

ABBOTSLEIGH

AUGUST 2010
YEAR 12
ASSESSMENT 4
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.

Outcomes assessed

HSC course

- E1** appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- E2** chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- E3** uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- E4** uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
- E5** uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
- E6** combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
- E7** uses the techniques of slicing and cylindrical shells to determine volumes
- E8** applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
- E9** communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

From the Extension 1 Mathematics Course

Preliminary course

- PE1** appreciates the role of mathematics in the solution of practical problems
- PE2** uses multi-step deductive reasoning in a variety of contexts
- PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- PE5** determines derivatives that require the application of more than one rule of differentiation
- PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- HE1** appreciates interrelationships between ideas drawn from different areas of mathematics
- HE2** uses inductive reasoning in the construction of proofs
- HE3** uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
- HE4** uses the relationship between functions, inverse functions and their derivatives
- HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- HE6** determines integrals by reduction to a standard form through a given substitution
- HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

QUESTION 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \frac{x^2}{(1+x^3)^2} dx$. **2**

(b) Find $\int \frac{x^2+4}{x^2+1} dx$. **2**

(c) Use integration by parts to evaluate $\int_0^1 x e^{-3x} dx$. **3**

(d) (i) Find real numbers a , b and c such that

$$\frac{x}{(x-1)^2(x-2)} \equiv \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x-2}.$$
2

(ii) Evaluate $\int \frac{x}{(x-1)^2(x-2)} dx$. **2**

(e) Use the substitution $x = \sin \theta$ to evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$. **4**

QUESTION 2 (15 marks) Use a SEPARATE writing booklet.

- (a) Let $z = 3 - i$ and $w = 2 + i$. Express the following in the form $x + iy$, where x and y are real numbers:

(i) $\frac{z}{w}$ **2**

(ii) $\overline{-2iz}$ **2**

(b) Let $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

(i) Express z in modulus-argument form. **2**

(ii) Show that $z^6 = 1$. **2**

(iii) Hence, or otherwise, graph all the roots of $z^6 - 1 = 0$ on an Argand diagram. **2**

- (c) The complex numbers α, β, γ and δ are represented on an Argand diagram by the points A, B, C and D respectively.

(i) Describe the point that represents $\frac{1}{2}(\alpha + \gamma)$. **1**

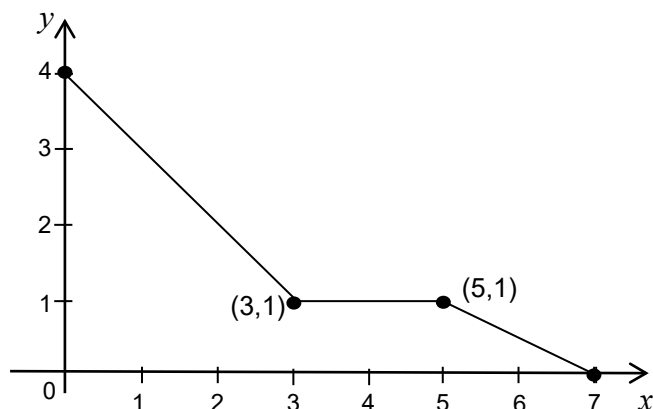
(ii) Deduce that if $\alpha + \gamma = \beta + \delta$ then $ABCD$ is a parallelogram. **2**

- (d) Let $z = x + iy$. Find the points of intersection of the curves given by:

$|z - i| = 1$ and $\operatorname{Re}(z) = \operatorname{Im}(z)$. **2**

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

- (a) The diagram below shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

- | | | |
|-------|---------------------|----------|
| (i) | $y = f(x)$ | 2 |
| (ii) | $y = f(2 - x)$ | 2 |
| (iii) | $y = \log_e f(x)$. | 2 |
-
- | | | |
|-----|---|----------|
| (b) | Sketch the graph of $y = \frac{1}{x(x-2)}$, without the use of calculus. | 3 |
|-----|---|----------|
-
- | | | |
|-----|---|----------|
| (c) | (i) Find the value of g for which $P(x) = 9x^4 - 25x^2 + 10gx - g^2$ is divisible by both $x-1$ and $x+2$. | 3 |
| | (ii) With this value of g , solve the equation $9x^4 - 25x^2 + 10gx - g^2 = 0$. | 3 |

QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

- (a) The area bounded by the curve $y = x^2 + 2$ and the line $y = 4 - x$ is rotated about the line $y = 1$.

(i) Find the points of intersection of the two curves. 2

(ii) By considering slices perpendicular to the x axis, show that the area, $A(x)$ of a typical slice is given by:

$$A(x) = \pi(8 - 6x - x^2 - x^4). \quad \text{2}$$

(iii) Find the volume of the solid formed. 2

- (b) Show that for all real x , $0 < \frac{1}{x^2 + 2x + 2} \leq 1$. 3

- (c) (i) If $I_n = \int x^3 (\log_e x)^n dx$, show that $I_n = \frac{x^4}{4} (\log_e x)^n - \frac{n}{4} I_{n-1}$. 3

(ii) Hence, or otherwise, evaluate $\int_1^2 x^3 (\log_e x)^2 dx$. 3

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

- (a) (i) Factorise the polynomial $z^3 - 1$ over the rational field. **1**
- (ii) If w is a complex root of 1, show that $1 + w + w^2 = 0$. **1**
- (iii) Hence, or otherwise, simplify $(1 + w^2)(1 + w^4)(1 + w^8)(1 + w^{10})$. **2**
- (b) Prove that if $a \neq c$ there are always two real values of k which will make $ax^2 + 2bx + c + k(x^2 + 1)$ a perfect square. **3**
- (c) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are two variable points on the hyperbola $xy = c^2$ which move so that the points P , Q and $S(c\sqrt{2}, c\sqrt{2})$ are always collinear. The tangents to the hyperbola at P and Q meet at the point R .
- (i) Show that the equation of the chord PQ is $x + pqy = c(p + q)$ **2**
- (ii) Hence show that $p + q = \sqrt{2}(1 + pq)$. **1**
- (iii) Show that R is the point $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. You may assume that the tangent at any point $T\left(ct, \frac{c}{t}\right)$ has equation $x + t^2y = 2ct$. (Do NOT prove this) **3**
- (iv) Hence find the equation of the locus of R . **2**

QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

- (a) (i) Prove that if x and y are positive numbers then $(x + y)^2 \geq 4xy$. **2**

- (ii) Deduce that if a, b, c and d are positive numbers then

$$\frac{1}{4}(a + b + c + d)^2 \geq ac + ad + bc + bd. \quad \text{2}$$

- (b) Scientists use a pressure gauge which measures depth as it sinks towards the ocean floor. The gauge of mass 2 kg is released from rest at the ocean's surface. As it sinks in a vertical line, the water exerts a resistance to its motion of $4v$ Newtons, where $v \text{ ms}^{-1}$ is the velocity of the gauge.

Let x be the displacement of the ball measured vertically downwards from the ocean's surface, t be the time in seconds elapsed after the gauge is released, and g be the constant acceleration due to gravity.

- (i) Show that $\frac{d^2x}{dt^2} = g - 2v$. **2**

- (ii) Hence show that $t = \frac{1}{2} \log_e \left(\frac{g}{g - 2v} \right)$. **3**

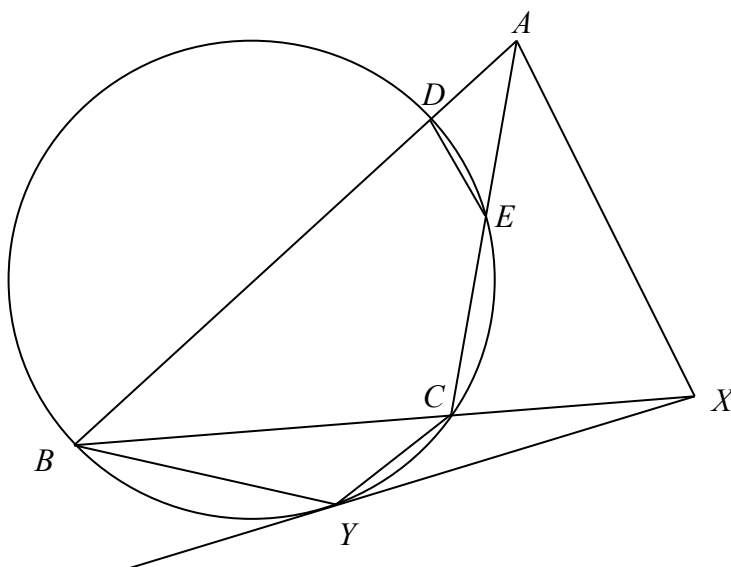
- (iii) Show that $v = \frac{g}{2} (1 - e^{-2t})$. **2**

- (iv) Write down the limiting (terminal) velocity of the gauge. **1**

- (v) At a particular location, the gauge takes 180 seconds to hit the ocean floor. Using $g = 10 \text{ ms}^{-2}$, calculate the depth of the ocean at that location, giving your answer correct to the nearest metre. **3**

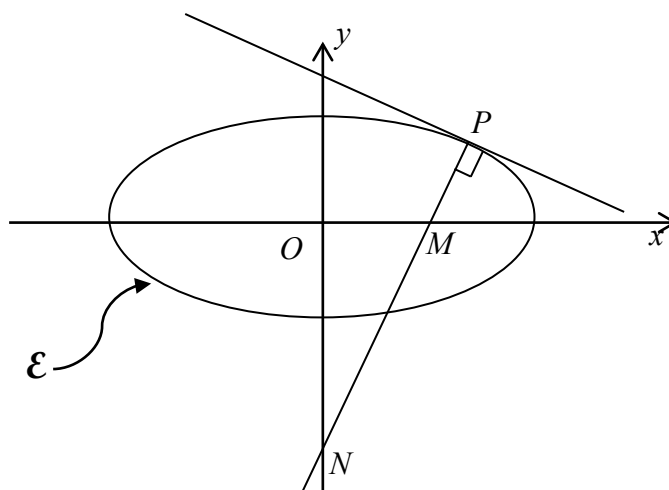
QUESTION 7 (15 marks) Use a SEPARATE writing booklet.

- (a) In the diagram XY is a tangent to the circle and $XY = XA$.



- (i) Show that $\triangle XCY \parallel \triangle XBY$. 2
- (ii) Hence explain why $\frac{XY}{BX} = \frac{CX}{XY}$. 1
- (iii) Show that $\triangle AXC \parallel \triangle AXB$. 3
- (iv) Prove that $DE \parallel AX$. 2
- (b) Consider the function $y = f(x)$ in the interval $1 \leq x \leq n$.
- (i) Sketch a possible graph of $y = f(x)$ given $f(x) \geq 0$ and $f''(x) < 0$. 1
- (ii) Show, by comparing the area under the curve $y = f(x)$ between $x = 1$ and $x = n$, with the area of a region found using repeated applications of the Trapezoidal Rule, each of width 1 unit, that
- $$\int_1^n f(x) \, dx > \frac{1}{2}f(1) + \frac{1}{2}f(n) + \sum_{r=2}^{n-1} f(r).$$
- 2
- (iii) By taking $f(x) = \log_e x$ in the inequality from (b) part (ii) above, deduce that if n is a positive integer, then
- $$n! < n^{\frac{n+1}{2}} e^{-n+1}.$$
- 4

(a)



The ellipse $\mathcal{E}$ has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The point $P(a \cos \theta, b \sin \theta)$ lies on the ellipse.

(i) Show that the equation of the normal to the ellipse at P is

$$y - b \sin \theta = \frac{a \sin \theta}{b \cos \theta} (x - a \cos \theta).$$

2

(ii) The normal at P meets the x axis at M and the y axis at N as shown in the diagram above.

Prove that $\frac{PM}{PN} = 1 - e^2$ where e is the eccentricity of $\mathcal{E}$.

3

(b) If $A(x) = \frac{1}{2} + \frac{1}{3} \binom{n}{1} x + \frac{1}{4} \binom{n}{2} x^2 + \dots + \frac{1}{n+2} x^n$,

(i) Show that $\frac{d}{dx} \{x^2 A(x)\} = x(1+x)^n$.

3

(ii) Show that $x(1+x)^n = (1+x)^{n+1} - (1+x)^n$.

1

(iii) Hence show that $x^2 A(x) = \frac{(1+x)^{n+2} - 1}{n+2} - \frac{(1+x)^{n+1} - 1}{n+1}$.

3

(iv) Deduce that $\sum_{r=0}^n \frac{1}{r+2} \binom{n}{r} = \frac{n \cdot 2^{n+1} + 1}{(n+1)(n+2)}$.

3

End of paper

ABBOTSLEIGH EXTENSION 2 TRIAL 2010 SOLUTIONS

QUESTION 1

$$\begin{aligned} (a) \int \frac{x^2}{(1+x^3)^2} dx &= \frac{1}{3} \int 3x^2 (1+x^3)^{-2} dx \\ &= \frac{1}{3} \frac{(1+x^3)^{-1}}{-1} + C \\ &= \frac{-1}{3(1+x^3)} + C \end{aligned}$$

$$\begin{aligned} (b) \int \frac{x^2+4}{x^2+1} dx &= \int \frac{x^2+1+3}{x^2+1} dx \\ &= \int \left(1 + \frac{3}{x^2+1}\right) dx \\ &= x + 3 \tan^{-1} x + C \end{aligned}$$

$$\begin{aligned} (c) \text{ Let } I &= \int_0^1 x e^{-3x} dx \\ \text{Let } u &= x \quad du = dx \\ dv &= e^{-3x} \quad v = -\frac{1}{3} e^{-3x} \end{aligned}$$

$$\begin{aligned} \therefore I &= \left[x \times -\frac{1}{3} e^{-3x} \right]_0^1 - \int_0^1 -\frac{1}{3} e^{-3x} dx \\ &= -\frac{1}{3} e^{-3} + 0 + \left[\frac{1}{3} \times \frac{e^{-3x}}{-3} \right]_0^1 \\ &= -\frac{1}{3e^3} + \frac{1}{9} (e^{-3} - e^0) \\ &= \frac{1}{9} \times \left(\frac{-1}{e^3} - \frac{3}{e^3} + 1 \right) \\ &= \frac{e^3 - 4}{9e^3} \end{aligned}$$

$$\begin{aligned} (d) (i) \quad x &= a(x-1)(x-2) + b(x-2) + c(x-1)^2 \\ \text{Let } x &= 1, \quad 1 = -b \Rightarrow b = -1 \\ \text{Let } x &= 2, \quad 2 = c \Rightarrow c = 2 \\ \text{Let } x &= 0, \quad 0 = a \times 2 - 1 \times 2 + 2 \times 1 \Rightarrow \\ &\quad \therefore a = -2 \end{aligned}$$

$$\begin{aligned} (ii) \therefore \int \frac{x dx}{(x-1)^2(x-2)} &= \int \left\{ \frac{-2}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x-2} \right\} dx \\ &= -2 \ln(x-1) + \frac{1}{x-1} + 2 \ln(x-2) \\ &= \frac{1}{x-1} + 2 \ln \left(\frac{x-2}{x-1} \right) + C \end{aligned}$$

$$\begin{aligned} Q(e) \text{ Let } I &= \int_{-1/2}^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx \\ &= 2 \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx \quad (\text{even fn}) \end{aligned}$$

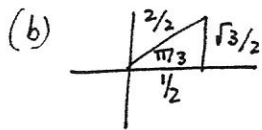
$$\begin{aligned} \text{Let } x &= \sin \theta & x &= 1/2, & \theta &= \pi/6 \\ dx &= \cos \theta d\theta & x &= 0, & \theta &= 0 \end{aligned}$$

$$\begin{aligned} \therefore I &= 2 \int_0^{\pi/6} \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \times \cos \theta d\theta \\ &= 2 \int_0^{\pi/6} \frac{\sin^2 \theta}{\cos \theta} \times \cancel{\cos \theta} d\theta \\ &= 2 \int_0^{\pi/6} \frac{1}{2} (1 - \cos 2\theta) d\theta \\ &= \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/6} \\ &= \left(\frac{\pi}{6} - \frac{\sin \pi/3}{2} \right) - \left(0 - \frac{\sin 0}{2} \right) \\ &= \frac{\pi}{6} - \frac{\sqrt{3}/2}{2} - 0 \\ &= \frac{1}{12} (2\pi - 3\sqrt{3}) \end{aligned}$$

Q2 (a)

$$\begin{aligned} (i) \frac{z}{w} &= \frac{3-i}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{6-5i-1}{4+1} \\ &= \frac{5-5i}{5} \\ &= 1-i \end{aligned}$$

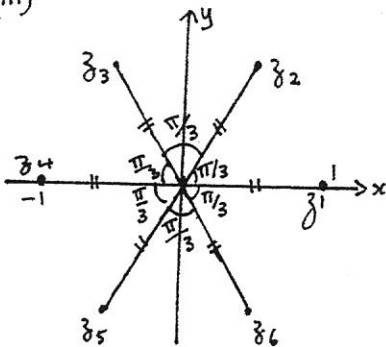
$$\begin{aligned} (ii) \overline{-2iz} &= \overline{-2i(3-i)} \\ &= \overline{-6i-2} \\ &= -2+6i \end{aligned}$$



$$(i) z = 1 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$

$$\begin{aligned} (ii) z^6 &= \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^6 \\ &= \cos 2\pi + i \sin 2\pi \\ &= 1 + i \times 0 \\ &= 1 \end{aligned}$$

(iii)



(c) (i) $\frac{1}{2}(\alpha+\delta)$ represents the midpoint of AC.

$$\begin{aligned} (ii) \text{ if } \alpha+\delta &= \beta+\delta \\ \text{ then } \frac{\alpha+\delta}{2} &= \frac{\beta+\delta}{2} \end{aligned}$$

$\therefore$ midpoint of AC = midpoint of BD

$\therefore$ ABCD is a parallelogram since its diagonals bisect each other.

Q2 (d) $|z-i|=1$ represents the circle $x^2+(y-1)^2=1$ ①
 $\operatorname{Re}(z) = \operatorname{Im}(z)$ represents the line $x=y$ ②

Substitute ② into ①

$$y^2 + (y-1)^2 = 1$$

$$y^2 + y^2 - 2y + 1 = 1$$

$$2y^2 - 2y = 0$$

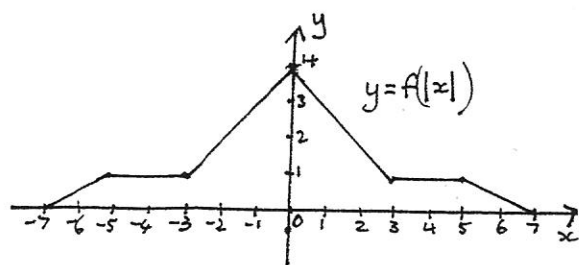
$$y(y-1) = 0$$

$$y = 0 \text{ or } 1$$

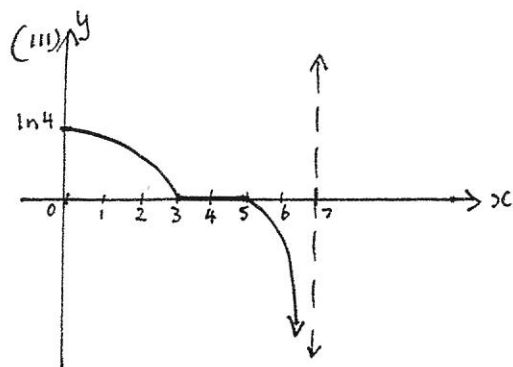
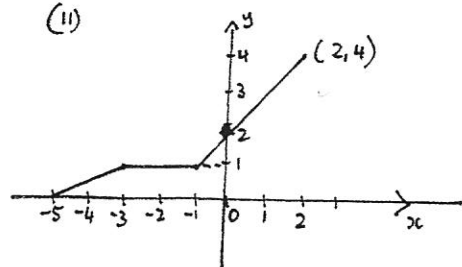
$$y=0, x=0 \text{ or } y=1, x=1$$

$\therefore$ Pts of intersection are $(0,0)$ and $(1,1)$

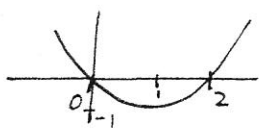
Q 3(a) (i)



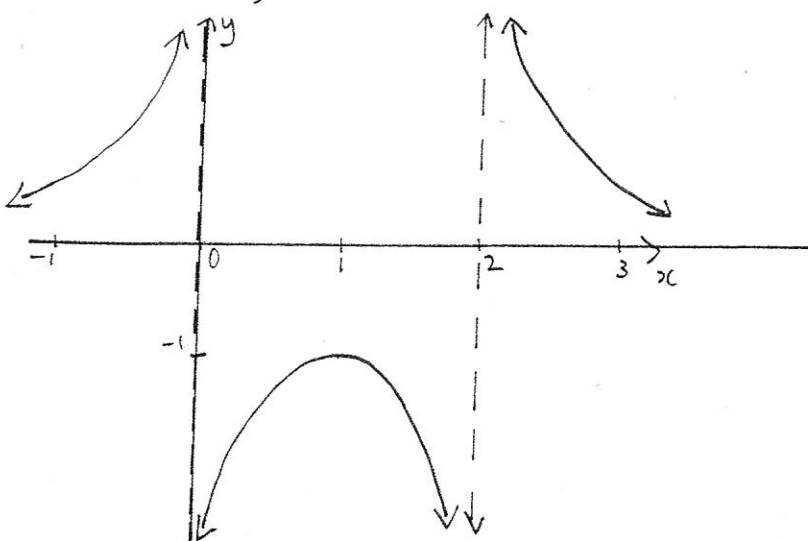
(ii)



(b) $y = x(x-2)$



$$y = \frac{1}{x(x-2)}$$



Q 3 (c) (i) $P(1) = P(2) = 0$

$$\therefore 9 - 25 + 10g - g^2 = 0$$

$$g^2 - 10g + 16 = 0$$

$$(g-8)(g-2) = 0$$

$$\therefore g = 8 \text{ or } 2$$

$$\text{also } 144 - 100 - 20g - g^2 = 0$$

$$g^2 + 20g - 44 = 0$$

$$(g+22)(g-2) = 0$$

$$g = -22 \text{ or } 2$$

$$\therefore P(1) = P(-2) = 0 \text{ only}$$

$$\text{if } g = 2$$

$$(ii) 9x^4 - 25x^2 + 20x - 4 = (x-1)(x+2)R(x)$$

$$= (x^2 + x - 2)(9x^2 + mx + 2)$$

$$x^3 \text{ term: } 0 = 9 + m$$

$$\therefore m = -9$$

$$\therefore P(x) = (x-1)(x+2)(9x^2 - 9x + 2)$$

$$= (x-1)(x+2)(3x-1)(3x-2)$$

$$\therefore \text{Solutions are } x = 1, -2, \frac{1}{3}, \frac{2}{3}$$

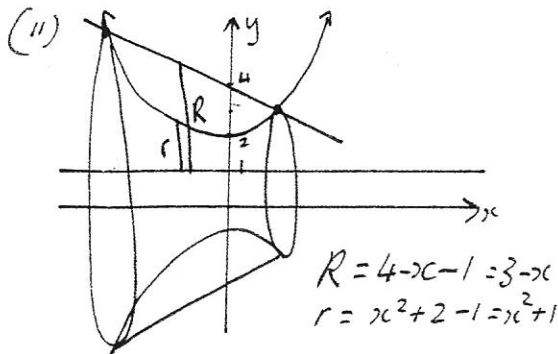
Q4(a) (i) $x^2 + 2 = 4 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$\therefore$ Pts of intersection are

$(1, 3)$ and $(-2, 6)$



Area of slice $= \pi R^2 - \pi r^2$

$A(x) = \pi(3-x)^2 - \pi(x^2+1)^2$

$= \pi \{ 9 - 6x + x^2 - x^4 - 2x^2 - 1 \}$

$= \pi \{ 8 - 6x - x^2 - x^4 \}$

(iii) Vol of slice $= A(x) \delta x$

Vol of solid $= \sum_{x=-2}^1 \pi(8-6x-x^2-x^4) \delta x$

$= \pi \int_{-2}^1 (8-6x-x^2-x^4) dx$

$= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^1$

$= \pi \left(8 - 3 - \frac{1}{3} - \frac{1}{5} \right) - \pi \left(-16 - 12 + \frac{8}{3} + \frac{32}{5} \right)$

$= \pi \left(5 - \frac{8}{15} + 28 - \frac{136}{15} \right)$

$= \frac{117\pi}{5}$ cubic units

b) $x^2 + 2x + 2 = (x+1)^2 + 1$

min value of $x^2 + 2x + 2$ is 1

$\therefore$ max value of $\frac{1}{x^2 + 2x + 2}$ is 1

Also, $\lim_{x \rightarrow \infty} \frac{1/x^2}{1 + 2/x + 1/x^2} = \frac{0}{1+0+0} = 0$

$\therefore$ min value of $\frac{1}{x^2 + 2x + 2} = 0$ (as a limit)

$\therefore 0 < \frac{1}{x^2 + 2x + 2} \leq 1$

4(c) (i) $I_n = \int x^3 (\log_e x)^n dx$ let $u = (\log_e x)^n$
 $du = n(\log_e x)^{n-1} \cdot \frac{1}{x} \cdot dx$
 $dv = x^3 dx$
 $v = \frac{x^4}{4}$

$\therefore I_n = (\log_e x)^n \frac{x^4}{4} - \int \frac{x^4}{4} \times n(\log_e x)^{n-1} \times \frac{1}{x} dx$

$= \frac{x^4}{4} (\log_e x)^n - \frac{n}{4} \int x^3 (\log_e x)^{n-1} dx$

$I_n = \frac{x^4}{4} (\log_e x)^n - \frac{n}{4} I_{n-1}$

(ii) $I_2 = \left[\frac{x^4}{4} (\log_e x)^2 \right]_1^2 - \frac{1}{2} I_1$

$= \frac{16}{4} (\log_e 2)^2 - 0 - \frac{1}{2} \left\{ \left[\frac{x^4}{4} (\log_e x) \right]_1^2 - \frac{1}{4} I_0 \right\}$

$= 4(\log_e 2)^2 - \frac{1}{8} (16(\log_e 2) - 0) + \frac{1}{8} \int_1^2 x^3 dx$

$= 4(\log_e 2)^2 - 2\log_e 2 + \left[\frac{x^4}{32} \right]_1^2$

$= 4(\log_e 2)^2 - 2\log_e 2 + \frac{1}{2} - \frac{1}{32}$

$= 4(\log_e 2)^2 - 2\log_e 2 + \frac{15}{32}$

Q5 (a) (i) $z^3 - 1 = (z-1)(z^2+z+1)$

(ii) If ω is a complex root then

$$\omega - 1 = 0 \Rightarrow \omega = 1 \text{ is not complex}$$

$$\therefore \omega^2 + \omega + 1 = 0$$

$$\text{also } \omega^3 = 1$$

$$\begin{aligned} \text{(iii)} & (1+\omega^2)(1+\omega)(1+\omega^8)(1+\omega^{10}) \\ &= (-\omega)(1+\omega \cdot \omega^3)(1+(\omega^3)^2 \cdot \omega^2)(1+(\omega^3)^3 \cdot \omega) \\ &= (-\omega)(1+\omega)(1+\omega^2)(1+\omega) \\ &= -\omega \times -\omega^2 \times -\omega \times -\omega^2 \\ &= \omega^3 \times \omega^3 \\ &= 1 \end{aligned}$$

(b) Re-arranging quadratic,

$$\begin{aligned} ax^2 + 2bx + c + k(x^2 + 1) \\ = x^2(a+k) + 2bx + (c+k) \end{aligned}$$

For a p.s., $\Delta = 0$

$$4b^2 - 4(a+k)(c+k) = 0$$

$$b^2 - ac - ak - ck - k^2 = 0$$

$$k^2 + k(a+c) + ac - b^2 = 0$$

If there are 2 values of k ,

$$\text{this } \Delta > 0$$

$$\begin{aligned} \Delta &= (a+c)^2 - 4(ac - b^2) \\ &= a^2 + 2ac + c^2 - 4ac + 4b^2 \\ &= a^2 - 2ac + c^2 + 4b^2 \\ &= (a-c)^2 + 4b^2 \end{aligned}$$

which is always > 0 if $a \neq c$

$\therefore$ There are always 2 values of k which make the quadratic a perfect square.

5 (c) (i) $P(cp, \frac{c}{p}) \quad Q(cq, \frac{c}{q})$

$$\frac{y - \frac{c}{p}}{x - cp} = \frac{\frac{c}{q} - \frac{c}{p}}{cq - cp}$$

$$\therefore = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{q-p}{pq}} \times \frac{pq}{pq}$$

$$= \frac{p-q}{pq(q-p)}$$

$$= -\frac{1}{pq}$$

$$\therefore pqy - cq = -x + cp$$

$$\therefore x + pqy = c(p+q)$$

is the equation of PQ

(ii) since $S(c\sqrt{2}, c\sqrt{2})$ lies on PQ,

$$c\sqrt{2} + pq \times c\sqrt{2} = c(p+q)$$

$$\therefore \sqrt{2}(1+pq) = p+q$$

(iii) tangent at P is $x + p^2y = 2cp$ ①

tangent at Q is $x + q^2y = 2cq$ ②

$$\text{①} - \text{②} \quad y(p^2 - q^2) = 2c(p - q)$$

$$y = \frac{2c(p-q)}{(p-q)(p+q)}$$

$$\therefore y = \frac{2c}{p+q} \text{ into ①}$$

$$x + p^2 \times \frac{2c}{p+q} = 2cp$$

$$\begin{aligned} x &= 2cp - \frac{2cp^2}{p+q} \\ &= \frac{2cp^2 + 2cpq - 2cp^2}{p+q} \\ &= \frac{2cpq}{p+q} \end{aligned}$$

$$\therefore R\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

$$\text{(iv)} \quad x = \frac{2cpq}{p+q} \text{ ①} \quad y = \frac{2c}{p+q} \text{ ②}$$

$$\text{①} + \text{②} \quad x + y = \frac{2c}{p+q}(pq+1)$$

$$\begin{aligned} x + y &= \frac{2c}{p+q} \times \frac{p+q}{\sqrt{2}} \text{ (from (ii))} \\ &= \frac{2c}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \end{aligned}$$

$$\therefore x + y = c\sqrt{2}$$

Q6 (a) (i) $(x-y)^2 \geq 0$

$$x^2 - 2xy + y^2 \geq 0$$

$$x^2 + y^2 \geq 2xy$$

$$x^2 + 2xy + y^2 \geq 4xy$$

$$\therefore (x+y)^2 \geq 4xy$$

(ii) let $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$

$$\therefore \left(\frac{a+b+c+d}{2} \right)^2 \geq 4 \left(\frac{a+b}{2} \right) \left(\frac{c+d}{2} \right)$$

$$\frac{1}{4} (a+b+c+d)^2 \geq ac+ad+bc+bd$$

(b) (i) $F = ma$

$$2g - 4v = 2 \times \frac{d^2x}{dt^2}$$

$$\therefore \frac{d^2x}{dt^2} = g - 2v$$

(ii) $\therefore \frac{dv}{dt} = g - 2v$

$$\frac{dt}{dv} = \frac{1}{g-2v}$$

$$t = \int_0^v \frac{1}{g-2v} dv \quad \text{since starts at rest.}$$

$$= \left[-\frac{1}{2} \ln(g-2v) \right]_0^v$$

$$= -\frac{1}{2} \ln(g-2v) + \frac{1}{2} \ln g$$

$$\therefore t = \frac{1}{2} \ln \frac{g}{g-2v}$$

(iii) From (ii) $2t = \ln \frac{g}{g-2v}$

$$e^{2t} = \frac{g}{g-2v}$$

$$\frac{g-2v}{g} = e^{-2t}$$

$$g-2v = g e^{-2t}$$

$$2v = g(1 - e^{-2t})$$

$$\therefore v = \frac{g}{2} (1 - e^{-2t})$$

(iv) From (iii) as $t \rightarrow \infty$ $e^{-2t} \rightarrow 0$

$$\therefore v \rightarrow \frac{g}{2} \text{ m/s (terminal velocity)}$$

(v) From (iii) $\frac{dx}{dt} = \frac{g}{2} (1 - e^{-2t})$, $g = 10$

$$x = 5 \int_0^{180} (1 - e^{-2t}) dt$$

$$= \left[5t + \frac{5e^{-2t}}{2} \right]_0^{180}$$

$$= 5 \times 180 + 2.5 e^{-360} - 0 - \frac{5}{2} e^0$$

$$\doteq 897.5$$

$$\doteq 898 \text{ metres deep.}$$

Q7(a) (i) In $\Delta^s XCY$ and XBY ,

$\angle X$ is common

$\angle CYX = \angle CBY$ ($\angle$ in the alternate segment thm)

$\therefore \Delta XCY \parallel \Delta XBY$ (equiangular)

(ii) $\frac{XY}{BX} = \frac{CX}{XY}$ because they are pairs of corresponding sides in the similar Δ^s in part (i).

(iii) $XY = AX$ (given)

$$\therefore \frac{AX}{BX} = \frac{CX}{AX}$$

also, in $\Delta^s AXC$ and AXB ,
 $\angle X$ is common,

$\therefore \Delta AXC \parallel \Delta AXB$ (2 pairs of sides in same ratio and the included angle is equal)

(iv) $\angle ACX = \angle BAX$

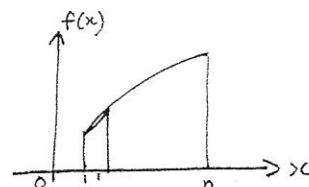
(Corresp. angles in similar Δ^s)

$\angle BDE = \angle ACX$ (ext $\angle$ of a cyclic quad. = opp. interior $\angle$)

$$\therefore \angle BAX = \angle BDE$$

$\therefore DE \parallel AX$ (corresp. $\angle^s$ are equal)

(b) (i)



$$\begin{aligned} \text{(ii)} \int_1^n f(x) dx &> \frac{1}{2} \times 1 \times (f(1) + f(2)) + \frac{1}{2} \times 1 \times (f(2) + f(3)) \\ &+ \dots + \frac{1}{2} \times 1 \times (f(n-1) + f(n)) \\ &= \frac{1}{2} [f(1) + f(n)] + f(2) + f(3) + \dots + f(n-1) \\ &= \frac{1}{2} f(1) + \frac{1}{2} f(n) + \sum_{r=1}^{n-1} f(r) \end{aligned}$$

7b) (iii) Let $f(x) = \log_e x$ in (ii)

$$\int_1^n \log_e x \, dx > \frac{1}{2} \ln 1 + \frac{1}{2} \ln n + \ln 2 + \ln 3 + \dots + \ln(n-1) \\ = 0 + \frac{1}{2} \ln n + \ln 2 \times 3 \times \dots \times (n-1)$$

Now Let $I = \int_1^n \log_e x \, dx$ $u = \log_e x$ $du = \frac{1}{x} dx$ $v = x$

$$\therefore \int_1^n \log_e x \, dx = [x \log_e x]_1^n - \int_1^n x \times \frac{1}{x} dx \\ = n \log_e n - 1 \times \log_e 1 - \int_1^n dx \\ = n \log_e n - [x]_1^n \\ = n \log_e n - n + 1$$

$$\therefore n \log_e n - n + 1 > \frac{1}{2} \log_e n + \log_e (n-1)!$$

$$\ln n^n - n + 1 > \ln n^{1/2} + \ln (n-1)!$$

$$\therefore -n + 1 > \ln n^{1/2} + \ln (n-1)! - \ln n^n \\ = \ln \left(\frac{n^{1/2} \times (n-1)!}{n^n} \right)$$

$$\therefore e^{-n+1} > \frac{n^{1/2-n} \times n!}{n^n} \\ = n^{-1/2-n} \times n!$$

$$n! < e^{1-n} \times n^{1/2+n}$$

$$\therefore n! < n^{n+1/2} \cdot e^{-n+1}$$

Q8 (a) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

at $P(a \cos \theta, b \sin \theta)$, m. of tang:

$$\frac{2a \cos \theta}{a^2} + \frac{2b \sin \theta}{b^2} \times \frac{dy}{dx} = 0$$

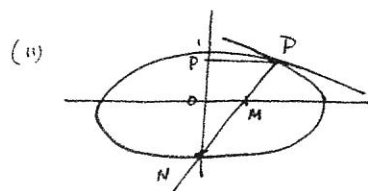
$$\frac{2 \cos \theta}{a} + \frac{2 \sin \theta}{b} \times \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\cos \theta}{a} \times \frac{b}{\sin \theta}$$

$$\therefore \text{m of normal at } P = \frac{a \sin \theta}{b \cos \theta}$$

$\therefore$ eq'n of normal at P is:

$$y - b \sin \theta = \frac{a \sin \theta}{b \cos \theta} (x - a \cos \theta)$$



$$\frac{PM}{PN} = \frac{P'O}{P'N} \quad (\text{ratio of intercepts on parallel lines})$$

$$P'(0, b \sin \theta)$$

at N, $x = 0$ on normal PN

$$\therefore y - b \sin \theta = \frac{a \sin \theta}{b \cos \theta} x - a \cos \theta$$

$$y = b \sin \theta - \frac{a^2 \sin \theta}{b}$$

$$= \frac{\sin \theta}{b} (b^2 - a^2)$$

$$\therefore N(0, \frac{\sin \theta}{b} (b^2 - a^2))$$

$$\therefore \frac{P'O}{P'N} = \frac{b \sin \theta}{b \sin \theta - \frac{\sin \theta}{b} (b^2 - a^2)}$$

$$= \frac{b^2}{b^2 - (b^2 - a^2)}$$

$$= \frac{b^2}{a^2} \\ = 1 - e^2$$

$$\therefore \frac{PM}{PN} = 1 - e^2$$

8 (b) (i) Show $\frac{d}{dx} \{x^2 A(x)\} = x(1+x)^n$

$$\begin{aligned} \text{LHS} &= \frac{d}{dx} \left\{ x^2 \left(\frac{1}{2} + \frac{1}{3} \binom{n}{1} x + \frac{1}{4} \binom{n}{2} x^2 + \dots + \frac{1}{n+2} x^n \right) \right\} \\ &= \frac{d}{dx} \left\{ \frac{x^2}{2} + \frac{1}{3} \binom{n}{1} x^3 + \frac{1}{4} \binom{n}{2} x^4 + \dots + \frac{1}{n+2} x^{n+2} \right\} \\ &= \frac{2x}{2} + \frac{1}{3} \binom{n}{1} 3x^2 + \frac{1}{4} \binom{n}{2} 4x^3 + \dots + \frac{1}{n+2} (n+2) x^{n+1} \\ &= x + \binom{n}{1} x^2 + \binom{n}{2} x^3 + \dots + x^{n+1} \\ \text{RHS} &= x \left\{ 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \dots + x^n \right\} \\ &= x + \binom{n}{1} x^2 + \binom{n}{2} x^3 + \dots + x^{n+1} \\ &= \text{LHS} \end{aligned}$$

(ii) Show $x(1+x)^n = (1+x)^{n+1} - (1+x)^n$

$$\begin{aligned} \text{RHS} &= (1+x)^n (1+x-1) \\ &= x(1+x)^n \\ &= \text{LHS} \end{aligned}$$

(iii) $\therefore$ From (i) $\frac{d}{dx} \{x^2 A(x)\} = (1+x)^{n+1} - (1+x)^n$

Int b.s. $x^2 A(x) = \frac{(1+x)^{n+2}}{n+2} - \frac{(1+x)^{n+1}}{n+1} + C$

Let $x=0$ $0 = \frac{1}{n+2} - \frac{1}{n+1} + C$

$$\therefore C = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\begin{aligned} \therefore x^2 A(x) &= \frac{(1+x)^{n+2}}{n+2} - \frac{(1+x)^{n+1}}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} \\ &= \frac{(1+x)^{n+2} - 1}{(n+2)} - \frac{(1+x)^{n+1} - 1}{(n+1)} \end{aligned}$$

(iv) Let $x=1$ in (iv)

$$A(1) = \frac{2^{n+2} - 1}{n+2} - \frac{2^{n+1} - 1}{n+1}$$

Also let $x=1$ in original eq'n

$$A(1) = \frac{1}{2} + \frac{1}{3} \binom{n}{1} + \frac{1}{4} \binom{n}{2} + \dots + \frac{1}{n+2} \binom{n}{n}$$

$$\therefore \frac{1}{2} + \frac{1}{3} \binom{n}{1} + \frac{1}{4} \binom{n}{2} + \dots + \frac{1}{n+2} \binom{n}{n} = \frac{2^{n+2} - 1}{n+2} - \frac{2^{n+1} - 1}{n+1}$$

$$\begin{aligned} \therefore \sum_{r=0}^n \frac{1}{r+2} \binom{n}{r} &= \frac{(2^{n+2} - 1)(n+1) - (2^{n+1} - 1)(n+2)}{(n+2)(n+1)} \\ &= \frac{2^{n+2}(n+1) - (n+1) - 2^{n+1}(n+2) + (n+2)}{(n+2)(n+1)} \\ &= \frac{2^{n+1} \times (2n+2) - 2^{n+1}(n+2) - n - 1 + n+2}{(n+2)(n+1)} \\ &= \frac{2^{n+1}(n) + 1}{(n+1)(n+2)} \end{aligned}$$