Trinity Grammar School

Mathematics Department

2017

HALF-YEARLY
EXAMINATION

HSC ASSESSMENT TASK 3

Year 12

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- NESA approved calculators for this course are allowed in this task
- A Reference Sheet will be supplied
- In Questions 11 – 16 show all necessary mathematical reasoning
- Write your NESA Student Number (Year 12 HSC)) and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating **N/A** and your Student Number or Name.

- **HSC Assessment Weighting:** 30%

NESA Student Number
(Year 12 only)

Class Teacher:

……………………………………………
Name:

……………………………………………

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

Date of examination:

Friday, 5 May 2017

Total marks – 100

Section I

10 marks

- Attempt Questions 1-10
- Allow about **15** minutes for this section

Section II

90 marks

- Attempt Questions 11 **-** 16
- Allow about **2** hours **45** minutes for this section

Section I **10 marks**

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section
- Shade the correct response on the multiple-choice answer sheet for Questions 1 – **10**
- Each question is worth 1 mark

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1. What is the indefinite integral for $\int\left(\cos^2 x+\tan^2 x\right)dx$?

(A) $\frac{1}{4}\sin 2x+\tan x+\frac{1}{2}x+c$

(B) $-\frac{1}{4}\sin 2x+\tan x+\frac{1}{2}x+c$

(C) $\frac{1}{4}\sin 2x+\tan x-\frac{1}{2}x+c$

(D) $-\frac{1}{4}\sin 2x+\tan x-\frac{1}{2}x+c$

2. If $i=\sqrt{-1}$ and a and b are non-zero real numbers, which expression is equivalent to $\frac{1}{a+bi}$?

(A) $\frac{a+bi}{a^2+b^2}$

(B) $\frac{a-bi}{a^2+b^2}$

(C) $\frac{a+bi}{a^2-b^2}$

(D) $\frac{a-bi}{a^2-b^2}$

3. How many solutions does the equation $\sec(x)=\operatorname{cosec}(x)$ have in the domain $-2\pi\le x\le 2\pi$?

(A) 1
(B) 2
(C) 3
(D) 4

4. The graph of $f(x) = \dfrac{1}{x^2 + mx - n}$, where m and n are real constants, has no vertical asymptotes if

(A) $m^2 < 4n$

(B) $m^2 = 4n$

(C) $m^2 = -4n$

(D) $m^2 < -4n$

5. Given the curve $y = f(x)$, the curve $y = f(|x|)$ is represented by

(A) A reflection of $y = f(x)$ in the y-axis

(B) A reflection of $y = f(x)$ in the x-axis

(C) A reflection of $y = f(x)$ for $x \geq 0$ in the y-axis

(D) A reflection of $y = f(x)$ for $x \geq 0$ in the x-axis

6. Use implicit differentiation on the equation $y^3 = x^2 + xy$, then $\dfrac{dy}{dx}$ would equal

(A) $\dfrac{3y^2 - 2x}{x}$

(B) $\dfrac{2x + y}{3y^2 - x}$

(C) $\dfrac{2x + y}{3y^2 + y}$

(D) $\dfrac{2x}{3y^2 + y}$

7. The solution to the equation $z^2 = i\overline{z}$?

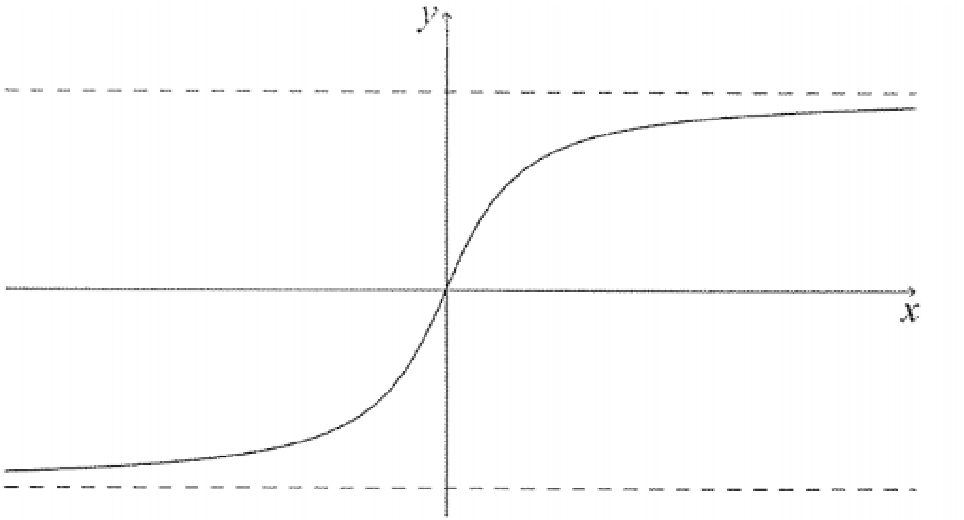
(A) (0,0) and (0,1)

(B) (0,0) and (0,-1)

(C) (0,0), (0,1), $(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2})$ and $(\dfrac{-\sqrt{3}}{2}, \dfrac{1}{2})$

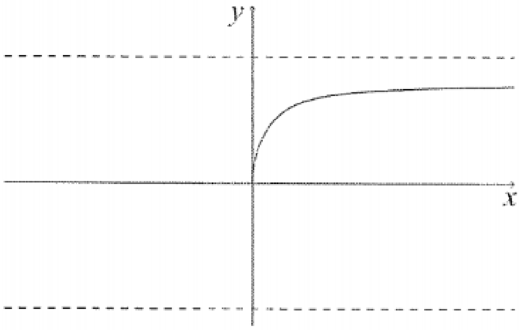
(D) (0,0), (0,-1), $(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2})$ and $(\dfrac{-\sqrt{3}}{2}, \dfrac{1}{2})$

8. The graph of $y = f(x)$ is shown below.

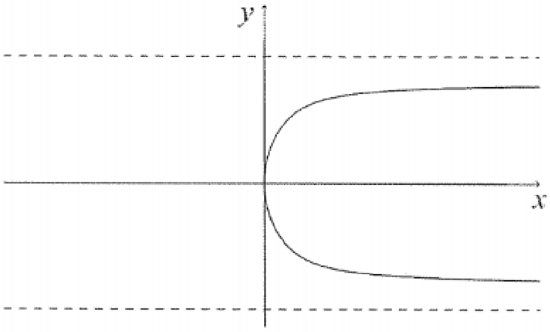


Which of the following graphs best represent $y^2 = f(x)$?

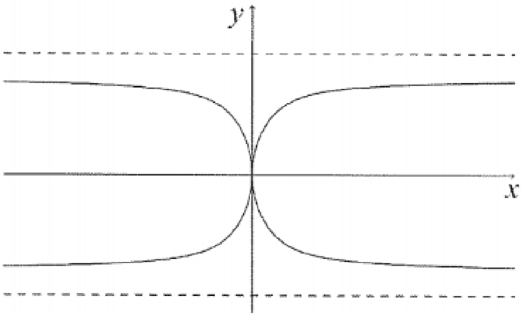
(A)



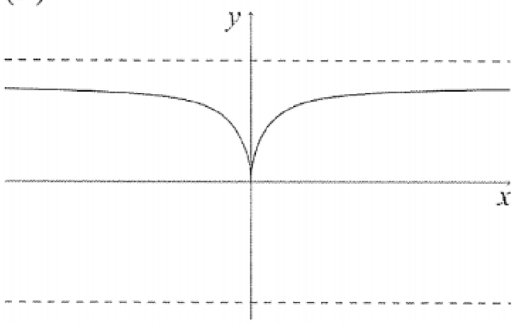
(B)



(C)



(D)



9. If ω is one of the complex roots of $z^3 = 1$, what is the value of

$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

(A) 9
(B) 6
(C) 3
(D) 0

10. For $z = a + bi$. Let $\lambda = \frac{1}{2}(-1+\sqrt{3}i)$.

Which of the following is a correct expression for $|\omega|$, where $\omega = a + b\lambda$?

(A) $\sqrt{(a-b)^2 - ab}$
(B) $\sqrt{(a-b)^2 - 2ab}$
(C) $\sqrt{(a-b)^2 + ab}$
(D) $\sqrt{(a-b)^2 + 2ab}$

End of Section I
Section II commences on the next page

Section II 90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section
- Answer each question in a SEPARATE writing booklet or answer sheet. Extra writing booklets are available.
- In Questions 11 – 16, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given $Z_1 = 1 + 2i$, $Z_2 = 2 - i$ and $Z_3 = 1 - \sqrt{3}i$.

Express the following in the form $a + bi$ where a and b are real.

(i) $Z_1 + Z_2$ **1**

(ii) $\frac{1}{Z_2}$ **1**

(iii) $(Z_3)^3$ **2**

(b) (i) Expand and simplify that $\sin(A - B) + \sin(A + B) = 2\sin A\cos B$ **1**

(ii) Hence, or otherwise, find $\int \sin 3x \cos x dx$ **2**

(c) Write $x^2 - 12x + 48$ as the product of two linear factors. **2**

(d) Graph the region in the Argand diagram which simultaneously satisfies **3**

$1 \le |z - i| \le 2$ and $\operatorname{Im} z \ge 0$.

(e) (i) Express $-1 + \sqrt{3}i$ in Modulus-Argument form. **1**

(ii) Hence evaluate $(-1 + \sqrt{3}i)^6$ **2**

End of Question 11

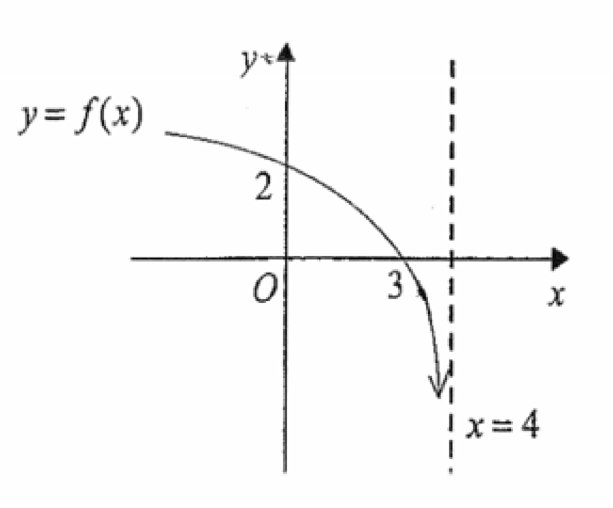
Question 12 **(15 marks)** Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $f(x) = \dfrac{x}{\sqrt{1-x^2}}$.

(i) Write down the domain of $f(x)$. **1**

(ii) Find $f^{-1}(x)$, the inverse function of $f(x)$. **3**

(b) The diagram shows the graph of the curve $y = f(x)$. The diagram is NOT to scale.



On separate diagrams, sketch the graphs of the curves listed below, showing clearly intercepts on the coordinate axes and the equations of any asymptotes :

(i) $y = |f(x)|$. **1**

(ii) $y = f(|x|)$. **1**

(iii) $y = \dfrac{1}{f(x)}$. **2**

(iv) $y = f(2x)$. **1**

Question 12 continues on the next page

Question 12 continued **Marks**

=

(c) Find the roots of $z^5 - i = 0$ and sketch them on an argand diagram. **3**

(d) One root of the equation $x^3 + ax^2 + bx + c = 0$ is equal to the sum of the other two roots. **3**

Show that $a^3 - 4ab + 8c = 0$.

End of Question 12

Question 13 **(15 marks)** Use a SEPARATE writing booklet. **Marks**

(a) Use the substitution $u=\sqrt{1-x}$ to evaluate $\int_0^1 x^2\sqrt{1-x}dx$. **3**

(b) Consider $f(x)=\dfrac{2-x}{2+x}$. Sketch the following graphs:

(i) $y=f(x)$. **2**

(ii) $y=\left[f(x)\right]^2$. **3**

(iii) $y=\sqrt{f(x)}$. **2**

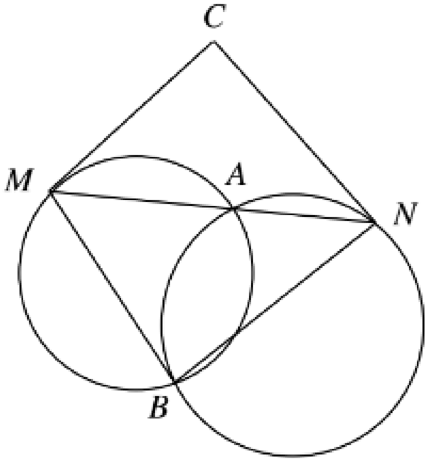
(iv) $y=\ln\left[\mathrm{f(x)}\right]$. **2**

(c) Solve the equation $\dfrac{1}{z+1}+\dfrac{1}{1-2i}=1$. **3**

End of Question 13

Question 14 **(15 marks)** Use a SEPARATE writing booklet. **Marks**

(a) Two circles intersect at A and B. A line through A cuts the circles at M and N. The tangents at M and N intersect at C.



(i) Prove that $\angle CMA + \angle CNA = \angle MBN$. **3**

(ii) Prove *M, C, N* and *B* are concyclic points. **2**

(b) Consider the function $f(x) = \sqrt{3 - \sqrt{x}}$.

(i) Find the domain of $f(x)$. **1**

(ii) Show that $f(x)$ is a decreasing function and deduce the range of $f(x)$. **2**

(c) Simplify $\dfrac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$. **2**

(d) (i) By letting $z = x + iy$, find the locus of points in the complex plane which **2**

satisfy $\text{Re}(\text{z} - \dfrac{1}{\overline{\text{z}}}) = 0$.

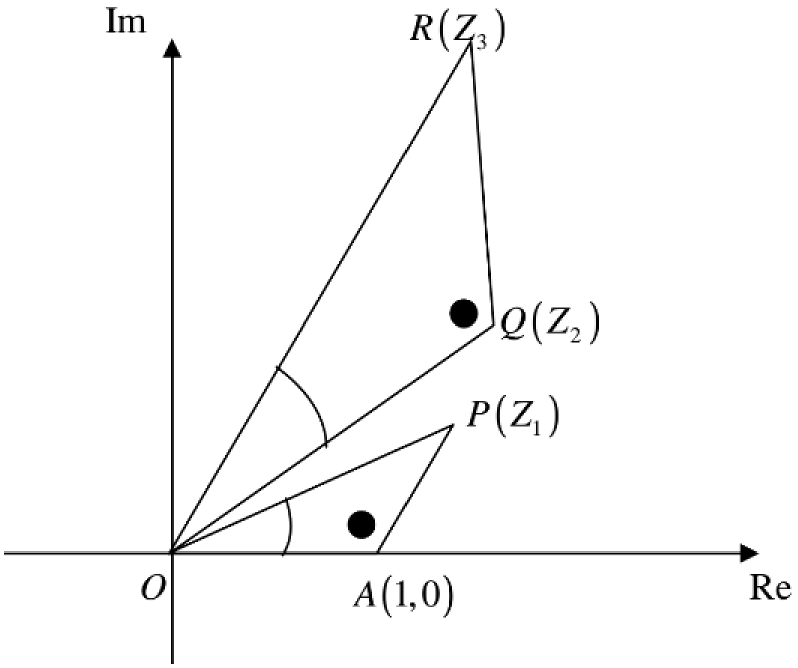
(ii) Sketch the locus. **1**

(e) If $z = \cos\theta + i\sin\theta$, use de Moivre’s Theorem to show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the figure below, the points *P*, *Q* and *A* represent the complex numbers Z_1, Z_2 and (1,0) respectively. Given $\angle OAP = \angle OQR$ and $\angle AOP = \angle QOR$.



Explain why the point $R(Z_3)$ is represented by the complex number Z_1Z_2. **3**

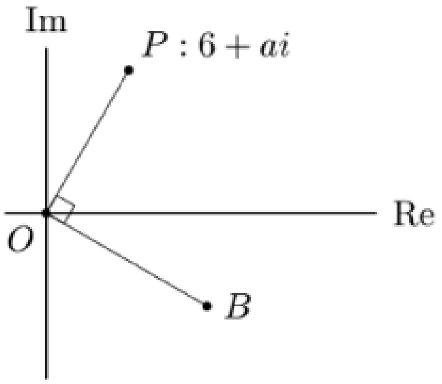
You must support your answer with clear and complete mathematical reasons.

(b) Find the equation of the tangent to the curve $\cos 2x + \sin y = 1$ at the point $x = \frac{\pi}{6}$. **3**

Question 15 continues on the next page

Question 15 continued **Marks**

(c) In the following Argand diagram, *P* represents the point 6 + *ai*, and *O* is the origin. **3**



Find the complex number represented by the point *B*, given $\angle POB = 90^\circ$ and $2|OB| = 3|OP|$.

(d) Sketch $y = \sqrt{\cos 3x}$ for $-\pi \le x \le \pi$. **2**

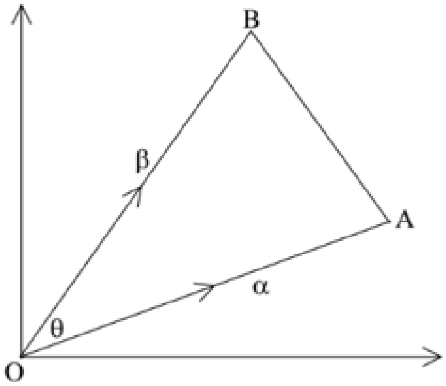
(e) By applying de Moivre's Theorem and by also expanding $(\cos\theta + i\sin\theta)^4$, obtain an expression for $\cos 4\theta$ in terms of $\cos\theta$. **4**

End of Question 15

Question 16 **(15 marks)** Use a SEPARATE writing booklet. **Marks**

(a) (i) Use the results $z+\overline{z}=2\,\mathrm{Re}(z)$ and $\left|z\right|^2=z\overline{z}$ for complex numbers z to show that $\left|\alpha\right|^2+\left|\beta\right|^2-\left|\alpha-\beta\right|^2=2\,\mathrm{Re}(\alpha\overline{\beta})$. **3**

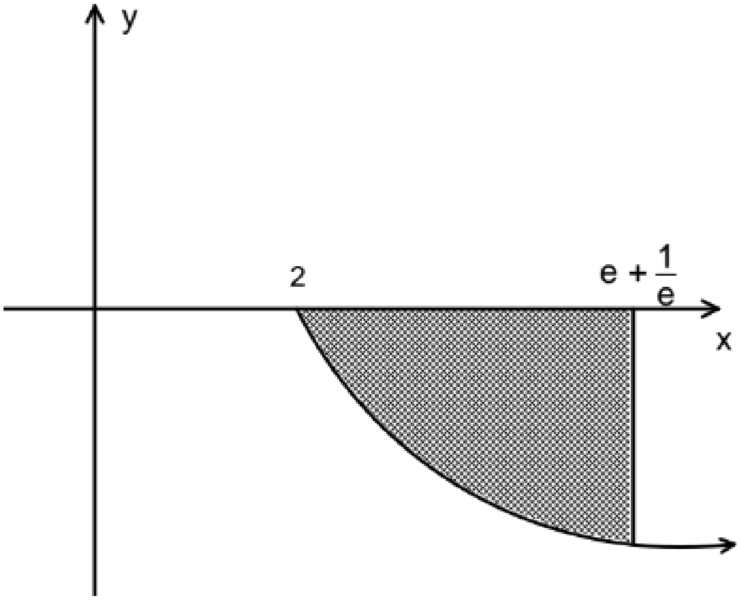
(ii) The diagram below shows the angle θ between the complex numbers α and β. **2**



Prove that $\left|\alpha\right|\left|\beta\right|\cos\theta=\mathrm{Re}(\alpha\overline{\beta})$.

Question 16 continues on the next page

(b) Below is a graph of $y = \ln\left(\dfrac{x-\sqrt{x^2-4}}{2}\right)$. The diagram is NOT to scale.



(i) Show that the equation of the inverse function is given by $y = e^x + e^{-x}$. **2**

(ii) Hence, find the area of the shaded region above. **3**

(c) Two points $P\,(x_1, y_1)$ and $Q\ (x_1 + a, y_2)$ lie on the parabola $x^2 = 4ay$.

(i) Show that the midpoint *C*, of *PQ* is given by : **2**

$$(\frac{2x_1+a}{2},\frac{2x_1^2+2ax_1+a^2}{8a})$$

(ii) Show that the locus of *C* is a parabola and state the vertex. **3**

End of Question 16

End of Section II

End of Examination

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Trinity Grammar School

Year 12 Mathematics Extension 2 Half Yearly Examination 2017

Suggested Solutions

Section I

1. C 2. B 3. D 4. D 5. C 6. B 7. D 8. B 9. A 10. C

Question 7

Let $z = x + iy$ and $\overline{z} = x - iy$

$$z^2 = i\overline{z}$$

$$(x+iy)^2 = i(x-iy)$$

$$x^2 - y^2 + 2xyi = y + ix$$

Equating the real and imaginary parts

$x^2 - y^2 = y \qquad (1)$

$2xy = x \qquad (2)$

Rearranging eqn (2)

$x(2y-1) = 0$

$x = 0$ or $y = \frac{1}{2}$

Substitute $x = 0$ into eqn (1)

$-y^2 = y$

$y(y+1) = 0$

$y = 0$ or $y = -1$

Substitute $y = \frac{1}{2}$ into eqn (1)

$x^2 - \frac{1}{4} = \frac{1}{2}$

$x^2 = \frac{3}{4},\ x = \pm\frac{\sqrt{3}}{2}$

Solution is $(0,0)$, $(0,-1)$, $(\frac{\sqrt{3}}{2},\frac{1}{2})$ and $(-\frac{\sqrt{3}}{2},\frac{1}{2})$

Question 9

$$(1-w)(1-w^2)(1-w^4)(1-w^8)$$

$$(1-w)(1-w^2)(1-w)(1-w^2)$$

$$\left[(1-w)(1-w^2)\right]^2$$

$$= \left[1-w-w^2+w^3\right]^2$$

$$= \left[2-w-w^2\right]^2$$

but $1+w+w^2=0$ so

$$-w-w^2=1$$

$$= [2+1]^2$$

$= 9$ Ⓐ

Question 10

$$\begin{aligned} a+b\lambda &= a+\tfrac{1}{2}\left(-1+i\sqrt{3}\right)b \\ &= \left(a-\tfrac{1}{2}b\right)+i\left(\tfrac{\sqrt{3}}{2}b\right) \end{aligned}$$

$$\begin{aligned} |\,w\,| &= \sqrt{\left(a-\tfrac{1}{2}b\right)^2+\left(\tfrac{\sqrt{3}}{2}b\right)^2} \\ &= \sqrt{a^2-ab+\tfrac{1}{4}b^2+\tfrac{3}{4}b^2} \\ &= \sqrt{a^2-ab+b^2} \\ &= \sqrt{\left(a^2-2ab+b^2\right)+ab} \\ &= \sqrt{\left(a-b\right)^2+ab} \end{aligned}$$

Section II

Question 11

(a)

(i) $z_1 + z_2 = 3 + i$

(ii) $\frac{1}{z_2} = \frac{1}{2-i} \frac{\times(2+i)}{\times(2+i)}$

$= \frac{2}{5} + \frac{1}{5}i$

(iii) $(z_3)^3 = (1-\sqrt{3}i)^3$

$= 1 - 3\sqrt{3}i + 3(\sqrt{3}i)^2 - (\sqrt{3}i)^3$

$= 1 - 3\sqrt{3}i - 9 + 3\sqrt{3}i$

$= -8$

(b)

i) $\sin A\cos B - \sin B\cos A + \sin A\cos B + \sin B\cos A$

$= 2\sin A\cos B$ Q.E.D.

ii) $\frac{1}{2}\int(\sin 2x + \sin 4x)\,dx$

$= -\frac{\cos 2x}{4} - \frac{\cos 4x}{8} + c$

(c)

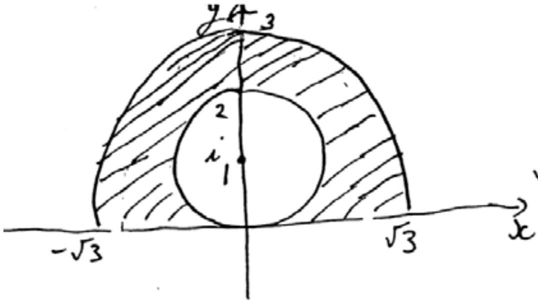
$x^2 - 12x + 48$

$(x-6)^2 + 12$

$(x-6)^2 - (2\sqrt{3}i)^2$

$(x - 6 - 2\sqrt{3}i)(x - 6 + 2\sqrt{3}i)$

(d)



✓ annulus (0,1) centre

✓ above x axis

✓ correct shading (radius between 1 and 2)

(e)

$$\text{i)}\quad |-1+\sqrt{3}i| = \sqrt{1^2+(\sqrt{3})^2} = 2$$

$$\text{ARG}(-1+\sqrt{3}i) = \theta \quad \text{WHERE} \quad \tan\theta = -\sqrt{3}$$

$$\therefore \quad \theta = \frac{2\pi}{3}$$

$$\therefore\ -1+\sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$\text{(ii)}\ \left(-1+\sqrt{3}i\right)^6 = \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^6$$

$$= 2^6\left[\cos 4\pi + i\sin 4\pi\right]$$

$$= 2^6\left[1+0\right]$$

$$= 64$$

Question 12

(a)

(i) $1-x^2>0 \qquad -1<x<1$

(ii) If $y=f(x)$, the inverse function is

$$x=\frac{y}{\sqrt{1-y^2}}$$

$$x^2=\frac{y^2}{1-y^2}$$

$$x^2-x^2y^2=y^2$$

$$y^2\left(1+x^2\right)=x^2$$

$$y^2=\frac{x^2}{1+x^2}$$

$$f^{-1}(x)=\frac{x}{\sqrt{1+x^2}} \quad \text{(odd function)}$$

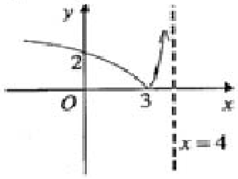
(b)

Marking Guidelines

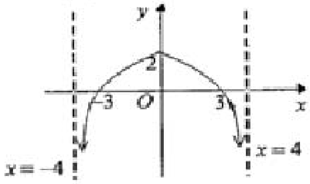
Criteria	Marks
i • copies curve for $x \le 3$ and reflects section of curve for $x > 3$ in x-axis	1
ii • copies curve for $x \ge 0$ and includes reflection of this section of curve in the y-axis	1
iii • shows vertical asymptote $x = 3$ and sketches left hand branch correctly	1
• sketches right hand branch correctly showing nature at $x = 4$	1
iv • Shows correct x intercept of 1.5 and asymptote of x = 2 as well as correct y-intercept of 2	1

Answ

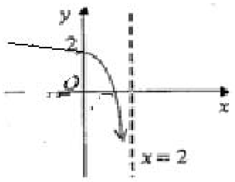
i. $y = \lvert f(x) \rvert$



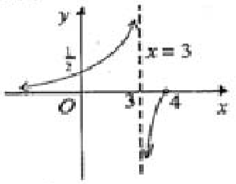
ii. $y = f(\lvert x \rvert)$



iv



iii. $y = \dfrac{1}{f(x)}$



(c)

$z^5 - i = 0 \rightarrow z^5 = i$

Let $z = \cos\theta + i\sin\theta$

$\therefore z^5 = \cos 5\theta + i\sin 5\theta = i$

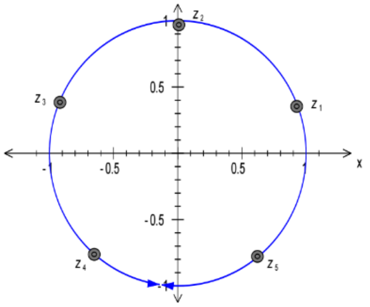
Equate real and imaginary parts:

$\cos 5\theta = 0$ and $\sin 5\theta = 1$

$\therefore 5\theta = \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}, \dfrac{13\pi}{2}, \dfrac{17\pi}{2}$

and $\theta = \dfrac{\pi}{10}, \dfrac{\pi}{2}, \dfrac{9\pi}{10}, \dfrac{13\pi}{10}, \dfrac{17\pi}{10}$

$\therefore z_1 = cis\dfrac{\pi}{10}, z_2 = i, z_3 = cis\dfrac{9\pi}{10}, z_4 = cis\dfrac{13\pi}{10}, z_5 = cis\dfrac{17\pi}{10}$



1 mark for 1 correct value of z

1 mark for 5 correct values of z

1 mark for plotting the values, showing equal moduli and equal imaginary parts for z_1, z_3 and z_4, z_5

Comment: This question was marked very generously

(d)

$x^3 + ax^2 + bx + c = 0$

roots $\alpha, \beta, \alpha+\beta$

$$2\alpha + 2\beta = -a \qquad \alpha\beta + \alpha^2 + \alpha\beta + \alpha\beta + \beta^2 = b$$

$$\alpha + \beta = -\frac{a}{2} \qquad \alpha^2 + 3\alpha\beta + \beta^2 = b$$

$$(\alpha+\beta)^2 + \alpha\beta = b$$

$$\alpha\beta(\alpha+\beta) = -c$$

$$\alpha\beta\left(-\frac{a}{2}\right) = -c$$

$$\alpha\beta = \frac{2c}{a}$$

$$(\alpha+\beta)^2 + \alpha\beta = b$$

$$\frac{a^2}{4} + \frac{2c}{a} = b$$

$$a^3 + 8c = 4ab$$

$$\underline{\underline{a^3 - 4ab + 8c = 0}}$$

Question 13

(a)

$u = \sqrt{1-x}$ when $x = 0$, $u = 1$

$x = 1 - u^2$ and $x = 1$, $u = 0$

and $dx = -2u\,du$

$$\int_0^1 x^2\sqrt{1-x}\;dx = \int_1^0 (1-u^2)^2 . u . -2u\,du$$

$$= \int_0^1 2u^2(1-u^2)^2\,du$$

$$= \int_0^1 2u^2 - 4u^4 + 2u^6\,du$$

$$= \left[\frac{2u^3}{3} - \frac{4u^5}{5} + \frac{2u^7}{7}\right]_0^1$$

$$= \frac{16}{105}$$

(b)

$$\begin{array}{r} -1 \\ 2+x\,\overline{)\,2-x} \end{array}$$

(a) $\frac{-(-2-x)}{4}$

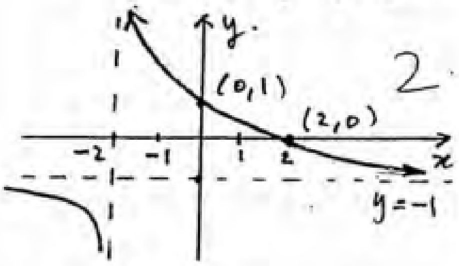
(i) ∴ $\frac{2-x}{2+x} = -1 + \frac{4}{2+x}$.

When $x = 0,\ y = 1$

$x = 2,\ y = 0$.

$\lim_{x \to \pm\infty} \left(\frac{2-x}{2+x}\right) = -1$.

∴ graph of $y = f(x)$:



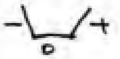
2.

(ii). $y = \left(\frac{2-x}{2+x}\right)^2$.

$\frac{dy}{dx} = 2\left(\frac{2-x}{2+x}\right) \cdot \left[\frac{-(2+x)-(2-x)}{(2+x)^2}\right]$

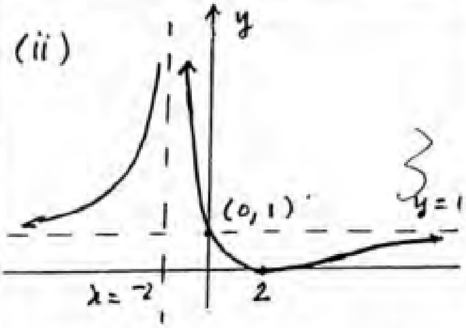
$= \frac{-8(2-x)}{(2+x)^3}$.

x	1	2	3
$\frac{dy}{dx}$	−	0	+



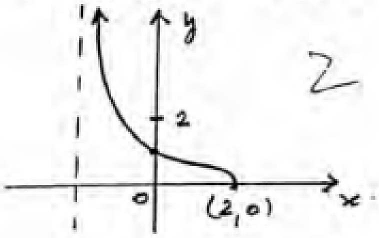
Note: The x-intercepts. and the x-coordinates of stationary points of $y = f(x)$ give the stationary points of $y = [f(x)]^2$.

(ii)



(iii) x intercept of $y = \frac{2-x}{2+x}$ is (2,0)

∴ (2,0) is a critical point of $y = \sqrt{\frac{2-x}{2+x}}$.



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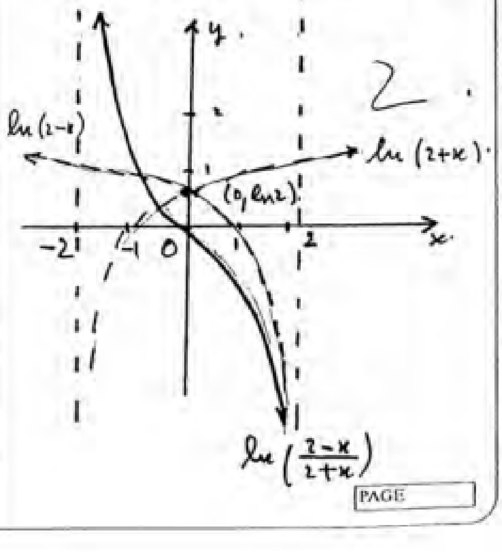
(iv) $y = \ln\left(\frac{2-x}{2+x}\right)$

$= \ln(2-x) - \ln(2+x)$.



2.

PAGE

(c)

$$\frac{1}{z+1}+\frac{1}{1-2i}=1$$

$$\frac{1}{z+1}=1-\frac{(1+2i)}{5}$$

$$\frac{1}{z+1}=1-\frac{1}{5}-\frac{2i}{5}$$

$$\frac{1}{z+1}=\frac{4-2i}{5}$$

$$z+1=\frac{5}{4-2i}\times\frac{4+2i}{4+2i}$$

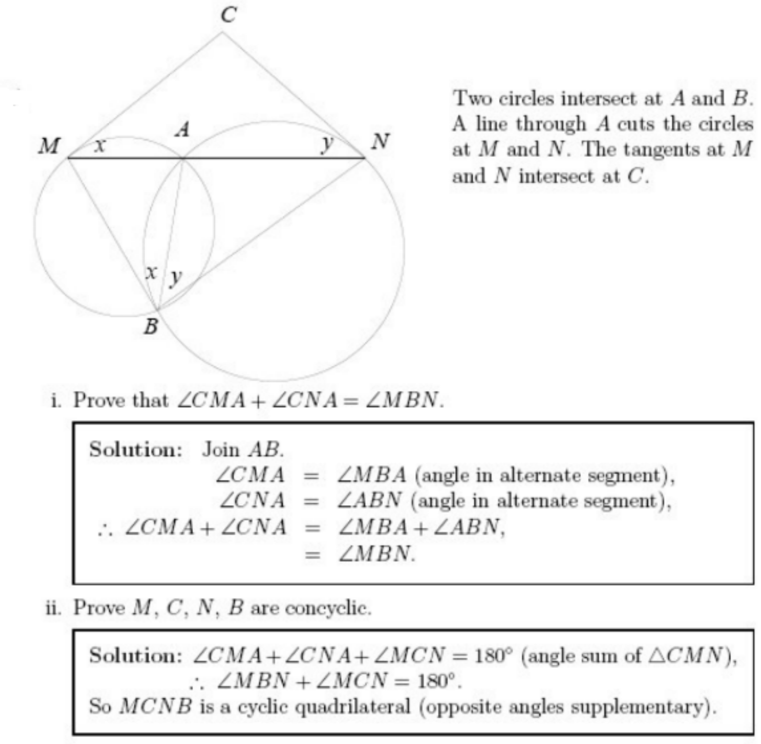
$$z+1=\frac{20+10i}{20}$$

$$z+1=1+\frac{i}{2}$$

$$z=\frac{i}{2}$$

Question 14

(a)



Two circles intersect at A and B. A line through A cuts the circles at M and N. The tangents at M and N intersect at C.

i. Prove that $\angle CMA + \angle CNA = \angle MBN$.

Solution: Join AB.

$$\begin{aligned} \angle CMA &= \angle MBA \text{ (angle in alternate segment)},\\ \angle CNA &= \angle ABN \text{ (angle in alternate segment)},\\ \therefore\ \angle CMA + \angle CNA &= \angle MBA + \angle ABN,\\ &= \angle MBN. \end{aligned}$$

ii. Prove M, C, N, B are concyclic.

Solution: $\angle CMA + \angle CNA + \angle MCN = 180^\circ$ (angle sum of $\triangle CMN$),

$\therefore\ \angle MBN + \angle MCN = 180^\circ$.

So $MCNB$ is a cyclic quadrilateral (opposite angles supplementary).

(b)

$$f(x) = \sqrt{3-\sqrt{x}} = \left(3 - x^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

(i) $3 - \sqrt{x} \geqslant 0$ and $x \geqslant 0$

$3 \geqslant \sqrt{x}$

$9 \geqslant x$ and $x \geqslant 0$

$\therefore$ Domain is $0 \leq x \leq 9$

(ii) $f'(x) = \frac{1}{2}\left(3 - x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \times -\frac{1}{2}x^{-\frac{1}{2}}$ (Chain rule)

$$= -\frac{1}{4} \times \frac{1}{\sqrt{x}\sqrt{3-\sqrt{x}}}$$

$$= -\frac{1}{4} \times \frac{1}{\sqrt{3x - x\sqrt{x}}}$$

Since $\sqrt{3x - x\sqrt{x}} \geqslant 0$ for all x in the domain
$f'(x) < 0$ for $0 < x < 9$ and
$f'(x)$ is undefined at $x = 0$ and $x = 9$.
as $x \to 0$ or $x \to 9$ $f'(x) \to -\infty$

$\therefore$ $f(x)$ is a decreasing function
$\therefore$ $f(x)_{max} = \sqrt{3}$ (when $x = 0$)
$f(x)_{min} = 0$ (when $x = 9$)

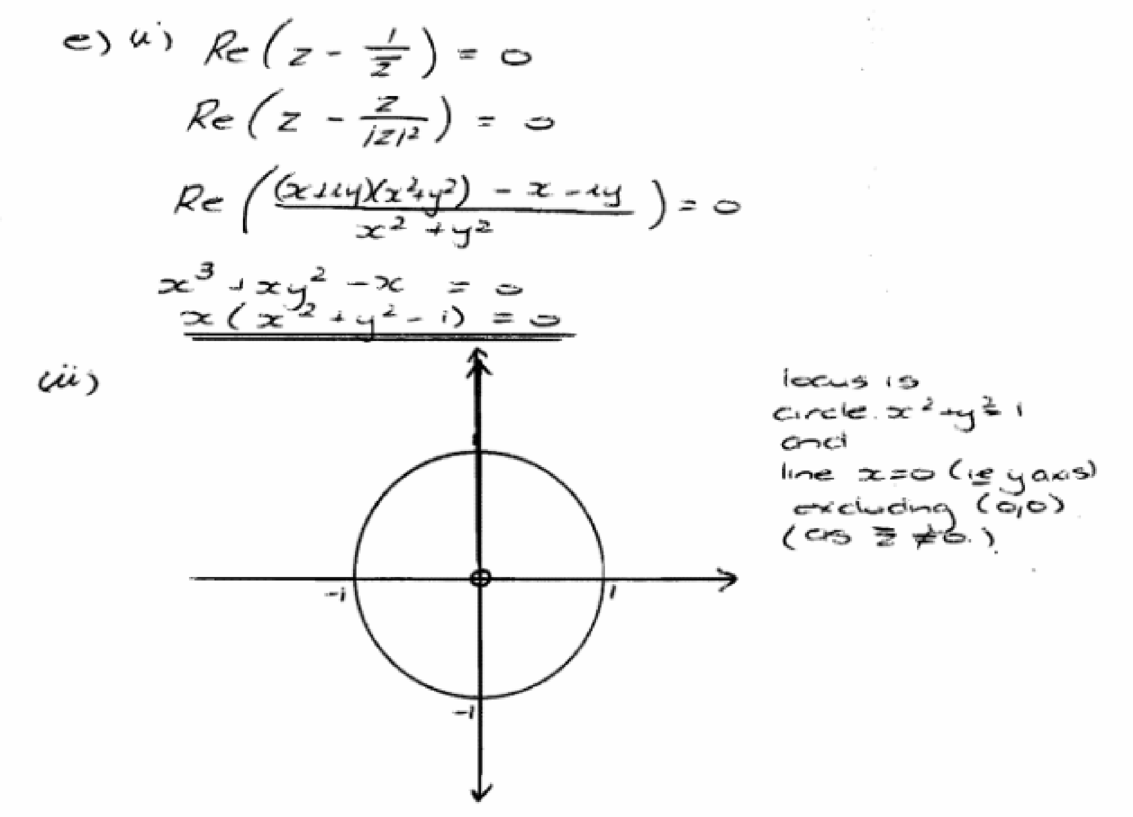
(c)

c) $$\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta} = \frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta} \times \frac{\cos 2\theta + i\sin 2\theta}{\cos 2\theta + i\sin 2\theta}$$

$$= \frac{\cos 5\theta + i\sin 5\theta}{\cos^2 2\theta + \sin^2 2\theta}$$

☐ $$= \underline{\underline{cis\, 5\theta}}$$

(d)



e) i) $Re\left(z - \frac{1}{z}\right) = 0$

$Re\left(z - \frac{z}{|z|^2}\right) = 0$

$Re\left(\frac{(x+iy)(x^2+y^2) - x - iy}{x^2+y^2}\right) = 0$

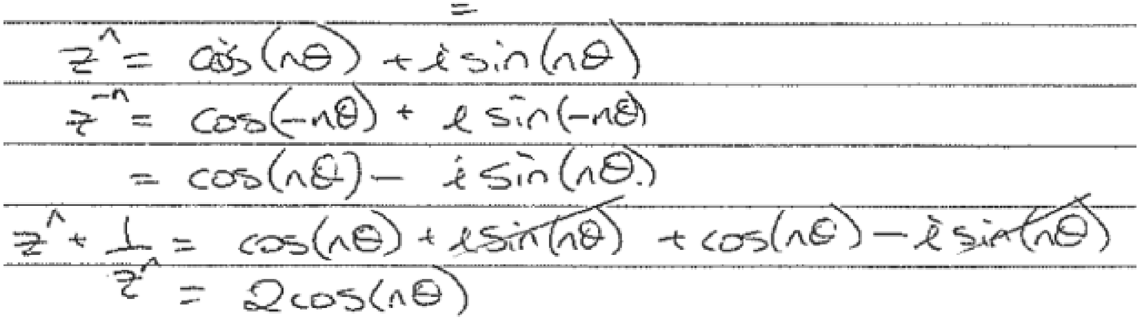
$x^3 + xy^2 - x = 0$

$\underline{\underline{x(x^2+y^2-1) = 0}}$

(ii)

locus is circle $x^2+y^2=1$ and line $x=0$ (ie y axis) excluding (0,0) (as $z \neq 0$.)

(e)



$z^n = \cos(n\theta) + i\sin(n\theta)$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta)$

$z^n + \frac{1}{z^n} = \cos(n\theta) + \cancel{i\sin(n\theta)} + \cos(n\theta) - \cancel{i\sin(n\theta)}$

$= 2\cos(n\theta)$

Question 15

(a)

$$\begin{aligned}\angle AOR &= \angle AOQ + \angle QOR \\ &= \angle AOQ + \angle AOP\end{aligned}$$

i.e. $\arg z_3 = \arg z_2 + \arg z_1$ **1**

the triangles ORQ and OPA are equiangular and hence similar

$\therefore$ their sides are proportional

$$\frac{OR}{OP} = \frac{OQ}{OA} = \frac{|z_3|}{|z_1|} = \frac{|z_2|}{|1|}$$ **1**

$$\begin{aligned}|z_3| &= \frac{|z_2||z_1|}{1} \\ &= |z_2||z_1|\end{aligned}$$

$\therefore z_3 = z_2.z$ *i.e* R represents the complex number $z_2 z_1$ **1**

(b)

slope tangent $= \frac{dy}{dx}$: (differentiating implicitly)

$$-2\sin 2x + \cos y \frac{dy}{dx} = 0$$

$$\cos y \frac{dy}{dx} = 2\sin 2x$$

$$\frac{dy}{dx} = \frac{2\sin 2x}{\cos y}$$ ☑

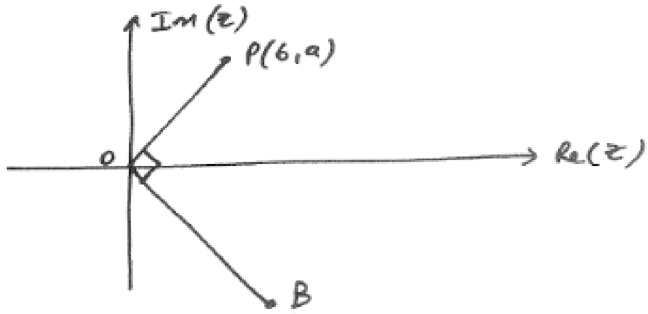
At $(\frac{\pi}{6}, \frac{\pi}{6})$, slope of tangent = 2 ☑

Equation of tangent is

$$y - \frac{\pi}{6} = 2(x - \frac{\pi}{6})$$

$$y = 2x - \frac{\pi}{6}$$ ☑

(c)



We're told $|\overrightarrow{OB}| = \frac{3}{2} \cdot |\overrightarrow{OP}|$

Thus $\overrightarrow{OB}$ is obtained from $\overrightarrow{OP}$ by a clockwise rotation of $\frac{\pi}{2}$ and a stretching by a factor of $\frac{3}{2}$.

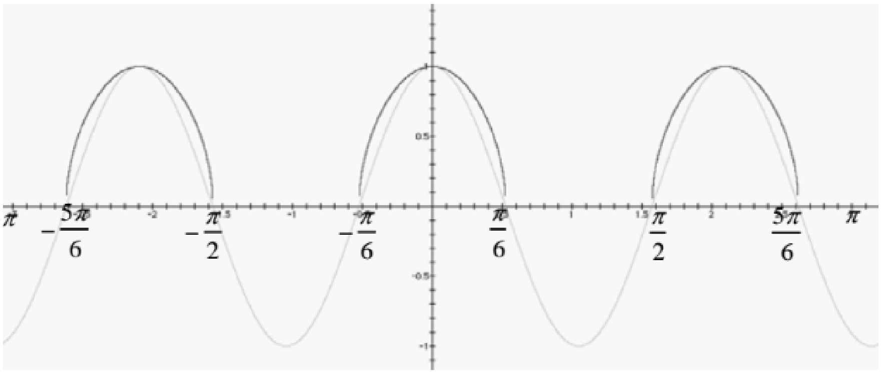
∴ $\overrightarrow{OB}$ represents $\frac{3}{2} \cdot -i \cdot (6+ai)$ ✓✓

$= -9i + \frac{3a}{2}$

∴ B represents $\frac{3a}{2} - 9i$. ✓ (final answer)

(d)

$y = \cos 3x$ ▬ $y = \sqrt{\cos 3x}$ ▬



(e)

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta \quad \text{by De-M.}$$

$$\begin{aligned}(\cos\theta + i\sin\theta)^4 &= \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta \\ &\quad + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)\end{aligned}$$

Equating reals

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

Question 16

(a)

(i) $|\alpha|^2 + |\beta|^2 - |\alpha-\beta|^2$

$$\begin{aligned}&= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)\overline{(\alpha-\beta)} \\ &= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)(\bar{\alpha}-\bar{\beta}) \\ &= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha\bar{\alpha} - \alpha\bar{\beta} - \bar{\alpha}\beta + \beta\bar{\beta}) \\ &= \alpha\bar{\beta} + \bar{\alpha}\beta \\ &= \alpha\bar{\beta} + \overline{(\alpha\bar{\beta})} = 2\,Re(\alpha\bar{\beta})\end{aligned}$$

(ii) $$\cos\theta = \frac{|\alpha|^2 + |\beta|^2 - |\alpha-\beta|^2}{2|\alpha||\beta|} \quad \text{since } \overrightarrow{BA} = \alpha-\beta$$

$$= \frac{2\,Re(\alpha\bar{\beta})}{2|\alpha||\beta|}$$

$$\Rightarrow |\alpha||\beta|\cos\theta = Re(\alpha\bar{\beta})$$

(b)

(4c) $y = \ln\left(\frac{x - \sqrt{x^2-4}}{2}\right)$

Inverse

✓ $x = \ln\left(\frac{y - \sqrt{y^2-4}}{2}\right)$

$e^x = \frac{y - \sqrt{y^2-4}}{2}$

$2e^x = y - \sqrt{y^2-4}$

$\sqrt{y^2-4} = y - 2e^x$ ✓

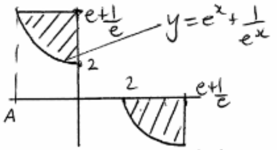
$\cancel{y^2} - 4 = \cancel{y^2} - 4ye^x + 4e^{2x}$

$4ye^x = 4e^{2x} + 4$

$y = \frac{4e^{2x}}{4e^x} + \frac{4}{4e^x}$

$y = e^x + \frac{1}{e^x}$

$y = e^x + e^{-x}$ ✓



2 areas are equal because of symmetry.

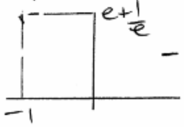
At A $y = e + \frac{1}{e}$

$\therefore e + \frac{1}{e} = e^x + \frac{1}{e^x}$

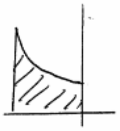
✓ $x = \pm 1$ by inspection.

here $x = -1$

$\therefore$ Area =



$-$



$= \left(e+\frac{1}{e}\right)\times 1 - \int_{-1}^{0} e^x + e^{-x}$ ✓

$= e + \frac{1}{e} - \left(e^x - e^{-x}\right)_{-1}^{0}$

$= e + \frac{1}{e} - \left(e^0 - e^0 - (e^{-1} - e^1)\right)$

$= e + \frac{1}{e} - \left(\cancel{1} - \cancel{1} - \frac{1}{e} + e\right)$

$= e + \frac{1}{e} + \frac{1}{e} - e$

$= \frac{2}{e}$ ✓

NB Award 2 for $\int_{-1}^{0} e^x + e^{-x} = e - \frac{1}{e}$

Award 2 $\int_{0}^{1} e^x + e^{-x}$

$= e - \frac{1}{e}.$

(c)

i) $P(x_1, y_1)$ $Q(x_1 + a, y_2)$ since $x^2 = 4ay$

$y = \frac{x^2}{4a}$

midpt c $\left(\frac{x_1 + x_1 + a}{2}, \frac{\frac{x_1^2}{4a} + \frac{(x_1+a)^2}{4a}}{2}\right)$ $y_2 = \frac{(x_1+a)^2}{4a}$ ✓

$= \left(\frac{2x_1 + a}{2}, \frac{x_1^2 + x_1^2 + 2ax_1 + a^2}{8a}\right)$ ✓

$= \left(\frac{2x_1 + a}{2}, \frac{2x_1^2 + 2ax_1 + a^2}{8a}\right)$

ii)

$x = \frac{2x_1+a}{2}$

$2x = 2x_1 + a$

$x_1 = \frac{2x-a}{2}$

$y = \frac{2x_1^2+2ax_1+a^2}{8a}$

$y = \frac{2\left(\frac{2x-a}{2}\right)^2 + \cancel{2}a\left(\frac{2x-a}{\cancel{2}}\right) + a^2}{8a}$ ✓

$= \frac{\cancel{2}\frac{(4x^2-4ax+a^2)}{\cancel{4}2} + 2ax - \cancel{a^2} + \cancel{a^2}}{8a}$

$= \frac{4x^2 - 4\cancel{ax} + a^2 + \cancel{4ax}}{16a}$

$= \frac{4x^2+a^2}{16a}.$ ✓

$\therefore 16ay = 4x^2+a^2$

$\underline{4x^2 = 16ay - a^2}$

$\therefore$ a <u>parabola</u> (the vertex is $(0, \frac{a}{16})$) (