Student's Name: ____________________

Student Number: | | | | | | | | | |

Teacher's Name: ____________________

TEMPUS CELERIUS RADIO FUGIT

ABBOTSLEIGH

2024

HIGHER SCHOOL CERTIFICATE

Assessment 4

Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen.
- **NESA approved** calculators may be used.
- **NESA approved** reference sheet is provided.
- All necessary working should be shown in every question.
- Make sure your HSC candidate Number is on the front cover of each booklet.
- Start a new booklet for Each Question.
- Answer the Multiple Choice questions on the answer sheet provided.
- If you do not attempt a whole question, you must still hand in the Writing Booklet, with the words **'NOT ATTEMPTED'** written clearly on the front cover.

Total marks – 100

- Attempt Sections I and II.
- All questions are of equal value.

Section I Pages 3 - 8

10 marks

- Attempt Questions 1–10.
- Allow about 15 minutes for this section.

Section II Pages 9 - 17

90 marks

- Attempt Questions 11– 16.
- Allow about 2 hrs and 45 minutes for this section.

Outcomes to be assessed:

Mathematics Extension 2

HSC :
A student

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

MEX12-3 uses vectors to model and solve problems in two and three dimensions

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

MEX12-5 applies techniques of integration to structured and unstructured problems

MEX12-6 uses mechanics to model and solve practical problems

MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

SECTION I

10 marks
Attempt Questions 1 – 10
Use the multiple-choice answer sheet
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

(A) ○ (B) ● (C) ○ (D) ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) ● (B) ~~●~~ (C) ○ (D) ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

correct

(A) ~~●~~ (B) ~~●~~ (C) ○ (D) ○

---

1. Consider the statement: "If you understand the chain rule for differentiation it will reduce the number of formulas for derivatives you need to remember".

The converse of this statement is:

A. If you don't want to remember formulas, understand the chain rule.

B. If you have reduced the number of formulas for derivatives that you remember, then you understand the chain rule.

C. If you haven't reduced the number of derivative formulas you need to remember, you don't understand the chain rule.

D. If you don't understand the chain rule for differentiation you will need to remember a lot of formulas.

2. What is the magnitude of the vector

$$\sin\theta \underset{\sim}{i} + \cot\theta \underset{\sim}{j} + \cos\theta \underset{\sim}{k}, \quad 0 < \theta < \frac{\pi}{2}?$$

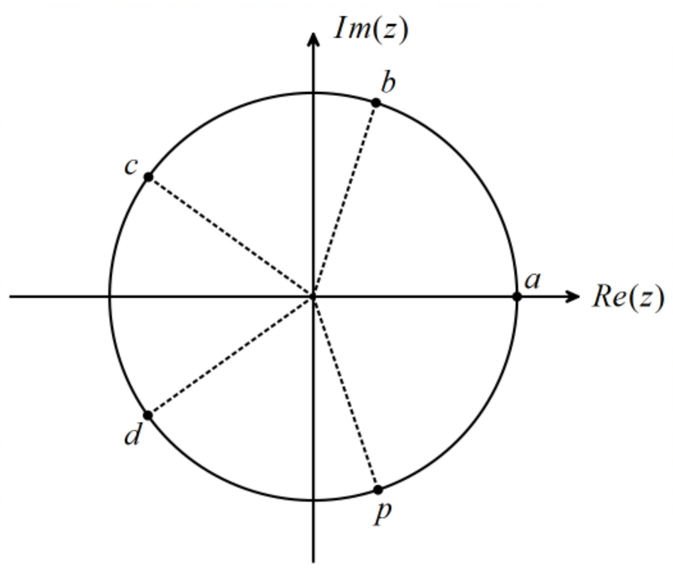
A. 1

B. $\operatorname{cosec}\theta$

C. $\cot\theta$

D. $\sec\theta$

3. If a, b, c, d and p are the fifth roots of unity as indicated on the diagram below, which of the points represents the square root of p?



A. a

B. b

C. c

D. d

4. Which of the following is a correct expression for $\int \frac{2x+1}{\sqrt{9-x^2}}\,dx$

A. $\sin^{-1}\left(\frac{x}{3}\right)-\sqrt{9-x^2}+C$

B. $\cos^{-1}\left(\frac{x}{3}\right)-2\sqrt{9-x^2}+C$

C. $\sin^{-1}\left(\frac{x}{3}\right)-2\sqrt{9-x^2}+C$

D. $-\cos^{-1}\left(\frac{x}{3}\right)-\sqrt{9-x^2}+C$

5. z satisfies the equation $\left|z-\sqrt{3}+i\right|=1$. The minimum value of $\arg(z)$ is

A. 0°

B. -30°

C. -60°

D. -75°

6. $OACB$ is a rectangle with $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 4 \\ b \end{pmatrix}$.

Which of the following is the vector $\overrightarrow{OC}$?

A. $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$

B. $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

C. $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

D. $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

7. Which of the following is a symbolic representation of the statement:

"At least one of your friends is perfect"

Let $P(x)$ be "x is perfect" and let $F(x)$ be "x is your friend" and let the domain be all people.

A. $\forall x \left(F(x) \rightarrow P(x)\right)$

B. $\forall x \left(F(x) \cap P(x)\right)$

C. $\exists x \left(F(x) \cap P(x)\right)$

D. $\exists x \left(F(x) \rightarrow P(x)\right)$

8. A particle of mass 1 kg is projected vertically upwards with a speed u from the origin O. The particle is subject to a constant gravitational force and a resistance that is proportional to half the square of its velocity $v\ ms^{-1}$, the constant of proportionality being k.

Let x be the displacement in metres of the particle above O at time t seconds after the particle is projected and g be the acceleration due to gravity.

Which of the following expressions gives the maximum height reached by the particle?

A. $$\int_u^0 \frac{dv}{-g-\frac{k}{2}v^2}$$

B. $$\int_u^0 \frac{v\ dv}{-g-\frac{k}{2}v^2}$$

C. $$\int_u^0 \frac{dv}{-g+\frac{k}{2}v^2}$$

D. $$\int_u^0 \frac{v\ dv}{-g+\frac{k}{2}v^2}$$

9. What is the value of $\int_1^e x^3 \ln(x)\, dx$

A. $\frac{3e^2}{2}-1$

B. $\frac{3e^2}{2}+1$

C. $\frac{1}{16}\left(3e^4-1\right)$

D. $\frac{1}{16}\left(3e^4+1\right)$

10. Consider the spheres:

$$S_1:(x+1)^2+(y-3)^2+(z+2)^2=4$$
$$S_2:(x-1)^2+(y+1)^2+(z-2)^2=r^2$$

If these spheres are tangential to each other, what are the possible values of r?

A. $r=2$ or 4

B. $r=4$ or 8

C. $r=2$ or 8

D. $r=4$ or 6

End of Section I

SECTION II

Total Marks – 90
Attempt Questions 11 - 16
All questions are of equal value

Answer each question in a **SEPARATE** writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve: $z^2 + z + 3 = 0, \quad z \in \mathbb{C}$, giving your answers in Cartesian form. **2**

(b) Find the equation of the line that is parallel to $r(\lambda) = \left(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}\right) + \lambda\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}\right)$

that passes through the point $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. **2**

(c) Prove that $n^3 - n$ is divisible by 6. **3**

(d) By considering Euler's representation of a complex number, find the exact value of

$$\sqrt{-1}^{\sqrt{-1}}.$$ **2**

(e) Prove that the sum of two irrational numbers is an irrational number or disprove by counter-example. **2**

(f) Find $\displaystyle\int \frac{x+16}{x^2+2x-8}\,dx$ **4**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove that $\log_2 5$ is irrational. **3**

(b) Find **3**

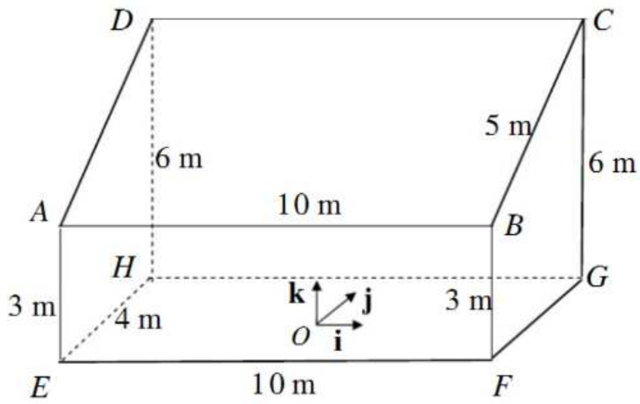
$$\int_0^{\pi} \left(\frac{\cot\theta + 1}{\operatorname{cosec}\theta} - \frac{\sec\theta}{\tan\theta + \cot\theta} \right) d\theta.$$

(c) Given $i(\omega + 2) = z$ and $\omega z = -4i$ where $\operatorname{Im}(\omega) < 0$ **3**

Find the exact values of w and z in the form $a + ib$ where a and b are real.

Question 12 continues on page 11

(d) The diagram below shows a schematic of the design of a garage.



NOT TO SCALE

ABCD is an inclined rectangular roof, where $AB = 10$ m and $AD = 5$ m. *EFGH* is a rectangle on horizontal ground, where $EF = 10$ m and $EH = 4$ m. The points *A* and *B* are 3 m directly above *E* and *F* respectively. The points *C* and *D* are 6m directly above *G* and *H* respectively.

The point *O* is at the centre of *EFGH* and is taken to be the origin. Perpendicular unit vectors $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$ are in the directions of $\overrightarrow{EF}, \overrightarrow{EH}$ and $\overrightarrow{EA}$ respectively.

(i) Show that $\overrightarrow{AC} = \begin{pmatrix} 10 \\ 4 \\ 3 \end{pmatrix}$ and find a vector equation of the line passing through *A* and *C*. **2**

(ii) A spotlight is erected on a vertical pillar at *Q* with coordinates $\begin{pmatrix} 15 \\ 6 \\ 0 \end{pmatrix}$. **1**

Find the height of the pillar if the spotlight is collinear with *A* and *C*.

(iii) The main electrical supply for the garage runs along the line between *A* and *C*. Find the coordinates of the point, *X*, on *AC* such that the distance from a light mounted at *D* to the main electric supply is minimised. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \ln\left(x+\sqrt{x^2-4}\right)\,dx$ **3**

(b) A series, $u_1 + u_2 + u_3 + \ldots$ has terms defined by:

$$u_1 = 2 \text{ and } u_{r+1} = u_r + \frac{1}{2^r}$$

Prove by mathematical induction that $u_n = 3-\left(\frac{1}{2}\right)^{n-1}$ for all $n \in \mathbb{Z}^+$. **3**

(c) The acceleration of a particle moving along the $x-$axis is given by: **4**

$$a = 2x^3 - 5x \text{ ms}^{-2}$$

If the particle has velocity, $v = 2$ ms^{-2} when $x = 0$, find an expression for v in terms of x and determine what values x can take.

(d) Given the polynomial $P(x) = x^4 - 2x^2 - 3x + a$ has a factor of $x^2 + x + 1$, find the value of a and hence find the roots of $P(x)$. **3**

(e) Let $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{k}$. Find the unit vector that is perpendicular to $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find an expression for $\int \frac{2x\ln\left(2x^2+1\right)}{2x^2+1}\,dx$ **3**

(b) Given $a,b,x,y,z \in \mathbb{R}^+$

(i) Prove the AM - GM inequality **1**

$$\frac{a+b}{2} \geq \sqrt{ab}$$

(ii) Further, if $x+y+z=1$, prove **3**

$$\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y} \geq 1$$

(iii) Given $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$, $a,b,c,d \in \mathbb{R}^+$, prove **3**

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

(c) A cube has diagonally opposite vertices at the origin, O, and $A(2a,4a,6a)$.

(i) Prove that the vertices of the cube lie on the sphere with equation

$$x^2+y^2+z^2 = 2ax + 4ay + 6az$$

Note: The edges of the cube are not parallel to the axes. **4**

(ii) Find the coordinates of the point on the sphere that has the greatest negative x-value. **1**

End of Question 14

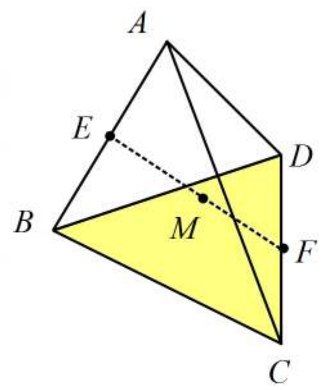
Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) For $n \geq 0$, define: $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$

(i) Show that $I_n = \dfrac{\left(\sqrt{2}\right)^{n-2}}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$ for $n \geq 2$, **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \sec^6 x \, dx$. **2**

(b) Consider the tetrahedron shown with vertices *A, B, C* and *D*. Let each vertex be represented by the position vector $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ and $\underset{\sim}{d}$ respectively. (i.e $\overrightarrow{OA} = \underset{\sim}{a}$ etc).



M is the midpoint of the line $\overrightarrow{EF}$ where *E* and *F* are the midpoints of *AB* and *CD* respectively. **4**
Find an expression for $\overrightarrow{OM}$ in terms of the position vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ and $\underset{\sim}{d}$.

Question 15 continues on page 15

(c) (i) Given $P_n(x) = (x - a_1)(x - a_2)...(x - a_n)$ prove by mathematical induction that **3**

$$\frac{P_n\,'(x)}{P_n(x)} = \frac{1}{(x-a_1)} + \frac{1}{(x-a_2)} + ... + \frac{1}{(x-a_n)} \quad \text{for } n \geq 2.$$

(ii) Consider ω is an nth root of unity. **3**

Let $P(z) = (z - \omega)(z - \omega^2)...(z - \omega^{n-1})$.

By finding another expression for $P(z)$ or otherwise, show that

$$\frac{1}{1-\omega} + \frac{1}{1-\omega^2} + ... + \frac{1}{1-\omega^{n-1}} = \frac{n-1}{2}$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given $x \geq 0$

(i) Prove that $1 - x \leq \dfrac{1}{1+x} \leq 1$. **2**

(ii) For $x > 0$, show that $1 - \dfrac{1}{2x} \leq x \ln\left(1 + \dfrac{1}{x}\right) \leq 1$. **3**

(iii) Hence deduce that $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$. **2**

Question 16 continues on page 17.

(b) An object of mass 1 kg is dropped from a height, h metres above the ground. There is a constant gravitational force of g and a resistive force of kv newtons, where $v\,\mathrm{ms}^{-1}$ is the velocity of the object. Hence $a = g - kv$.

(i) Show that the velocity of the particle at time t is $v = \frac{g}{k}\left(1 - e^{-kt}\right)$ **2**

(ii) Show that the position of the object below the point of release at time t is

$$x = \frac{gt}{k} + \frac{g}{k^2}\left(e^{-kt} - 1\right)$$ **2**

At the same time the object is released a second, identical object, is projected vertically upward from the ground with velocity $u\,\mathrm{ms}^{-1}$. Taking the origin as the ground and the upward direction as positive, the velocity of the second particle is given by:

$$v = \frac{g + ku}{k}e^{-kt} - \frac{g}{k}$$ (DO NOT PROVE THIS)

(iii) Show that the position, X, of the second particle below h is given by **2**

$$X = h + \frac{g + ku}{k^2}\left(e^{-kt} - 1\right) + \frac{g}{k}t$$

(iv) If the two objects collide at time T, show $T = \frac{1}{k}\ln\left(\frac{u}{u - kh}\right)$ **2**

End of paper

BLANK PAGE

2024 EXT 2 TRIAL SOLUTIONS

① Converse: "To reduce the number of formulas you need to remember, you need to understand the chain rule.

B

② $|\sin\theta\,\underset{\sim}{i} + \cot\theta\,\underset{\sim}{j} + \cos\theta\,\underset{\sim}{k}| = \sqrt{\sin^2\theta + \cot^2\theta + \cos^2\theta}$

$= \sqrt{1+\cot^2\theta}$

$= \sqrt{\operatorname{cosec}^2\theta}$

$= \operatorname{cosec}\theta$

B

③ To square a unit vector double its arg.

$\therefore\ a^2 = a,\ b^2 = c,\ c^2 = p,\ d^2 = b,\ p^2 = d$

C

④ $\int \frac{2x+1}{\sqrt{9-x^2}}\,dx = -\int \frac{-2x-1}{\sqrt{9-x^2}}\,dx$

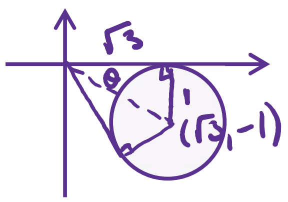
$= -\left[\int \frac{-2x}{\sqrt{9-x^2}}\,dx - \int \frac{1}{\sqrt{9-x^2}}\,dx\right]$

$= -\frac{(9-x^2)^{1/2}}{1/2} + \sin^{-1}(x/3) + C$

$= \sin^{-1}\left(\frac{x}{3}\right) - 2\sqrt{9-x^2} + C$

C

⑤ $|z-(\sqrt{3}-i)| = 1 \implies$ circle, centre $(\sqrt{3}, -1)$
radius = 1



min $\arg(z) = -2\theta$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = 30°$

$\therefore$ min $\arg(z) = -60$

C

(6) if OACB is a rectangle, then $\vec{OA}\cdot\vec{OB}=0$

as angles in a rectangle are $90°$

$$\therefore \begin{pmatrix}2\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\b\end{pmatrix}=0$$

i.e. $-2-4+3b=0$

$3b=6$

$b=2$

$$\vec{OC}=\vec{OA}+\vec{OB}$$

$$=\begin{pmatrix}2\\-1\\3\end{pmatrix}+\begin{pmatrix}-1\\4\\2\end{pmatrix}$$

$$=\begin{pmatrix}1\\3\\5\end{pmatrix}$$

D

(7) $\exists\ F(x)\ \&\ P(x) \Rightarrow \exists x\,(F(x)\cap P(x))$

C

(8) $\ddot{x}=-g-\frac{1}{2}kv^2$

$v\frac{dv}{dx}=-g-\frac{1}{2}kv^2$

$\int_u^0 \frac{v\,dv}{-g-\frac{1}{2}kv^2}=\int_0^H dx$ where H is max. height.

B

(9) $I=\int_1^e x^3\ln x\,dx$ let $u=\ln x$ $\quad v'=x^3$

$u'=\frac{1}{x}$ $\quad v=\frac{x^4}{4}$

$=\left[\frac{x^4}{4}\ln x\right]_1^e-\int_1^e\frac{x^3}{4}dx$

$=\frac{e^4}{4}-\left[\frac{x^4}{16}\right]_1^e$

$=\frac{e^4}{4}-\left(\frac{e^4}{16}-\frac{1}{16}\right)$

$=\frac{3e^4}{16}+\frac{1}{16}$

D

(10) Let D be the distance between the centres of the spheres.

$$d = \sqrt{(1--1)^2 + (-3-1)^2 + (2--2)^2}$$
$$= \sqrt{4+16+16}$$
$$= 6$$

For S_1 & S_2 to touch externally, the sum of their radii must equal 6

$$\therefore r + 2 = 6$$
$$r = 4$$

For S_1 & S_2 to touch internally, the difference of their radii must equal 6

$$r - 2 = 6$$
$$r = 8$$

B.

SECTION II

QUESTION 11

a) $z^2 + z + 3 = 0 \quad z \in \mathbb{C}$

$$z^2 + z + \frac{1}{4} = -3 + \frac{1}{4} \qquad (1)$$
$$(z + \tfrac{1}{2})^2 = -\frac{11}{4}$$
$$z + \tfrac{1}{2} = \pm \frac{\sqrt{11}\, i}{2}$$
$$z = -\frac{1}{2} \pm \frac{\sqrt{11}\, i}{2} \qquad (1)$$

b) $r(\lambda) = (2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}) + \lambda(-\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k})$

Direction vector $(-\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k})$ $\therefore$ line // through $(2, -1, 3)$

$$p(\lambda) = (2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) + \lambda(-\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k})$$

(1) (1)

c) Prove n^3-n is divisible by 6

$$n^3-n = n(n^2-1)$$
$$= n(n-1)(n+1)$$
$$= (n-1)n(n+1) \quad (1)$$

Hence, n^3-n is the product of 3 consecutive numbers. So, one of the numbers must be (1) divisible by 3 & (at least) one of the numbers is even so must be divisible by 2

$\Rightarrow n^3-n$ is divisible by 2 & 3 (1)

$\Rightarrow n^3-n$ is divisible by 6.

d) $$\sqrt{-1} = i$$
$$= e^{\frac{i\pi}{2}} \quad (1)$$

so $$\sqrt{-1}^{\sqrt{-1}} = \left(e^{i\pi/2}\right)^{i}$$
$$= e^{i^2\frac{\pi}{2}}$$
$$= e^{-\pi/2}$$ (a real number) (1)

e) consider $2+\sqrt{3}$ & $2-\sqrt{3}$, both irrational (1)

then $2+\sqrt{3} + 2-\sqrt{3} = 4$, a rational number

$\therefore$ The sum of two irrational numbers is not necessarily irrational. (1)

f) $$I = \int \frac{x+16}{x^2+2x-8}\,dx$$

$$= \int \frac{x+16}{(x+4)(x-2)}\,dx \quad (1)$$

consider $\dfrac{x+16}{(x+4)(x-2)} = \dfrac{A}{x+4} + \dfrac{B}{x-2}$

i.e. $x+16 = A(x-2) + B(x+4)$

let $x=2$: $18 = 6B$

$B = 3$

let $x=-4$ $12 = -6A$

$A = -2$ ①

$\therefore \quad I = \int \left(\dfrac{-2}{x+4} + \dfrac{3}{x-2} \right) dx$ ①

$= -2\ln|x+4| + 3\ln|x-2| + c$

$= \ln\left[\dfrac{|x-2|^3}{(x+4)^2}\right] + c$ ①

QUESTION 12

a) assume that $\log_2 5$ is rational

i.e. let $\log_2 5 = p/q$ where p & q are mutually prime integers > 0 ①

so $5 = 2^{p/q}$ from definition of logs

or $5^q = 2^p$ ①

but all powers of 5 are odd & all powers of 2 are even ①

$\therefore 5^q \neq 2^p$

so $\log_2 5$ cannot be rational (proof by contradiction)

b) $I = \int_0^{\pi} \left(\frac{\cot\theta + 1}{\operatorname{cosec}\theta} - \frac{\sec\theta}{\tan\theta + \cot\theta} \right) d\theta$

now $\frac{\cot\theta + 1}{\operatorname{cosec}\theta} - \frac{\sec\theta}{\tan\theta + \cot\theta} = \frac{\frac{\cos\theta}{\sin\theta} + 1}{\frac{1}{\sin\theta}} - \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}$ 1

$= \sin\theta\left(\frac{\cos\theta}{\sin\theta} + 1\right) - \frac{1}{\cos\theta}\left(\frac{1}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}\right.$

$= \cos\theta + \sin\theta - \frac{1}{\cos\theta}\left(\frac{\cos\theta\sin\theta}{\cos^2\theta + \sin^2\theta}\right)$

$= \cos\theta + \sin\theta - \sin\theta \quad (\text{as } \cos^2\theta + \sin^2\theta = 1)$

$= \cos\theta$ 1

$\therefore I = \int_0^{\pi} \cos\theta \, d\theta$

$= [\sin\theta]_0^{\pi}$

$= 0$ 1

c) $z = i(w+2)$ (1) $\qquad wz = -4i$ (2)

Sub (1) in (2)

$w(\cancel{i}(w+2)) = -4\cancel{i}$

$w^2 + 2w + 1 = -4 + 1$

$(w+1)^2 = -3$

$w + 1 = \pm\sqrt{3}i$ 1

$w = -1 \pm \sqrt{3}i$

Given $\operatorname{Im}(w) < 0 \Rightarrow w = -1 - \sqrt{3}i$ 1

Sub in (1): $z = i(-1 - \sqrt{3}i + 2)$

$= i(1 - \sqrt{3}i)$

$= \sqrt{3} + i$ 1

d) (i) For the given position of O

$$\vec{OA} = -5\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$$
$$\vec{OC} = 5\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$$
$$\vec{AC} = \vec{OC} - \vec{OA}$$
$$= 5\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k} - (-5\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}) \quad (1)$$
$$= 10\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k} \quad \text{as required.}$$

$\therefore$ Equation of AC

$$\underset{\sim}{r}(\lambda) = -5\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k} + \lambda(10\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}) \quad (1)$$

(ii) We need to find λ such that $\underset{\sim}{r}(\lambda) = 15\underset{\sim}{i} + 6\underset{\sim}{j} + h\underset{\sim}{k}$
where h is the height of the pillar.
Consider $\underset{\sim}{i}$ component:

$$-5\underset{\sim}{i} + 10\lambda\underset{\sim}{i} = 15\underset{\sim}{i}$$
$$\text{i.e.} \quad 10\lambda = 20$$
$$\lambda = 2$$
$$\therefore \ r(2) = 15\underset{\sim}{i} + 6\underset{\sim}{j} + (3 + 2(3))\underset{\sim}{k}$$
$$= 15\underset{\sim}{i} + 6\underset{\sim}{j} + 9\underset{\sim}{k} \quad (1)$$

$\therefore$ height of pillar is 9 m

(iii) $\vec{OD} = -5\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$

$$\vec{AD} = \vec{OD} - \vec{OA}$$
$$= 4\underset{\sim}{j} + 3\underset{\sim}{k} \quad (1)$$

To find X, find projection of $\vec{AD}$ on $\vec{AC}$

$$\text{Proj}_{\vec{AC}}\vec{AD} = \frac{(4\underset{\sim}{j} + 3\underset{\sim}{k})\cdot(10\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k})}{(10\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k})\cdot(10\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k})}(10\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k})$$

$$= \frac{16+9}{100+16+9}(10\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k})$$

$$= \frac{1}{5}(10\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}) \quad 1$$

$$\therefore \quad \vec{OX} = (-5\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}) + \frac{1}{5}(10\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k})$$

$$= -3\underset{\sim}{i} - \frac{6}{5}\underset{\sim}{j} + \frac{18}{5}\underset{\sim}{k} \quad 1$$

QUESTION 13

a) $I = \int \ln(x + \sqrt{x^2-4})\,dx$

let $u = \ln(x + \sqrt{x^2-4})$ $\qquad v' = 1$

$$u' = \frac{1}{x+\sqrt{x^2-4}}\cdot\left(1 + \frac{1}{2}(x^2-4)^{-1/2}\cdot 2x\right) \qquad v = x$$

$$= \frac{1}{x+\sqrt{x^2-4}}\cdot\left(1 + \frac{x}{\sqrt{x^2-4}}\right)$$

$$= \frac{1}{x+\sqrt{x^2-4}}\left(\frac{\sqrt{x^2-4}+x}{\sqrt{x^2-4}}\right) \quad 1$$

$$= \frac{1}{\sqrt{x^2-4}}$$

$$\therefore \quad I = x\ln(x+\sqrt{x^2-4}) - \int \frac{x}{\sqrt{x^2-4}}\,dx \quad 1$$

$$= x\ln(x+\sqrt{x^2-4}) - \frac{1}{2}\int \frac{2x}{\sqrt{x^2-4}}\,dx$$

$$= x\ln(x+\sqrt{x^2-4}) - \frac{1}{2}\,\frac{(x^2-4)^{1/2}}{1/2} + c$$

$$= x\ln(x+\sqrt{x^2-4}) - \sqrt{x^2-4} + c \quad 1$$

b) $u_1 = 2$ $\qquad u_{r+1} = u_r + \frac{1}{2^r}$

Prove $\quad u_n = 3 - (\frac{1}{2})^{n-1}$ for $n \in \mathbb{Z}^+$

Step 1: Prove true for $n=1$

$u_1 = 2$ (given) $\qquad u_1 = 3 - (\frac{1}{2})^{1-1}$

$= 3 - 1$

$= 2$ (1)

$\therefore$ true for $n=1$

Step 2: assume true for $n=k$

i.e. $\quad u_k = 3 - (\frac{1}{2})^{k-1}$

Step 3: Prove true for $n=k+1$ given true for $n=k$

i.e. RTP $\quad u_{k+1} = 3 - (\frac{1}{2})^k$

$u_{k+1} = u_k + \frac{1}{2^k}$ from definition.

$= 3 - (\frac{1}{2})^{k-1} + \frac{1}{2^k}$ by assumption (1)

$= 3 - \left[\frac{1}{2^{k-1}} - \frac{1}{2^k}\right]$

$= 3 - \left[\frac{1}{2^{k-1}}(1 - \frac{1}{2})\right]$

$= 3 - \left[\frac{1}{2^{k-1}} \cdot \frac{1}{2}\right]$

$= 3 - \frac{1}{2^k}$ (1)

$= 3 - (\frac{1}{2})^k$ as required.

Step 4: So the result holds by the inductive process.

c) $a = 2x^3 - 5x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 - 5x$

$\frac{1}{2}v^2 = \frac{2x^4}{4} - \frac{5x^2}{2} + c$

$v^2 = x^4 - 5x^2 + 2c$ 1

when $x=0$, $v=2$.

$\therefore \; 2^2 = 0 - 0 + 2c$

$2c = 4$

$\therefore \; v^2 = x^4 - 5x^2 + 4$ 1

$v = \pm\sqrt{x^4 - 5x^2 + 4}$

Since $v=2$ when $x=0$, take positive root

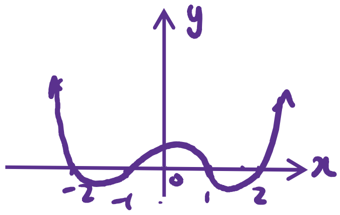
so $v = \sqrt{x^4 - 5x^2 + 4}$ 1

For domain: $x^4 - 5x^2 + 4 \geq 0$

$(x^2-1)(x^2-4) \geq 0$

critical values: $x^2 = 1, 2$

$x = \pm 1, \pm 2$



as $x=0$ is part of the domain.

$-1 \leq x \leq 1$ 1

d) $p(x) = x^4 - 2x^2 - 3x + a$

$$
\begin{array}{r|l}
 & x^2 - x - 2 \\
\hline
x^2+x+1 & x^4 + 0x^3 - 2x^2 - 3x + a \\
 & x^4 + x^3 + x^2 \\
\hline
 & -x^3 - 3x^2 - 3x \\
 & -x^3 - x^2 - x \\
\hline
 & -2x^2 - 2x + a \\
 & -2x^2 - 2x - 2 \\
\hline
 & a+2
\end{array}
$$

1

as x^2+x+1 is a factor, the remainder is 0

$\therefore a+2=0$

$a=-2$ 1

so $P(x)=x^4-2x^2-3x-2$

$=(x^2+x+1)(x^2-x-2)$

$=(x^2+x+1)(x-2)(x+1)$ 1

$\therefore$ for roots $x=2,-1$ or $x^2+x+1=0$

$x^2+x+\frac{1}{4}=-1+\frac{1}{4}$

$(x+\frac{1}{2})^2=-\frac{3}{4}$

$x+\frac{1}{2}=\pm\frac{\sqrt{3}i}{2}$

$x=-\frac{1}{2}\pm\frac{\sqrt{3}i}{2}$

$\therefore$ roots: $x=-1, 2, \frac{-1\pm\sqrt{3}i}{2}$ 1

e) let $\underset{\sim}{w}=a\underset{\sim}{i}+b\underset{\sim}{j}+c\underset{\sim}{k}$ be $\perp$ to $\underset{\sim}{u}$ & $\underset{\sim}{v}$

$\therefore \underset{\sim}{u}\cdot\underset{\sim}{w}=0$ & $\underset{\sim}{v}\cdot\underset{\sim}{w}=0$

$(\underset{\sim}{i}+\underset{\sim}{j})\cdot(a\underset{\sim}{i}+b\underset{\sim}{j}+c\underset{\sim}{k})=0$ $\quad$ $(\underset{\sim}{i}+2\underset{\sim}{k})\cdot(a\underset{\sim}{i}+b\underset{\sim}{j}+c\underset{\sim}{k})=0$

$a+b=0$ $\quad$ $a+2c=0$

$b=-a$ $\quad$ $c=-\frac{a}{2}$

$\therefore \underset{\sim}{w}=a\underset{\sim}{i}-a\underset{\sim}{j}-\frac{a}{2}\underset{\sim}{k}$ 1

$=\frac{a}{2}(2\underset{\sim}{i}-2\underset{\sim}{j}-\underset{\sim}{k})$

$=\lambda(2\underset{\sim}{i}-2\underset{\sim}{j}-\underset{\sim}{k})$

$\hat{\underset{\sim}{w}}=\frac{\lambda(2\underset{\sim}{i}-2\underset{\sim}{j}-\underset{\sim}{k})}{\sqrt{(2\lambda)^2+(-2\lambda)^2+(\lambda)^2}}$

$=\frac{\lambda(2\underset{\sim}{i}-2\underset{\sim}{j}-\underset{\sim}{k})}{\sqrt{9\lambda^2}}$

$=\frac{1}{3}(2\underset{\sim}{i}-2\underset{\sim}{j}-\underset{\sim}{k})$ 1

QUESTION 14

a) $I = \int \frac{2x\ln(2x^2+1)}{2x^2+1}\,dx$

let $u = \ln(2x^2+1)$

$\frac{du}{dx} = \frac{1}{2x^2+1} \times 4x$

$= \frac{4x}{2x^2+1}$

$du = \frac{4x}{2x^2+1}\,dx$ (1)

$\therefore I = \frac{1}{2}\int \frac{4x\ln(2x^2+1)}{2x^2+1}\,dx$

$= \frac{1}{2}\int u\,du$ (1)

$= \frac{1}{2} \times \frac{u^2}{2} + c$

$= \frac{1}{4}\left(\ln(2x^2+1)\right)^2 + c$ (1)

b) (i) For $a, b \geq 0$, we have

$(\sqrt{a} - \sqrt{b})^2 \geq 0$

$a - 2\sqrt{ab} + b \geq 0$

$a + b \geq 2\sqrt{ab}$

$\frac{a+b}{2} \geq \sqrt{ab}$ # (1)

(ii) $x + y + z = 1$

consider $\frac{xy}{z} + \frac{yz}{x} \geq 2\sqrt{\frac{xy^2z}{xz}}$

$= 2y$ (1) (1)

Similarly $\frac{xy}{z} + \frac{zx}{y} \geq 2x$ (2)

& $\frac{yz}{x} + \frac{zx}{y} \geq 2z$ (3) (1)

adding ①, ② + ③ we have:

$$\frac{xy}{z}+\frac{yz}{x}+\frac{xy}{z}+\frac{zx}{y}+\frac{yz}{x}+\frac{zx}{y}\geq 2y+2x+2z$$

$$2\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)\geq 2(x+y+z) \quad ①$$

$$\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\geq 1 \quad \text{as } x+y+z=1$$

(iii) Given $\dfrac{a+b+c+d}{4}\geq \sqrt[4]{abcd}$ $\quad (\bigstar)$

Let $a=a,\ b=b,\ c=c,\ d=\dfrac{a+b+c}{3}$ ①

Sub in $(\bigstar)$:

$$\frac{a+b+c+\frac{a+b+c}{3}}{4}\geq\left[abc\left(\frac{a+b+c}{3}\right)\right]^{1/4} \quad ①$$

$$\frac{\frac{3a+3b+3c+a+b+c}{3}}{4}\geq\left[abc\left(\frac{a+b+c}{3}\right)\right]^{1/4}$$

$$\frac{a+b+c}{3}\geq(abc)^{1/4}\left(\frac{a+b+c}{3}\right)^{1/4}$$

$$\left(\frac{a+b+c}{3}\right)^{3/4}\geq(abc)^{1/4} \quad \text{as } a,b,c>0$$

$$\left[\left(\frac{a+b+c}{3}\right)^{3/4}\right]^{4/3}\geq\left[(abc)^{1/4}\right]^{4/3} \quad ①$$

$$\frac{a+b+c}{3}\geq\sqrt[3]{abc}$$

#

OR $\quad \therefore a+b+c=3d$ ①

sub in $(\bigstar)$: $\quad \dfrac{3d+d}{4}\geq\sqrt[4]{abcd}$ ①

$$d\geq(abcd)^{1/4}$$

$$d^{3/4}\geq(abc)^{1/4} \quad ①$$

$$d\geq(abc)^{1/3}$$

$$\frac{a+b+c}{3}\geq\sqrt[3]{abc}$$

OR

c) (i) If the vertices of the cube lie on a sphere, the centre of the sphere must be the midpoint of OA:

$$M = \left(\frac{0+2a}{2}, \frac{0+4a}{2}, \frac{0+6a}{2}\right)$$

$$= (a, 2a, 3a) \quad (1)$$

$$\text{Radius} = |\vec{OM}|$$

$$= \sqrt{a^2+(2a)^2+(3a)^2}$$

$$= a\sqrt{14} \quad (1)$$

$\therefore$ Sphere is: $\left|\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a \\ 2a \\ 3a \end{pmatrix}\right| = a\sqrt{14}$ (1)

$$\therefore \sqrt{(x-a)^2+(y-2a)^2+(z-3a)^2} = a\sqrt{14}$$

$$x^2-2ax+a^2+y^2-4ay+4a^2+z^2-6az+9a^2 = 14a^2$$

$$x^2+y^2+z^2-2ax-4ay-6az+14a^2 = 14a^2$$

$$\text{ie} \quad x^2+y^2+z^2 = 2ax+4ay+6az \quad (1)$$

(ii) Greatest negative x-value will occur at the centre minus the radius in the x-direction

$$\text{ie. } P = \begin{pmatrix} a \\ 2a \\ 3a \end{pmatrix} - \begin{pmatrix} \sqrt{14}\,a \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} a(1-\sqrt{14}) \\ 2a \\ 3a \end{pmatrix} \quad (1)$$

QUESTION 15

a) $I_n = \int_0^{\pi/4} \sec^n x \, dx \quad , n \geq 0$

(i) $I_n = \int_0^{\pi/4} (\sec^{n-2} x \cdot \sec^2 x) \, dx$

Let $u = \sec^{n-2} x \qquad v' = \sec^2 x$ 1

$u' = (n-2)(\sec^{n-3} x)(\sec x \tan x) \qquad v = \tan x$

$\therefore I_n = \left[\sec^{n-2} x \tan x\right]_0^{\pi/4} - (n-2)\int_0^{\pi/4} \sec^{n-2} x \tan^2 x \, dx$

$= (\sqrt{2})^{n-2} \cdot 1 - 0 - (n-2)\int_0^{\pi/4} \sec^{n-2} x (\sec^2 x - 1) \, dx$ 1

$= \sqrt{2}^{\,n-2} - (n-2)\{ I_n - I_{n-2} \}$

$\therefore I_n + (n-2) I_n = \sqrt{2}^{\,n-2} + (n-2) I_{n-2}$

$(n-1) I_n = \sqrt{2}^{\,n-2} + (n-2) I_{n-2}$ 1

$I_n = \frac{\sqrt{2}^{\,n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ #

(ii) $I_6 = \int_0^{\pi/4} \sec^6 x \, dx$

$= \frac{(\sqrt{2})^4}{5} + \frac{4}{5} I_4$ 1

$= \frac{4}{5} + \frac{4}{5}\left(\frac{(\sqrt{2})^2}{3} + \frac{2}{3} I_2\right)$

$= \frac{4}{5} + \frac{8}{15} + \frac{8}{15}\left(\frac{\sqrt{2}^{\,0}}{1} + 0\right)$

$= \frac{12 + 8 + 8}{15}$

$= \frac{28}{15}$ 1

b) $\vec{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$$\vec{OE} = \vec{OA} + \tfrac{1}{2}\vec{AB}$$
$$= \underset{\sim}{a} + \tfrac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$$
$$= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) \quad ①$$

$$\vec{CD} = \underset{\sim}{d} - \underset{\sim}{c}$$
$$\vec{OF} = \vec{OC} + \tfrac{1}{2}\vec{CD}$$
$$= \underset{\sim}{c} + \tfrac{1}{2}(\underset{\sim}{d} - \underset{\sim}{c})$$
$$= \tfrac{1}{2}(\underset{\sim}{c} + \underset{\sim}{d}) \quad ①$$

$$\vec{OM} = \tfrac{1}{2}(\vec{OE} + \vec{OF}) \quad ①$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b}) + \tfrac{1}{2}(\underset{\sim}{c}+\underset{\sim}{d})\right]$$
$$= \tfrac{1}{4}(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d}) \quad ①$$

OR: $\vec{EF} = \vec{OF} - \vec{OE}$
$$= \tfrac{1}{2}(\underset{\sim}{c}+\underset{\sim}{d}) - \tfrac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})$$
$$= \tfrac{1}{2}(\underset{\sim}{c}+\underset{\sim}{d}-\underset{\sim}{a}-\underset{\sim}{b}) \quad ①$$
$$\vec{OM} = \vec{OE} + \tfrac{1}{2}\vec{EF}$$
$$= \tfrac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b}) + \tfrac{1}{4}(\underset{\sim}{c}+\underset{\sim}{d}-\underset{\sim}{a}-\underset{\sim}{b})$$
$$= \tfrac{1}{4}(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d}) \quad ①$$

c)(i) $P_n(x) = (x-a_1)(x-a_2)\ldots(x-a_n)$

Prove $\dfrac{P_n'(x)}{P_n(x)} = \dfrac{1}{x-a_1} + \dfrac{1}{x-a_2} + \cdots + \dfrac{1}{x-a_n}$ $\quad n \geq 2$

Step 1: Prove for $n=2$

$$P_2(x) = (x-a_1)(x-a_2)$$
$$P_2'(x) = (x-a_1) + (x-a_2) \quad \text{(Product rule)}$$
$$\therefore \frac{P_2'(x)}{P_2(x)} = \frac{(x-a_1)+(x-a_2)}{(x-a_1)(x-a_2)}$$
$$= \frac{x-a_1}{(x-a_1)(x-a_2)} + \frac{x-a_2}{(x-a_1)(x-a_2)} \quad ①$$
$$= \frac{1}{x-a_2} + \frac{1}{x-a_1}$$

$\therefore$ true for $n=1$

Step 2: assume true for $n=k$

i.e. $$\frac{P_k'(x)}{P_k(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \dots + \frac{1}{x-a_k}$$

Step 3: Prove true for $n=k+1$ if true for $n=k$

RTP $$\frac{P_{k+1}'(x)}{P_{k+1}(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \dots + \frac{1}{x-a_{k+1}}$$

Now $$P_{k+1}(x) = (x-a_1)(x-a_2)\dots(x-a_k)(x-a_{k+1})$$
$$= P_k(x)(x-a_{k+1})$$
$$P_{k+1}'(x) = P_k(x).1 + (x-a_{k+1})P_k'(x)$$
$$\therefore \frac{P_{k+1}'(x)}{P_{k+1}(x)} = \frac{P_k(x) + (x-a_{k+1})P_k'(x)}{P_{k+1}(x)} \quad 1$$
$$= \frac{P_k(x)}{P_k(x)(x-a_{k+1})} + \frac{(x-a_{k+1})P_k'(x)}{P_k(x)(x-a_{k+1})}$$
$$= \frac{1}{x-a_{k+1}} + \frac{P_k'(x)}{P_k(x)}$$
$$= \frac{1}{x-a_{k+1}} + \frac{1}{x-a_1} + \frac{1}{x-a_2} + \dots + \frac{1}{x-a_k} \quad 1$$

from assumption

$$= RHS$$

(ii) $P(z) = (z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$

now $z^n-1 = (z-1)(z^{n-1}+z^{n-2}+\dots+z+1)$

so $$\frac{z^n-1}{z-1} = (z^{n-1}+z^{n-2}+\dots+z+1)$$

also, since $1, \omega, \omega^2 \dots \omega^{n-1}$ are the roots of z^n-1

$$z^n-1 = (z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1}) \quad 1$$

or $\dfrac{z^n-1}{z-1} = (z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$

$\therefore\ P(z) = 1+z+z^2+\dots+z^{n-1}$

& $\dfrac{P'(z)}{P(z)} = \dfrac{1}{z-\omega} + \dfrac{1}{z-\omega^2} + \dots + \dfrac{1}{z-\omega^{n-1}}$

from (i)

note

$\dfrac{P'(1)}{P(1)} = \dfrac{1}{1-\omega} + \dfrac{1}{1-\omega^2} + \dots + \dfrac{1}{1-\omega^{n-1}}$ (*) 1

we also have:

$P(z) = 1 + z + \dots + z^{n-1}$

with $P'(z) = 0 + 1 + 2z + 3z^2 + \dots + (n-1)z^{n-2}$

$= 1 + 2 + 2z^2 + \dots + (n-1)z^{n-2}$

Here: $P(1) = 1 + 1^1 + 1^2 \dots + 1^{n-1}$

$= n \times 1$

$= n$ 1

& $P'(1) = 1 + 2 + 3 + \dots + (n-1)$

$= \frac{n-1}{2}(1 + (n-1))$ Sum of A.P.

$= \dfrac{n(n-1)}{2}$ 1

$\therefore\ \dfrac{P'(1)}{P(1)} = \dfrac{n(n-1)}{2} \div n$

$= \dfrac{n-1}{2}$

$= \dfrac{1}{1-\omega} + \dfrac{1}{1-\omega^2} + \dots + \dfrac{1}{1-\omega^{n-1}}$ from (*)

QUESTION 16

a) (i) $x \geq 0$

$\therefore \quad -x^2 \leq 0 \leq x$ 1

$1 - x^2 \leq 1 \leq 1 + x$

since $x \geq 0$, $1 + x > 0$ $\therefore$ can divide by $1 + x$

$\frac{1-x^2}{1+x} \leq \frac{1}{1+x} \leq 1$ 1

$1 - x \leq \frac{1}{1+x} \leq 1$ as required.

(ii) Integrating the result in (i)

$\int_0^x (1-x)\,dx \leq \int_0^x \frac{1}{1+x}\,dx \leq \int_0^x 1\,dx$ for $x > 0$

$\left[x - \frac{x^2}{2}\right]_0^x \leq \left[\ln|1+x|\right]_0^x \leq \left[x\right]_0^x$ 1

$x - \frac{x^2}{2} \leq \ln|1+x| \leq x$

$\div x$: $\quad 1 - \frac{x}{2} \leq \frac{1}{x}\ln|1+x| \leq 1$ 1

now let $x = \frac{1}{n}$ $\quad 1 - \frac{1}{2}\left(\frac{1}{n}\right) \leq \frac{1}{1/n}\ln\left|1 + \frac{1}{n}\right| \leq 1$ 1

$1 - \frac{1}{2n} \leq n\ln\left(1 + \frac{1}{n}\right) \leq 1$ (abs value not required as $x > 0$)

as n is just a dummy variable, we could write

$1 - \frac{1}{2x} \leq x\ln\left(1 + \frac{1}{x}\right) \leq 1$

(iii) we can rewrite the result from (ii) as

$1 - \frac{1}{2x} \leq \ln\left(1 + \frac{1}{x}\right)^x \leq 1$

raising e: $\quad e^{1 - 1/2x} \leq e^{\ln\left(1+\frac{1}{x}\right)^x} \leq e^1$ 1

$e^{1 - \frac{1}{2x}} \leq \left(1 + \frac{1}{x}\right)^x \leq e$

taking $\lim_{x \to \infty}$ $\quad \lim_{x \to \infty} e^{1 - \frac{1}{2x}} \leq \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x \leq \lim_{x \to \infty} e$

$\lim_{x \to \infty} \frac{1}{2x} = 0 \quad \Rightarrow \quad e^1 \leq \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x \leq e$ 1

$\therefore \quad \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x = e$ as it is sandwiched.

b) (i) $a = g - kv$

$$\frac{dv}{dt} = g - kv$$

$$\int_0^v \frac{dv}{g-kv} = \int_0^t dt$$

$$-\frac{1}{k}\int_0^v \frac{-k\,dv}{g-kv} = [t]_0^t \quad 1$$

$$-\frac{1}{k}\left[\ln|g-kv|\right]_0^v = t$$

$$\ln|g-kv| - \ln g = -kt$$

$$\ln\left|\frac{g-kv}{g}\right| = -kt$$

$$\frac{g-kv}{g} = e^{-kt}$$

$$g - kv = ge^{-kt} \quad 1$$

$$kv = g - ge^{-kt}$$

$$v = \frac{g}{k}(1-e^{-kt}) \quad \#$$

(ii) so $\frac{dx}{dt} = \frac{g}{k}(1-e^{-kt})$

$$\int_0^x dx = \frac{g}{k}\int_0^t (1-e^{-kt})\,dt$$

$$x = \frac{g}{k}\left[t + \frac{1}{k}e^{-kt}\right]_0^t$$

$$= \frac{g}{k}\left[t + \frac{1}{k}e^{-kt} - (0 + \frac{1}{k}e^0)\right] \quad 1$$

$$= \frac{g}{k}\left[t + \frac{1}{k}e^{-kt} - \frac{1}{k}\right]$$

$$= \frac{gt}{k} + \frac{g}{k^2}(e^{-kt}-1) \quad 1 \quad \#$$

(iii) $v = \frac{g+ku}{k}e^{-kt} - \frac{g}{k}$

so $\frac{dx}{dt} = \frac{g+ku}{k}e^{-kt} - \frac{g}{k}$

$\int_0^x dx = \int_0^t \left(\frac{g+ku}{k}e^{-kt} - \frac{g}{k} \right)$

$x = \left[\frac{g+ku}{k} \cdot \left(-\frac{1}{k}\right) e^{-kt} - \frac{gt}{k} \right]_0^t$ 1

$= -\frac{g+ku}{k^2}e^{-kt} - \frac{gt}{k} - \left(-\frac{g+ku}{k^2}e^0 - 0\right)$

$= \frac{g+ku}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$ 1

so $X = h - x$

$= h - \left[\frac{g+ku}{k^2}(1-e^{-kt}) - \frac{gt}{k} \right]$

$= h + \frac{g+ku}{k^2}(e^{-kt}-1) + \frac{gt}{k}$ #

(iv) For collision $x = X$ where x is position of object 1

$\therefore \cancel{\frac{gt}{k}} + \frac{g}{k^2}(e^{-kt}-1) = h + \frac{g+ku}{k^2}(e^{-kt}-1) + \cancel{\frac{gt}{k}}$

$(e^{-kt}-1)\left\{ \frac{g}{k^2} - \frac{g+ku}{k^2} \right\} = h$ 1

$(e^{-kt}-1)\left(-\frac{u}{k}\right) = h$

$e^{-kt} - 1 = -\frac{hk}{u}$

$e^{-kt} = 1 - \frac{hk}{u}.$

$e^{-kt} = \frac{u-hk}{u}.$

$e^{kt} = \frac{u}{u-hk}$

$kt = \ln\left|\frac{u}{u-hk}\right|$

$t = \frac{1}{k}\ln\left|\frac{u}{u-hk}\right|$ 1

#