BOSTES Student Number

Class Teacher:

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Trinity Grammar School

Mathematics Department

2015

HALF-YEARLY EXAMINATION

HSC ASSESSMENT TASK 3

Year 12

Mathematics Extension 2

Do NOT write solutions to be marked on this question paper. Any working on this question paper will NOT be marked.

Date of Examination:
Wednesday 22 April 2015

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- BOSTES approved calculators for this course may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- Write your BOSTES Student Number on this question paper and on any answer sheets or writing booklets used to write your responses
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating N/A and your BOSTES Student Number
- HSC Assessment Weighting: 30%

Total marks – 100

Section I

10 marks

- Attempt Questions 1–10
- Allow about **15** minutes for this section

Section II

90 marks

- Attempt Questions 11–16
- Allow about **2** hours **45** minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Shade the correct response on the multiple-choice answer sheet for Questions 1–10

Each question is worth 1 mark

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1 If $\sin x = e^y$ for $0 < x < \pi$, what is $\dfrac{dy}{dx}$ in terms of x ?

(A) $-\tan x$

(B) $-\cot x$

(C) $\cot x$

(D) $\tan x$

2 If n is a non-negative integer, then $\displaystyle\int_0^1 x^n \; dx = \int_0^1 (1-x)^n \; dx$ for

(A) n even, only

(B) n odd, only

(C) non-zero n, only

(D) all n

3 If $\dfrac{d}{dx}f(x) = g(x)$ and $\dfrac{d}{dx}g(x) = f(x^2)$, then $\dfrac{d^2}{dx^2}f(x^3) =$

(A) $6x^2 f(x^6) + x^3 g(x^6)$

(B) $9x^4 g(x^3) + 3x^2 f(x^3)$

(C) $9x^4 f(x^3) + 3x^2 g(x^3)$

(D) $9x^4 f(x^6) + 6xg(x^3)$

4 If $\int x^2 \cos x \ \ dx = f(x) - \int 2x \sin x \ \ dx$, then $f(x) =$

(A) $2\sin x + 2x\cos x + c$

(B) $x^2 \sin x + c$

(C) $2x\cos x - x^2 \sin x + c$

(D) $4\cos x - 2x\sin x + c$

5 Evaluate $\lim_{x \to 0} \dfrac{1-\cos^2(2x)}{4x^2}$.

(A) 0

(B) 1

(C) 2

(D) 4

6 $\int_1^3 \dfrac{x^2 - 2x + 4}{x+1} dx =$

(A) $-2 + 7\ln 2$

(B) $\ln 2$

(C) $7\ln 2$

(D) $2 + 7\ln 2$

Question 7 commences on the next page

7 Find $\int \frac{dx}{(x-1)(x+3)}$.

(A) $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|+c$

(B) $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right|+c$

(C) $\frac{1}{2}\ln\left|(x-1)(x+3)\right|+c$

(D) $\ln\left|(x-1)(x+3)\right|+c$

8 Which pair of equations give the directrices of the hyperbola defined parametrically by $x = 5\sec\theta,\ y = 2\tan\theta$?

(A) $x = \pm\frac{1}{\sqrt{29}}$

(B) $x = \pm\frac{25}{\sqrt{29}}$

(C) $y = \pm\frac{2}{5}x$

(D) $y = \pm\frac{5}{2}x$

9 Let z represent a complex number with $\arg z = -\frac{\pi}{3}$ and $|\,z\,| = 2$. The argument **and** modulus of the complex number $\frac{2i}{z}$ **respectively** are

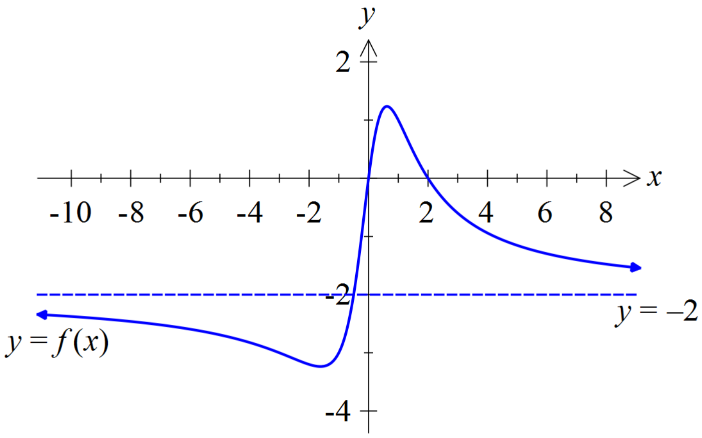
(A) $-\frac{5\pi}{6},\ \frac{1}{2}$

(B) $-\frac{\pi}{6},\ \frac{1}{2}$

(C) $\frac{\pi}{6},\ 1$

(D) $\frac{5\pi}{6},\ 1$

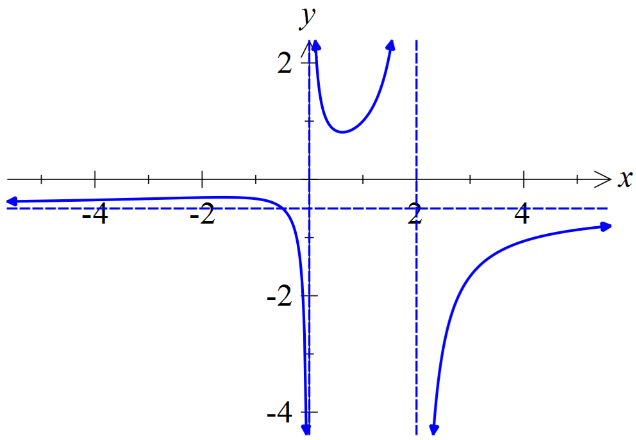
10



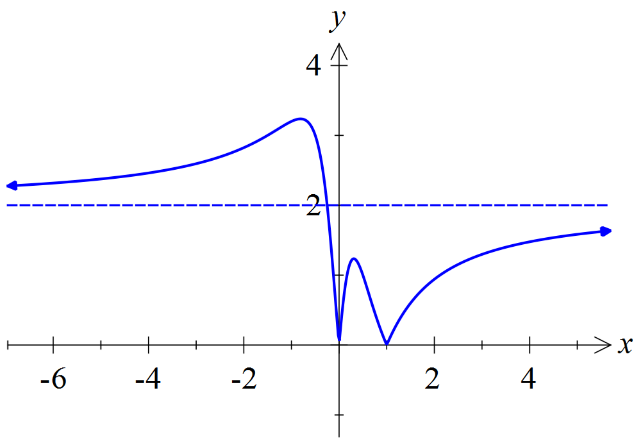
The diagram above represents the graph of $y = f(x)$.

Which diagram below best represents the graph of $y = \dfrac{1}{f(2x)}$?

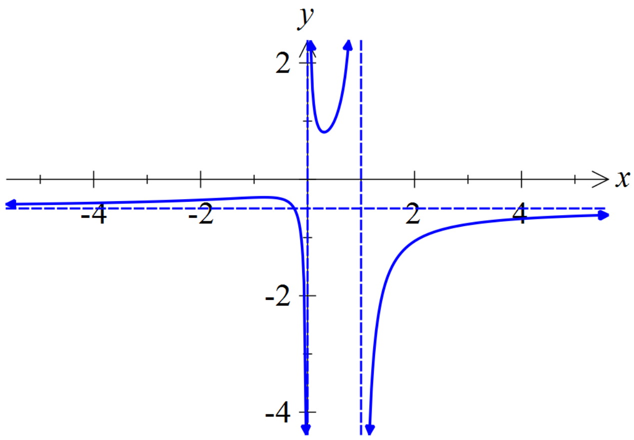
(A)



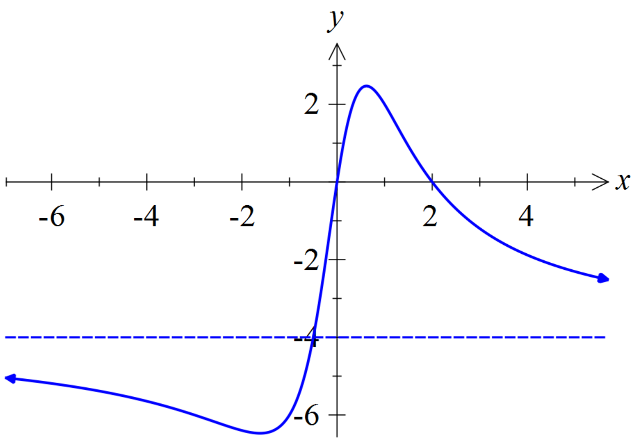
(B)



(C)



(D)



End of Section I
Section II commences on the next page

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Let $z = 5 + 5i$ and $w = 2 + i$.

Find:

(i) $2z + 3\overline{w}$; **2**

(ii) $z(w-1)$; **2**

(iii) $\dfrac{z}{w}$, **2**

(iv) $\arg z - \arg \overline{z}$. **1**

(b) Using De Moivre's theorem, simplify the expression $\dfrac{(1+\sqrt{3}i)^6}{(1-i)^4}$. **3**

(c) If ω is the complex cube root of unity, evaluate $(1-3\omega+\omega^2)(1+\omega-8\omega^2)$. **2**

(d) Sketch in the Argand plane the region satisfying the following pair of inequalities simultaneously. **3**

$$\begin{cases} |z-1-2i| \le 2 \\ \dfrac{\pi}{4} < \arg z \le \dfrac{\pi}{2} \end{cases}$$

End of Question 11

Please turn over

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Using De Moivre's theorem or otherwise, find all six roots of the equation $z^6 = -64$. **3**

(b) Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x \ \ dx$. **3**

(c) (i) Find real constants A, B and C such that **3**

$$\frac{x^2+x+1}{x^3+3x^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}.$$

(ii) Hence, find $\int \frac{x^2+x+1}{x^3+3x^2}\, dx$. **2**

(d) (i) Express x^2+x+1 in the form $(x+b)^2+c$, where b and c are real numbers. **1**

(ii) Hence, by using your result in (i) and the table of standard integrals or otherwise, find: **1**

$$\int \frac{1}{x^2+x+1}\, dx$$

(iii) Find $\int \frac{dx}{2+\sin x}$ using the substitution $t = \tan\frac{x}{2}$. **2**

End of Question 12

Please turn over

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) The variable point $P\left(ct, \dfrac{c}{t}\right)$ lies on the hyperbola $xy = c^2$.

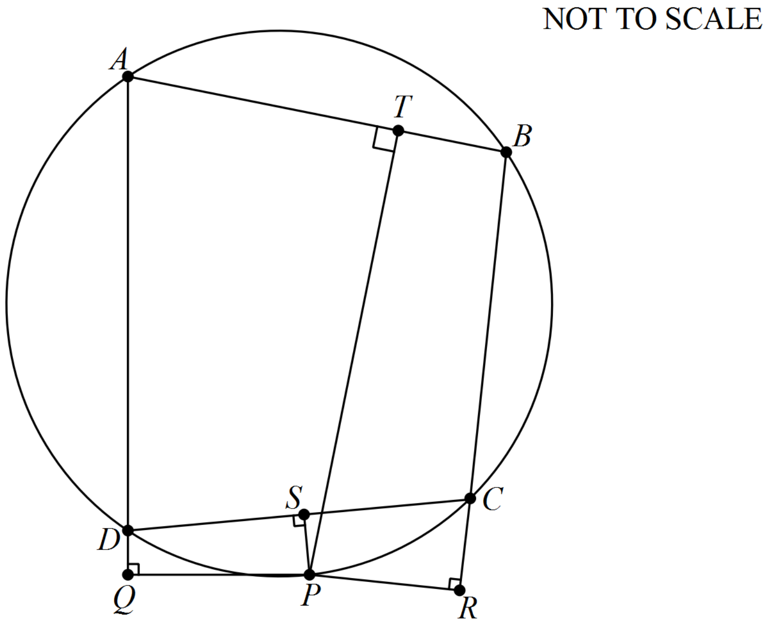
(i) Prove that the equation of the tangent to the hyperbola at P is given by: **2**
$x + yt^2 = 2ct$

The perpendicular (from the origin O) meets the tangent at P, at the point N.

(ii) Prove that the coordinates of N are given by $\left(\dfrac{2ct}{1+t^4}, \dfrac{2ct^3}{1+t^4}\right)$. **2**

(iii) Prove that the locus of N as P moves on the hyperbola is the curve defined by $(x^2 + y^2)^2 = 4c^2xy$. **3**

(b)



In the diagram, $ABCD$ is a cyclic quadrilateral. P is a point on the circle through A, B, C and D. PQ, PR, PS and PT are the perpendiculars from P to AD produced, BC produced, CD and AB respectively. ***Copy this diagram into your writing booklet.***

(i) Explain why $SPRC$ and $AQPT$ are cyclic quadrilaterals. **2**

(ii) Hence show that $\angle SPR = \angle QPT$ and $\angle PRS = \angle PTQ$. **2**

(iii) Prove that $\Delta SPR \,|||\, \Delta QPT$. **2**

(iv) Hence show that:

(α) $PS \times PT = PQ \times PR$; **1**

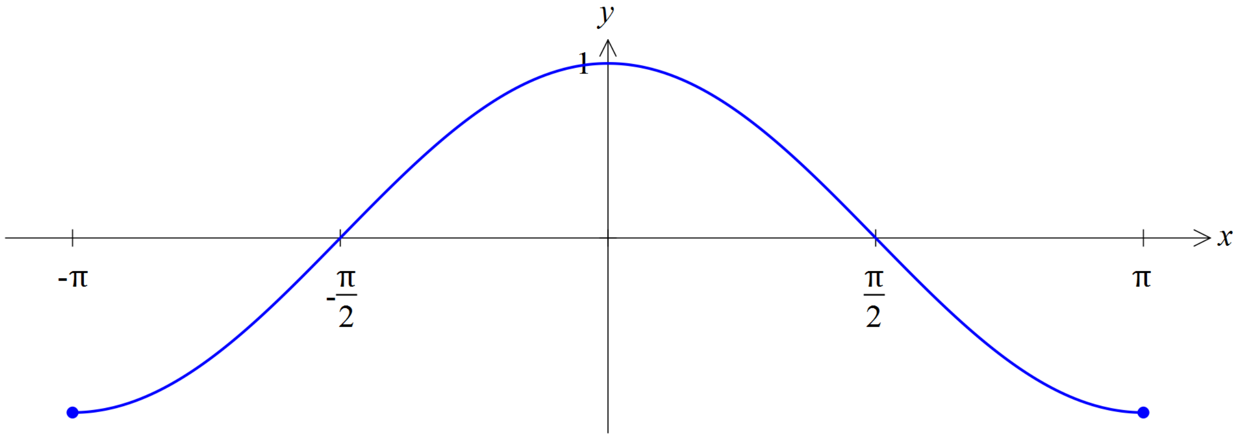
(β) $\dfrac{PS \times PR}{PQ \times PT} = \dfrac{SR^2}{QT^2}$. **1**

End of Question 13

Please turn over

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) The diagram below represents the graph (not to scale) of $f(x) = \cos x$ in the interval $-\pi \le x \le \pi$.



On separate, one-third page number planes, sketch the graphs of the following:

(i) $y = f(x)^2$ **2**

(ii) $y = \dfrac{1}{f\left(x - \dfrac{\pi}{2}\right)}$ **2**

(iii) $y^3 = f(x)$ **2**

(iv) $y = \ln f(x)$ **2**

(v) $y = f(\sqrt{|x|})$ **2**

(b) A locus definition of an ellipse is given by $PS + PS' = 2a$, where P is a variable point (x, y) on the ellipse and S and S' represent the foci of the ellipse and $|a|$ is the length of its semi-major axis. Suppose P is represented by the complex variable $z = x + yi$, where x and y are real numbers. **5**

Using the above statement or otherwise, find the eccentricity **and** the Cartesian equation of the ellipse determined by

$$|z-1| + |z+1| = 6.$$

(Leave your answer in the Standard form)

End of Question 14

Please turn over

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) If z is a complex number so that $|z| = 3$ and $\arg z = \dfrac{\pi}{6}$, **mark clearly on the same Argand diagram** the **points** represented by the following complex numbers:

(i) z; **1**

(ii) $-i^{25}z$; **1**

(iii) $z\bar{z}$; **1**

(iv) $z^2 + z$; and **1**

(v) $z^2 - z$. **1**

(b) Sketch the locus in the Argand plane of the variable point z, where $z = x + yi$ such that $|z-1| = \mathrm{Re}(z)$ **and state** the locus of z in Cartesian form. **2**

(c) For the hyperbola H: $\dfrac{y^2}{4} - \dfrac{x^2}{12} = 1$,

(α) Find:

(i) its eccentricity; **1**

(ii) the coordinates of its foci; **1**

(iii) the equations of its directrices; **1**

(iv) the equations of its asymptotes. **1**

(β) Sketch the graph of H, showing the features above including its vertices. **2**

(d) Find $\displaystyle\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$. **2**

End of Question 15

Please turn over

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) The points P and Q with respective parameters θ and $\theta + \frac{\pi}{2}$ both lie on the ellipse E: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(i) Show that Q has coordinates $(-a\sin\theta, b\cos\theta)$. **2**

(ii) If O is the centre of the ellipse E prove that $OP^2 + OQ^2 = a^2 + b^2$. **1**

(iii) Show that the midpoint M of PQ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$. **1**

(iv) State the equation, in parametric form, of the tangent at P on E. **1**

(v) Determine the point of intersection T of the tangent at P and Q to E. **2**

(vi) Show that T lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$. **1**

(b) The sum of the first k positive integers can be written as **3**

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}.$$

Given that n is a positive integer **prove** that the sum of the integers between 1 and $15n$ inclusive **which are not** divisible by 3 or 5 is $60n^2$.

(c) (i) If $\sin(x - \theta) = k\sin(x + \theta)$, show that $\tan x = \frac{-(k+1)}{k-1}\tan\theta$. **2**

(ii) Hence, for $k = \frac{1}{2}$ and $\theta = 210°$ solve for x in the interval $0° \le x \le 360°$. **2**

End of Question 16

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \;, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

Trinity Grammar School
Mathematics Department
2015
Year 12 Mathematics Extension 2

HSC Assessment Task 3
Section I
Objective Response Answer Sheet

Suggested Solutions & marking scheme

BOSTES Student Number

Class Teacher: EH

- Write your answers for **Section I** on this answer sheet.
- ***Shade completely only ONE answer for each question.***
- To change an answer, erase your previous mark completely and then record your new answer.

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D
6	A	B	C	D
7	A	B	C	D
8	A	B	C	D
9	A	B	C	D
10	A	B	C	D

A table of Standard Integrals is printed on the reverse side of this answer sheet

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Shade the correct response on the multiple-choice answer sheet for Questions 1–10

Each question is worth 1 mark

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1 If $\sin x = e^y$ for $0 < x < \pi$, what is $\frac{dy}{dx}$ in terms of x ?

(A) $-\tan x$

(B) $-\cot x$

$\boxed{\text{(C)}}$ $\cot x$

(D) $\tan x$

$y = \ln(\sin x)$

$\therefore y' = \frac{\cos x}{\sin x}$

$= \cot x$

$\frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$

2 If n is a non-negative integer, then $\int_0^1 x^n \; dx = \int_0^1 (1-x)^n \; dx$ for

(A) n even, only

(B) n odd, only

(C) non-zero n, only

$\boxed{\text{(D)}}$ all n

$RHS = \int_0^1 (1-x)^n \, dx$

Let $u = 1-x \quad \therefore du = -dx$

when $x = 0, \; u = 1$

$x = 1, \; u = 0$

$\therefore \int_0^1 (1-x)^n \, dx = \int_1^0 u^n \, -du$

$= \int_0^1 u^n du$

$= \int_0^1 x^n \, dx$

Or $RHS = \int_0^1 (1-x)^n dx.$

$= \frac{(1-x)^{n+1}}{-(n+1)}\Big|_0^1 = 1/n+1.$

3 If $\frac{d}{dx} f(x) = g(x)$ and $\frac{d}{dx} g(x) = f(x^2)$, then $\frac{d^2}{dx^2} f(x^3) =$

(A) $6x^2 f(x^6) + x^3 g(x^6)$

(B) $9x^4 f(x^3) + 3x^2 g(x^3)$

(C) $9x^4 f(x^3) + 3x^2 g(x^3)$

$\boxed{\text{(D)}}$ $9x^4 f(x^6) + 6xg(x^3)$

Given $f'(x) = g(x) \; \therefore f'(x^3) = g(x^3)$

$g'(x) = f(x^2) \therefore f''(x^3) = g'(x^3)$

$= f((x^3)^2)$

$= f(x^6)$

Let $y = f(x^3)$

$\therefore y' = 3x^2 f'(x^3)$

$y'' = f'(x^3) \cdot 6x + 3x^2 f''(x^3) \cdot 3x^2$

$= 9x^4 f''(x^3) + 6x f'(x^3)$

$= 9x^4 f(x^6) + 6x g(x^3)$

4 If $\int x^2 \cos x \ dx = f(x) - \int 2x \sin x \ dx$, then $f(x) =$

(A) $2\sin x + 2x\cos x + c$

$\boxed{\text{(B)}}$ $x^2 \sin x + c$

(C) $2x\cos x - x^2 \sin x + c$

(D) $4\cos x - 2x\sin x + c$

let $u = x^2$ $dv = \cos x$
$du = 2x$ $v = \sin x$
$\therefore I = uv - \int v \, du$
$= \underbrace{x^2 \sin x}_{f(x)} - \int 2x \sin x \, dx$

5 Evaluate $\lim_{x \to 0} \frac{1-\cos^2(2x)}{4x^2}$.

(A) 0

$\boxed{\text{(B)}}$ 1

(C) 2

(D) 4

$= \lim_{x \to 0} \left[\frac{\sin^2(2x)}{4x^2}\right]$
$= \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right) \cdot \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)$
$= 1 \times 1$
$= 1$

6 $\int_1^3 \frac{x^2 - 2x + 4}{x+1} \, dx =$

$\boxed{\text{(A)}}$ $-2 + 7\ln 2$

(B) $\ln 2$

(C) $7\ln 2$

(D) $2 + 7\ln 2$

$= \int_1^3 (x-3) + \frac{7}{x+1} \, dx$ (long division or by inspection)
$= \frac{x^2}{2} - 3x + 7\ln(x+1) \Big|_1^3$
$= \frac{9}{2} - 9 + \underbrace{7\ln 4}_{14\ln 2} - \frac{1}{2} + 3 - 7\ln(2)$
$= -2 + 7\ln 2$

Question 7 commences on the next page

7 Find $\int \frac{dx}{(x-1)(x+3)}$. $= \int \frac{\frac{1}{4}}{x-1} - \frac{\frac{1}{4}}{x+3}\, dx$

$= \frac{1}{4}\left[\ln|x-1| - \ln|x+3|\right]$

$\boxed{\text{(A)}}$ $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + c$

(B) $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right| + c$

(C) $\frac{1}{2}\ln|(x-1)(x+3)| + c$

(D) $\ln|(x-1)(x+3)| + c$

NB: $\frac{A}{x-1} + \frac{B}{x+3} \equiv \frac{1}{(x-1)(x+3)}$

$A(x+3) + B(x-1) \equiv 1$

Let $x = 1 \rightarrow A = 1/4$
$x = -3 \rightarrow B = -1/4$ } etc.

8 Which pair of equations give the directrices of the hyperbola defined parametrically by $x = 5\sec\theta,\ y = 2\tan\theta$? $(a > b)$

$\frac{x^2}{25} - \frac{y^2}{4} = 1$

(A) $x = \pm\frac{1}{\sqrt{29}}$

$\boxed{\text{(B)}}$ $x = \pm\frac{25}{\sqrt{29}}$

(C) $y = \pm\frac{2}{5}x$

(D) $y = \pm\frac{5}{2}x$

$x = \pm\frac{a}{e} = \pm\frac{5}{e}$

$b^2 = a^2(e^2-1) \quad e > 1$

$4 = 25(e^2-1)$

$\frac{4}{25} + 1 = e^2$

$e^2 = \frac{29}{25} \quad \therefore e = \frac{\sqrt{29}}{5}$

$\therefore x = \pm\frac{5}{\frac{\sqrt{29}}{5}} = \pm\frac{25}{\sqrt{29}}$

9 Let z represent a complex number with $\arg z = -\frac{\pi}{3}$ and $|z| = 2$. The argument **and** modulus of the complex number $\frac{2i}{z}$ **respectively** are

$z = 2\,\text{cis}(-\pi/3)$

$\Rightarrow \frac{1}{z} = \frac{1}{2}\text{cis}\left(\frac{\pi}{3}\right)$

$\Rightarrow \frac{i}{z} = \frac{1}{2}\text{cis}\left(\frac{\pi}{3}+\frac{\pi}{2}\right) = \frac{1}{2}\text{cis}\left(\frac{5\pi}{6}\right)$

$\therefore \frac{2i}{z} = \text{cis}\left(\frac{5\pi}{6}\right)$

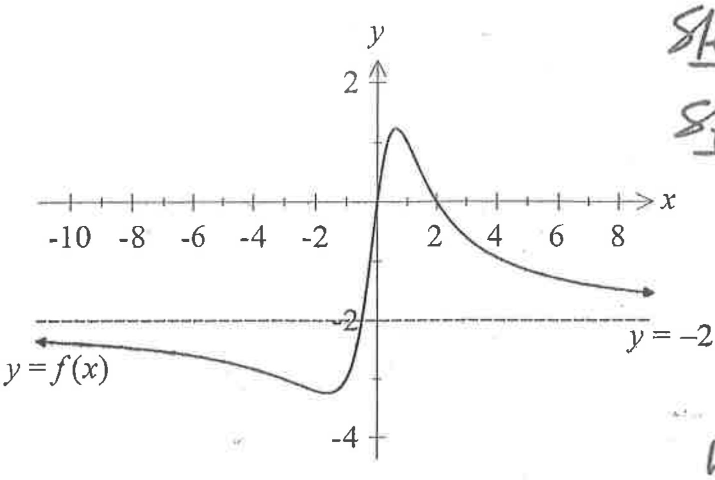
(A) $-\frac{5\pi}{6}, \frac{1}{2}$

(B) $-\frac{\pi}{6}, \frac{1}{2}$

(C) $\frac{\pi}{6}, 1$

$\boxed{\text{(D)}}$ $\frac{5\pi}{6}, 1$

10



Step1: Draw $y = f(2x)$

Step2: Draw $y = \frac{1}{f(2x)}$.

In Step1:
* Same shape but x-ordinates are halved at intercepts
y-int is maintained.

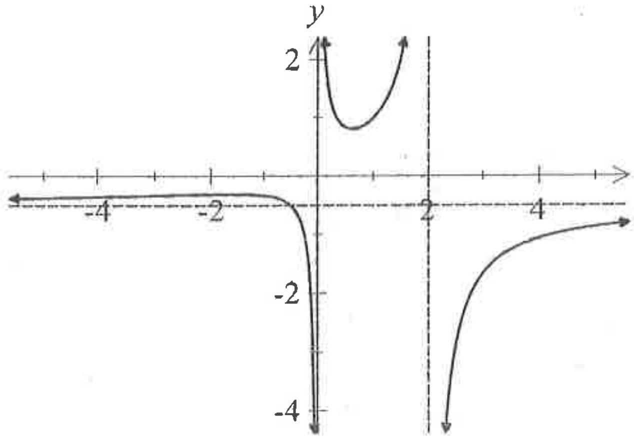
In Step2: V. Asymptotes at $x = 1$ and $x = 0$.

$\therefore$ $\boxed{C}$.

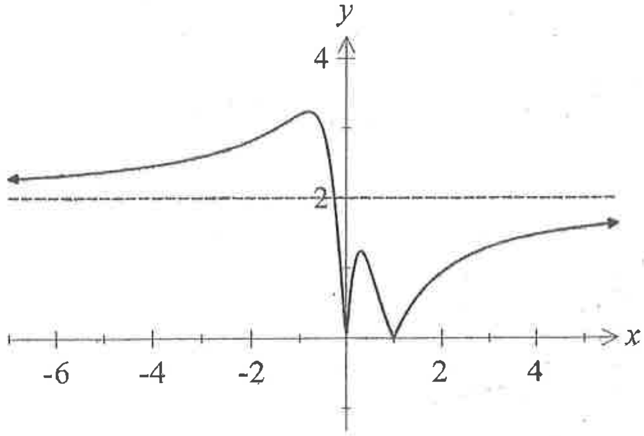
The diagram above represents the graph of $y = f(x)$.

Which diagram below best represents the graph of $y = \dfrac{1}{f(2x)}$?

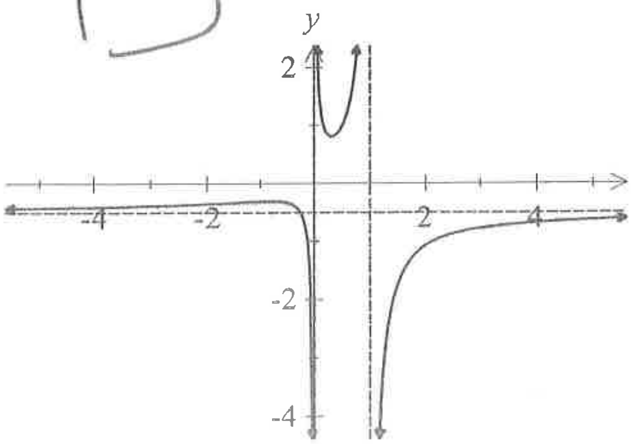
(A)



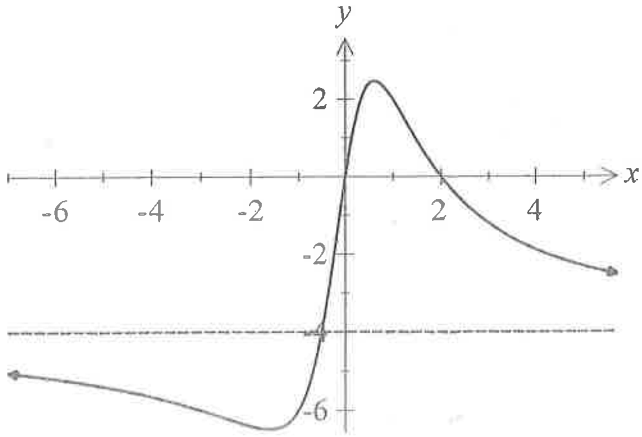
(B)



(C)



(D)



End of Section I

Section II commences on the next page

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 11 | **Marker comments**

a) i) $2z + 3\bar{w} = 2(5+5i) + 3(2-i)$

$= 10 + 10i + 6 - 3i$

$= 16 + 7i$ ✓✓

ii) $z(w-1) = (5+5i)(1+i)$

$= 5 + 5i + 5i - 5$

$= 10i$ ✓✓

iii) $\dfrac{z}{w} = \dfrac{5+5i}{2+i} = \dfrac{(5+5i)(2-i)}{(2+i)(2-i)}$ ✓

$= \dfrac{10 - 5i + 10i + 5}{4+1}$

$= \dfrac{15+5i}{5}$

$= 3 + i$ ✓

iv) $\arg z - \underbrace{\arg \bar{z}}_{(-\arg z)} = \tan^{-1}\left(\frac{5}{5}\right) + \tan^{-1}\left(\frac{5}{5}\right)$

$= \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$ ✓

---

b) $\dfrac{(1+\sqrt{3}i)^6}{(1-i)^4} = \dfrac{\left(2 \operatorname{cis} \frac{\pi}{3}\right)^6}{\left(\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^4}$

$= \dfrac{2^6 \operatorname{cis} 2\pi}{2^2 \operatorname{cis}(-\pi)}$ ✓

$= 2^4 \operatorname{cis} 3\pi$ ✓

$= 16 \times (-1)$

$= -16.$ ✓

Question 11 ctd

Marker comments

c) Let $z^3 = 1$ and ω is a root $\omega \in \mathbb{C}$.

$\therefore z^3 - 1 = (z-1)(z^2+z+1)$

$\Rightarrow \omega^2 + \omega + 1 = 0$ ① since ω is a root ✓

$\Rightarrow \omega^3 = 1$ ②

$(1 - 3\omega + \omega^2)(1 + \omega - 8\omega^2)$

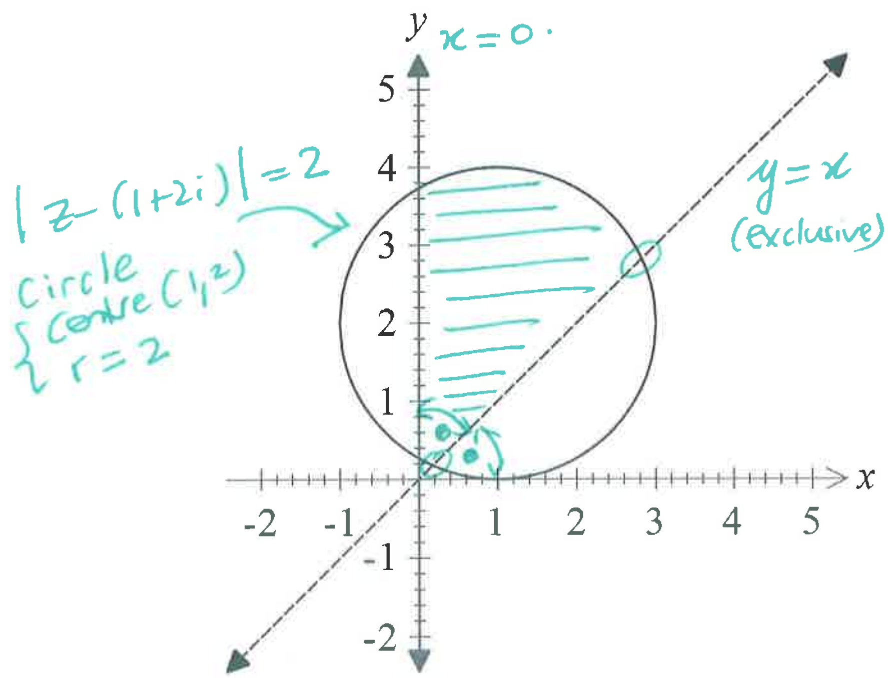
$= (1 - 3\omega + (-\omega - 1))(-\omega^2 - 8\omega^2)$ from ①

$= (-4\omega)(-9\omega^2)$

$= (-4)(-9)\omega^3$

$= 36$ ✓

d)



✓ → circle

✓ → y = x (excluding boundary)

✓ → shaded region.

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 12

Marker comments

a) $z^6 = -64$

$= 64 \operatorname{cis} \pi.$

$\therefore z = 64^{1/6} (\operatorname{cis} \pi)^{1/6}$

$= (2^6)^{1/6} \operatorname{cis} (\pi + 2k\pi)^{1/6}$

$= 2 \operatorname{cis} \left(\frac{\pi + 2k\pi}{6}\right) \quad k \in \mathbb{Z}.$

$z_1 = 2 \operatorname{cis} \pi/6 \quad (k=0)$

$z_2 = 2 \operatorname{cis} \left(\frac{3\pi}{6}\right) = 2 \operatorname{cis} \frac{\pi}{2} \quad (k=1)$

$z_3 = 2 \operatorname{cis} \left(-\frac{\pi}{6}\right) \quad (k=-1) \rightarrow$ could use $k=5 \Rightarrow z = 2\operatorname{cis}\left(\frac{11\pi}{6}\right)$

$z_4 = 2 \operatorname{cis} \left(\frac{5\pi}{6}\right) \quad (k=2)$

$z_5 = 2 \operatorname{cis} \left(-\frac{\pi}{2}\right) \quad (k=-2) \rightarrow$ could use $k=4, \Rightarrow z = 2\operatorname{cis}\frac{3\pi}{2}$

$z_6 = 2 \operatorname{cis} \left(\frac{7\pi}{6}\right) \quad (k=3) \rightarrow$ could use $k=-3, \Rightarrow z = 2\operatorname{cis}\left(-\frac{5\pi}{6}\right).$

[NB: all roots are equally spaced on the circle of radius 2, centre (0,0) by $\frac{\pi}{3}$ at the centre].

---

b) $\int_0^{\pi/2} \sin^3 x \, dx = \int_0^{\pi/2} \sin^2 x \cdot \sin x \, dx$

$= \int_0^{\pi/2} (1 - \cos^2 x) \sin x \, dx$

$= \int_0^{\pi/2} \sin x - \sin x (\cos x)^2 \, dx$ ✓

$= \left[-\cos x + \frac{\cos^3 x}{3}\right]_0^{\pi/2}$ ✓

$= \left[-\cos \pi/2 + \frac{(\cos \pi/2)^3}{3} + \cos 0 - \frac{\cos 0}{3}\right]$

$= 0 + 0 + 1 - 1/3 = 2/3$ ✓

Question 12 ctd.

Marker comments

c) i) $x^2+x+1 \equiv Ax(x+3)+B(x+3)+Cx^2$

let $x=0$, $\Rightarrow 1 = 3B$

$\therefore B=\frac{1}{3}$

let $x=-3$, $\Rightarrow 9-3+1=9C$

$\therefore C = 7/9$

let $x=1$, $\Rightarrow 3 = A\times1\times4+B\times4+C$

ie $3 = 4A + \frac{4}{3} + \frac{7}{9}$

$\Rightarrow A = 2/9$

$\therefore (A,B,C) = (\frac{2}{9}, \frac{1}{3}, \frac{7}{9})$

ii) $\int \frac{x^2+x+1}{x^3+3x^2}\,dx = \int \frac{\frac{2}{9}}{x} + \frac{\frac{1}{3}}{x^2} + \frac{\frac{7}{9}}{x+3}\,dx$ from (i)

$= \frac{2}{9}\ln|x| - \frac{1}{3x} + \frac{7}{9}\ln|x+3| + c$

d) i) $x^2+x+1 = x^2+x+(\frac{1}{2})^2+1-(\frac{1}{2})^2$

$= (x+\frac{1}{2})^2 + 3/4$

ii) $\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+3/4}\,dx$

$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2(x+\frac{1}{2})}{\sqrt{3}} + C$

iii) $t = \tan x/2$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(\tan^2 x/2 + 1)$

$\therefore \frac{dt}{dx} = \frac{1}{2}(t^2+1)$

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 12(d) ctd

Marker comments

$$\therefore \quad dx = \frac{2}{1+t^2}\,dt$$

hence $\displaystyle\int \frac{1}{2+\sin x}\,dx$

$$= \int \frac{1}{2+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{1}{t^2+t+1}\,dt$$

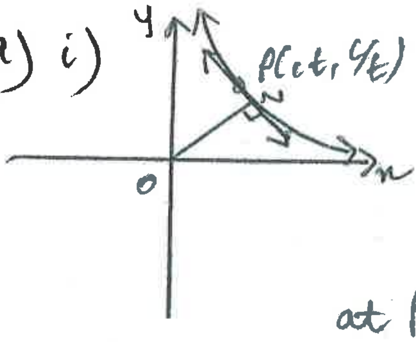
$$= \frac{2}{\sqrt{3}} \tan^{-1}\left[\frac{2(t+\frac{1}{2})}{\sqrt{3}}\right] + C \quad \text{from (ii)}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{(2\tan^{x}/_{2}+1)}{\sqrt{3}} + C$$

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 13 | Marker comments

a) i)



$$xy = c^2$$

$$\therefore y(1) + xy' = 0$$

ie $\dfrac{dy}{dx} = -\dfrac{y}{x}$

at P, $\dfrac{dy}{dx} = \dfrac{c/t}{ct}$

$$= -\frac{1}{t^2}$$

$\therefore$ Equation of tangent is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

ie $yt^2 - ct = -x + ct$

ie $x + yt^2 = 2ct$

ii) Equation of ON is $y = t^2 x$ ①

($m_1 \times m_2 = -1$ at N $\therefore$ slope of ON $= t^2$)

Solving: $\begin{cases} y = t^2 x & ① \\ x + yt^2 = 2ct & ② \end{cases}$

Sub ① into ② $\Rightarrow x + (t^2 x)t^2 = 2ct$

$\Rightarrow x(1 + t^4) = 2ct$

ie $x = \dfrac{2ct}{1+t^4}$ ③

Sub ③ into ① $\therefore y = t^2\left(\dfrac{2ct}{1+t^4}\right)$

$y = \dfrac{2ct^3}{1+t^4}$ ④

$\therefore N = \left(\dfrac{2ct}{1+t^4}, \dfrac{2ct^3}{1+t^4}\right)$.

Question 13 (a) ctd.

Marker comments

iii) From ③ and ④
$$\begin{cases} x = \dfrac{2ct}{1+t^4} & \to ③ \\ y = \dfrac{2ct^3}{1+t^4} & \to ④ \end{cases}$$

and from ① $\dfrac{y}{x} = t^2$

ie $t^4 = \dfrac{y^2}{x^2} \to ⑤$

in ③ $t = \dfrac{x(1+t^4)}{2c}$

$= \dfrac{x\left(1+\frac{y^2}{x^2}\right)}{2c}$ from ⑤

$t = \dfrac{x^2+y^2}{2cx}$

hence in ④

$y(1+t^4) = 2c\,t^3$

ie $y\left[1+\dfrac{y^2}{x^2}\right] = 2c\left[\dfrac{x^2+y^2}{2cx}\right]^3$

$\dfrac{y(x^2+y^2)}{x^2} = \dfrac{2c}{8c^3x^3}(x^2+y^2)^3$

ie $\dfrac{y}{x^2} = \dfrac{1}{4c^2x^3}(x^2+y^2)^2$

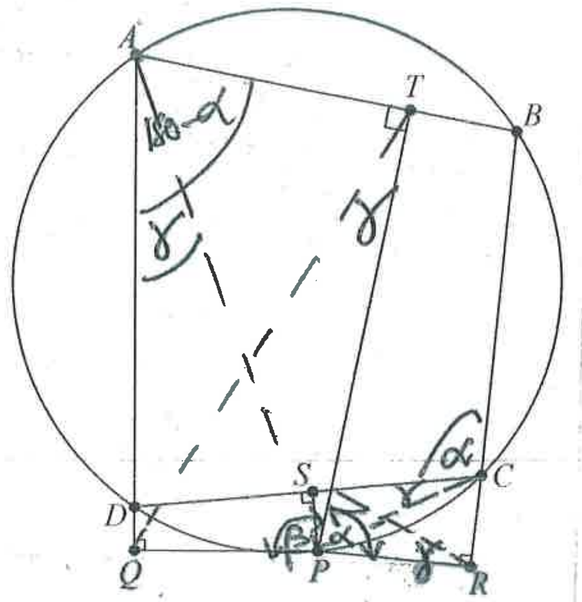
ie $4c^2xy = (x^2+y^2)^2$

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 13(b)

Marker comments

i)



In quad SPRC, $\angle DSP = \angle CRP = 90°$ (exterior angle of a quad)

As the ext $\angle$ at S is equal to the interior opposite angle at R, then SPRC is a cyclic quad.

Also $\angle ATP = \angle PQA = 90°$

Since $\angle ATP + \angle PQA = 90 + 90$
$= 180°$

then AQPT is a cyclic quad (opp. $\angle$'s are supplementary).

ii) Let $\angle SPR = \alpha$, $\angle QPT = \beta$

Join SR, PC, TQ, PA

$\angle BCS = \alpha$ (ext $\angle$ of cyclic quad SPRC)

$\therefore \angle DAB = 180 - \alpha$ (opp. supp. $\angle$'s in cyclic quad ABCD)

$\therefore \angle TPQ = \alpha$ (opp supp. $\angle$'s, AQPT cyclic)
$= \beta$

ie $\alpha = \beta$ $\quad \therefore \angle SPR = \angle QPT$

Question 13(b) ctd.

Marker comments

Let $\angle PRS = \gamma$

$\therefore \angle SCP = \gamma$ ($\angle$'s in same segment SPRC cyclic)

also $\angle DAP = \gamma$ ($\angle$'s in same segment ACPD cyclic)

$= \angle QTP$ ($\angle$'s in same segment AQPT cyclic)

$\therefore \angle PRS = \angle QTP$

iii) RTP: $\Delta SPR \parallel\!| \Delta QPT$

1. $\angle SPR = \angle QPT = \alpha$ (see ii)
2. $\angle PRS = \angle PTQ = \gamma$ (see ii)

$\therefore \Delta SPR \parallel\!| \Delta QPT$ (equiangular)

iv) (α) Since corresponding sides in $\parallel\!|$ Δ's are in proportion then

$$\frac{PS}{PQ} = \frac{PR}{PT}$$

ie $(PS)(PT) = (PR)(PQ)$

(β) also: $$\frac{PS}{PQ} = \frac{PR}{PT} = \frac{SR}{QT}$$ (corresponding sides proportional)

$$\therefore \frac{PS}{PQ} \cdot \frac{PR}{PT} = \frac{SR}{QT} \cdot \frac{SR}{QT}$$

$$\text{ie } \frac{PS}{PQ} \cdot \frac{PR}{PT} = \left(\frac{SR}{QT}\right)^2$$

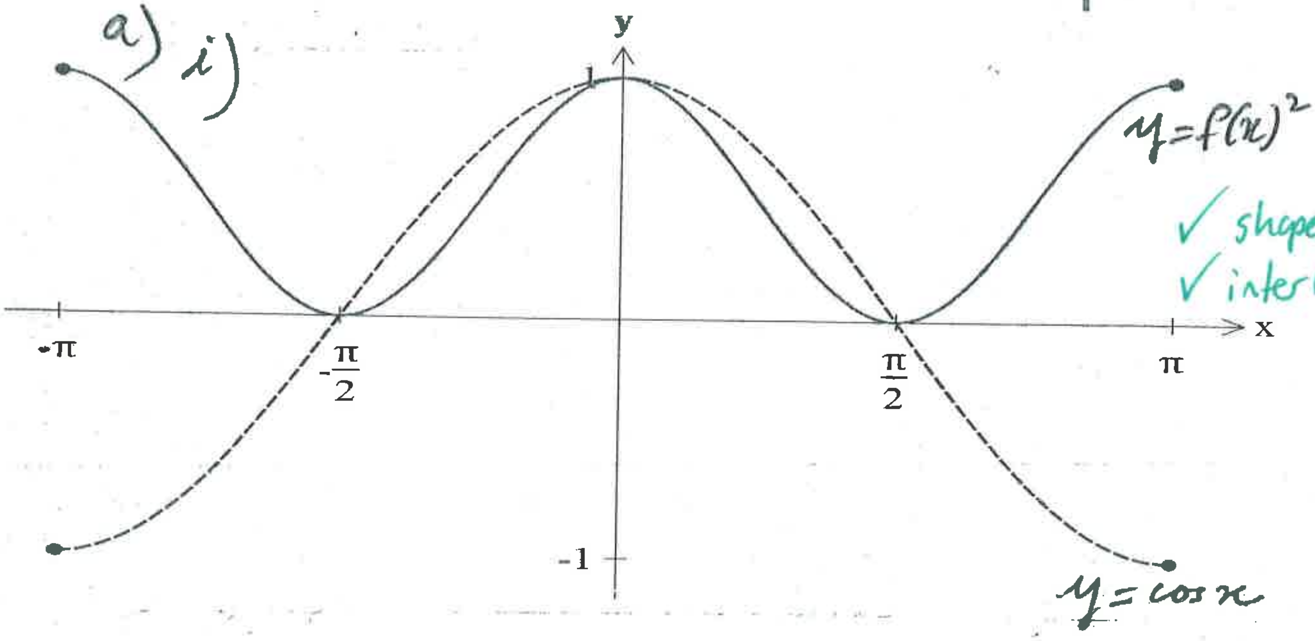
2015 MAXX Half-Yearly Task 3 Suggested solutions

* All graphs must be within the specified domain * .

Question 14

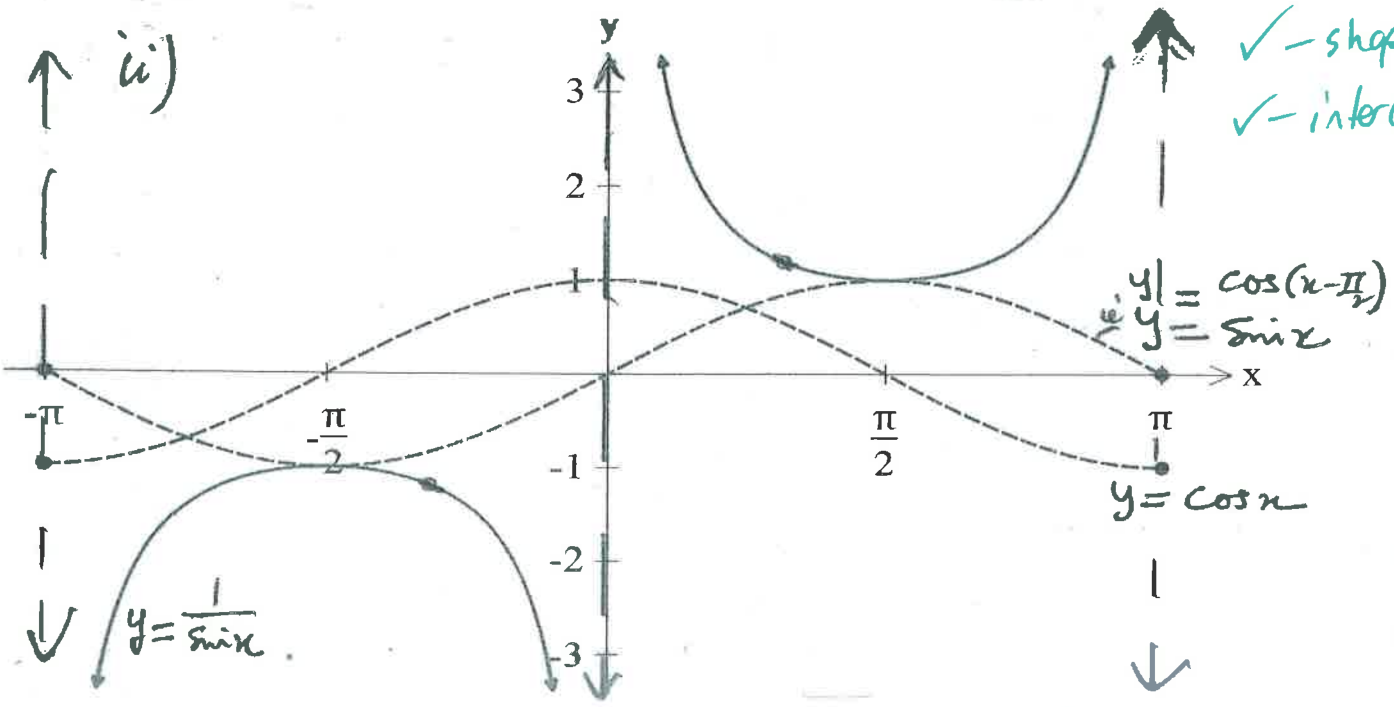
Marker comments

a) i)



✓ shape/position
✓ intercepts .

ii)



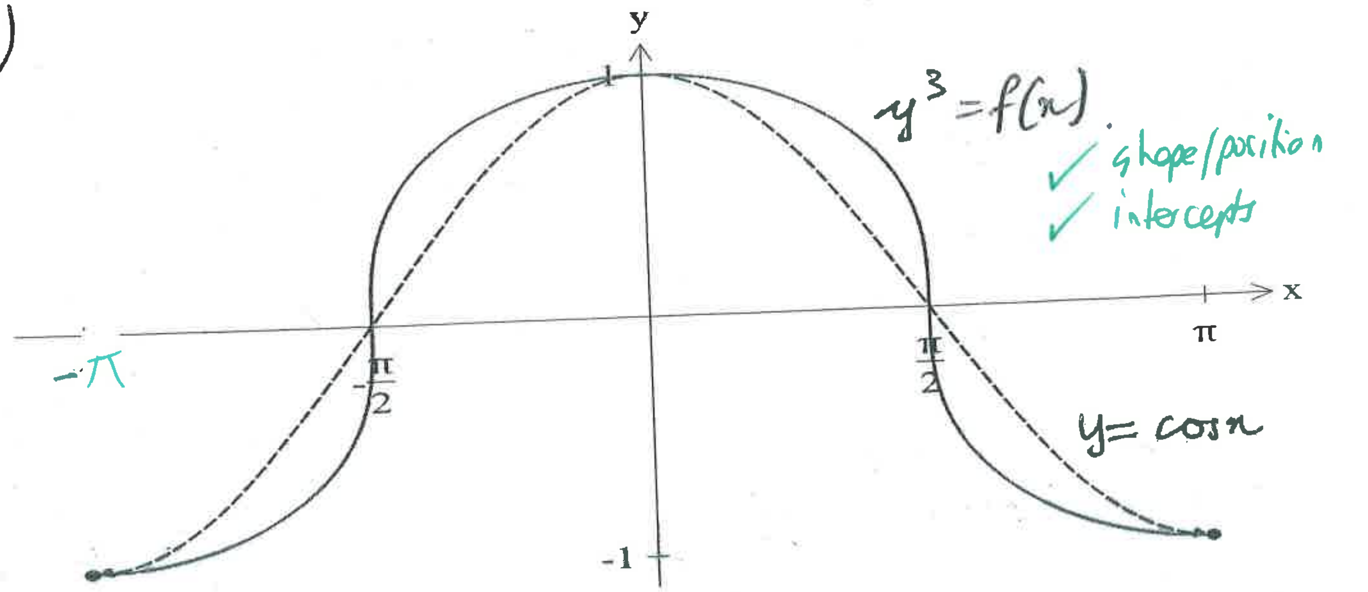
✓ – shape/position
✓ – intercepts

2015 MAXX Half-Yearly Task 3 Suggested solutions

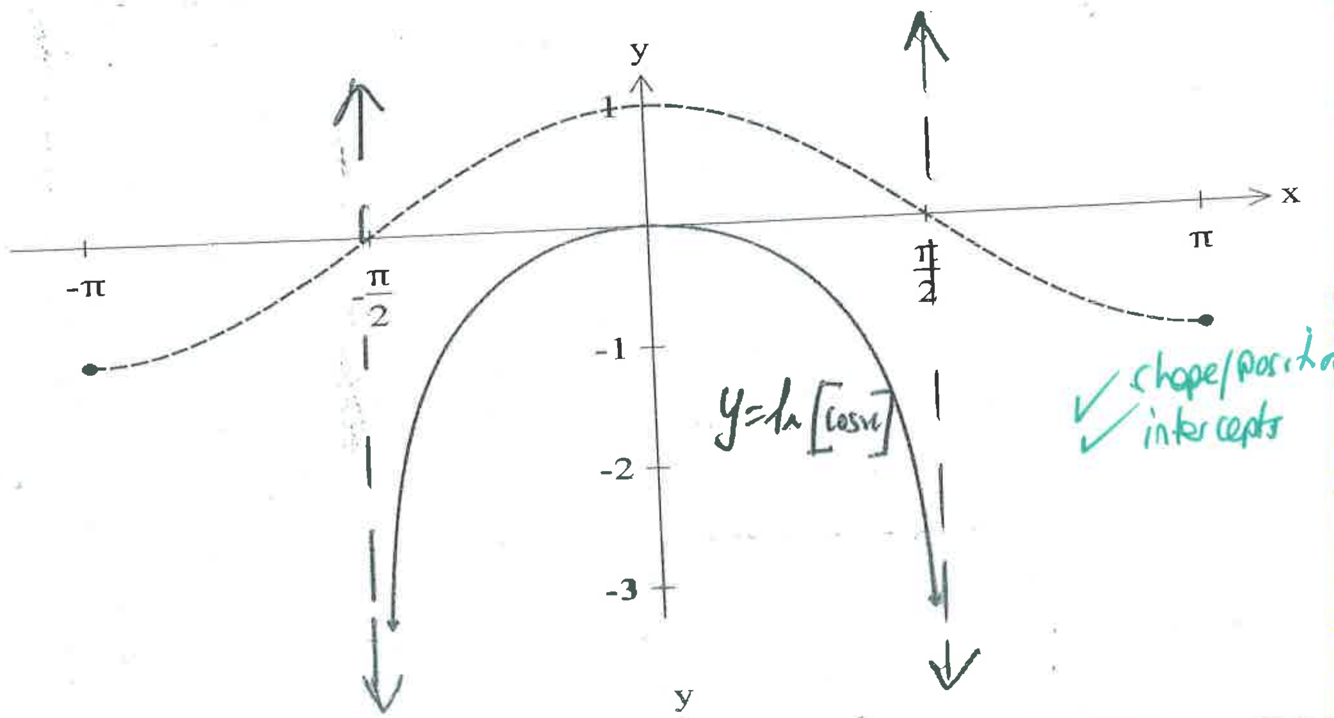
Question 14(a) ctd.

Marker comments

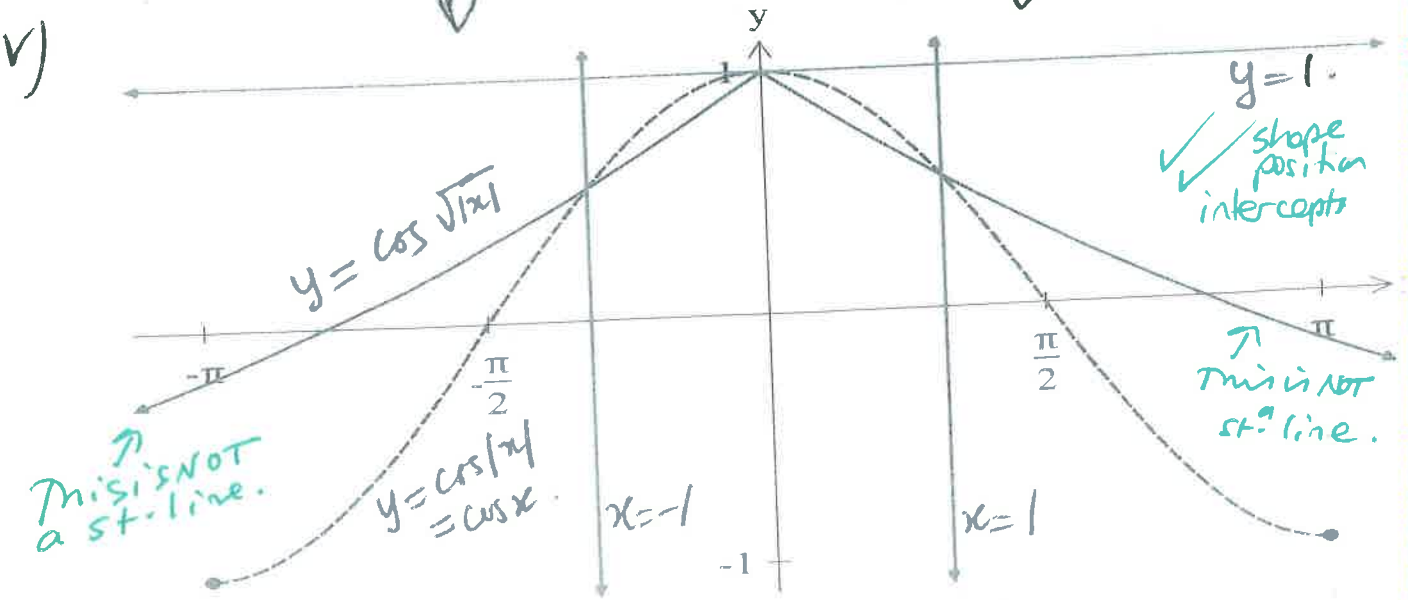
iii)



iv)



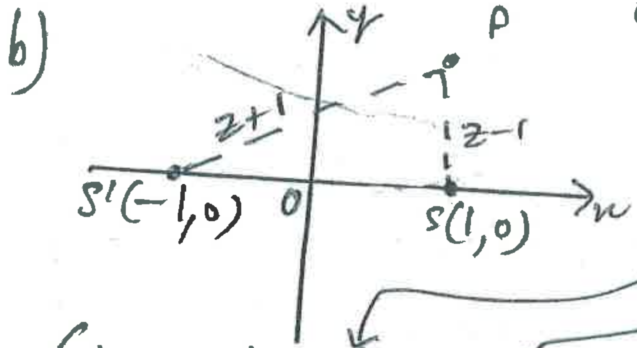
v)



2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 14(b)

b)



Given definition

$PS + PS' = 2a$

Marker comments

Given $|z-1| + |z+1| = 6$

$\therefore \quad 2a = 6$

$\Rightarrow a = 3$

we know $\quad ae = 1$

$\therefore e = \frac{1}{a}$

ie $\quad e = \frac{1}{3}$

Also $\quad b^2 = a^2(1-e^2) \quad [0<e<1]$

$\therefore \quad b^2 = 9\left(1 - \frac{1}{9}\right)$

$b^2 = 9 - 1$

$b^2 = 8$

hence $\quad \frac{x^2}{9} + \frac{y^2}{8} = 1$

[If the given definition was not used a maximum of 2 marks is awarded]

[Alternate: $|(x-1)+yi| + |(x+1)+yi| = 6$

$\Rightarrow \sqrt{(x-1)^2+y^2} + \sqrt{(x+1)^2+y^2} = 6 \quad$ (1st 36)

$\therefore \left(\sqrt{(x-1)^2+y^2} - 6\right)^2 = \left(-\sqrt{(x+1)^2+y^2}\right)^2$

$\Rightarrow \quad (9-x) = 3\sqrt{(x-1)^2+y^2}$

$\Rightarrow \quad 9x^2 - 18x + 9 + 9y^2 = 0 \quad$ etc....

2015 MAXX Half-Yearly Task 3 Suggested solutions

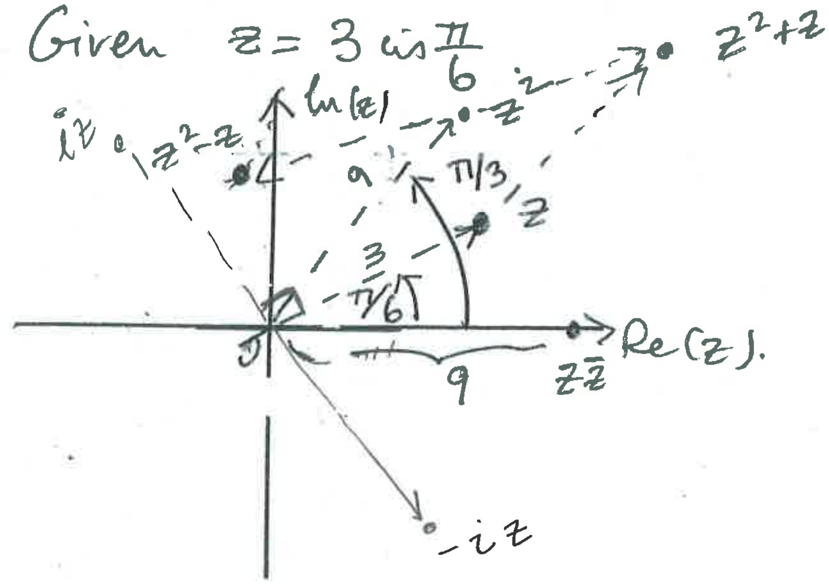
Question 15

Marker comments

✓ z
✓ $-i^{25}z$
✓ $z\bar{z}$
✓ z^2+z
✓ z^2-z

a) Given $z = 3\operatorname{cis}\frac{\pi}{6}$

i) (v)



$-i^{25}z = -iz$ [$\because 25 \div 4 = 6 \text{ r } 1$]

$iz = 3\operatorname{cis}\left(\frac{\pi}{6}+\frac{\pi}{2}\right) = 3\operatorname{cis}\left(\frac{2\pi}{3}\right)$

$\therefore -iz = 3\operatorname{cis}\left(-\frac{\pi}{3}\right) = \left(3\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)$

$z\bar{z} = |z|^2 = 3^2$
$= 9$ [or $9\operatorname{cis}0$].

iv) $z^2 = 9\operatorname{cis}\pi/3$

$\therefore$ z^2+z is drawn by vector addition
z^2-z see diagram

b) $|z-1| = \operatorname{Re}(z)$

$\Longrightarrow |(x-1)+yi| = x$

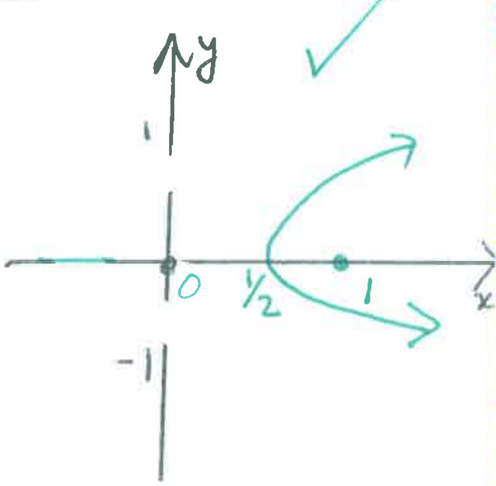
ie $(x-1)^2+y^2 = x^2$

$x^2-2x+1+y^2=x^2$

ie $y^2 = 2x-1$ (on expansion) ✓

$y^2 = 2(x-\frac{1}{2})$ or $y^2 = 4(\frac{1}{2})(x-\frac{1}{2})$

This is a parabola $V(\frac{1}{2},0)$ and $|a| = \frac{1}{2}$.



✓

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 15(c)

Marker comments

c) $H: \quad \frac{y^2}{4} - \frac{x^2}{12} = 1$

α) i) let $a^2 = 4$

$b^2 = 12$

$\therefore b^2 = a^2(e^2 - 1)$ hyperbola

$12 = 4(e^2 - 1)$

$e^2 = 4$

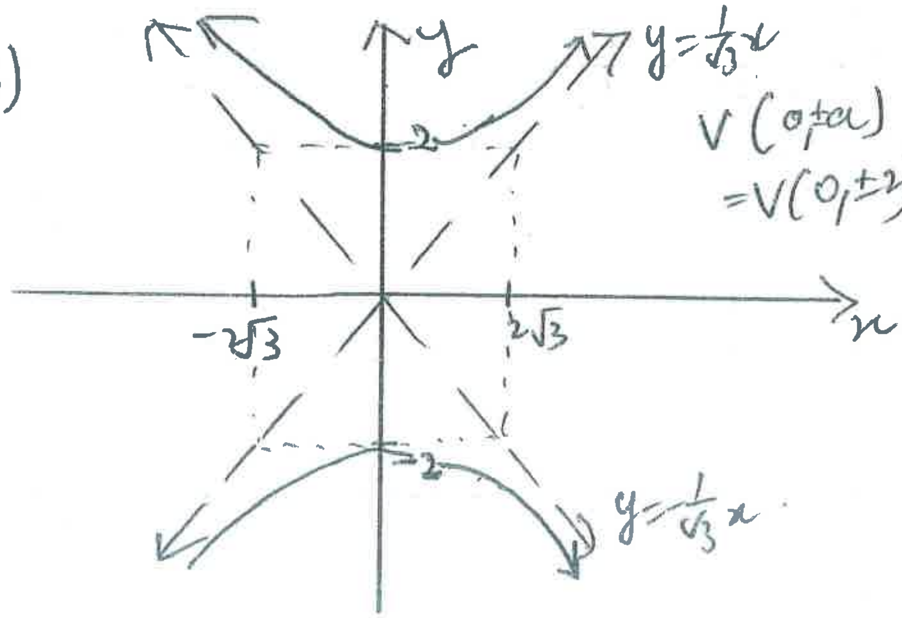
$\therefore e = 2$ ✓

ii) $S(0, \pm ae) = S(0, \pm 4)$ ✓

iii) $y = \pm \frac{a}{e} = \pm 1$ ✓

iv) $y = \pm \frac{a}{b}x = \pm \frac{1}{\sqrt{3}}x$ ✓

β)



$V(0, \pm a)$
$= V(0, \pm 2)$

✓ shape/position

✓ labelled intercepts/asymptotes.

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 15(d)

Marker comments

d) $\displaystyle\int_1^2 \frac{1}{x^2} e^{\frac{1}{x}}\,dx$

$= \displaystyle\int_1^2 -\frac{d}{dx}\left(\frac{1}{x}\right) e^{\frac{1}{x}}\,dx$

$= -e^{\frac{1}{x}}\Big|_1^2$

$= -\left(e^{\frac{1}{2}} - e^{1}\right)$

$= -\left(\sqrt{e} - e\right)$

$= e - \sqrt{e}.$

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 16.

Marker comments

a) $P = [a\cos\theta, b\sin\theta]$

i) $Q = [a\cos(\theta+\frac{\pi}{2}), b\sin(\theta+\frac{\pi}{2})]$

$= [a(\cos\theta\cos\frac{\pi}{2} - \sin\theta\sin\frac{\pi}{2}), b(\sin(\theta)\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos\theta)]$

$= [-a\sin\theta, b\cos\theta]$

No marks for substituting Q into E.

Only 1 mark for subst $(\theta+\pi/2)$ instead of θ and getting straight to solution

(OR)

$Q = (a\cos(\frac{\pi}{2}-(-\theta)), b\sin(\frac{\pi}{2}-(-\theta)))$

$= (a\sin(-\theta), b\cos(-\theta))$

$= (a\sin\theta, b\cos\theta)$.

ii) $LHS = OP^2 + OQ^2$

$= a^2\cos^2\theta + a^2\sin^2\theta + b^2\sin^2\theta + b^2\cos^2\theta$

$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$

$= a^2 \cdot 1 + b^2 \cdot 1$

$= a^2 + b^2$

$= RHS$

iii) $M = \left[\frac{-a\sin\theta + a\cos\theta}{2}, \frac{b\cos\theta + b\sin\theta}{2}\right]$

ie $M = \left[\frac{a}{2}(\cos\theta - \sin\theta), \frac{b}{2}(\cos\theta + \sin\theta)\right]$

needed to be substituted into equation

$LHS = \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2}{4a^2}(\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta) + \frac{b^2}{4b^2}(\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta)$

$= \frac{1}{4}[1 + 1 + 0]$

$= 2/4$

$= 1/2 = RHS$.

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 16 a) ctd.

Marker comments

iv) $T_P: \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ ①

v) $T_Q: -\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 1$ ②

Solving ① & ② for x, y

① (x) $-\frac{\sin\theta}{a} \Rightarrow \frac{-x}{a^2}\sin\theta\cos\theta - \frac{y\sin^2\theta}{ab} = \frac{-\sin\theta}{a}$ ③

② (x) $\frac{\cos\theta}{a} \Rightarrow -\frac{x\sin\theta\cos\theta}{a^2} + \frac{y\cos^2\theta}{ab} = \frac{\cos\theta}{a}$ ④

④ − ③ $\frac{y}{ab}\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1} = \frac{1}{a}(\cos\theta+\sin\theta)$

$\therefore y = b(\cos\theta + \sin\theta)$ ⑤

Sub ⑤ into ①

$\frac{x}{a}\cos\theta + \frac{b(\cos\theta+\sin\theta)\sin\theta}{b} = 1$

$\frac{x\cos\theta}{a} + \cos\theta\sin\theta + \sin^2\theta = 1$

$x\cos\theta = a(1-\sin^2\theta - \cos\theta\sin\theta)$

$x\cos\theta = a(\cos^2\theta - \cos\theta\sin\theta)$

$x\cos\theta = a(\cos\theta)(\cos\theta - \sin\theta)$

$\therefore x = a(\cos\theta - \sin\theta)$

$\therefore$ The point of intersection T is

$[a(\cos\theta - \sin\theta),\ b(\cos\theta + \sin\theta)]$

Question 16(a) ctd

Marker comments

vi) $LHS = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$= \frac{a^2}{a^2}(\cos\theta - \sin\theta)^2 + \frac{b^2}{b^2}(\sin\theta + \cos\theta)^2$

$= \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta + \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$

$= 2(\cos^2\theta + \sin^2\theta)$

$= 2 \times 1$

$= 2$

$= RHS$

---

b) Given $1+2+3+\dots+k = \frac{k(k+1)}{2}$ ①

let $k = 15n$ from ①

$\therefore \sum_{k=1}^{15n} k = \frac{15n(15n+1)}{2}$ ——— ②

[This is the sum of all integers from $1 \longrightarrow 15n$ inclusive].

We must now eliminate from this sum the sum of integers divisible by 3 or 5.

By 3: $3+6+9+\dots+15n$ ← since $15n$ is ÷ by 3

Number of terms given by $T_k = a+(k-1)d$

$15n = 3+(k-1)3$

$\therefore k = 5n$ (terms)

2015 MAXX Half-Yearly Task 3 Suggested solutions

Question 16(b) ctd.

Marker comments

$\therefore\ 3+6+9+\cdots+15n$

$= 3(1+2+3+\cdots+5n)$

$= 3\sum_{k=1}^{5n} k$

$= 3\left(\frac{5n(5n+1)}{2}\right)$

$= \frac{15n(5n+1)}{2}$ —— (3)

Similarly the sum of the integers ÷ by 5 are:

$5+10+\cdots+15n$

$= 5(1+2+3+\cdots+3n)$ (3n terms)

$= 5\times\sum_{k=1}^{3n} k$

$= 5\left(\frac{3n(3n+1)}{2}\right)$

$= \frac{15n(3n+1)}{2}$ —— (4)

However, by subtracting (3) & (4) from (2) results in removing that are divisible by 15 twice so, we need to add one set back in.

So, $15+30+45+\cdots+15n$

$= 15(1+2+3+\cdots n)$

$= 15\sum_{k=1}^{n} k$

Question 16(b) | Marker comments

$$= \frac{15n(n+1)}{2} \quad \text{——} \quad (5)$$

Hence, the sum of the integers which are not divisible by 3 or 5 is given by

$$\left[\frac{15n(15n+1)}{2}\right] - \left[\frac{15n(3n+1)}{2}\right] - \left[\frac{15n(5n+1)}{2}\right] + \left[\frac{15n(n+1)}{2}\right]$$

$$= \frac{15n}{2}\left[15n+1-3n-1-5n-1+n+1\right]$$

$$= \frac{15n}{2}(8n)$$

$$= 60n^2.$$

---

c) i) $\sin(x-\theta) = k\sin(x+\theta)$

$$\sin x\cos\theta - \sin\theta\cos x = k(\sin x\cos\theta + \sin\theta\cos x)$$

$$\sin x\cos\theta - k\sin x\cos\theta = k\sin\theta\cos x + \sin\theta\cos x$$

$$\sin x\cos\theta(1-k) = \sin\theta\cos x(k+1)$$

$$\Rightarrow \frac{\sin x}{\cos x} = \frac{\sin\theta}{\cos\theta}\,\frac{(k+1)}{(1-k)}$$

ie $\tan x = \tan\theta\,\dfrac{(k+1)}{-(k-1)}$

ie $\tan x = -\dfrac{(k+1)}{k-1}\tan\theta.$

Question 16(c) ctd.

Marker comments

ii) $k = \frac{1}{2}$, $\theta = 210°$

$\therefore \tan x = \dfrac{-\left(\frac{3}{2}\right) \tan 210°}{\left(-\frac{1}{2}\right)}$ ✓

Substitution needed.

$\tan x = 3 \times \tan 30°$

$\tan x = \dfrac{3}{\sqrt{3}}$

$= \dfrac{3}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$\tan x = \sqrt{3}$

$\Rightarrow x = 60°, 240°$. ✓

(Radians not accepted).