Trinity Grammar School

Mathematics Department

2016

HALF-YEARLY
EXAMINATION

HSC ASSESSMENT TASK 3

Year 12

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- BOSTES calculators for this course are allowed in this task
- A Reference Sheet will be supplied
- In Questions 11 – 16 show all necessary mathematical reasoning
- Write your BOSTES Student Number (Year 12 HSC)) and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating **N/A** and your Student Number or Name.

- **HSC Assessment Weighting:** 30%

BOSTES Student Number
(Year 12 only)

Class Teacher:

……………………………………………

Name:

……………………………………………

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

Date of examination:

Thursday, 28 April 2016

Total marks – 100

Section I

10 marks

- Attempt Questions 1-10
- Allow about **15** minutes for this section

Section II

90 marks

- Attempt Questions 11 - 16
- Allow about **2** hours **45** minutes for this section

Section I **10 marks**

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section
- Shade the correct response on the multiple-choice answer sheet for Questions 1 – **10**
- Each question is worth 1 mark

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1 If $w = 2cis\frac{\pi}{3}$ and $z = 4cis\frac{\pi}{3}$, what is the value of wz?

(A) $-4(1+i\sqrt{3})$

(B) $-4(1-i\sqrt{3})$

(C) $4(1+i\sqrt{3})$

(D) $-4(\sqrt{3}+i)$

2 The polynomial $P(x) = 2x^3 + 5x^2 + 6x + 2$ has factors $2x+1$ and $x+1-i$. What is the third factor of $P(x)$?

(A) $2x-1$

(B) $x-1+i$

(C) $x+1+i$

(D) $x-1-i$

3 Let $z = 4+i$ and $w = 3-2i$.

What is the value of $\bar{z}w$?

(A) $10-11i$

(B) $10+11i$

(C) $14-11i$

(D) $14+11i$

4 Which of the following is an expression for $\int \cot x\, dx$?

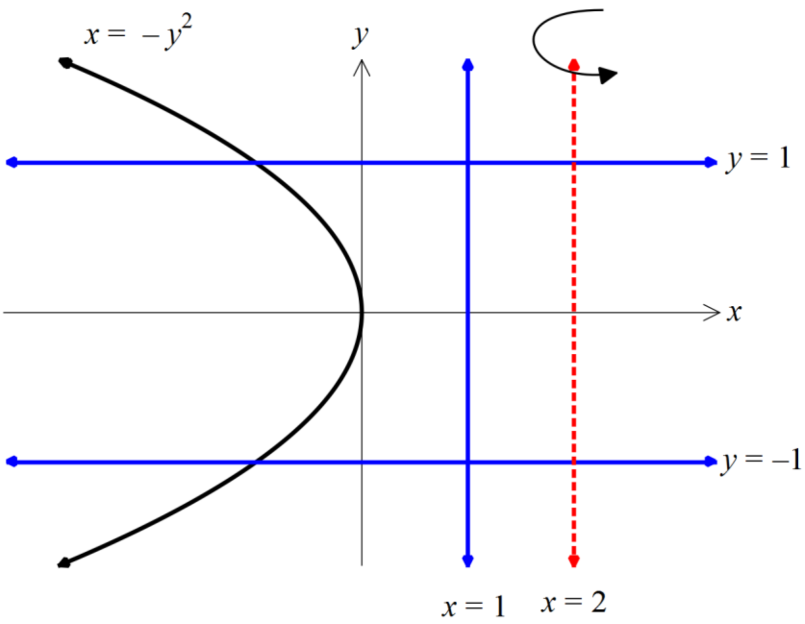
(A) $\ln(\tan x) + c$

(B) $\ln(\cos x) + c$

(C) $\ln(\sin x) + c$

(D) $\ln(\cot x) + c$

5 The region bounded by the lines $x = 1$, $y = 1$, $y = -1$ and by the curve $x = -y^2$. The region is rotated through 360° about the line $x = 2$ to form a solid. What is the correct expression for the volume of this solid?



(A) $V = \int_{-1}^{1} \pi\left(y^4 - 4y^2 + 3\right)dy$

(B) $V = \int_{-1}^{1} \pi\left(y^4 + 4y^2 + 3\right)dy$

(C) $V = \int_{-1}^{1} \pi\left(y^4 - 4y^2 + 4\right)dy$

(D) $V = \int_{-1}^{1} \pi\left(y^4 + 4y^2 + 4\right)dy$

6 The polynomial $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$ has $x = 1$ as a root of multiplicity 3 and $x = i$ as a root. Which of the following expressions is a factorised form of $P(x)$ over the complex numbers?

(A) $P(x) = (x+1)^3(x-1)(x+1)$

(B) $P(x) = (x-1)^3(x-1)(x+1)$

(C) $P(x) = (x+1)^3(x-i)(x+i)$

(D) $P(x) = (x-1)^3(x-i)(x+i)$

7 If $f(x) = 2x^2 \cos^{-1} 2x$, then $f'(x) =$

(A) $\dfrac{-8x}{\sqrt{1-2x^2}}$

(B) $\dfrac{-8x}{\sqrt{1-4x^2}}$

(C) $\dfrac{-4x^2}{\sqrt{1-2x^2}} + 4x\cos^{-1} 2x$

(D) $-\dfrac{4x^2}{\sqrt{1-4x^2}} + 4x\cos^{-1} 2x$

8 Which of the following is an expression for $\int \dfrac{e^x}{1+e^{2x}}\,dx$?

(A) $e^x \tan^{-1} e^x + c$

(B) $e^x \tan^{-1} e^{2x} + c$

(C) $\tan^{-1} e^x + c$

(D) $\tan^{-1} e^{2x} + c$

9 The polynomial equation $x^3 - 3x^2 - x + 2 = 0$ has roots $\alpha,\ \beta$ and γ.
Which of the following polynomial equations have roots $2\alpha + \beta + \gamma$, $\alpha + 2\beta + \gamma$ and $\alpha + \beta + 2\gamma$?

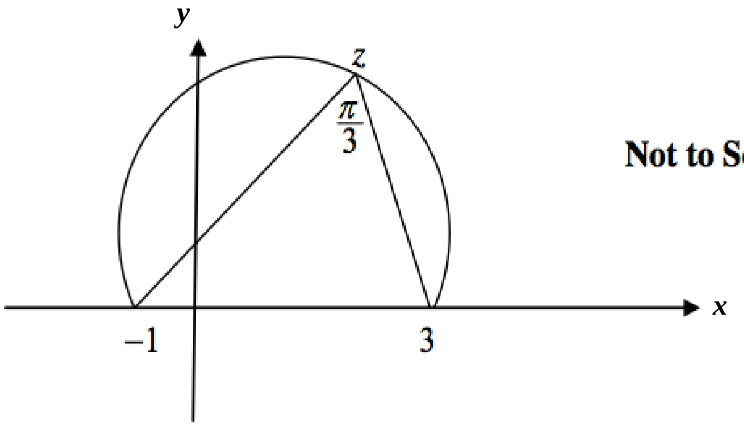
(A) $x^3 - 12x^2 + 44x - 49 = 0$

(B) $x^3 - 6x^2 + 44x - 49 = 0$

(C) $x^3 + 3x^2 + 36x + 5 = 0$

(D) $x^3 + 6x^2 + 36x + 5 = 0$

10



The diagram shows the locus of points z in the complex number plane such that $\arg(z-3) - \arg(z+1) = \dfrac{\pi}{3}$. This locus forms part of a circle. The angle between the lines from -1 to z and from 3 to z is $\dfrac{\pi}{3}$ as shown. The coordinates of the centre of the circle are?

(A) $\left(1, \dfrac{1}{\sqrt{3}}\right)$

(B) $\left(1, \dfrac{2}{\sqrt{3}}\right)$

(C) $(1, 0)$

(D) $(1, 1)$

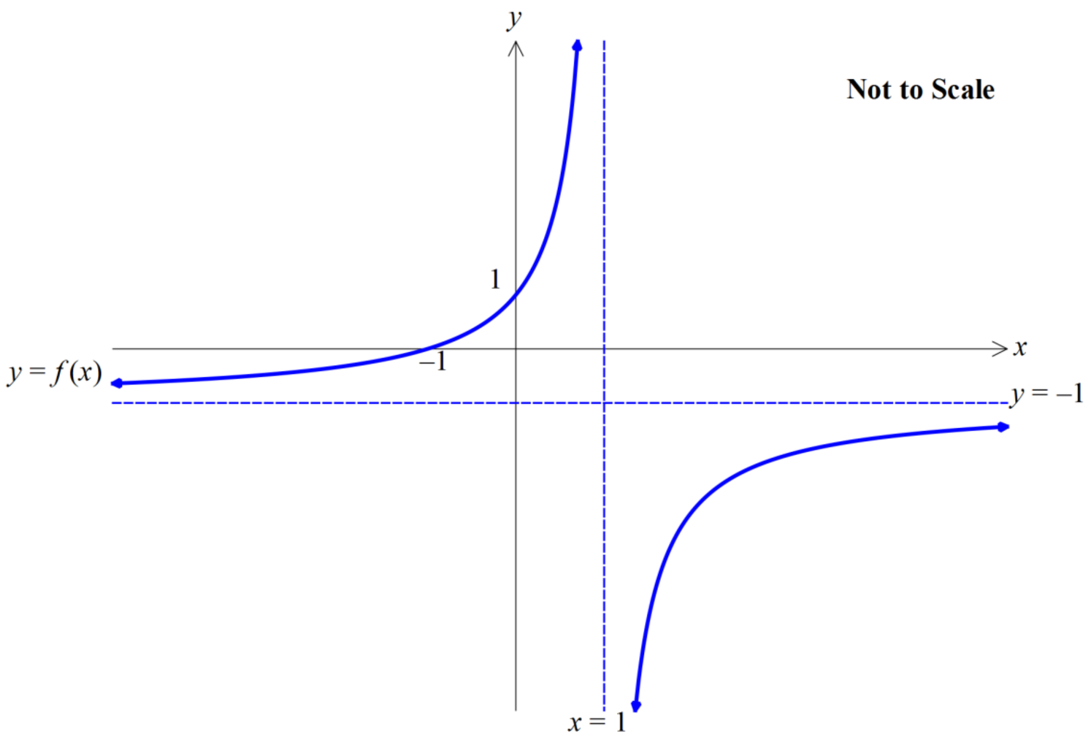
End of Section I
Section II commences on the next page

Section II **90 marks**

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section
- Answer each question in a SEPARATE writing booklet or answer sheet. Extra writing booklets are available.
- In Questions 11 – 16, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 **(15 marks)** Use a SEPARATE writing booklet.

(a) The graph for $y = f(x)$ is shown below.



On the separate answer sheet provided at the end of the question paper, sketch the graphs of the following:

(i) $y^2 = f(x)$ **2**

(ii) $|y| = f(x)$ **2**

(iii) $|y| = f(|x|)$ **2**

Question 11 continues on the next page

Question 11 continued

(b) If $z = 2 - i$,

(i) Find $\dfrac{1}{z}$ in the form $a + bi$. **1**

(ii) Find the real numbers p and q such that $pz + \dfrac{q}{z} = 1$. **2**

(c) Use De Moivre's theorem to evaluate $(1+i)^7 + (1-i)^7$. **3**

(d) If $|\, z - 2 - i \,| = 2$, find the maximum **and** the minimum values of $|\, z \,|$. **3**

End of Question 11

Question 12 commences on the next page

Question 12 **(15 marks)** Use a SEPARATE writing booklet.

(a) Sketch the locus of z for each of the following:

(i) $2(z+\bar{z})-5i(z-\bar{z})=20$ **2**

(ii) $\frac{\pi}{4} \le \arg\left(\frac{1}{\bar{z}}\right) \le \frac{\pi}{2}$ **2**

(b) If $1, \omega, \omega^2$ are the three cube roots of 1, evaluate

(i) $(1+\omega)^9$ **3**

(ii) $\omega^2+\omega+1+\frac{1}{\omega}+\frac{1}{\omega^2}$. **2**

(c) (i) If k is real, over $\mathbb{R}$, explain why $x^5+k=0$ must have one real root, but $x^6+k=0$ need not have any real roots. **2**

(ii) Find the roots of $x^6+1=0$ over $\mathbb{C}$ **2**

(iii) Factorise x^6+1 over the real numbers. **2**

End of Question 12

Question 13 commences on the next page

Question 13 **(15 marks)** Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int x\sin(x^2+3)\,dx$ **2**

(b) Using integration by parts, evaluate $\displaystyle\int_0^{\frac{1}{2}} \cos^{-1} x\,dx$ **3**

(c) Using the substitution $u^2 = e^x + 1$, find $\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx$ **4**

(d) Find $\displaystyle\int \frac{dx}{4x^2-1}$ **3**

(e) Evaluate $\displaystyle\int_3^4 \frac{x^2+x-4}{x-2}\,dx$ **3**

End of Question 13

Question 14 commences on the next page

Question 14 **(15 marks)** Use a SEPARATE writing booklet.

(a) The curves $y = k - x^2$ and $y = \dfrac{1}{x}$ intersect at the points P, Q and R where $x = \alpha$, β and γ respectively.

(i) Find the value of $\alpha + \beta + \gamma$. **2**

(ii) Show that the monic cubic equation with roots α^2, β^2 and γ^2 is $x^3 - 2kx^2 + k^2x - 1 = 0$ **3**

(iii) Hence or otherwise find the monic cubic equation in terms of k, with roots $\dfrac{1}{\alpha^2}$, $\dfrac{1}{\beta^2}$ and $\dfrac{1}{\gamma^2}$. **3**

(b) (i) Calculate $\lim\limits_{x \to 2}\left(\dfrac{x-2}{x^2-4}\right)$ **1**

(ii) Sketch the graph of $y = \dfrac{x-2}{x^2-4}$. indicating any special features. **2**

(c) A solid has as its base as the parabola $y = x^2$ in the XY plane with $0 \le y \le 9$.

Cross sections taken perpendicular to the Y axis are squares.

Determine the volume of this solid. **4**

End of Question 14

Question 15 commences on the next page

Question 15 **(15 marks)** Use a SEPARATE writing booklet.

(a) For the curve $y = \cos^{-1}(e^x)$

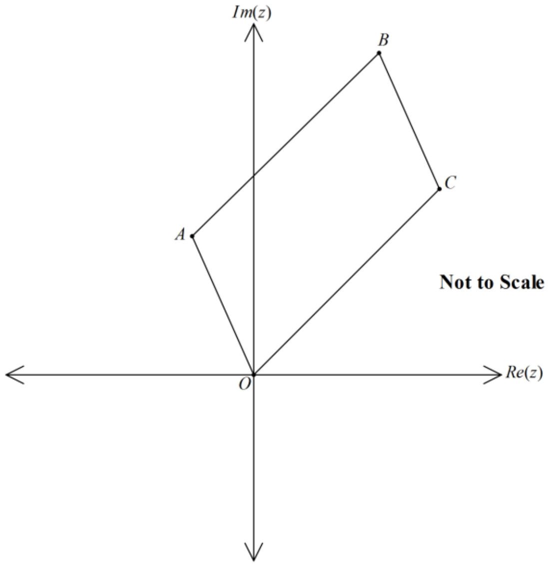
(i) State the domain of the function. **1**

(ii) Find any intercepts, equations of vertical tangents **and** asymptotes. **2**

(iii) Sketch the graph showing the above features. **1**

(iv) State the range of $y = \cos^{-1}(e^x)$. **1**

(b)



In the diagram, $OABC$ is a parallelogram with $|OA| = \frac{1}{2}|OC|$.

The point A represents the complex number $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

If $\angle AOC = 60^\circ$, what complex number does C represent?
Give your answer in the form $x + yi$. **3**

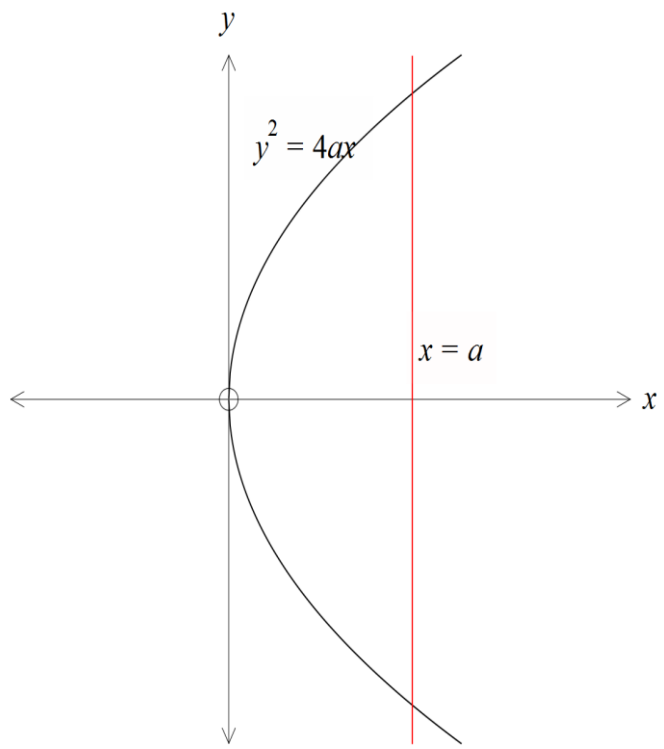
(c) (i) Simplify $\cos(A - B) - \cos(A + B)$. **1**

(ii) Hence find $\int \sin 6x \sin 2x \, dx$. **2**

Question 15 continues on the next page

Question 15 continued

(d) A solid is formed by rotating the region enclosed by the parabola $y^2 = 4ax$, its vertex (0,0) and the line $x = a$, about the x axis as shown in the diagram.



By using the method of cylindrical shells, prove that the volume of the solid formed is given by $V = 2\pi a^3$ cubic units. **4**

End of Question 15

Question 16 commences on the next page

Question 16 **(15 marks)** Use a SEPARATE writing booklet.

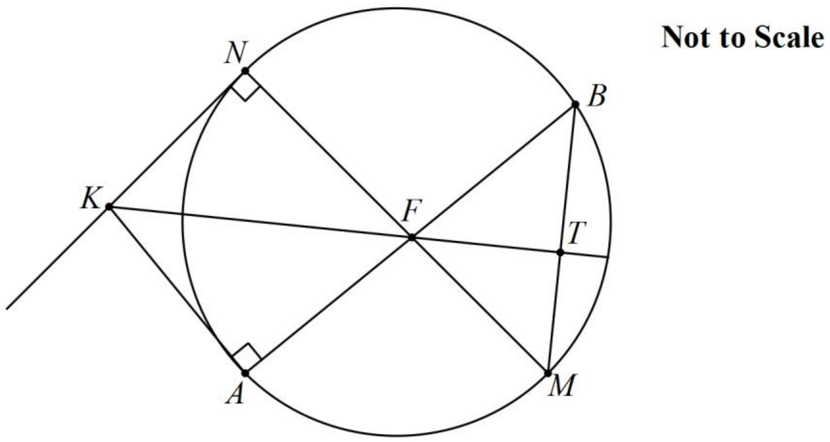
(a) Find the solution to the equation $\tan^{-1}(4x) - \tan^{-1}(3x) = \tan^{-1}\left(\frac{1}{7}\right)$. **3**

(b) (i) Let $I_n = \int \cos^n x \, dx$ **4**

Prove that $I_n = \frac{1}{n}\sin x \cos^{n-1} x + \left(\frac{n-1}{n}\right) I_{n-2}$.

(ii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$. **4**

(c) A circle has two chords AB and MN intersecting internally at F. Perpendiculars are drawn to these chords at A and at N intersecting externally at K. KF produced meets MB at T as shown in the diagram below.



Copy or trace this diagram into your writing booklet.

Prove, with reasons, that KT is perpendicular to MB. (Hint: Join AN). **4**

End of Section II

End of Examination

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Task # 3 – 2016 Half Yearly

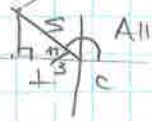
Extension 2

Multiple Choice:

1.) $w = 2\text{cis}\frac{\pi}{3}$ $\quad z = 4\text{cis}\frac{\pi}{3}$

$wz = \left(2\text{cis}\frac{\pi}{3}\right)\left(4\text{cis}\frac{\pi}{3}\right)$

$= 8\text{cis}\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$



$= 8\text{cis}\frac{2\pi}{3}$

$= 8\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$= 8\left[\left(-\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)\right]$

$= -4 + i4\sqrt{3}$

$= -4\left(1 - i\sqrt{3}\right)$ B

2.) $P(x) = 2x^3 + 5x^2 + 6x + 2$

$= (2x+1)\left(x - (-1+i)\right)\underbrace{\left(x - (-1-i)\right)}_{\text{conjugate}}$

Real coefficients

$x + 1 + i$

C

3.) $z = 4 + i$ $\quad \bar{z} = 4 - i$

$w = 3 - 2i$

$(4 - i)(3 - 2i)$

$= 12 - 8i - 3i + 2(-1)$

$= 10 - 11i$ A

4.) $\int \cot x \, dx$

$= \int \frac{\cos x}{\sin x} \, dx$

$= \ln|\sin x| + C$ C

5.) Area of the slice is an annulus. Inner radius is 1 and outer radius is $2 + y^2$ and height y.

$\delta A = \pi(R^2 - r^2)$

$= \pi\left((2 - y^2)^2 - (1)^2\right)$

$= \pi\left(4 - 4y^2 + y^4 - 1\right)$

$= \pi\left(y^4 - 4y^2 + 3\right)$

$\delta V = \delta A \cdot \delta y$

$V = \int_{-1}^{1} \pi\left(y^4 - 4y^2 + 3\right) \delta y$

A

6.) $P(x) = (x-1)^3(x-i)(x+i)$

$x = 1$ multiplicity 3 and $x = i$ is a root

D

7.) $f(x) = 2x^2\cos^{-1}2x$

$f'(x) = 2x^2\left(\frac{-1}{\sqrt{\frac{1}{4} - x^2}}\right) + 4x\cos^{-1}2x$

$= -\frac{4x^2}{\sqrt{1 - 4x^2}} + 4x\cos^{-1}2x$

D

8.) $\int \frac{e^x}{1 + e^{2x}} \, dx$

$= \tan^{-1}(e^x) + c$ C.

9.) The equation $x^3 - 3x^2 - x + 2 = 0$

$\alpha + \beta + \gamma = 3$ $\therefore$ $x = 2\alpha + \beta + \gamma = \alpha + 3$

$x = \alpha + 2\beta + \gamma = \beta + 3$

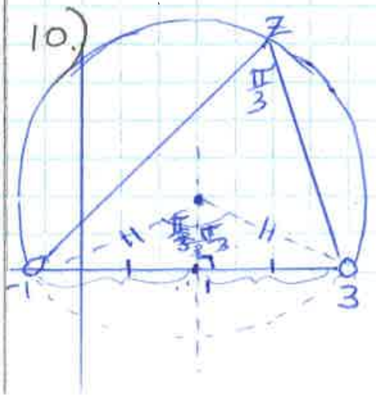
$x = \alpha + \beta + 2\gamma = \gamma + 3$

$\therefore \alpha = x - 3$

$\therefore (x-3)^3 - 3(x-3)^2 - (x-3) + 2 = 0$

$x^3 - 12x^2 - 44x + 49 = 0$ A

10.)



$\frac{2}{y} = \tan\frac{\pi}{3}$

$y = \frac{2}{\tan\frac{\pi}{3}}$

$y = \frac{2}{\sqrt{3}}$

$\therefore \left(1, \frac{2}{\sqrt{3}}\right)$ B

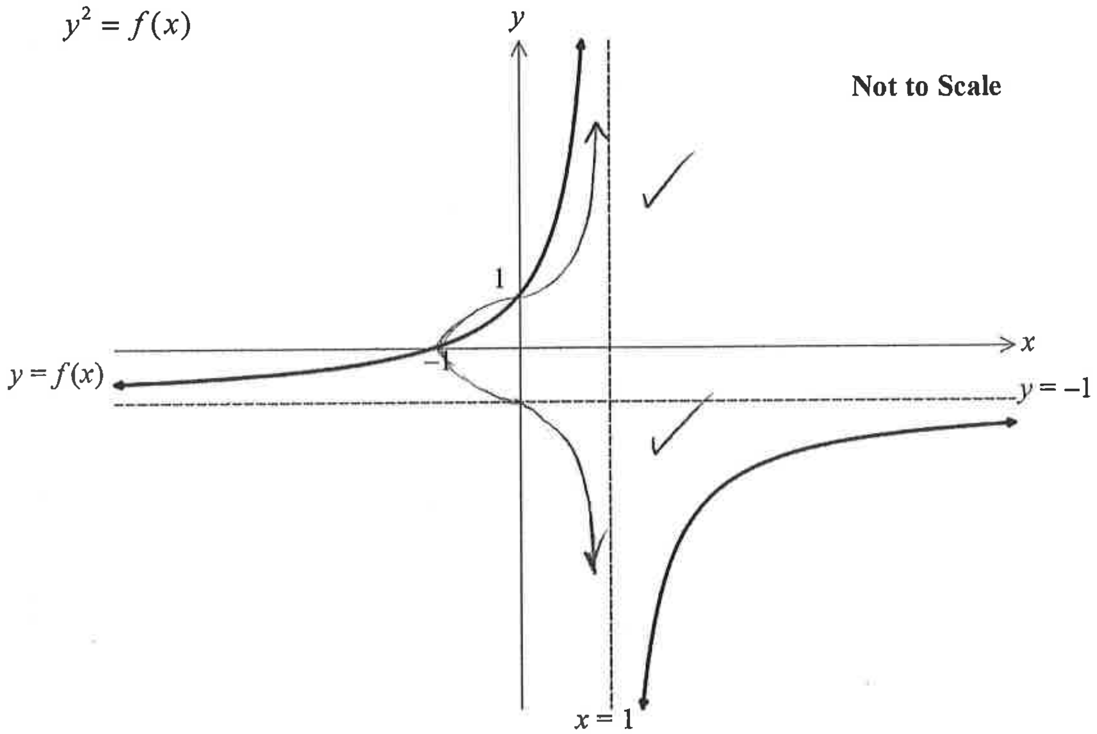
Trinity Grammar School
2016 Year 12 Mathematics Extension 2
Task 3 Half Yearly Examination

Answer sheet: **Question 11a**

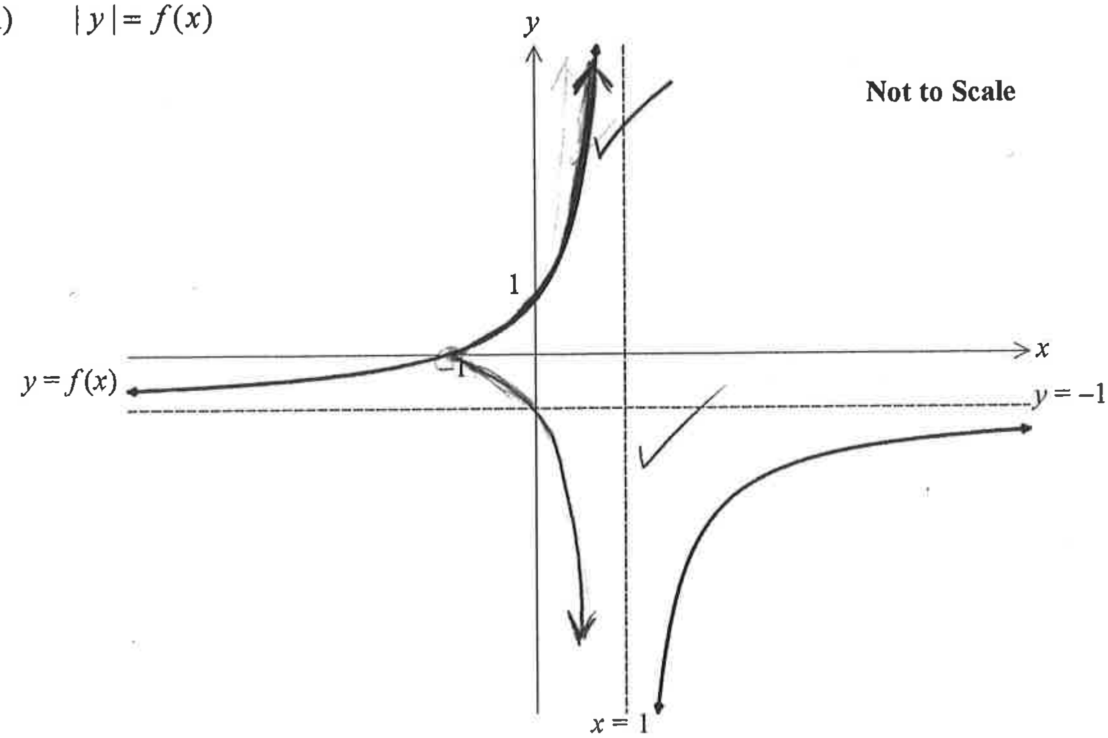
Teacher: ______________________________

BOSTES Number: ______________________________

(i) $y^2 = f(x)$ **2**

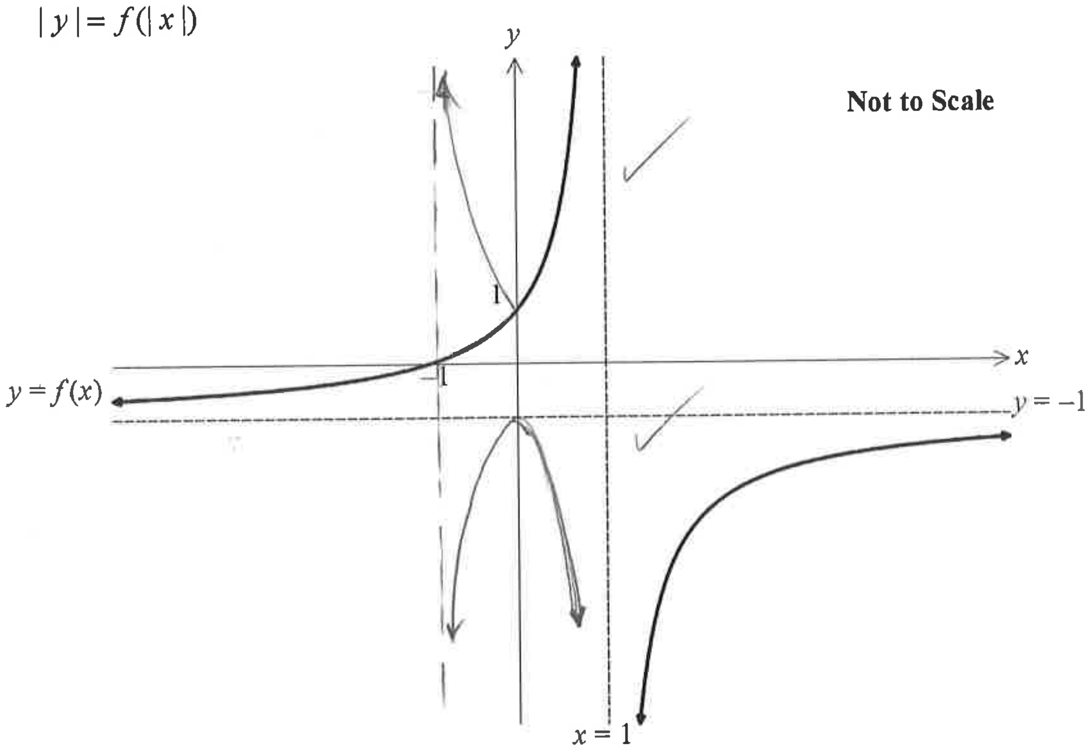


(ii) $|y| = f(x)$ **2**



(Question 11a continued)

(iii) $|y| = f(|x|)$ **2**



End of Question 11a

YOU MUST INSERT THIS ANSWER SHEET INTO YOUR WRITING BOOKLET FOR QUESTION 11

Question 11 continued...

b) $z = 2 - i$

i) $\dfrac{1}{z} = \dfrac{1}{2-i} \times \dfrac{2+i}{2+i}$

$= \dfrac{2+i}{4+1}$

$= \dfrac{2}{5} + \dfrac{1}{5}i$ (1)

ii) $pz + \dfrac{q}{z} = 1$

$p(2-i) + \dfrac{q}{2-i}$

$= 2p - pi + q\left(\dfrac{2}{5} + \dfrac{i}{5}\right)$

$= 2p + \dfrac{2q}{5} + \dfrac{q}{5}i - pi$

but $pz + \dfrac{q}{z} = 1 + 0i$

$\therefore\ 2p + \dfrac{2q}{5} = 1$ and $\dfrac{q}{5} - p = 0$

$\therefore p = \dfrac{q}{5}$

$2\left(\dfrac{q}{5}\right) + \dfrac{2q}{5} = 1$

$\dfrac{4q}{5} = 1$

$q = \dfrac{5}{4}$

$p = \dfrac{1}{4}$ (2)

c) $(1+i) = \sqrt{2}\,\text{cis}\,\dfrac{\pi}{4}$

$(1-i) = \sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right)$

$(1+i)^7 + (1-i)^7$

$= (\sqrt{2})^7\,\text{cis}\,\dfrac{7\pi}{4} + (\sqrt{2})^7\,\text{cis}\left(-\dfrac{7\pi}{4}\right)$

$= (\sqrt{2})^7\left[2\cos\dfrac{7\pi}{4}\right]$

NOTE: $\cos\left(-\dfrac{7\pi}{4}\right) = \cos\dfrac{7\pi}{4}$

$\sin\left(-\dfrac{7\pi}{4}\right) = -\sin\dfrac{7\pi}{4}$

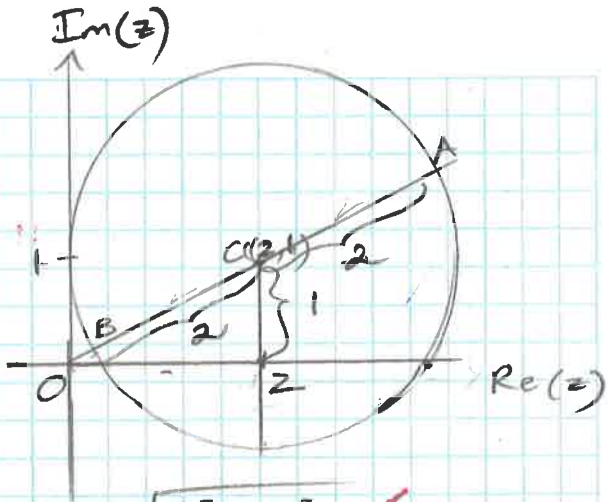
$= 16\sqrt{2}\left(\cos\dfrac{7\pi}{4}\right)$

$= 16\sqrt{2}\left(\cos\left(2\pi - \dfrac{\pi}{4}\right)\right)$

$= 16\sqrt{2}\left[\cos\dfrac{\pi}{4}\right]$

$= 16\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = 16$ (3)

d)



$OC = \sqrt{1^2 + 2^2}$

$= \sqrt{5}$

At A : max $= \sqrt{5} + 2$

At B : min $= \sqrt{5} - 2$

(3)

Question 12:

a)i) If $z = x + iy$
$\bar{z} = x - iy$

$z + \bar{z} = 2x$
$z - \bar{z} = 2iy$

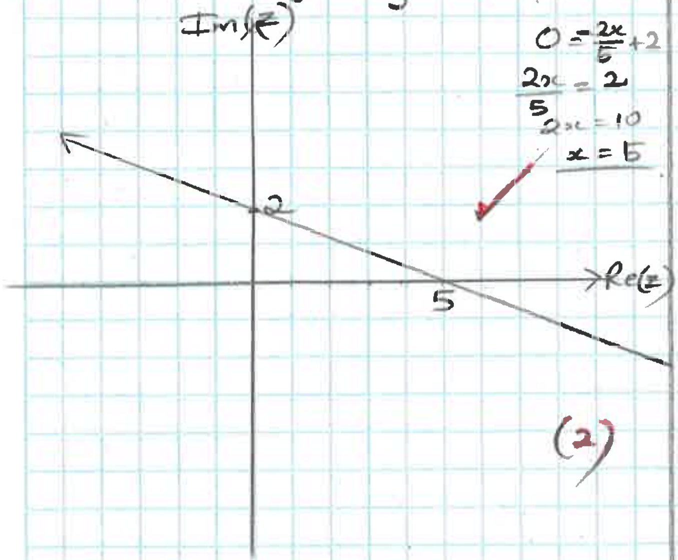
$2(z+\bar{z}) - 5i(z-\bar{z}) = 20$

$4x + 10y = 20$

$2x + 5y = 10$
$y = 2 - \frac{2x}{5}$

$y = \frac{-2x}{5} + 2$

$0 = \frac{-2x}{5} + 2$
$\frac{2x}{5} = 2$
$2x = 10$
$x = 5$



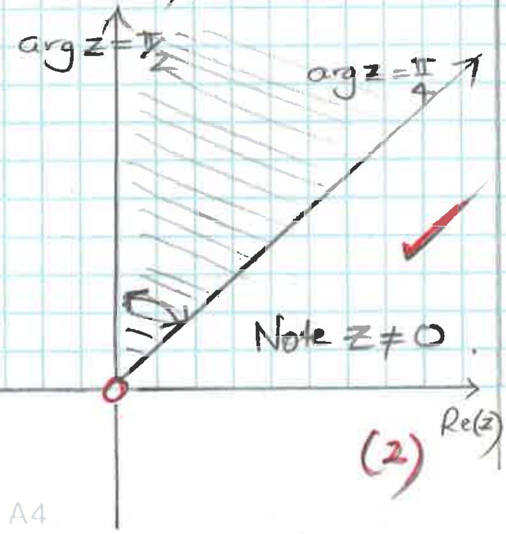
(2)

a)ii) $\frac{\pi}{4} \le \arg\left(\frac{1}{\bar{z}}\right) \le \frac{\pi}{2}$

$\frac{\pi}{4} \le \arg 1 - \arg \bar{z} \le \frac{\pi}{2}$

$\frac{\pi}{4} \le 0 - (-\arg z) \le \frac{\pi}{2}$

$\frac{\pi}{4} \le \arg z \le \frac{\pi}{2}$



(2)

$(z-1)(z^2+z+1) = 0$

b)i) $\omega \in \mathbb{C}$ of $z^3 - 1 = 0$
$1 + \omega + \omega^2 = 0$
$1 + \omega = -\omega^2$
$(1+\omega)^9 = (-\omega^2)^9$
$= -\omega^{18}$
$= -(\omega^3)^6$; $\omega^3 = 1$
$= -(1)^6$
$= -1$

(3)

ii) $\omega^2 + \omega + 1 + \frac{1}{\omega} + \frac{1}{\omega^2} = 0 + \frac{1}{\omega} + \frac{1}{\omega^2}$
$= 0 + \frac{\omega^2}{\omega^3} + \frac{\omega}{\omega^3}$
$= 0 + \frac{\omega^2}{1} + \frac{\omega}{1}$
$= \omega^2 + \omega$
$= -1$

(2)

c)i) $x^5 + k = 0$ has 5 roots and complex unreal roots must occur in conjugate pairs
∴ even number of complex unreal roots leaving one real root. A polynomial of odd degree has at least 1 real root.
$x^6 + k = 0$ has 6 roots
All 6 of the roots may be complex unreal roots in conjugate pairs.

(2)

ii) $x^6 = -1$
$|x^6| = |-1|$
$x^6 = 1$
$x = 1$
$x = \cos\theta + i\sin\theta$
$x^6 = \cos 6\theta + i\sin 6\theta = -1 + 0i$

$\cos 6\theta = -1$ $\quad$ $\sin 6\theta = 0$

$6\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi, 11\pi$

$\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}$

$\text{cis}\frac{11\pi}{6} = \text{cis}\left(-\frac{\pi}{6}\right) = \text{cis}\frac{\pi}{6}$

$\text{cis}\frac{9\pi}{6} = \text{cis}\left(-\frac{3\pi}{6}\right) = \text{cis}\frac{3\pi}{6}$

$\text{cis}\frac{7\pi}{6} = \text{cis}\left(-\frac{5\pi}{6}\right) = \text{cis}\frac{5\pi}{6}$

$\alpha = \text{cis}\frac{\pi}{6}$; $\bar{\alpha} = \text{cis}\left(-\frac{\pi}{6}\right)$; $\beta = \text{cis}\frac{3\pi}{6}$

$\gamma = \text{cis}\frac{5\pi}{6}$; $\bar{\gamma} = \text{cis}\left(-\frac{5\pi}{6}\right)$; $\bar{\beta} = \text{cis}\left(-\frac{3\pi}{6}\right)$

(2)

5mm A4

Question 12 cont:

$$x^6+1=(x-\alpha)(x-\bar{\alpha})(x-\beta)(x-\bar{\beta})(x-\gamma)(x-\bar{\gamma})$$

Note that: $(x-\alpha)(x-\bar{\alpha})$

$$= x^2-(\alpha-\bar{\alpha})x+\bar{\alpha}\alpha$$
$$= x^2-2\cos\frac{\pi}{6}x+\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}$$
$$= x^2-2\cos\frac{\pi}{6}x+1$$

$$\therefore x^6+1=\left(x^2-2\cos\frac{\pi}{6}x+1\right)\left(x^2-2\cos\frac{3\pi}{6}x+1\right)\left(x^2-2\cos\frac{5\pi}{6}+1\right)$$
$$=\left(x^2-2\left(\frac{\sqrt{3}}{2}\right)x+1\right)\left(x^2-2(0)x+1\right)\left(x^2-2\left(\frac{-\sqrt{3}}{2}\right)x+1\right.$$
$$=\left(x^2-\sqrt{3}x+1\right)\left(x^2+1\right)\left(x^2+\sqrt{3}x+1\right)$$

(2)

Solutions	Mark	Comments
Question 13 :		
(a) $\int x \sin\left(x^2+3\right)dx = \frac{-1}{2}\cos\left(x^2+3\right)+c$ (2)	2	Deduct 1 for no 'c'
(b) $\int \cos^{-1} x\, dx = x\cos^{-1} x - \int x \frac{-1dx}{\sqrt{1-x^2}}$ $\quad u = \cos^{-1}x \quad v' = 1$	1	
$= x\cos^{-1} x - \sqrt{1-x^2} + c$ $\quad u' = \frac{-1}{\sqrt{1-x^2}} \quad v = x$		
$\therefore \int_0^{\frac{1}{2}} \cos^{-1} x\, dx = \left[x\cos^{-1} x - \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$	1	
$= \left(\frac{1}{2}\cdot\frac{\pi}{3} - \sqrt{\frac{3}{4}}\right) - (0-1)$		
$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$ (3) $\quad \frac{\pi - 2\sqrt{3} + 6}{6}$	1	
(c) $u^2 = e^x + 1 \rightarrow e^x = u^2 - 1$		
$2u\frac{du}{dx} = e^x \qquad \therefore dx = \frac{2u\,.\,du}{e^x}$		
$= \frac{2u\,du}{u^2-1}$		
$\therefore \int \frac{e^{2x}dx}{\sqrt{e^x+1}} = \int \frac{\left(u^2-1\right)^2}{u} \cdot \frac{2u\,du}{u^2-1}$	1	
$= \int 2\left(u^2-1\right)du$		
$= \frac{2}{3}u^3 - 2u + c$ (4	2	
Resub:		
$= \frac{2}{3}\sqrt{\left(e^x+1\right)^3} - 2\sqrt{e^x+1} + c$	1	Full marks to here
$= \sqrt{e^x+1}\left[\frac{2}{3}\left(e^x+1\right) - 2\right] + c$		
$= \sqrt{e^x+1}\left[\frac{2}{3}e^x - \frac{4}{3}\right]$ (4)		

Question 13 Cont'd

(d) Let $\dfrac{A}{2x+1}+\dfrac{B}{2x-1}=\dfrac{1}{4x^2-1}$

$A(2x-1)+B(2x+1)=1$

$x=\frac{1}{2}$ $\quad \therefore 2B=1 \quad \therefore B=\frac{1}{2}$

$x=-\frac{1}{2}$ $\quad \therefore -2A=1 \quad \therefore A=-\frac{1}{2}$ **1**

$$\int\frac{dx}{4x^2-1}=\int\left[\frac{-\frac{1}{2}}{2x+1}+\frac{\frac{1}{2}}{2x-1}\right]dx$$

$$=-\frac{1}{4}\ln(2x+1)+\frac{1}{4}\ln(2x-1)+c$$

OR ↑

$$=\frac{1}{4}\ln\left(\frac{2x-1}{2x+1}\right)+c$$

(3) **2** No reduction for omission of 'c'.

(e)

$$\begin{array}{r l} & x+3 \\ x-2 \big) & x^2+x-4 \\ & \underline{x^2-2x} \\ & 3x-4 \\ & \underline{3x-6} \\ & 2 \end{array}$$

$$\int\frac{x^2+x-4}{x-2}\,dx=\int\left[x+3+\frac{2}{x-2}\right]dx$$ **1**

$$=\frac{1}{2}x^2+3x+2\ln(x-2)+c$$

$$\int_3^4\frac{x^2+x-4}{x-2}\,dx=\left[\frac{1}{2}x^2+3x+2\ln(x-2)\right]_3^4$$ **1**

$$=(8+12+2\ln 2)-(4\tfrac{1}{2}+9+2\ln 1)$$

$$=\frac{13}{2}+2\ln 2$$

(3) **1**

Question 14:

a i) $k - x^2 = \frac{1}{x}$

$kx - x^3 = 1$

$x^3 - kx + 1 = 0$

$\alpha + \beta + \gamma = \frac{-0}{1} = 0$

(2)

ii) Use: $x^3 - kx + 1 = 0$

Now if $y = x^2$ then $x = \sqrt{y}$

$\therefore (\sqrt{y})^3 - k\sqrt{y} + 1 = 0$

$y\sqrt{y} - k\sqrt{y} = -1$

$\sqrt{y}\,(y - k) = -1$

$y(y-k)^2 = (-1)^2$

$y(y^2 - 2yk + k^2) = 1$

$y^3 - 2ky^2 + k^2y - 1 = 0$

OR $x^3 - 2kx^2 + k^2x - 1 = 0$

↑ given

(3)

iii) Use: $x^3 - 2kx^2 + k^2x - 1 = 0$

$x = \frac{1}{y}$

$y = \frac{1}{x}$

$\left(\frac{1}{x}\right)^3 - 2k\left(\frac{1}{x}\right)^2 + k^2\left(\frac{1}{x}\right) - 1 = 0$

$\frac{1}{x^3} - 2k\frac{1}{x^2} + \frac{k^2}{x} - 1 = 0$

$1 - 2kx + k^2x^2 - x^3 = 0$

OR $x^3 - k^2x^2 + 2kx - 1 = 0$

(3)

b i) $\lim_{x \to 2}\left(\frac{x-2}{x^2-4}\right)$

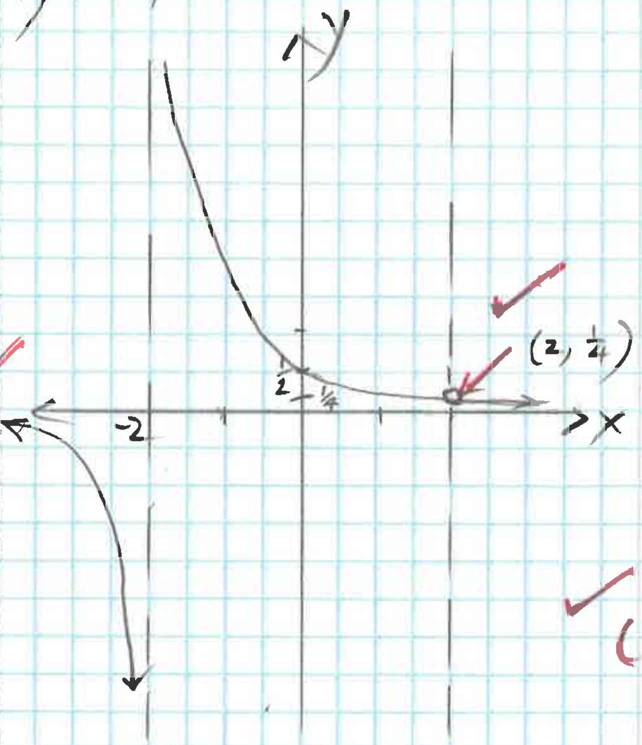
$= \lim_{x \to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$

$= \lim_{x \to 2} \frac{1}{x+2}$

$= \frac{1}{4}$

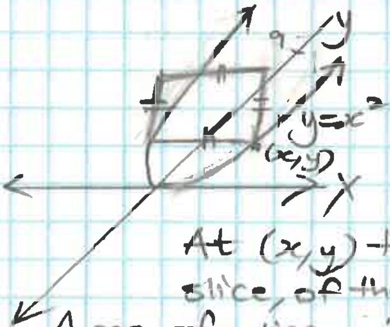
(1)

ii) $x \neq \pm 2$



(2)

c)



At (x, y) take a thin slice, of thickness δy

Area of slice $= 2x \times 2x$

$\delta V = 2x \times 2x \times \delta y$

$= 2\sqrt{y} \times 2\sqrt{y} \times \delta y$

$= 4y\,\delta y$

$V = \int_0^9 4y\,dy$

$= \left[2y^2\right]_0^9$

$= 2(81) - 2(0)$

$= 162 \text{ units}^3$

(4)

Question 15:

FROM OUTER FUNCTION, IN

a) i) Domain of $y = \cos^{-1}(e^{x})$

$\Rightarrow -1 \leq e^{x} \leq 1$

but $0 < e^{x} \leq 1$

since $e^{x} > 0$ for all x

if $e^{x} \leq 1$ it means that $x \leq \ln 1$ $\therefore$ $\boxed{x \leq 0}$ ← Domain (1)

ii) $y = \cos^{-1}(e^{x})$ is a decreasing function

let $x = 0$ $\therefore$ $y = \cos^{-1}(e^{x}) = 0$

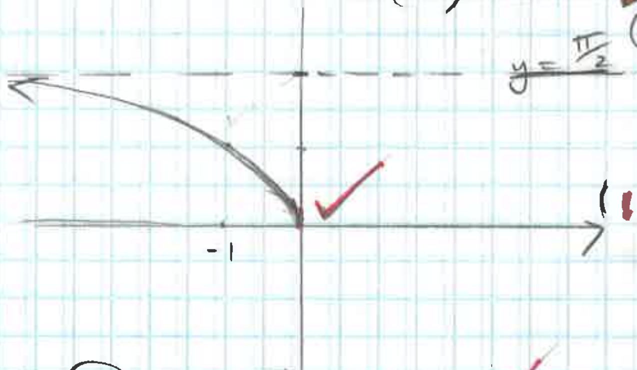
Now let $x = -1$ $\therefore$ $y = 1.19$

A vertical tangent at $x = 1$ since $\frac{dy}{dx} = -\frac{1}{\sqrt{1-e^{2x}}} \cdot e^{x}$

$e^{2x} \neq 1$ $\therefore$ $x = 0$ is the vertical tangent

$x \neq 0$

As $x \to -\infty$ $\cos^{-1}(e^{x}) \to \frac{\pi}{2}$ ← Asymp (2)



(1)

iii) Range: $0 \leq y < \frac{\pi}{2}$ (1)

b) $|\vec{OA}| = \frac{1}{2}|\vec{OC}|$

$\therefore |\vec{OC}| = 2|\vec{OA}|$

Now $|OA| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \underline{\underline{1}}$

$\therefore |OC| = \underline{\underline{2}}$

$\text{Arg}(A) = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)$

$\text{Arg } A = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$

$\therefore \text{Arg } C = \frac{2\pi}{3} - \frac{\pi}{3}$

$= \frac{\pi}{3}$

$\therefore C = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \underline{1 + \sqrt{3}i}$ (3)

c) i) $\cos(A-B) - \cos(A+B)$

$= \cos A\cos B + \sin A\sin B$

$- (\cos A\cos B - \sin A\sin B)$ SUBTRACT

$\underline{2\sin A\sin B}$ (1)

ii) $\therefore$ $\sin 6x \cdot \sin 2x$

(let $A = 6x$ $B = 2x$)

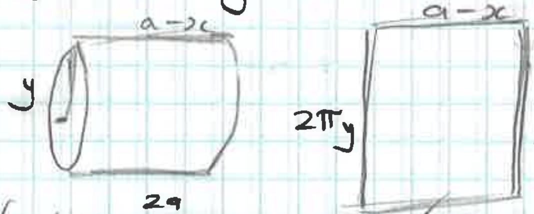
$= \frac{1}{2}(\cos 4x - \cos 8x)$

$\therefore \int \sin 6x \cdot \sin 2x \, dx$

$= \frac{1}{2}\int \cos 4x - \cos 8x \, dx$

$= \frac{1}{2}\left[\frac{1}{4}\sin 4x - \frac{1}{8}\sin 8x\right] + C$ (2)

d) cylindrical shells radius is y and height $a - x = a - \frac{y^2}{4a}$



δy is thickness

$y^2 = 4a(a)$

$y = 2a$

limit

$V = \lim_{\delta y \to 0} \sum_{y=0}^{2a} 2\pi y\left(a - \frac{y^2}{4a}\right)\delta y$

$= 2\pi\int_0^{2a} y\left(a - \frac{y^2}{4a}\right)\delta y$

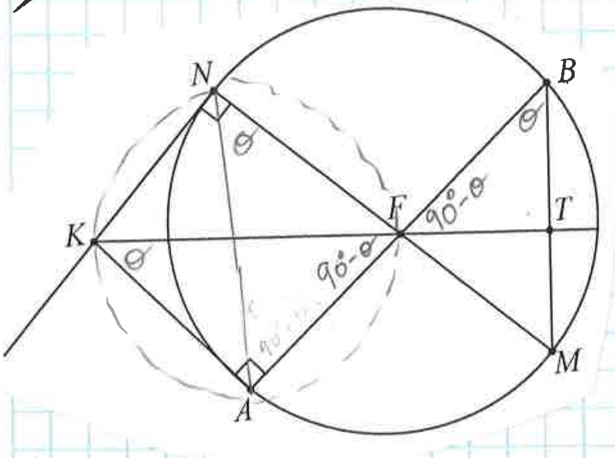
$= 2\pi\int_0^{2a} ya - \frac{y^3}{4a} \, \delta y$

$= 2\pi\left[\frac{ay^2}{2} - \frac{y^4}{16a}\right]_0^{2a}$

$= 2\pi\left[\left(\frac{4a^3}{2} - \frac{16a^4}{16a}\right) - 0\right]$

$= \underline{2\pi a^3}$ (4)

c)



Join AN and let ∠ANF = θ

∠ANM = ∠ABM = θ (Angles in same seg)

∠KNF = ∠KAF = 180°

∴ KNFA is a cyclic quadrilateral

∠AKF = ∠ANF = θ (∠'s in same segment in cyclic quad KNFA)

∠KFA = 90° − θ (Angle sum of △ AKF)

∴ ∠BFT = ∠KFA = 90° − θ (Vertically opposite angles)

∴ ∠FTB = 180° − (∠BFT + ∠FBM)

∠ABM = ∠FBM

∴ ∠FBM = 180° − [(90 − θ) + θ]

= 90°

∴ KF is perpendicular to MB.

5mm A4

Question 16:

a) Let $\tan A = 4x$ and $\tan B = 3x$

$A - B = \tan^{-1}\left(\frac{1}{7}\right)$

$\therefore \tan(A-B) = \frac{1}{7}$

$\therefore \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{1}{7}$ ✓

$\frac{4x - 3x}{1 + 4x \times 3x} = \frac{1}{7}$ ✓

$\frac{x}{1+12x^2} = \frac{1}{7}$

$7x = 1 + 12x^2$

$12x^2 - 7x + 1 = 0$

$x = \frac{1}{3}$ OR $x = \frac{1}{4}$ ✓

(3)

b)i) $I_n = \int \cos^n x \, dx$

$= \int \cos x \cdot \cos^{n-1} x \, dx$ ✓

$\left(\begin{array}{l|l} u = \cos^{n-1} x & v' = \cos x \\ u' = (n-1)\cos^{n-2} x \cdot (-\sin x) & v = \sin x \end{array} \right)$

$= \sin x \cdot \cos^{n-1} x - \int (n-1)\sin x \cdot \cos^{n-2} x (-\sin x) \, dx$ ✓

$= \sin x \cdot \cos^{n-1} x + (n-1) \int \sin^2 x \cdot \cos^{n-2} x \, dx$

* sign ✓

$= \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cdot \cos^{n-2} \, dx$

$= \sin x \cdot \cos^{n-1} x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx$

$= \sin x \cdot \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$ ✓

$I_n + (n-1) I_n = \sin x \cdot \cos^{n-1} x + (n-1) I_{n-2}$

$n I_n = \sin x \cdot \cos^{n-1} x + (n-1) I_{n-2}$

$I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{(n-1)}{n} I_{n-2}$

(4)

ii) $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$

$I_4 = \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} I_2$ ✓

now

$I_2 = \frac{1}{2} \sin x \cos x + \frac{1}{2} I_0$

$I_0 = \int_0^{\frac{\pi}{2}} \cos^0 x \, dx$

$= \int_0^{\frac{\pi}{2}} 1 \, dx$

$= [x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$ ✓

$\therefore I_2 = \frac{1}{2} \sin x \cos x + \frac{\pi}{4}$

$\therefore I_4 = \frac{1}{4} \sin x \cos^3 x + \frac{3}{4}\left(\frac{1}{2} \sin x \cos x + \frac{\pi}{4}\right)$

$= \frac{1}{4} \sin x \cos^3 x + \frac{3}{8} \sin x \cos x + \frac{3\pi}{16}$

$\therefore \int_0^{\frac{\pi}{2}} \cos^4 x \, dx$ ✓

$= \left[\frac{1}{4} \sin x \cos^3 x + \frac{3}{8} \sin x \cos x + \frac{3\pi}{16}\right]_0^{\frac{\pi}{2}}$

$= \frac{3\pi}{16} - 0$

$= \frac{3\pi}{16}$ ✓

(4)