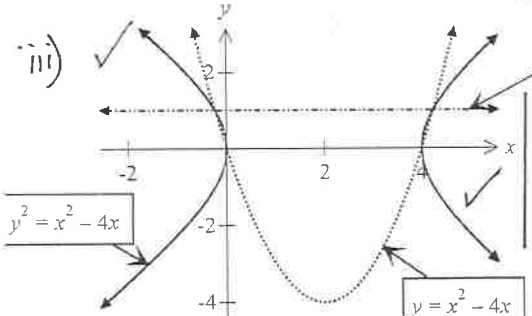
i)



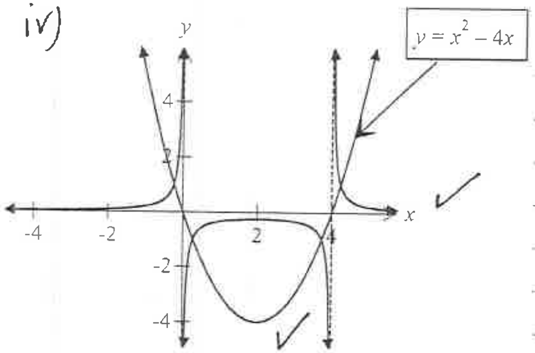
ii)



iii)



iv)



b) $z_1 = \frac{7}{2} + 2i \quad z_2 = -1 - i$

Given $2|z - z_1| = |z - z_2|$

ie $2\left|x + yi - \frac{7}{2} - 2i\right| = |x + yi + 1 + i|$

ie $2\left|\left(x - \frac{7}{2}\right) + i(y-2)\right| = |(x+1) + i(y+1)|$

ie $4\left[\left(x - \frac{7}{2}\right)^2 + (y-2)^2\right] = (x+1)^2 + (y+1)^2$

$4\left[x^2 - 7x + \frac{49}{4} + y^2 - 4y + 4\right] = x^2 + 2x + 1 + y^2 + 2y + 1$

$4x^2 - 28x + 49 + 4y^2 - 16y + 16 = x^2 + 2x + y^2 + 2y + 2$

$3x^2 + 3y^2 - 30x - 18y = -63$

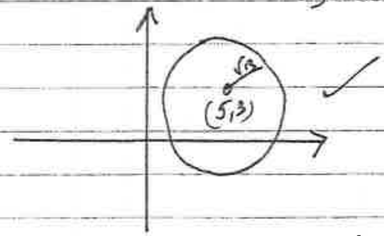
$x^2 + y^2 - 10x - 6y = -21$

$(x-5)^2 + (y-3)^2 = -21 + 25 + 9$

$(x-5)^2 + (y-3)^2 = 13.$

Circle centre $(5,3)$

radius $= \sqrt{13}$ {units}.



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13(c) $ax^4 + 4bx + c = 0.$

i) Let $p(x) = ax^4 + 4bx + c$

$p(x)$ has a double root (given)

$\therefore p'(\alpha) = p(\alpha) = 0.$

ie $4ax^3 + 4b = p'(x)$

$\therefore p'(\alpha) = 0$

ie $4a\alpha^3 = -4b$

$\alpha^3 = -b/a.$

ii) $27x^4 - 32x + 16 = 0$

$\therefore a = 27$ from (i)

$\therefore$ from i) $4b = -32$

$\therefore b = -8.$

hence $\alpha^3 = -b/a$

$\alpha^3 = 8/27$

$\alpha = 2/3$

hence $(3x-2)^2$ is a factor.

and $2/3, 2/3$ are two roots.

By division or otherwise it can be shown that $3x^2 + 4x + 4$ is another factor with zeros

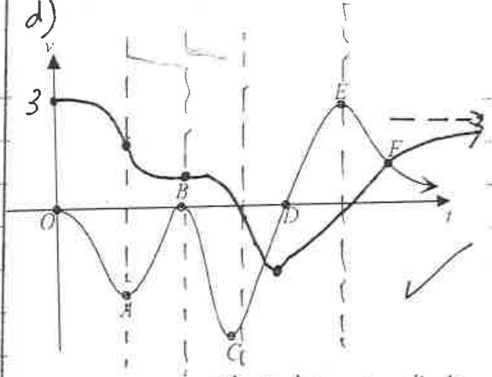
$x = \dfrac{-4 \pm \sqrt{16 - 4(3)(4)}}{6}$

$= \dfrac{-2}{3} \pm \dfrac{2\sqrt{2}i}{3}$

All the roots of $27x^4 - 32x + 16 = 0$ are

$2/3,\ 2/3,\ -\frac{2}{3} + \frac{2\sqrt{2}i}{3},\ -\frac{2}{3} - \frac{2\sqrt{2}i}{3}.$

d)



Variations by "c" other than at $(0,3)$

14)

a) $y = x^2 - 4 \quad y = \frac{1}{x}$

i) if P, Q, R are points of intersection then $x^2 - 4 = \frac{1}{x}$

ie $x^3 - 4x = 1$

$x^3 - 4x - 1 = 0$

The 3 roots of $x^3 - 4x - 1 = 0$ are therefore the x-coordinates of P, Q, and R $\rightarrow \alpha, \beta, \gamma.$

[A graph alone is insufficient]

ii) $x \rightarrow \sqrt{x}$ in $p(x) = x^3 - 4x - 1$

ie $(\sqrt{x})^3 - 4(\sqrt{x}) - 1 = 0$

$x\sqrt{x} - 4\sqrt{x} - 1 = 0$

$\sqrt{x}(x-4) = 1$

$x(x-4)^2 = 1$

$x(x^2 - 8x + 16) = 1$

$x^3 - 8x^2 + 16x - 1 = 0$

iii) "switching coefficients" or $x \rightarrow \frac{1}{x}$ in (ii).

$-x^3 + 16x^2 - 8x + 1 = 0$

or $x^3 - 16x^2 + 8x - 1 = 0$

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14 a)(iv) $P = \left(\alpha, \frac{1}{\alpha}\right)$, $Q = \left(\beta, \frac{1}{\beta}\right)$, $R = \left(\gamma, \frac{1}{\gamma}\right)$ — they also lie on $y = \frac{1}{x}$.

$\therefore OP^2 = (\alpha - 0)^2 + \left(\frac{1}{\alpha} - 0\right)^2$
$= \alpha^2 + \frac{1}{\alpha^2}$

Similarly $OQ^2 = \beta^2 + \frac{1}{\beta^2}$

and $OR^2 = \gamma^2 + \frac{1}{\gamma^2}$

$\therefore OP^2 + OQ^2 + OR^2$
$= \alpha^2 + \beta^2 + \gamma^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$
$= \underbrace{8}_{(ii)} + \underbrace{(16)}_{iii}$
$= 24$ as required.

b) $x^{1/2} + y^{1/2} = a^{1/2}$

i) $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$

$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0$

$y' = \dfrac{-\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}}$

ie $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$
$= -\sqrt{y/x}$ $(x \neq 0)$

ii) Slope of tangent at (p, q) is $-\sqrt{\frac{q}{p}}$. Equation of tangent is:

$y - y_1 = m(x - x_1)$
$y - q = -\sqrt{\frac{q}{p}}(x - p)$ $*$

at $(m, 0) \Rightarrow -q = -\sqrt{\frac{q}{p}}(m - p)$ $*$
$m\sqrt{\frac{q}{p}} = q + p\sqrt{\frac{q}{p}}$

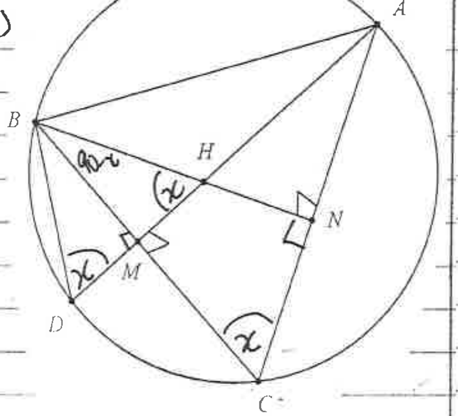
$m = q\sqrt{p/q} + p$
$m = \sqrt{pq} + p$

Similarly subs $(0, n)$ in $*$
$n - q = -\sqrt{\frac{q}{p}}(-p)$
$n - q = p\sqrt{\frac{q}{p}}$
$n - q = \sqrt{pq}$
$\left\{\begin{array}{l} n = q + \sqrt{pq} \\ m = p + \sqrt{pq} \end{array}\right\}$

iii) $m + n = p + 2\sqrt{pq} + q$
$= (\sqrt{p})^2 + 2\sqrt{p}\sqrt{q} + (\sqrt{q})^2$
$m + n = (\sqrt{p} + \sqrt{q})^2$

but $x^{1/2} + y^{1/2} = a^{1/2}$
$\therefore m + n = (a^{1/2})^2$
$= a$.

14(c)



If we can show ΔBDH is isosceles then we are done.

$\angle BNC = 90°$ (supp adj $\angle$)
Let $\angle BCN = x$
$\therefore \angle CBN = 90 - x$ (complementary $\angle$ of ΔBCN.

$\therefore \angle BHM = 90 - (90 - x)$
$= x$

but $\angle BDA = x$
(angles in the same segment st. on the same arc BA are $=$)

$\therefore \Delta BDH$ is isosceles.
Since $BM \perp DH$ in ΔBDH
$DM = MH$ (property of isosceles Δ).

Altitude of an isosceles Δ bisects (non-equal side) in ½.

There are alternative ~~methods~~ eg. CMHN is cyclic etc...

15.

a) i) Let $p(z) = z^4 - 4z^2 - 16z - 16$

$p(-1 + i\sqrt{3})$
$= (-1 + i\sqrt{3})^4 - 4(-1 + i\sqrt{3})^2 - 16(-1 + i\sqrt{3}) - 16$

NB: $z^2 = (-1 + i\sqrt{3})^2$
$= (-1)^2 - 2\sqrt{3}i - 3$
$= -2 - 2\sqrt{3}i$

$z^4 = (z^2)^2$
$= (-2 - 2\sqrt{3}i)^2$
$= (-2 - 2\sqrt{3}i)(-2 - 2\sqrt{3}i)$
$= 4 + 4\sqrt{3}i + 4\sqrt{3}i - 12$
$= -8 + 8\sqrt{3}i$

$\therefore p(z) = -8 + 8\sqrt{3}i - 4(-2 - 2\sqrt{3}i) - 16(-1 + i\sqrt{3}) - 16$
$= -8 + 8\sqrt{3}i + 8 + 8\sqrt{3}i + 16 - 16\sqrt{3}i - 16$
$= 0$ as expected.

Another method could be to assume
$p(-1 + \sqrt{3}i) = p(-1 - \sqrt{3}i) = 0$
$\therefore$ these are roots of $z^2 + 2z + 4$
(Can you see why?)
If these are roots of $p(z)$ then by division the Quotient is $z^2 - 2z - 4$ which means there is no remainder etc...
Investigate this further.

ii) Two roots of $p(z)=0$ are $-1 \pm i\sqrt{3}$ (conjugate root theorem). Therefore $z^2 - (z_1+z_2)z + z_1 z_2$ is a factor

ie z^2+2z+4 is a factor.

by division.

$$\begin{array}{r} z^2 - 2z - 4 \\ z^2+2z+4 \overline{) z^4 - 4z^2 - 16z - 16} \\ z^4 + 2z^3 + 4z^2 \\ \hline -2z^3 - 8z^2 - 16z - 16 \\ -2z^3 - 4z^2 - 8z \\ \hline -4z^2 - 8z - 16 \\ -4z^2 - 8z - 16 \\ \hline 0 \end{array}$$

Other roots are those of $z^2 - 2z - 4 = 0$

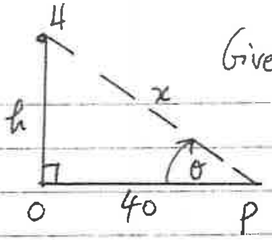
$$z = \frac{2 \pm \sqrt{4 - 4(1)(-4)}}{2}$$

$$z = \frac{2 \pm \sqrt{20}}{2}$$

$$= \frac{2 \pm 2\sqrt{5}}{2}$$

ie $z = 1 \pm \sqrt{5}$

$\therefore$ all the roots are $-1 \pm i\sqrt{3}$ and $1 \pm \sqrt{5}$

b) 

Given $\frac{dh}{dt} = 10$ m/s.

Let PH = x metres

i) $\tan\theta = \frac{h}{40}$

$h = 40\tan\theta \Rightarrow \frac{dh}{d\theta} = 40\sec^2\theta = 40(1+\tan^2\theta)$ ← or use implicit differentiation

We want $\frac{d\theta}{dt}$.

$$\frac{d\theta}{dt} = \frac{dh}{dt} \times \frac{d\theta}{dh}$$

$$= 10 \times \frac{1}{40(1+\tan^2\theta)}$$

$$= \frac{1}{4(1+\tan^2\theta)}$$

when $h = 30$, $\tan\theta = 3/4$.

$$\therefore \left.\frac{d\theta}{dt}\right|_{h=30} = \frac{1}{4(1+\frac{9}{16})}$$

$$= \frac{1}{4 + 9/4}$$

$$\left.\frac{d\theta}{dt}\right|_{h=30} = \frac{4}{25} \text{ rads/s.}$$

(0.16).

ii) $x^2 = h^2 + 40^2$

$$2x\frac{dx}{dh} = 2h$$

$$\frac{dx}{dh} = \frac{h}{x}$$

$$\therefore \frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$$

Q15 ii) ctd.

$$\frac{dx}{dt} = \frac{h}{x} \cdot 10.$$

when $h = 30$, $x^2 = 30^2 + 40^2$

$x^2 = 900 + 1600$

$x^2 = 2500$

$x = 50 \quad (x > 0)$

$$\therefore \frac{dx}{dt} = \frac{30}{50} \times 10$$

$= \frac{3}{5} \times 10$

$= 3 \times 2$

$= 6$ m/s as required.

c) $$I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$$

$$= \int_0^1 (1+x^2)^{-n}\, dx$$

let $u = (1+x^2)^{-n}$ $dv = 1$

$du = -2n(1+x^2)^{-n-1}x$ $v = x$

$$\therefore I_n = x(1+x^2)^{-n}\Big]_0^1 + 2n\int_0^1 \frac{x^2}{(x^2+1)^{n+1}}\,dx$$

$$= x(1+x^2)^{-n}\Big]_0^1 + 2n\int_0^1 \frac{x^2+1}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+1}}$$

$$I_n = x(1+x^2)^{-n}\Big]_0^1 + 2n\int_0^1 \frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}\,dx$$

$$I_n = 2^{-n} + 2n\left[I_n - I_{n+1}\right]$$

$$I_n = 2^{-n} + 2nI_n - 2nI_{n+1}$$

$$2nI_{n+1} = 2^{-n} + (2n-1)I_n$$

ie $I_{n+1} = \frac{1}{2n}\left[2^{-n} + (2n-1)I_n\right]$.

ii) $$I_5 = \int_0^1 \frac{dx}{(1+x^2)^5}$$

$n=4$ $I_5 = \frac{1}{8}\left[\frac{1}{16} + 7I_4\right]$

$n=3$ $I_4 = \frac{1}{6}\left[\frac{1}{8} + 5I_3\right]$

$n=2$ $I_3 = \frac{1}{4}\left[\frac{1}{4} + 3I_2\right]$

$n=1$ $I_2 = \frac{1}{2}\left[\frac{1}{2} + I_1\right]$

$$= \frac{1}{4} + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{4} + \frac{1}{2}(\pi/4)$$

$I_2 = \frac{1}{4} + \frac{\pi}{8}$

$$\therefore I_3 = \frac{1}{4}\left[\frac{1}{4} + 3\left(\frac{1}{4} + \frac{\pi}{8}\right)\right]$$

$$= \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right)$$

$$= \frac{1}{4} + \frac{3\pi}{32}$$

$$I_4 = \frac{1}{6}\left[\frac{1}{8} + 5\left(\frac{1}{4} + \frac{3\pi}{32}\right)\right]$$

$$= \frac{1}{48} + \frac{5}{6}\left(\frac{1}{4} + \frac{3\pi}{32}\right)$$

$$= \frac{1}{48} + \frac{5}{24} + \frac{15}{192}\pi$$

$$= \frac{11}{48} + \frac{15}{192}\pi$$

$$\therefore I_5 = \frac{1}{8}\left[\frac{1}{16} + 7\left(\frac{11}{48} + \frac{15}{192}\pi\right)\right]$$

$$= \frac{1}{8} \times \frac{1}{16} + \frac{7}{8}\left(\frac{11}{48} + \frac{15}{192}\pi\right)$$

$$= \frac{5}{24} + \frac{35}{512}\pi.$$

16) a) $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

i) $p(\alpha) = 0$ ✓

$\therefore a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0$

ie $a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha = -e$

$\alpha(a\alpha^3 + b\alpha^2 + c\alpha + d) = -e$

ie $\alpha k = -e$ (k is an integer) ✓

$\Rightarrow \alpha$ is a factor of e.

ii) $q(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$

$q(x)$ is a polynomial with integer coefficients $\therefore$ the only possible roots are factors of -3 ✓

ie $\pm 1, \pm 3$.

Testing:

$q(1) = 4 - 1 + 3 + 2 - 3 \neq 0$

$q(-1) = 4 + 1 + 3 - 2 - 3 \neq 0$

$q(3) = 324 - 27 + 27 + 6 - 3 \neq 0$

$q(-3) = 324 + 27 + 27 - 6 - 3 \neq 0$ ✓

$\therefore q(x)$ does not have integer roots from i).

b) i) $RHS = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{1}{2} \cdot 2v \frac{dv}{dx}$ ✓

$= v \cdot \frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$ ✓

$= \frac{dv}{dt}$

$= d^2x/dt^2 = LHS$.

ii) $\ddot{x} = 36 - 9x$ & SHM.

$\ddot{x} = -9(x - 4) \Rightarrow n = 3.$

α) $T = \frac{2\pi}{n} = \frac{2\pi}{3}s.$ ✓

$x = 4 \Rightarrow$ centre. ✓

β) Using (b)(i) means you need to show evidence of integration and cannot simply quote results such as

$\begin{cases} v^2 = n^2[a^2 - (x-k)^2] \text{ or} \\ x = a\cos(nt + \alpha) + k. \end{cases}$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9(x-4)$

$\frac{1}{2}v^2 = -\frac{9(x-4)^2}{2} + C_1$

when $t = 0, v = 0, x = 6$

$\therefore 0 = -\frac{9(2)^2}{2} + C_1$

$0 = -18 + C_1$

$\therefore C_1 = 18$

$\frac{1}{2}v^2 = -\frac{9}{2}(x-4)^2 + 18$

$v^2 = 36 - 9(x-4)^2$

$v^2 = 9[\underset{\uparrow a^2}{4} - (x-4)^2]$ ✓

$\therefore a = 2$ ✓ $\Rightarrow$ amplitude

(when $v = 0$)

16 c) i) let $u = a - x$ $\therefore \frac{du}{dx} = -1$ ie $du = -dx$

when $x = 0, u = a$

$x = a, u = 0$

$\therefore \int_0^a f(x)\,dx = \int_a^0 f(a-u) - du$ ✓

$= \int_0^a f(a-u)\,du$ ✓

$= \int_0^a f(a-x)\,dx$

ii) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$

$= \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$ ✓

ie $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}\,dx - \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$

$2\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$

$= \pi\int_0^\pi \frac{\sin x}{1+(\cos x)^2}\,dx$

$= \pi\int_0^\pi \frac{-\frac{d}{dx}(\cos x)}{1+(\cos x)^2}\,dx$ ✓

$= \pi\left[-\tan^{-1}(\cos x)\right]_0^\pi$

$= -\pi\left[\tan^{-1}(\cos\pi) - \tan^{-1}(\cos 0)\right]$

$= -\pi\left[-\pi/4 - \pi/4\right]$

$= \pi^2/2$

$\therefore \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{2} \times \frac{1}{2} = \frac{\pi^2}{4}$ ✓