ABBOTSLEIGH

Student's Name: ____________________

Student Number: | | | | | | | | |

Teacher's Name: ____________________

2025 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- Diagrams are not to scale unless indicated
- Ensure your HSC candidate number is on each booklet
- For questions in Section I, answer on the sheet provided
- For questions in Section II,
  - show all necessary working
  - start a new booklet for each question

Total marks: 100

Section I – 10 marks (pages 3–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8–16)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Year 12 Mathematics Extension 2 outcomes

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

MEX12-3 uses vectors to model and solve problems in two and three dimensions

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

MEX12-5 applies techniques of integration to structured and unstructured problems

MEX12-6 uses mechanics to model and solve practical problems

MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

Section I

10 Marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1. What is the value of i^{2025}?

A. 1

B. -1

C. i

D. $-i$

2. A cubic polynomial $P(z)$ has all real coefficients.

$z=3$ and $z=2+i$ are two roots of $P(z)=0$.

Which equation describes $P(z)$?

A. $P(z)=z^3-7z^2-17z+15$

B. $P(z)=z^3-7z^2+17z-15$

C. $P(z)=z^3+7z^2-35z+15$

D. $P(z)=z^3+7z^2+35z-15$

3. Given that $z = \dfrac{\sqrt{3}+i}{1+i}$, which of the following is equal to z^5?

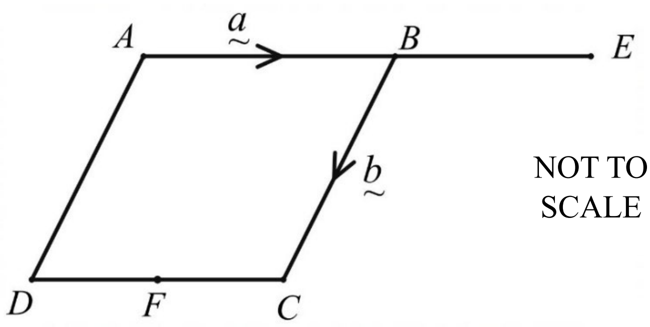
A. $2\sqrt{2}e^{\frac{\pi}{12}i}$

B. $4\sqrt{2}e^{\frac{\pi}{12}i}$

C. $2\sqrt{2}e^{\frac{19\pi}{12}i}$

D. $4\sqrt{2}e^{\frac{19\pi}{12}i}$

4. *ABCD* is a rhombus. It is given that $\overrightarrow{AB} = \underset{\sim}{a}$, and $\overrightarrow{BC} = \underset{\sim}{b}$.



$\overrightarrow{BE}$ is an extension of $\overrightarrow{AB}$ such that $\overrightarrow{AB} : \overrightarrow{BE} = 4:3$.

The point *F* is the midpoint of *CD*.

Which expression gives the vector $\overrightarrow{FE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$?

A. $\dfrac{11}{6}\underset{\sim}{a} + \underset{\sim}{b}$

B. $\dfrac{5}{4}\underset{\sim}{a} + \underset{\sim}{b}$

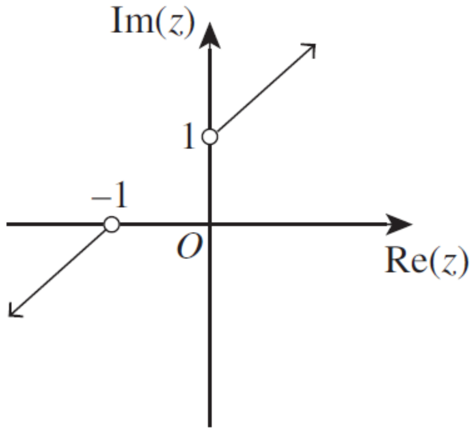
C. $\dfrac{11}{6}\underset{\sim}{a} - \underset{\sim}{b}$

D. $\dfrac{5}{4}\underset{\sim}{a} - \underset{\sim}{b}$

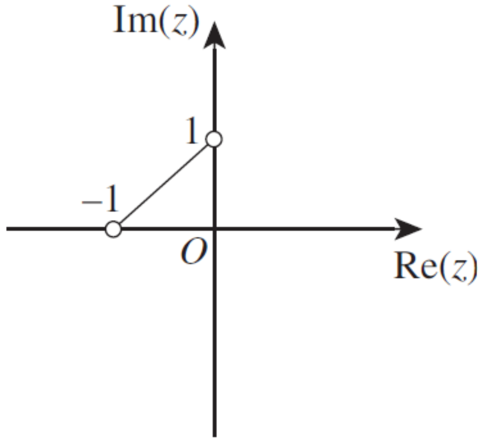
5. Which of the following Argand diagrams represents the solutions to the equation:

$$Arg(z-i) = Arg(z+1)?$$

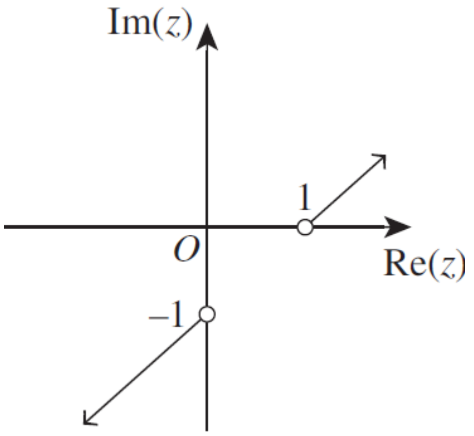
A.



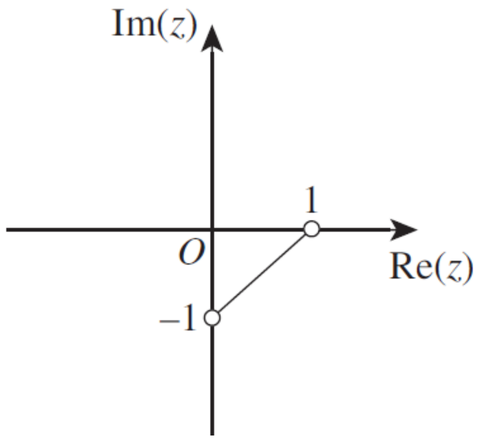
B.



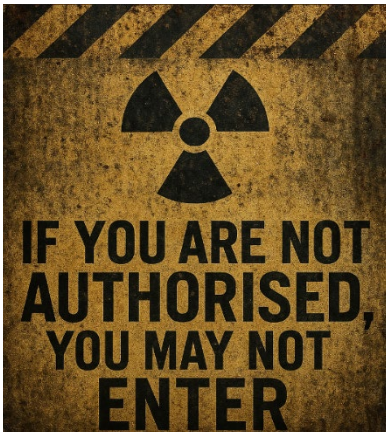
C.



D.



6. Which of the statements is logically equivalent to the one shown opposite.



A. If you may enter, then you are authorised.

B. If you are authorised, then you may enter.

C. If you are not authorised, then you may enter.

D. If you may not enter, then you are not authorised.

7. A particle of **unit** mass is projected vertically upwards from the ground with initial velocity of u m/s.

The particle is affected by a gravitational force, g, and a resistive force equivalent to $\frac{1}{2}kv^2$,

where the particle's velocity is v m/s. Let k be the constant of proportionality and x be the displacement, in metres, from the ground after t seconds.

Which of the following is the maximum height, H, of the particle?

A. $H = \displaystyle\int_u^0 \frac{v}{g + \frac{kv^2}{2}}\, dv$

B. $H = \displaystyle\int_0^u \frac{v}{g + \frac{kv^2}{2}}\, dv$

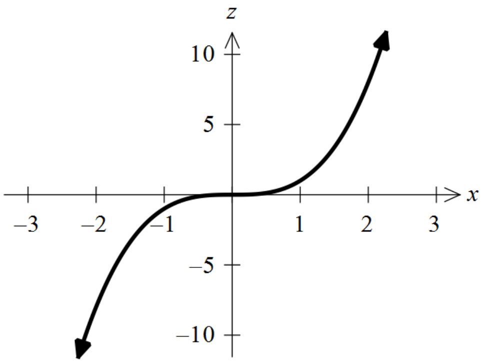
C. $H = \displaystyle\int_u^0 \frac{v}{g - \frac{kv^2}{2}}\, dv$

D. $H = \displaystyle\int_0^u \frac{v}{g - \frac{kv^2}{2}}\, dv$

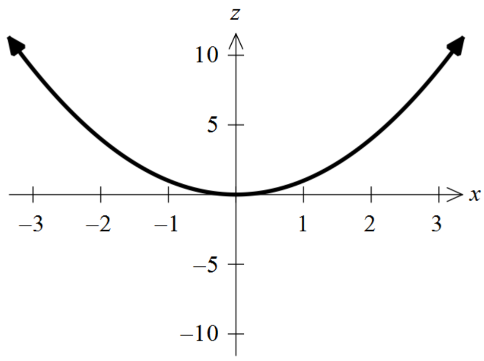
8. The curve $r(t) = \left(t^2, t^4, t^6\right)$ is defined for all real values of t.

Which of the following graphs best represents its **projection onto the xz-plane**?

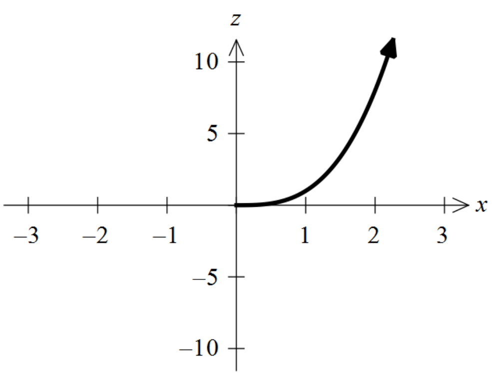
A.



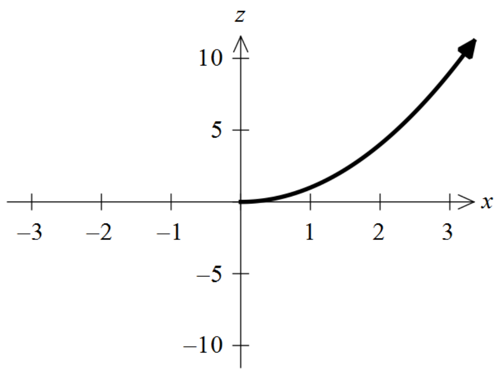
B.



C.



D.



9. A car journey is split into two parts.

During the first part the average speed is V km/h.

In the second part, the average speed is U km/h.

Kimie concludes that the average speed for the whole journey is $\dfrac{U+V}{2}$ km/h.

Which of the following is NOT a counterexample to this conclusion?

A. The car initially travels 100 km at 20 km/h, and then travels 100 km at 40 km/h.

B. The car initially travels 120 km at 60 km/h and 80 km at 40 km/h.

C. The car initially travels for one hour at 100 km/h and then two hours at 50 km/h.

D. The car initially travels 60 km in two hours and then travels 40 km in three hours.

10. Consider the definite integral, $I = \int_0^{\pi} x \sin x \, dx$.

Using the substitution $x = \pi - y$, what is the value of I?

A. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin y \, dy$

B. $\pi \int_0^{\pi} \sin y \, dy$

C. $\dfrac{\pi}{2} \int_0^{\pi} \sin y \, dy$

D. $\int_0^{\pi} \sin y \, dy$

End of Section I

Section II

90 marks
Attempt Questions 11 - 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a **SEPARATE** writing booklet. Extra writing booklets are available.

In Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z_1 = -1-5i$ and $z_2 = 2-i$.

(i) Calculate $i(z_1 - z_2)$. **1**

(ii) Write $\dfrac{a+ib}{z_2}$ in the form $\omega = x+iy$, in terms of a and b. **2**

(b) Consider the statement about integers **2**

If $k^2 - 14k$ is odd, then k is odd.

By considering the contrapositive prove that the statement is true.

(c) Let $P\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be any point that is equidistant from $Q\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ and $R\begin{pmatrix} -7 \\ 1 \\ 0 \end{pmatrix}$.

(i) Show that the equation of all such points P is $18x - 8y + 8z = -21$. **2**

(ii) Give a geometric description of P. **1**

(d) Solve $z^2 + (5+5i)z + (6+15i) = 0$. Give your answer in the form $z = x+iy$. **3**

(e) Use partial fractions to find $\displaystyle\int_{-2}^{1} \frac{1+7x}{(x-2)(x+3)}\, dx$. **4**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that if $S_1 = a$ and $S_n - S_{n-1} = ar^{n-1}$, $n \geq 2$, **3**

then $S_n = \dfrac{a\left(1-r^n\right)}{1-r}$, $n \geq 1$.

(b) Consider the complex number $z = 1 + i\sqrt{3}$.

(i) Write z in mod-arg form. **1**

(ii) Show $\sqrt{z^3} = 2\sqrt{2}i$. **1**

(iii) Find all values of n for which z^n is purely imaginary. **1**

(c) (i) Find the Cartesian equation for the line $\boldsymbol{l}$ that passes through the point $\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ **2**

and is parallel to the line with equation $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$.

(ii) Determine where the line $\boldsymbol{l}$ meets the ***x*-*z*** plane. **2**

(d) Prove that $\log_2 9$ is irrational. **2**

(e) If ω is a complex solution of $z^3 = 1$, simplify the expression $\left(1+\omega\right)^2 + \left(1+\omega^2\right)^2$. **2**

(f) Write the negation of the following statement $\exists x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+ : xy = 1$. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^1 \log_e\left(x^2+1\right)\,dx$. **4**

(b) Consider the spheres S_1 and S_2 **2**

$$S_1 : x^2 - 2x + y^2 + 4y + z^2 - 4z = 0$$
$$S_2 : x^2 + y^2 + z^2 + 10x + 2z + 10 = 0.$$

Explain why S_1 and S_2 intersect at a single point. (Do NOT find this point.)

(c) Consider the complex number $z = \cos\theta + i\sin\theta$.

(i) Show that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$. **1**

(ii) Hence, or otherwise, find the constants a, b and c such that **3**

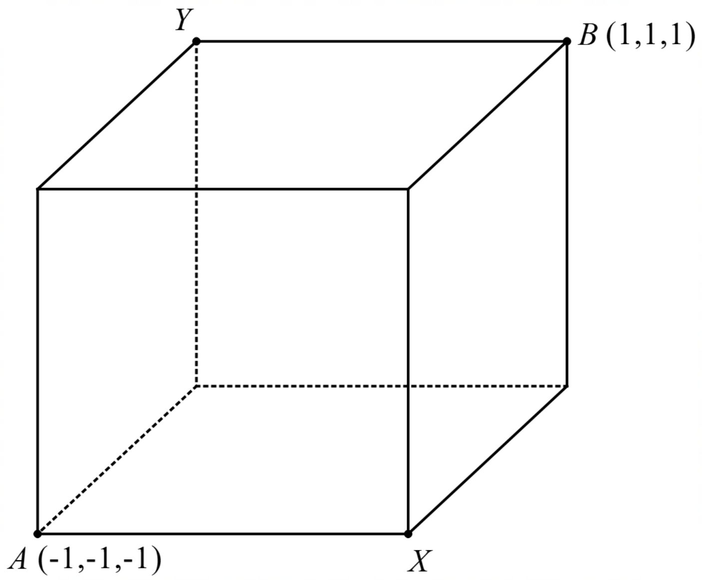
$$\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta.$$

(iii) Hence evaluate $16\int_0^{\frac{\pi}{2}} \sin^5\theta\; d\theta$. **1**

Question 13 continues over page

(d) A cube has a pair of opposite vertices $A\,(-1,-1,-1)$ and B (1, 1, 1). **2**

Another pair of opposite vertices are X and Y, as displayed in the diagram below.



If α is the acute angle between AB and XY, show that $\sin 2\alpha = \dfrac{2\sqrt{8}}{9}$.

(e) Three positive number a, b and c satisfy the following conditions **2**

$$a \geq b \geq c \text{ and } a+b+c \leq 1.$$

By considering $(a+b+c)^2$ or otherwise, prove that $a^2+3b^2+5c^2 \leq 1$.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution of $u = \sin x - x\cos x$, evaluate **3**

$$\int_{\frac{\pi}{2}}^{\pi} \frac{x}{1 - x\cot x}\, dx.$$

(b) Consider the following four equations, where $a > 0$ and $b > 0$. **3**

$$D = \frac{2ab}{a+b}$$

$$U = \sqrt{ab}$$

$$B = \frac{a+b}{2}$$

$$O = \sqrt{\frac{a^2 + b^2}{2}}$$

Show that $D \le U \le B \le O$.

(c) The triangle ABC in the complex plane is formed from the points $A(\alpha)$, $B(\beta)$ and $C(\gamma)$. **3**

The complex numbers α, β and γ satisfy these two conditions:

$$\frac{\gamma - \alpha}{\beta - \alpha} = 1 - i \quad \text{and} \quad \alpha + \beta + \gamma = 0.$$

Calculate $|\alpha| : |\beta|$.

Question 14 continues over page

Question 14 (continued)

(d) Two magpies, A and B, are observed flying. **3**

Their initial respective positions are:

$\left(\underset{\sim}{i}-2\underset{\sim}{j}+4\underset{\sim}{k}\right)$ metres and $\left(-2\underset{\sim}{i}+a\underset{\sim}{j}+6\underset{\sim}{k}\right)$ metres, where a is a constant.

They are travelling with constant velocities and these are respectively:

$\left(2\underset{\sim}{i}+3\underset{\sim}{j}+6\underset{\sim}{k}\right)$ m/s and $\left(3\underset{\sim}{i}+12\underset{\sim}{j}+4\underset{\sim}{k}\right)$ m/s.

Determine the constant a, if the distance between the two magpies is least after 5 seconds.

(e) By considering $(1+i)(4-i)^2$ prove that $\tan^{-1}\left(\dfrac{7}{23}\right)+2\tan^{-1}\left(\dfrac{1}{4}\right)=\dfrac{\pi}{4}$. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) A body of **unit** mass falls vertically from rest, under gravity, from a height of 100 metres above the ground through a resisting medium. The resistance to its motion is $\frac{v^2}{100}$, where v is the speed of the body (in m/s) after falling a distance x metres.

(i) Show that the acceleration $\ddot{x}$ of the body is given by $\ddot{x} = g - \frac{v^2}{100}$. **1**

(ii) Determine the terminal velocity V of the body. **1**

(iii) Show that the velocity v of the body, after it has fallen a vertical distance x, is given by **3**

$$v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right).$$

(iv) Determine the distance fallen when the velocity of the body first reaches $\frac{V}{2}$. **2**

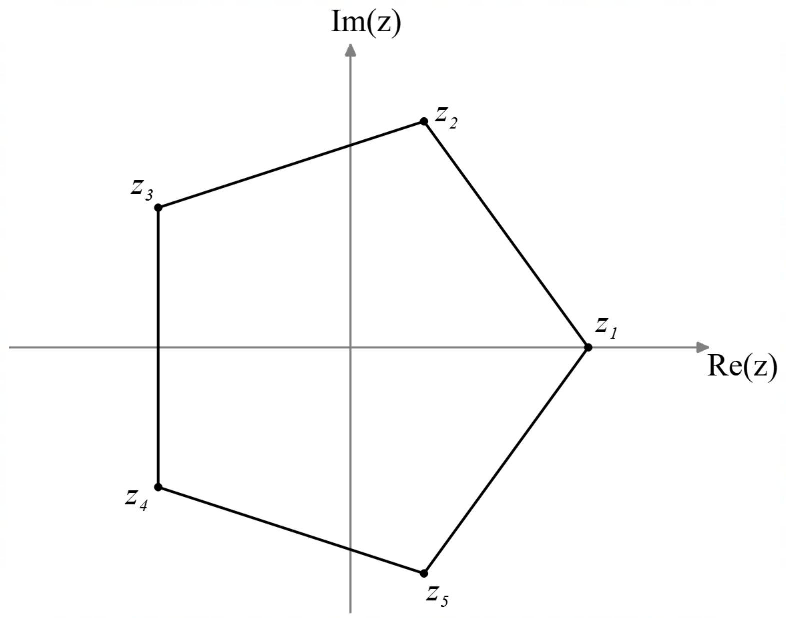
(b) Let $S_n = \sum_{p=1}^{n} \tan^{-1}\left(\frac{1}{2p^2}\right)$. **4**

Prove by induction that $\tan(S_n) = \frac{n}{n+1}$.

Question 15 continues over page

Question 15 (continued)

(c) The solutions of $z^5 = 1$ can be joined to form a regular pentagon in the complex plane, as shown below.



(i) For $n \geq 3$, show that the perimeter of the regular n-gon formed by the solutions of $z^n = 1$ is $P_n = 2n\sin\left(\dfrac{\pi}{n}\right)$. **2**

(ii) It is known that as $x \to 0$ then $\dfrac{\sin x}{x} \to 1$. (DO NOT prove this.) **2**

Using part (i), show that as $n \to \infty$, $P_n \to 2\pi$.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

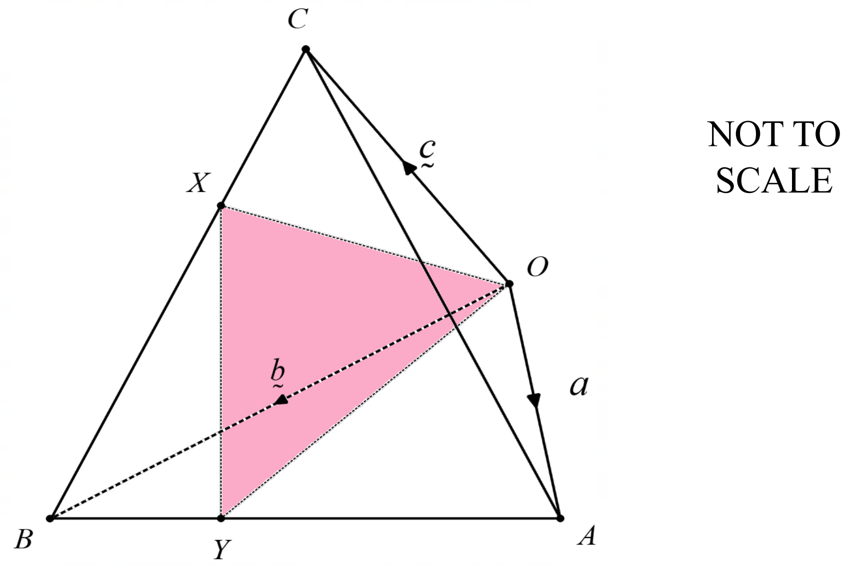
(a) Let $I_n = \int_0^{\frac{1}{4}} \frac{x^n}{e^x}\, dx$ where $n = 1, 2, 3, \ldots$.

(i) Calculate I_1. **2**

(ii) Show that $kI_k = I_{k+1} - I_k + \dfrac{\left(\frac{1}{4}\right)^{k+1}}{e^{\frac{1}{4}}}$. **3**

(iii) Deduce that $\sum_{k=1}^{n} kI_k = I_{k+1} - I_1 + \dfrac{1-\left(\frac{1}{4}\right)^{n}}{12e^{\frac{1}{4}}}$. **3**

(b) The diagram shows a regular tetrahedron $OABC$ with sides of length 1.



Let X and Y be points on the edges such that $\left|\overrightarrow{BX}\right| : \left|\overrightarrow{XC}\right| = \left|\overrightarrow{AY}\right| : \left|\overrightarrow{YB}\right| = x : 1-x$, where $0 < x < 1$.

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, and $\overrightarrow{OC} = \underset{\sim}{c}$.

(i) Show that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = \frac{1}{2}$. **1**

(ii) Show that $\left|\overrightarrow{OY}\right| = \sqrt{1 - x + x^2}$. **2**

(iii) Determine the range of values of $\angle YOX$. **4**

End of Paper

2025 Abbotsleigh Extension 2 Trial Solutions

1. What is the value of i^{2025}?

A. 1

B. -1

C. i

D. $-i$

As

$$i^4 = 1$$
$$i^{2024} = 1$$
$$i^{2025} = i$$

Answer is C

2. A cubic polynomial $P(z)$ has all real coefficients.

$z = 3$ and $z = 2 + i$ are two roots of $P(z) = 0$.

Which equation describes $P(z)$?

A. $P(z) = z^3 - 7z^2 - 17z + 15$

B. $P(z) = z^3 - 7z^2 + 17z - 15$

C. $P(z) = z^3 + 7z^2 - 35z + 15$

D. $P(z) = z^3 + 7z^2 + 35z - 15$

Roots are $z = 3, 2 + i, and\ 2 - i$ (complex conjugate).

Sum of roots,

$$-\frac{b}{a} = 3 + (2 + i) + (2 - i) = 7$$
$$b = -7$$

So, answer is either A or B.

Product of roots,

$$-\frac{d}{a} = 3(2 + i)(2 - i) = 15$$
$$d = -15$$

So, the answer is B.

3. Given that $z = \dfrac{\sqrt{3}+i}{1+i}$, which of the following is equal to z^5?

A. $2\sqrt{2}e^{\frac{\pi}{12}i}$

B. $4\sqrt{2}e^{\frac{\pi}{12}i}$

C. $2\sqrt{2}e^{\frac{19\pi}{12}i}$

D. $4\sqrt{2}e^{\frac{19\pi}{12}i}$

$$\sqrt{3}+i = 2\text{cis}\left(\frac{\pi}{6}\right)$$

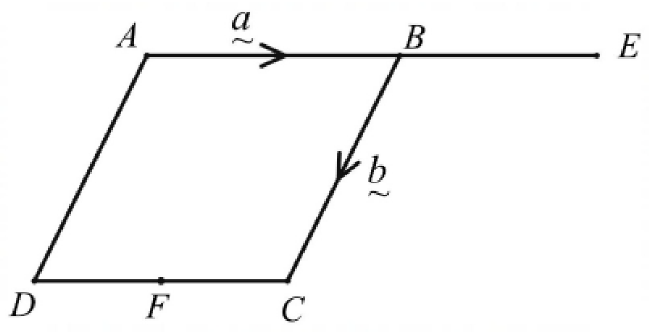
$$1+i = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

$$z = \frac{2\text{cis}\left(\frac{\pi}{6}\right)}{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)} = \frac{2}{\sqrt{2}}\text{cis}\left(\frac{\pi}{6}-\frac{\pi}{4}\right) = \sqrt{2}\text{cis}\left(-\frac{\pi}{12}\right)$$

$$z^5 = \sqrt{2}^5\text{cis}\left(-\frac{5\pi}{12}\right) = 4\sqrt{2}\text{cis}\left(-\frac{5\pi}{12}\right) = 4\sqrt{2}\text{cis}\left(\frac{19\pi}{12}\right)$$

So, the answer is D.

4. $ABCD$ is a rhombus. $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$.



$\overrightarrow{BE}$ is an extension of $\overrightarrow{AB}$, such that $\overline{AB} : \overline{BE} = 4 : 3$.

The point F is the midpoint of CD.

Which expression gives the vector $\overrightarrow{FE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$?

A. $\frac{11}{6}\underset{\sim}{a}+\underset{\sim}{b}$

B. $\frac{5}{4}\underset{\sim}{a}+\underset{\sim}{b}$

C. $\frac{11}{6}\underset{\sim}{a}-\underset{\sim}{b}$

D. $\frac{5}{4}\underset{\sim}{a}-\underset{\sim}{b}$

$$\overrightarrow{\boldsymbol{FE}} = \overrightarrow{\boldsymbol{FC}} + \overrightarrow{\boldsymbol{CB}} + \overrightarrow{\boldsymbol{BE}}$$

As $\boldsymbol{AB}:\boldsymbol{BE} = \frac{4}{3} \rightarrow \boldsymbol{BE} = \frac{3}{4}\boldsymbol{AB} = \frac{3}{4}\boldsymbol{a}$

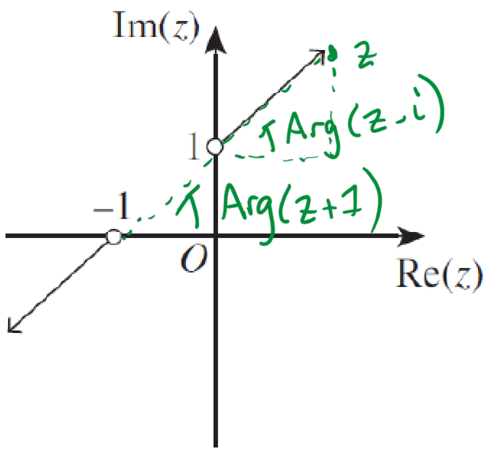
$$\overrightarrow{\boldsymbol{FE}} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{3}{4}\boldsymbol{a}$$

$$\overrightarrow{\boldsymbol{FE}} = \frac{5}{4}\boldsymbol{a} - \boldsymbol{b}$$

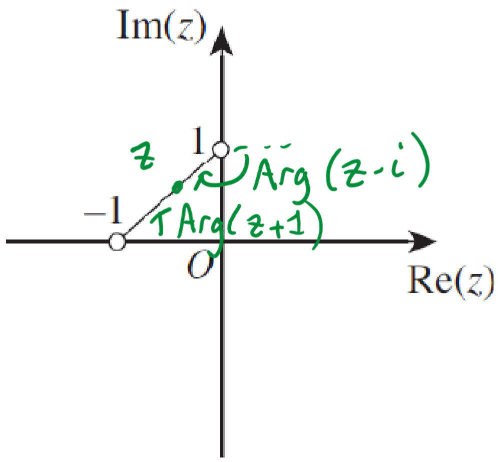
So, the answer is D.

5. Which of the following Argand diagrams represents the solutions to the equation: $Arg(z-i) = Arg(z+1)$?

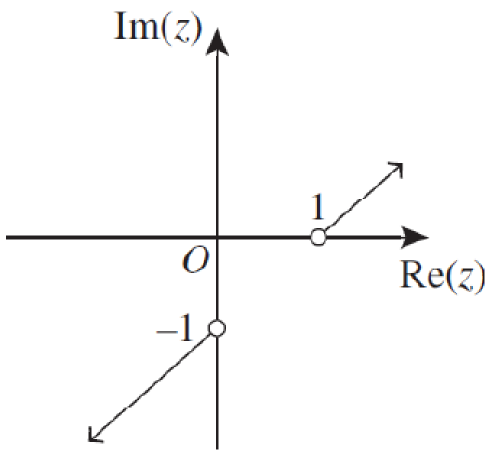
A.



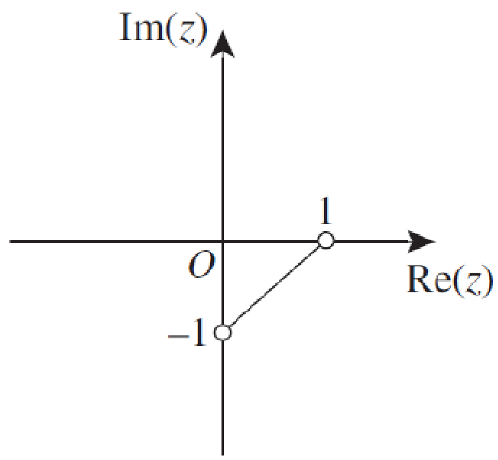
B.



C.



D.



The argument formed from *z* and the point $(0,1)$ is equal to the argument formed from *z* and the point $(-1,0)$.

From the diagram above you can see that B results in the arguments being supplementary rather than equal – hence the answer is A.

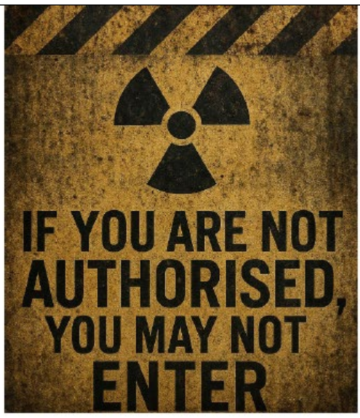
6. Which of the statements is logically equivalent to the one shown opposite.

A. If you may enter, then you are authorised

B. If you are authorised, then you may enter

C. If you are not authorised, then you may enter

D. If you may not enter, then you are not authorised



$$(\textbf{\textit{Not authorised}}) \rightarrow (\textbf{\textit{May not enter}})$$

Is equivalent to its contrapositive,

$$(\textbf{\textit{May enter}}) \rightarrow (\textbf{\textit{Authorised}})$$

So, the answer is A.

7. A particle of **unit** mass is projected vertically upwards from the ground with initial velocity of u m/s. The particle is affected by a gravitational force, g, and a resistive force equivalent to $\frac{1}{2}kv^2$, where the particle's velocity is v m/s.
Let k be the constant of proportionality and x be the displacement, in metres, from the ground after t seconds.
Which of the following is the maximum height, H, of the particle?

A. $H = \int_u^0 \frac{v}{g + \frac{kv^2}{2}}\,dv$

B. $H = \int_0^u \frac{v}{g + \frac{kv^2}{2}}\,dv$

C. $H = \int_u^0 \frac{v}{g - \frac{kv^2}{2}}\,dv$

D. $H = \int_0^u \frac{v}{g - \frac{kv^2}{2}}\,dv$

Equation of motion being projected upwards, so gravity and resistance are negative.

Finding x in terms of v.

$$a = -g - \frac{1}{2}kv^2$$

$$v\frac{dv}{dx} = -\left(g + \frac{1}{2}kv^2\right)$$

$$-\frac{v}{g + \frac{kv^2}{2}}dv = dx$$

Journey to max height is $x: 0 \rightarrow H$, and $v: u \rightarrow 0$.

$$-\int_u^0 \frac{v}{g + \frac{kv^2}{2}}dv = \int_0^H dx$$

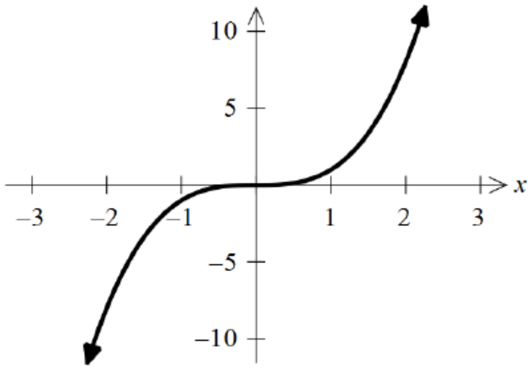
Flipping the sign of the integral with the negative,

The answer is B.

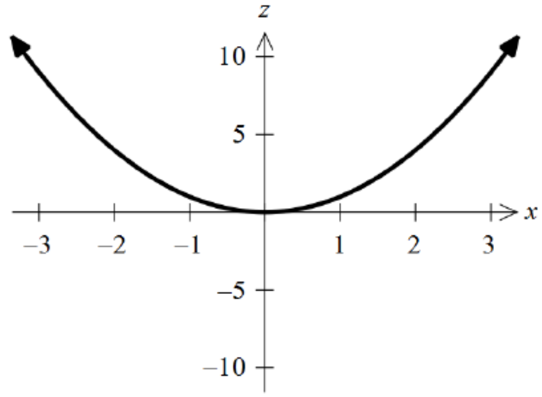
8. The curve $r(t) = (t^2, t^4, t^6)$ is defined for all real values of t.

Which of the following graphs best represents its **projection onto the xz-plane**?

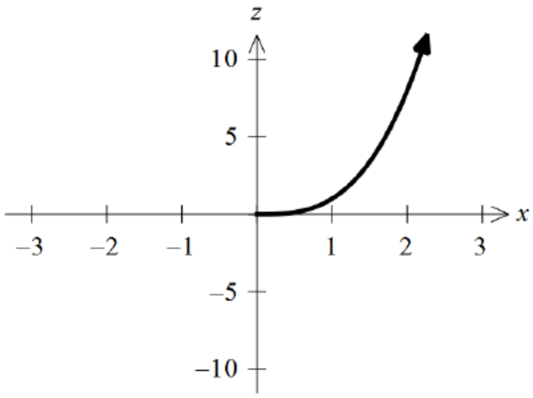
A.



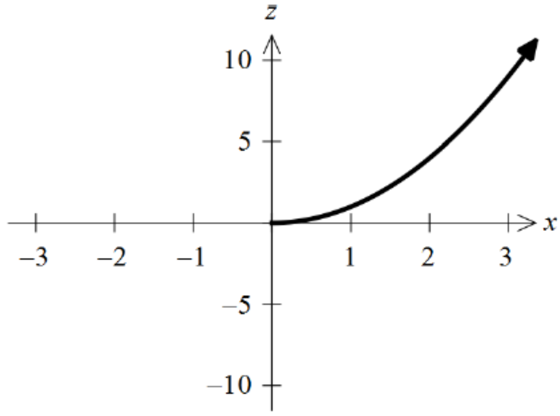
B.



C.



D.



Projection onto the *xz*-plane implies that $\boldsymbol{y = 0}$.

$$\boldsymbol{x = t^2, and\ z = t^6}$$

$$\boldsymbol{z = (t^2)^3 = x^3}$$

But, as $\boldsymbol{x = t^2}$, and *t* is a real value $\boldsymbol{x \geq 0}$.

So, **the answer is C**, as this BEST shows a cubic, compared to D.

9. A car journey is split into two parts.

During the first part the average speed is V km/h.

In the second part, the average speed is U km/h.

Kimie concludes that the average speed for the whole journey is $\frac{U+V}{2}$ km/h.

Which of the following is NOT a counterexample to this conclusion?

A. The car initially travels 100 km at 20 km/h, and then travels 100 km at 40 km/h.

B. The car initially travels 120 km at 60 km/h and 80 km at 40 km/h.

C. The car initially travels for one hour at 100 km/h and then two hours at 50 km/h.

D. The car initially travels 60 km in two hours and then travels 40 km in three hours.

Going through each option:

A:

$$\frac{U+V}{2} = 30km/h$$

Total distance, 200km. Total time, $\frac{100}{20} + \frac{100}{40} = 7.5 \ hours$

Average speed, $\frac{200}{7.5} = 26.\dot{6}km/h$Not, 30km/h, so this is a counterexample.

B:

$$\frac{60+40}{2} = 50km/h$$

Total distance, $120 + 80 = 200km$. Total time $\frac{120}{60} + \frac{80}{40} = 4 \ hours$

Average speed, $\frac{200}{4} = 50 \ km/h$So, this is NOT a counterexample!

The answer is B.

(Alternatively, thinking about it – if they travel for the same time in both legs then the average speed matches the equation).

10. Consider the definite integral, $I = \int_0^{\pi} x \sin x \, dx$.

Using the substitution $x = \pi - y$, what is the value of I?

A. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin y \, dy$

B. $\pi \int_0^{\pi} \sin y \, dy$

C. $\frac{\pi}{2} \int_0^{\pi} \sin y \, dy$

D. $\int_0^{\pi} \sin y \, dy$

$$x = \pi - y$$

$$\frac{dx}{dy} = -1$$

$$dx = -dy$$

$$x = \pi = \pi - y$$

$$y = 0$$

$$x = 0 = \pi - y$$

$$y = \pi$$

$$I = \int_0^{\pi} x \sin x \, dx = \int_{\pi}^{0} (\pi - y) \sin(\pi - y) \times -dy$$

Flipping the sign first, then applying obtuse angle fact, then expanding brackets.

$$I = \int_0^{\pi} (\pi - y) \sin(\pi - y) \, dy$$

$$I = \int_0^{\pi} (\pi - y) \sin(y) \, dy$$

$$I = \int_0^{\pi} \pi \sin(y) \, dy - \int_0^{\pi} y \sin(y) \, dy$$

Note, that second integral is what we started with!

$$I = \int_0^{\pi} \pi \sin(y) \, dy - I$$

$$I = \frac{\pi}{2} \int_0^{\pi} \sin(y) \, dy$$

Answer is C.

Question 11

(a) Let $z_1 = -1-5i$ and $z_2 = 2-i$.

(i) Calculate $i(z_1 - z_2)$. **1**

(ii) Write $\dfrac{a+ib}{z_2}$ in the form $\omega = x+iy$, in terms of a and b. **2**

$$i(z_1 - z_2) = i[(-1-5i)-(2-i)]$$
$$i(z_1 - z_2) = i(-3-4i) = 4-3i$$

$$\frac{a+ib}{z_2} = \frac{a+ib}{2-i}$$
$$\frac{a+ib}{2-i} \times \frac{2+i}{2+i} = \frac{2a+ai+2bi-b}{2^2+1^2}$$
$$\frac{2a-b}{5} + i\frac{a+2b}{5}$$

(b) Consider the statement about integers

If $k^2 - 14k$ is odd, then k is odd.

Prove by contrapositive that the statement is true. **2**

Contrapositive,

$$(k \text{ is even}) \rightarrow (k^2 - 14k \text{ is even})$$
$$\text{Assume } k = 2M, \text{where } M \text{ is an integer}$$
$$k^2 - 14k = (2M)^2 - 14 \times 2M$$
$$k^2 - 14k = 2(2M^2 - 14M)$$

Hence, $k^2 - 14k$ is even as M is an integer.

(c) Let $P\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be any point that is equidistant from $Q\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ and $R\begin{pmatrix} -7 \\ 1 \\ 0 \end{pmatrix}$. (i) Show that the equation of all such points P is $18x - 8y + 8z = -21$. (ii) Give a geometric description of P.	 **2** **1**

$$|QP| = |RP|$$

$$\sqrt{(x-2)^2 + (y+3)^2 + (z-4)^2} = \sqrt{(x+7)^2 + (y-1)^2 + (z)^2}$$

Squaring and expanding…

$$x^2 - 4x + 4 + y^2 + 6y + 9 + z^2 - 8z + 16 = x^2 + 14x + 49 + y^2 - 2y + 1 + z^2$$

Simplifying…

$$29 - 50 = 18x - 8y + 8z$$

$$18x - 8y + 8z = -21$$

As required.

P is a plane that bisects PQ and is perpendicular to PQ. (Plane was needed)

(d) Solve $z^2 + (5+5i)z + (6+15i) = 0$. Give your answer in the form $z = x + iy$.	**3**

Quadratic formula.

$$z = \frac{-(5+5i) \pm \sqrt{(5+5i)^2 - 4(6+15i)}}{2}$$

$$z = \frac{-5 - 5i \pm \sqrt{25 + 50i - 25 - 24 - 60i}}{2} = \frac{-5 - 5i \pm \sqrt{-24 - 10i}}{2}$$

Need to find $\sqrt{-24 - 10i} = a + ib$

$$-24 - 10i = a^2 + 2abi - b^2$$

$$a^2 - b^2 = -24, \qquad and -10 = 2ab$$

$$ab = -5$$

By observation, $a = -1, b = 5$ so…

$$z = \frac{-5 - 5i \pm (-1 + 5i)}{2}$$

$$z = \frac{-6}{2} = -3$$

Or

$$z = \frac{-4 - 10i}{2} = -2 - 5i$$

(e) Find $\displaystyle\int_{-2}^{1} \frac{1+7x}{(x-2)(x+3)} dx$.	4

Partial fractions…

$$\frac{1+7x}{(x-2)(x+3)} \equiv \frac{A}{x-2} + \frac{B}{x+3}$$

$$1+7x \equiv A(x+3) + B(x-2)$$

$$1+7x \equiv 3A - 2B + (A+B)x$$

$$1 = 3A - 2B, and\ 7 = A + B$$

$$B = 7 - A$$

$$1 = 3A - 2(7 - A)$$

$$1 = 5A - 14$$

$$A = 3 \rightarrow B = 4$$

$$\int_{-2}^{1} \frac{1+7x}{(x-2)(x+3)} dx = \int_{-2}^{1} \left(\frac{3}{x-2} + \frac{4}{x+3}\right) dx$$

$$\Big[3\ln|x-2| + 4\ln|x+3|\Big]_{-2}^{1}$$

$$(3\ln 1 + 4\ln 4) - (3\ln 4 + 4\ln 1) = \ln 4$$

Question 12

(a) Use mathematical induction to prove that if $S_1 = a$ and $S_n - S_{n-1} = ar^{n-1}$, $n \geq 2$, then $S_n = \dfrac{a\left(1-r^n\right)}{1-r}$, $n \geq 1$. **3**

Base case: $\boldsymbol{n = 1}$

$$S_1 = a\ (given)$$

Formula,

$$S_1 = \frac{a(1-r^1)}{1-r} = a\ (as\ required)$$

As required.

Assume: $\boldsymbol{n = k}$

$$S_k = \frac{a\left(1-r^k\right)}{1-r}$$

Prove: $\boldsymbol{n = k + 1}$

$$S_{k+1} - S_k = ar^{k+1-1}(by\ definition)$$

$$S_{k+1} = ar^k + \frac{a\left(1-r^k\right)}{1-r}(by\ assumption)$$

$$S_{k+1} = \frac{ar^k(1-r) + a\left(1-r^k\right)}{1-r}$$

$$S_{k+1} = \frac{ar^k - ar^{k+1} + a - ar^k}{1-r}$$

$$S_{k+1} = \frac{a\left(1-r^{k+1}\right)}{1-r}$$

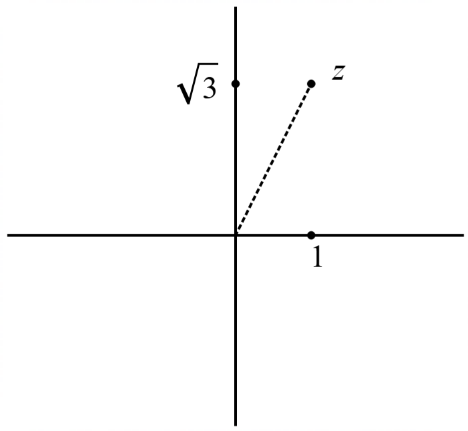
As required.

(b) Consider the complex number $z = 1 + i\sqrt{3}$.

(i) Write z in mod-arg form. **1**

(ii) Show $\sqrt{z^3} = 2\sqrt{2}i$ **1**

(iii) Find all values of n for which z^n is purely imaginary. **1**



$$\mathbf{Arg(z)} = \arctan\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

$$Mod(z) = \sqrt{1^2 + \sqrt{3}^2} = 2$$

$$z = 2\,\text{cis}\,\frac{\pi}{3}$$

$$z^3 = 2^3\,\text{cis}\,\frac{3\pi}{3} = 8\cos\pi + 8\sin\pi = -8$$

$$\sqrt{x^3} = \sqrt{-8} = 2\sqrt{2}i$$

$$z^n = 2^n\,\text{cis}\,\frac{\pi n}{3}$$

Purely imaginary means $\cos\left(\frac{\pi n}{3}\right) = 0$

$$\frac{\pi}{3}n = \frac{\pi}{2} + \pi k$$

Where k is an integer.

$$n = \frac{3}{2} + 3k = \frac{6k+3}{2}$$

(c) (i) Find the Cartesian equation for the line l that passes through the point $\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

and is parallel to the line with equation $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$. **2**

(ii) Determine where the line l meets the xz plane. **2**

$$l = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

$$x = 1 + 5\mu$$

$$\mu = \frac{x-1}{5}$$

$$y = -2 + 2\mu$$

$$\mu = \frac{y+2}{2}$$

$$z = 4 - \mu$$

$$\mu = 4 - z$$

$$\frac{x-1}{5} = \frac{y+2}{2} = 4 - z$$

Line meets xz plane when $y = 0$.

$$\mu = \frac{2}{2} = 1$$

$$x = 1 + 5 \times 1 = 6$$

$$z = 4 - 1 = 3$$

$$(6, 0, 3)$$

(d) Prove that $\log_2 9$ is irrational. **2**

Assume rational.

$$log_2 9 = \frac{a}{b}, where\ a\ and\ b\ are\ integers\ with\ no\ common\ factors > 1$$

$$2^{\frac{a}{b}} = 9$$

$$2^a = 9^b$$

But, as a and b are both integers, then $2^a = EVEN$ and $9^b = ODD$.

So, this cannot be true.

Therefore $log^2 9$ is not rational.

(e) If ω is a complex solution of $z^3 = 1$, simplify the expression $(1+\omega)^2 + (1+\omega^2)^2$. **2**

$$z^3 = 1$$

$$z^3 - 1 = 0$$

From factorizing, and knowing $z = 1$ is a root.

$$(z-1)(z^2+z+1) = 0$$

$$(\omega-1)(\omega^2+\omega+1) = 0$$

$$(\omega^2+\omega+1) = 0$$

$$\omega^2 + \omega = -1,\ \ (\omega+1) = -\omega^2\ and\ (\omega^2+1) = -\omega$$

$$(1+\omega)^2 + (1+\omega^2)^2 = (-\omega^2)^2 + (-\omega)^2$$

$$(1+\omega)^2 + (1+\omega^2)^2 = \omega^4 + \omega^2$$

$$(1+\omega)^2 + (1+\omega^2)^2 = \omega^3 \times \omega + \omega^2 = \omega + \omega^2 = -1$$

NOTE: Expanding the expressions early will reduce to the correct result, but this is not recommended as the powers could be larger (e.g. $(1+\omega^2)^{10}$, good luck expanding that!)

(f) Write the negation of the following statement $\exists x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+ : xy = 1$. **1**

$$\neg\exists x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+ : xy = 1$$

Is equivalent to…

$$\forall x \in \mathbb{R}^+, \neg\forall y \in \mathbb{R}^+ : xy = 1$$

Is equivalent to

$$\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+ : \neg(xy = 1)$$

$$\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+ : xy \neq 1$$

For all real numbers, there exists another number such that their product is not one.

Question 13

(a) Evaluate $\int_0^1 \log_e\left(x^2+1\right)dx$.	4

By parts.

$$u = \ln(x^2+1) \qquad v = x$$

$$\frac{du}{dx} = \frac{2x}{x^2+1} \qquad \frac{dv}{dx} = 1$$

$$\int_0^1 \ln(x^2+1)dx = \left(\ln(x^2+1)\times x\right)_0^1 - \int_0^1 \frac{2x}{x^2+1}\times x\,dx$$

$$\int_0^1 \ln(x^2+1)dx = (\ln 2 - \ln 1) - \int_1^e \frac{2x^2+2-2}{x^2+1}dx$$

$$\int_0^1 \ln(x^2+1)dx = \ln 2 - \int_0^1 \left(2-\frac{2}{x^2+1}\right)dx$$

$$\int_0^1 \ln(x^2+1)dx = \ln 2 - \left(2x - 2\tan^{-1}x\right)_0^1$$

$$\int_0^1 \ln(x^2+1)dx = \ln 2 - [2-2\tan^{-1}1-(0-2\tan^{-1}0)]$$

$$\int_0^1 \ln(x^2+1)dx = \ln 2 - 2 + \frac{\pi}{2}$$

(b) Consider the spheres S_1 and S_2, where

$$S_1 : x^2-2x+y^2+4y+z^2-4z=0$$

$$S_2 : x^2+y^2+z^2+10x+2z+10=0$$

Explain why S_1 and S_2 intersect at a single point. (Do NOT find this point.) **2**

$$S_1: (x-1)^2+(y+2)^2+(z-2)^2 = 1+4+4 = 9$$

Sphere with centre $(1,-2,2)$ and radius 3 units.

$$S_2: (x+5)^2+(y)^2+(z+1)^2 = -10+25+1 = 16$$

Sphere with centre $(-5,0,-1)$ and radius *4* units.

- **Distance between the centres?**

$$\sqrt{6^2+2^2+3^2} = 7$$

Equivalent to the sum of the radii – hence they will touch once.

(c) Consider the complex number $z = \cos\theta + i\sin\theta$.

(i) Show that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$. **1**

(ii) Hence, or otherwise, find the constants a, b and c such that **3**

$$\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta.$$

(iii) Hence evaluate $16\displaystyle\int_0^{\frac{\pi}{2}} \sin^5\theta\ d\theta$. **1**

$$z^n = \operatorname{cis}(n\theta)$$

$$\frac{1}{z^n} = \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$$

$$\frac{1}{z^n} = \frac{\cos n\theta - i\sin n\theta}{\cos^2\theta + \sin^2\theta} = \cos n\theta - i\sin n\theta$$

$$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2i\sin n\theta$$

Proofs involving De Moivre's are fine.

$$\left(z^1 - \frac{1}{z^1}\right)^5 = (2i\sin\theta)^5 = 2^5 i\sin^5\theta$$

Now, expanding the LHS.

$$\binom{5}{0}z^5 + \binom{5}{1}z^4 \times \left(-\frac{1}{z}\right)^1 + \binom{5}{2}z^3 \times \left(-\frac{1}{z}\right)^2 + \binom{5}{3}z^2 \times \left(-\frac{1}{z}\right)^3 + \binom{5}{4}z^1 \times \left(-\frac{1}{z}\right)^4 + \binom{5}{5}z^0 \times \left(-\frac{1}{z}\right)^5$$

$$z^5 - 5z^3 + 10z - 10\frac{1}{z} + 5\frac{1}{z^3} - \frac{1}{z^5} = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

Then, using the identity from part (i) again.

$$\left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

Combining the two results,

$$2^5 i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$\sin^5\theta = \frac{1}{2^4}\sin 5\theta - \frac{5}{2^4}\sin 3\theta + \frac{10}{2^4}\sin\theta$$

(iii) Hence evaluate $16\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta$. **1**

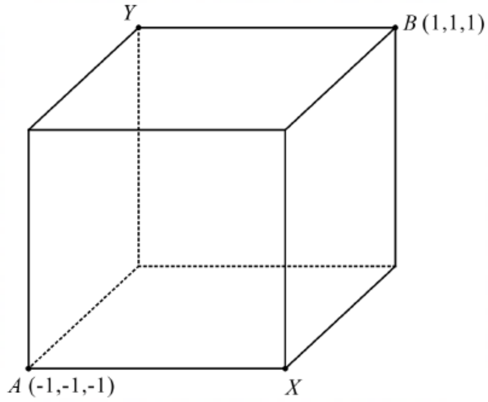
$$16\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = 16\int_0^{\frac{\pi}{2}} \left(\frac{1}{2^4}\sin 5\theta - \frac{5}{2^4}\sin 3\theta + \frac{10}{2^4}\sin\theta\right) d\theta$$

$$16\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta) d\theta$$

$$16\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \left(-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right)_0^{\frac{\pi}{2}}$$

$$16\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = (0 - 0 + 0) - \left(-\frac{1}{5} + \frac{5}{3} - 10\right) = \frac{128}{15}$$

(d) A cube has a pair of opposite vertices A (-1, -1, -1) and B (1, 1, 1).
Another pair of opposite vertices are X and Y, as displayed in the diagram below.



If α is the acute angle between AB and XY, show that

$$\sin 2\alpha = \frac{2\sqrt{8}}{9}.$$

2

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Without loss of generality we can assume that X and Y are formed to be parallel with the axes, so $\boldsymbol{X} = (\mathbf{1}, -\mathbf{1}, -\mathbf{1})$ and $\boldsymbol{Y} = (-\mathbf{1}, \mathbf{1}, \mathbf{1})$.

$$\overrightarrow{XY} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$$

$$|AB| = |XY| = \sqrt{12} = 2\sqrt{3}$$

$$\overrightarrow{AB} \cdot \overrightarrow{XY} = |\overrightarrow{AB}| \times |\overrightarrow{XY}|\cos\alpha = -4 + 4 + 4$$

$$12\cos\alpha = 4$$

$$\cos\alpha = \frac{1}{3}$$

From an acute triangle or $(\sin^2 x = 1 - \cos^2 x)$ **we can conclude that,**

$$\sin\alpha = \frac{\sqrt{8}}{3}$$

$$\sin 2\alpha = 2\cos\alpha\sin\alpha = 2 \times \frac{1}{3} \times \frac{\sqrt{8}}{3} = \frac{2\sqrt{8}}{9}$$

(e) Three positive number a, b and c satisfy the following conditions $a \geq b \geq c$ and $a+b+c \leq 1$.

By considering $(a+b+c)^2$ or otherwise, prove that $a^2+3b^2+5c^2 \leq 1$. **2**

Initially, following the instructions.

$$a+b+c \leq 1$$

$$(a+b+c)^2 \leq 1$$

$$a^2+ab+ac+b^2+ab+bc+c^2+ac+bc \leq 1$$

$$a^2+b^2+c^2+2ab+2ac+2bc \leq 1$$

Now, as

$$a \geq b \geq c$$

Then by multiplying by '2b'

$$2ab \geq 2b^2 \geq 2bc$$

Or multiplying by '2a'

$$2ac \geq 2bc \geq 2c^2$$

Which means,

$$2ab \geq 2b^2 \ldots and \ldots\ 2ac \geq 2c^2 \ldots and \ldots 2bc \geq c^2$$

$$a^2+b^2+c^2+2b^2+2c^2+2c^2 \leq a^2+b^2+c^2+2ab+2ac+2bc \leq 1$$

$$a^2+3b^2+5c^2 \leq 1$$

As required.

Question 14

(a) Use the substitution of $u = \sin x - x\cos x$, evaluate **3**

$$\int_{\frac{\pi}{2}}^{\pi} \frac{x}{1 - x\cot x}\,dx.$$

$$u = \sin x - x\cos x$$

$$\frac{du}{dx} = \cos x - \cos x + x\sin x = x\sin x$$

$$\int \frac{x}{1 - x\cot x}\,dx = \int \frac{x}{1 - x\dfrac{\cos x}{\sin x}}\,dx = \int \frac{x\sin x}{\sin x - x\cos x}\,dx$$

$$\int \frac{x\sin x}{\sin x - x\cos x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x - x\cos x| + C$$

So…

$$\int_{\frac{\pi}{2}}^{\pi} \frac{x}{1 - x\cot x}\,dx = \Big[\ln|\sin x - x\cos x|\Big]_{\frac{\pi}{2}}^{\pi}$$

$$= \ln|0 - \pi \times -1| - \ln\left|1 - \frac{\pi}{2} \times 0\right| = \ln \pi$$

Obviously changing the bounds is a valid approach.

(b) Consider the following four equations, where $a>0$ and $b>0$.

$$D=\frac{2ab}{a+b}$$

$$U=\sqrt{ab}$$

$$B=\frac{a+b}{2}$$

$$O=\sqrt{\frac{a^2+b^2}{2}}$$

Show that $D\le U\le B\le O$. **3**

$$\left(\sqrt{a}-\sqrt{b}\right)^2\ge 0$$

$$a-2\sqrt{ab}+b\ge 0$$

$$a+b\ge 2\sqrt{ab}$$

$$\frac{a+b}{2ab}\ge\frac{2\sqrt{ab}}{2ab}$$

$$\frac{a+b}{2ab}\ge\frac{1}{\sqrt{ab}}$$

$$\sqrt{ab}\ge\frac{2ab}{a+b}$$

$$U\ge D$$

From earlier,

$$a+b\ge 2\sqrt{ab}$$

$$\frac{a+b}{2}\ge\sqrt{ab}$$

$$B\ge U$$

If you can't see a pathway, contradiction works! Assume $B>O$

$$\frac{a+b}{2}>\sqrt{\frac{a^2+b^2}{2}}$$

$$\frac{a^2+2ab+b^2}{4}>\frac{a^2+b^2}{2}$$

$$0>\frac{2a^2+2b^2-a^2-2ab-b^2}{4}$$

$$0>\frac{a^2-2ab+b^2}{4}$$

$$0>\frac{(a-b)^2}{4}$$

Contradiction as $(a-b)^2\ge 0$**, therefore** $B\le O$ **as required!**

NOTE: Students should avoid assuming the AM-GM inequality in situations where it is naively similar.

(c) The triangle ABC in the complex plane is formed from the points $A(\alpha)$, $B(\beta)$ and $C(\gamma)$. The complex numbers α, β and γ satisfy these two conditions:

$$\frac{\gamma-\alpha}{\beta-\alpha}=1-i \quad \text{and} \quad \alpha+\beta+\gamma=0.$$

Calculate $|\alpha|:|\beta|$. **3**

$$\gamma=-\alpha-\beta$$

$$\frac{\gamma-\alpha}{\beta-\alpha}=1-i$$

$$\frac{-2\alpha-\beta}{\beta-\alpha}=1-i$$

$$-2\alpha-\beta=\beta-i\beta-\alpha+i\alpha$$

$$-2\beta+i\beta=\alpha+i\alpha$$

$$\beta(-2+i)=\alpha(1+i)$$

$$\frac{\alpha}{\beta}=\frac{-2+i}{1+i}$$

$$\left|\frac{\alpha}{\beta}\right|=\left|\frac{-2+i}{1+i}\right|$$

$$\frac{|\alpha|}{|\beta|}=\frac{|-2+i|}{|1+i|}$$

$$\frac{|\alpha|}{|\beta|}=\frac{\sqrt{5}}{\sqrt{2}}=\frac{\sqrt{10}}{2}$$

Methods that involved geometric implications sadly got very muddled and confused – as the length of the numbers are not equal, so, yes, when considered like vectors they appear as at a triangle with a vertex at the origin. (But there isn't much more that can be concluded due to different magnitudes)

(d) Two magpies, A and B, are observed flying. **3**

Their initial respective positions are:

$\left(\underset{\sim}{i}-2\underset{\sim}{j}+4\underset{\sim}{k}\right)$ metres and $\left(-2\underset{\sim}{i}+a\underset{\sim}{j}+6\underset{\sim}{k}\right)$ metres, where a is a constant.

They are travelling with constant velocities and these are respectively:

$\left(2\underset{\sim}{i}+3\underset{\sim}{j}+6\underset{\sim}{k}\right)$ m/s and $\left(3\underset{\sim}{i}+12\underset{\sim}{j}+4\underset{\sim}{k}\right)$ m/s.

Determine the constant a, if the distance between the two magpies is least when $t=5$ seconds.

$v_A = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$	$v_B = \begin{pmatrix} 3 \\ 12 \\ 4 \end{pmatrix}$
$x_A = \begin{pmatrix} 2t \\ 3t \\ 6t \end{pmatrix} + C$	$x_B = \begin{pmatrix} 3t \\ 12t \\ 4t \end{pmatrix} + C$
$t = 0, x_A = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$	$t = 0, x_B = \begin{pmatrix} -2 \\ a \\ 6 \end{pmatrix}$
$x_A = \begin{pmatrix} 2t+1 \\ 3t-2 \\ 6t+4 \end{pmatrix}$	$x_B = \begin{pmatrix} 3t-2 \\ 12t+a \\ 4t+6 \end{pmatrix}$

Distance between the two,

$$\left|\begin{pmatrix} 3t-2 \\ 12t+a \\ 4t+6 \end{pmatrix} - \begin{pmatrix} 2t+1 \\ 3t-2 \\ 6t+4 \end{pmatrix}\right| = \left|\begin{pmatrix} t-3 \\ 9t+a+2 \\ -2t+2 \end{pmatrix}\right|$$

$$DIST = \sqrt{(t-3)^2 + (9t+a+2)^2 + (2-2t)^2}$$

When max distance occurs, so does max $DIST^2$

$$DIST^2 = t^2 - 6t + 9 + 81t^2 + 9at + 18t + 9at + a^2 + 2a + 18t + 2a + 4 + 4 - 8t + 4t^2$$

$$DIST^2 = 86t^2 + 22t + 18at + 17$$

This is a quadratic, so the minimum distance occurs at the turning point, when the derivative is zero.

$$\frac{d(DIST^2)}{dt} = 172t + 22 + 18a = 0$$

$$t = 5, is\ when\ it\ occurs\ (from\ question)$$

$$882 + 18a = 0$$

$$a = -49$$

NOTE: Using $t = 5$ before finding the max distance results in an incorrect result as time is a variable and the nature of the function in terms of time needs to be considered.

(e)	By considering $(1+i)(4-i)^2$ prove that $\tan^{-1}\left(\frac{7}{23}\right)+2\tan^{-1}\left(\frac{1}{4}\right)=\frac{\pi}{4}$.	3

Expanding algebraically…

$$(1+i)(4-i)^2 = (1+i)(17-8i) = 23+7i$$

Then, considering the arguments

$$Arg(1+i) = \frac{\pi}{4}, Arg(4-i) = -\arctan\left(\frac{1}{4}\right), Arg\,(23+7i) = \arctan\left(\frac{7}{23}\right)$$

And so by applying the addition of arguments…

$$\frac{\pi}{4} + 2\times -\arctan\left(\frac{1}{4}\right) = \arctan\left(\frac{7}{23}\right)$$

$$\frac{\pi}{4} = \arctan\left(\frac{7}{23}\right) + 2\arctan\left(\frac{1}{4}\right)$$

Question 15

(a) A body of **unit** mass falls vertically from rest, under gravity, from a height of 100 metres above the ground through a resisting medium. The resistance to its motion is $\frac{v^2}{100}$, where v is the speed of the body (in m/s) after falling a distance x metres.

(i) Show that the acceleration $\ddot{x}$ of the body is given by $\ddot{x} = g - \frac{v^2}{100}$. **1**

(ii) Determine the terminal velocity V of the body. **1**

(iii) Show that the velocity v of the body, after it has fallen a vertical distance x, is given by

$$v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right).$$ **3**

(iv) Determine the distance fallen (in metres) when the velocity of the body first reaches $\frac{V}{2}$. **2**

$$F = ma$$

$$mg - \frac{v^2}{100} = ma$$

$$m = 1$$

$$a = mg - \frac{v^2}{100}$$

Terminal velocity occurs when acceleration is zero.

$$g - \frac{V^2}{100} = 0$$

$$V^2 = 100g$$

$$V = 10\sqrt{g}$$

$$v \times \frac{dv}{dx} = g - \frac{v^2}{100} = \frac{100g - v^2}{100}$$

$$\frac{v}{100g - v^2}dv = \frac{dx}{100}$$

$$\frac{-2v}{V^2 - v^2}dv = -\frac{dx}{50} \quad (using\ V^2\ definition\ from\ earlier)$$

$$\ln|V^2 - v^2| = -\frac{x}{50} + C$$

$$t = 0, x = 0, v = 0$$

$$\ln|V^2| = C$$

$$\ln|V^2 - v^2| = -\frac{x}{50} + \ln(V^2)$$

$$V^2 - v^2 = e^{-\frac{x}{50}} \times e^{ln(V^2)}$$

$$V^2 - v^2 = V^2 \times e^{-\frac{x}{50}}$$

$$v^2 = V^2 - V^2 \times e^{-\frac{x}{50}} = V^2\left(1 - e^{-\frac{x}{50}}\right)$$

$$v = \frac{V}{2}, x = ?$$

$$\left(\frac{V}{2}\right)^2 = V^2\left(1 - e^{-\frac{x}{50}}\right)$$

$$\frac{1}{4} = 1 - e^{-\frac{x}{50}}$$

$$e^{-\frac{x}{50}} = \frac{3}{4}$$

$$-\frac{x}{50} = \ln\frac{3}{4}$$

$$x = -50\ln\frac{3}{4} = 50\ln\frac{4}{3}$$

(b) Let $S_n = \sum_{p=1}^{n} \tan^{-1}\left(\frac{1}{2p^2}\right)$.

Prove by induction that $\tan(S_n) = \frac{n}{n+1}$. **4**

Base case, $n = 1$

$$S_1 = \arctan\left(\frac{1}{2}\right)$$

$$\tan(S_1) = \frac{1}{2} = \frac{1}{1+1}$$

Assume, $n = k$

$$\tan(S_k) = \frac{k}{k+1}$$

Prove, $n = k + 1$

$$S_{k+1} = \arctan\left(\frac{1}{2(k+1)^2}\right) + S_k$$

$$\tan(S_{k+1}) = \tan\left(\arctan\left(\frac{1}{2(k+1)^2}\right) + S_k\right)$$

$$\tan(S_{k+1}) = \frac{\tan\left(\arctan\left(\frac{1}{2(k+1)^2}\right)\right) + \tan(S_k)}{1 - \tan\left(\arctan\left(\frac{1}{2(k+1)^2}\right)\right) \times \tan(S_k)}$$

$$\tan(S_{k+1}) = \frac{\frac{1}{2(k+1)^2} + \frac{k}{k+1}}{1 - \frac{1}{2(k+1)^2} \times \frac{k}{k+1}} \quad (\textit{using the assumption and} \tan(\tan^{-1} x) = x)$$

Multiplying by $2(k+1)^3$

$$\tan(S_{k+1}) = \frac{(k+1) + 2k(k+1)^2}{2(k+1)^3 - k}$$

$$\tan(S_{k+1}) = \frac{(k+1)\big(1 + 2k(k+1)\big)}{2k^3 + 6k^2 + 6k + 2 - k}$$

$$\tan(S_{k+1}) = \frac{(k+1)(2k^2 + 2k + 1)}{2k^3 + 6k^2 + 5k + 2}$$

Now, knowing that the desired denominator is *k+2*

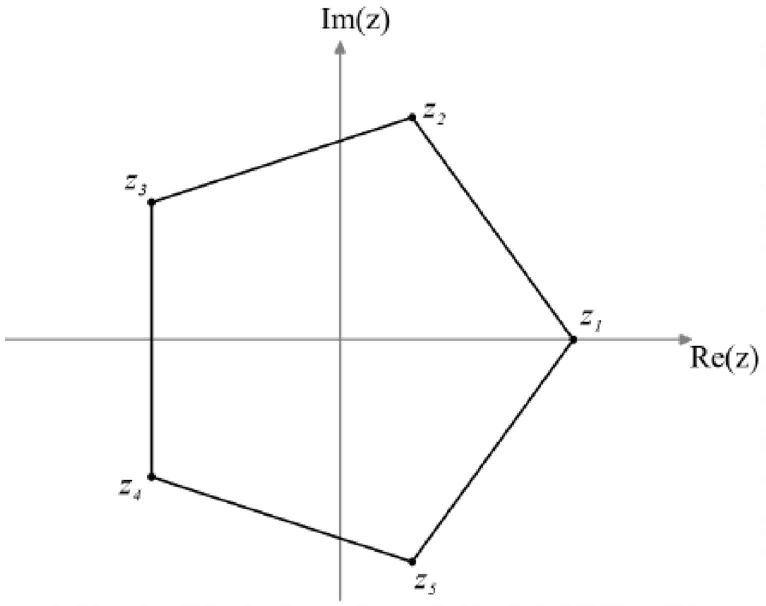
$$\tan(S_{k+1}) = \frac{(k+1)(2k^2 + 2k + 1)}{(k+2)(ak^2 + bk + c)}$$

Applying polynomial division / by observation

$$\tan(S_{k+1}) = \frac{(k+1)(2k^2 + 2k + 1)}{(k+2)(2k^2 + 2k + 1)} = \frac{k+1}{k+2}$$

As required!

(c) The solutions of $z^5 = 1$ can be joined to form a regular pentagon in the complex plane, as shown below.



(i) For $n \geq 3$, show that the perimeter of the regular n-gon formed by the solutions of $z^n = 1$ is $P_n = 2n\sin\left(\dfrac{\pi}{n}\right)$. **2**

(ii) It is known that as $x \to 0$ then $\dfrac{\sin x}{x} \to 1$. (Do NOT prove this.)
Using part (i), show that as $n \to \infty$, $P_n \to 2\pi$. **2**

The length of one side, S, can be found with the cosine rule.

The angle is $\frac{2\pi}{n}$ and the adjacent sides are 1.

$$S^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\left(\frac{2\pi}{n}\right)$$

$$S^2 = 2\left(1 - \cos\left(2 \times \frac{\pi}{n}\right)\right)$$

From the reference sheet,

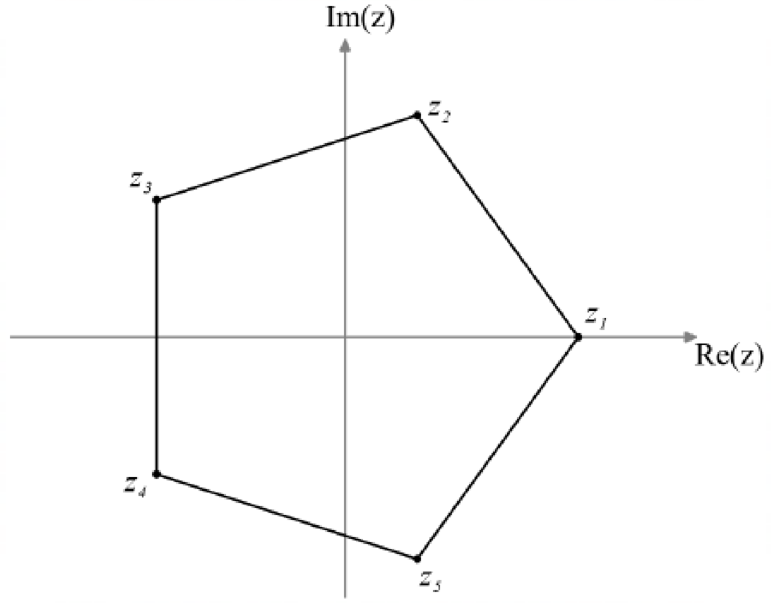
$$S^2 = 2 \times 2\sin^2\frac{\pi}{n}$$

$$S = 2\sin\frac{\pi}{n}$$

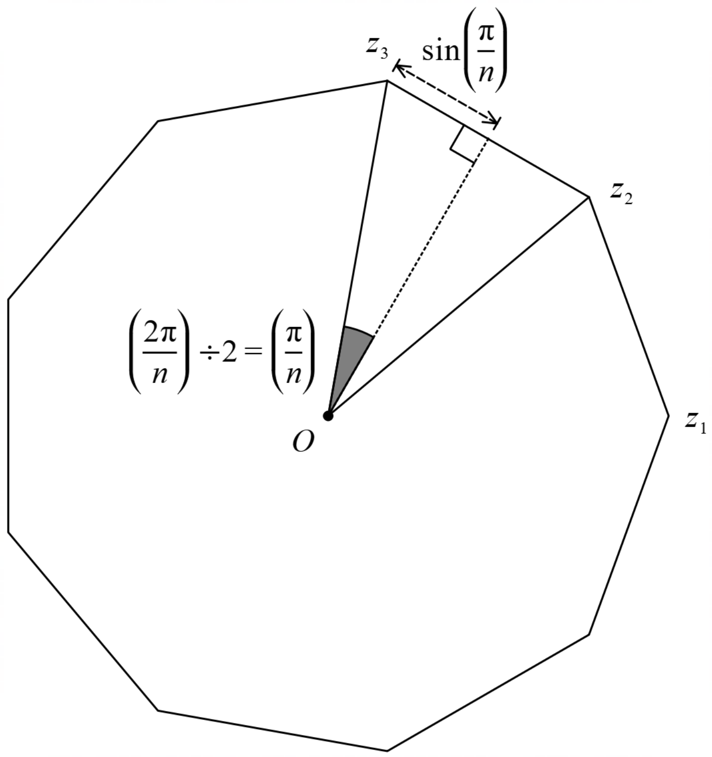
There are n sides, so

$$P_n = 2n\sin\frac{\pi}{n}$$

(c) The solutions of $z^5 = 1$ can be joined to form a regular pentagon in the complex plane, as shown below.



(i) For $n \geq 3$, show that the perimeter of the regular n-gon formed by the solutions of $z^n = 1$ is $P_n = 2n\sin\left(\frac{\pi}{n}\right)$. **2**



Alternatively, you could use right angled trig to find the length of half of one of those sides.

As, $|z_n| = 1$, then

$$\sin\left(\frac{\pi}{n}\right) = \frac{\textit{length}}{1}$$

This then concludes that the length of one side is $2\sin\left(\frac{\pi}{n}\right)$

And then $P_n = 2n\sin\left(\frac{\pi}{n}\right)$

(ii) It is known that as $x \to 0$ then $\frac{\sin x}{x} \to 1$. (Do NOT prove this.)

Using part (i), show that as $n \to \infty$, $P_n \to 2\pi$. **2**

As $n \to \infty$

$$\frac{\pi}{n} \to 0$$

So,

$$\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \to 1$$

$$\frac{n\sin\left(\frac{\pi}{n}\right)}{\pi} \to 1$$

$$n\sin\left(\frac{\pi}{n}\right) \to \pi$$

$$2n\sin\left(\frac{\pi}{n}\right) \to 2\pi$$

As required!

Question 16

(a) Let $I_n = \int_0^{\frac{1}{4}} \frac{x^n}{e^x}\,dx$ where $n = 1, 2, 3, \ldots$.

(i) Calculate I_1. **2**

$$I_1 = \int_0^{\frac{1}{4}} \frac{x}{e^x}\,dx = \int_0^{\frac{1}{4}} xe^{-x}\,dx$$

Integration by parts,

$u = x$	$v = -e^{-x}$
$u' = 1$	$v' = e^{-x}$

$$I_1 = \left(-xe^{-x}\right)_0^{\frac{1}{4}} - \int_0^{\frac{1}{4}} 1 \times -e^{-x}\,dx$$

$$I_1 = -\frac{1}{4} \times e^{-\frac{1}{4}} - 0 - \left(e^{-x}\right)_0^{\frac{1}{4}}$$

$$I_1 = -\frac{1}{4e^{\frac{1}{4}}} - \left(e^{-\frac{1}{4}} - e^0\right)$$

$$I_1 = 1 - \frac{1}{e^{\frac{1}{4}}} - \frac{1}{4e^{\frac{1}{4}}} = 1 - \frac{5}{4e^{\frac{1}{4}}}$$

(ii) Show that $kI_k = I_{k+1} - I_k + \frac{\left(\frac{1}{4}\right)^{k+1}}{e^{\frac{1}{4}}}$. **3**

$$I_{k+1} = \int_0^{\frac{1}{4}} \frac{x^{k+1}}{e^x}\,dx$$

By parts,

$u = x^{k+1}$	$v = -e^{-x}$
$u' = (k+1)x^k$	$v' = e^{-x}$

$$I_{k+1} = \left(-x^{k+1}e^{-x}\right)_0^{\frac{1}{4}} - \int_0^{\frac{1}{4}} (k+1)x^k \times -e^{-x}\,dx$$

$$I_{k+1} = -\left(\frac{1}{4}\right)^{k+1} e^{-\frac{1}{4}} - 0 + (k+1)\int_0^{\frac{1}{4}} \frac{x^k}{e^x}\,dx$$

$$I_{k+1} = \frac{-\left(\frac{1}{4}\right)^{k+1}}{e^{\frac{1}{4}}} + (k+1)I_k$$

$$I_{k+1} - I_k + \frac{\left(\frac{1}{4}\right)^{k+1}}{e^{\frac{1}{4}}} = kI_k$$

As required.

(Alternative approach on next page.)

(ii) Show that $kI_k = I_{k+1} - I_k + \frac{\left(\frac{1}{4}\right)^{k+1}}{e^{\frac{1}{4}}}$.	**3**

$$I_k = \int_0^{\frac{1}{4}} \frac{x^k}{e^x} dx$$

By parts,

$u = e^{-x}$	$v = \frac{x^{k+1}}{k+1}$
$u' = -e^{-x}$	$v' = x^k$

$$I_k = \left(e^{-x} \times \frac{x^{k+1}}{k+1}\right)_0^{\frac{1}{4}} - \int_0^{\frac{1}{4}} (-e^{-x}) \times \left(\frac{x^{k+1}}{k+1}\right) dx$$

$$I_k = \left[\frac{\left(\frac{1}{4}\right)^{k+1} e^{-\frac{1}{4}}}{k+1} - 0\right] + \frac{1}{k+1} \int_0^{\frac{1}{4}} \frac{x^{k+1}}{e^x} dx$$

$$I_k = \frac{\left(\frac{1}{4}\right)^{k+1} e^{-\frac{1}{4}}}{k+1} + \frac{I_{k+1}}{k+1}$$

$$(k+1)I_k = \frac{\left(\frac{1}{4}\right)^{k+1}}{e^{\frac{1}{4}}} + I_{k+1}$$

$$kI_k = I_{k+1} - I_k + \frac{\left(\frac{1}{4}\right)^{k+1}}{e^{\frac{1}{4}}}$$

As required.

NOTE, if you accidentally get an expression involving 'k-1' – you can substitute $u = k-1, u = k, and\ u + 1 = k + 1$

This should then get you to the desired result (if you are careful...)

(iii) Deduce that $\sum_{k=1}^{n} kI_k = I_{k+1} - I_1 + \frac{1-\left(\frac{1}{4}\right)^n}{12e^{\frac{1}{4}}}$. **3**

Induction: $n = 1$ **From part (ii)**

$$\sum_{k=1}^{1} kI_k = I_1 = I_2 - I_1 + \frac{\left(\frac{1}{4}\right)^2}{e^{\frac{1}{4}}} = I_2 - I_1 + \frac{\frac{1}{16}}{e^{\frac{1}{4}}} = I_2 - I_1 + \frac{1}{16e^{\frac{1}{4}}}$$

Desired result is,

$$\sum_{k=1}^{1} kI_k = I_2 - I_1 + \frac{1-\left(\frac{1}{4}\right)^1}{12e^{\frac{1}{4}}} = I_2 - I_1 + \frac{1}{16e^{\frac{1}{4}}}$$

As required.

Assume: $n = Q$

$$\sum_{k=1}^{Q} kI_k = I_{Q+1} - I_1 + \frac{1-\left(\frac{1}{4}\right)^Q}{12e^{\frac{1}{4}}}$$

Prove: $n = Q + 1$

$$\sum_{k=1}^{Q+1} kI_k = \sum_{k=1}^{Q} kI_k + (Q+1)I_{Q+1}$$

$$= I_{Q+1} - I_1 + \frac{1-\left(\frac{1}{4}\right)^Q}{12e^{\frac{1}{4}}} + (Q+1)I_{Q+1}\ (Assumption)$$

From part (ii)

$$(Q+1)I_{Q+1} = I_{Q+2} - I_{Q+1} + \frac{\left(\frac{1}{4}\right)^{Q+2}}{e^{\frac{1}{4}}}$$

$$\sum_{k=1}^{Q+1} kI_k = I_{Q+1} - I_1 + \frac{1-\left(\frac{1}{4}\right)^Q}{12e^{\frac{1}{4}}} + I_{Q+2} - I_{Q+1} + \frac{\left(\frac{1}{4}\right)^{Q+2}}{e^{\frac{1}{4}}}$$

$$\sum_{k=1}^{Q+1} kI_k = I_{Q+2} - I_1 + \frac{1-\left(\frac{1}{4}\right)^Q + 12\left(\frac{1}{4}\right)^{Q+2}}{12e^{\frac{1}{4}}}$$

$$\sum_{k=1}^{Q+1} kI_k = I_{Q+2} - I_1 + \frac{1-\left(\frac{1}{4}\right)^Q\left(1-12\left(\frac{1}{4}\right)^2\right)}{12e^{\frac{1}{4}}} = I_{Q+2} - I_1 + \frac{1-\left(\frac{1}{4}\right)^{Q+1}}{12e^{\frac{1}{4}}}$$

As required!

Alternatively, the sum is in fact telescoping – that is.

$$\sum_{k=1}^{n} kI_k = 1I_1 + 2I_2 + 3I_3 + \cdots + (n-1)I_{n-1} + nI_n$$

Using part (ii)

$$\sum_{k=1}^{n} kI_k = \left(I_2 - I_1 + \frac{\left(\frac{1}{4}\right)^2}{e^{\frac{1}{4}}}\right) + \left(I_3 - I_2 + \frac{\left(\frac{1}{4}\right)^3}{e^{\frac{1}{4}}}\right) + \left(I_4 - I_3 + \frac{\left(\frac{1}{4}\right)^4}{e^{\frac{1}{4}}}\right) + \cdots + \left(I_n - I_{n-1} + \frac{\left(\frac{1}{4}\right)^n}{e^{\frac{1}{4}}}\right) + \left(I_{n+1} - I_n + \frac{\left(\frac{1}{4}\right)^{n+1}}{e^{\frac{1}{4}}}\right)$$

So, as this sum keeps going you see that the middle I_k terms cancel out! So, we are left with.

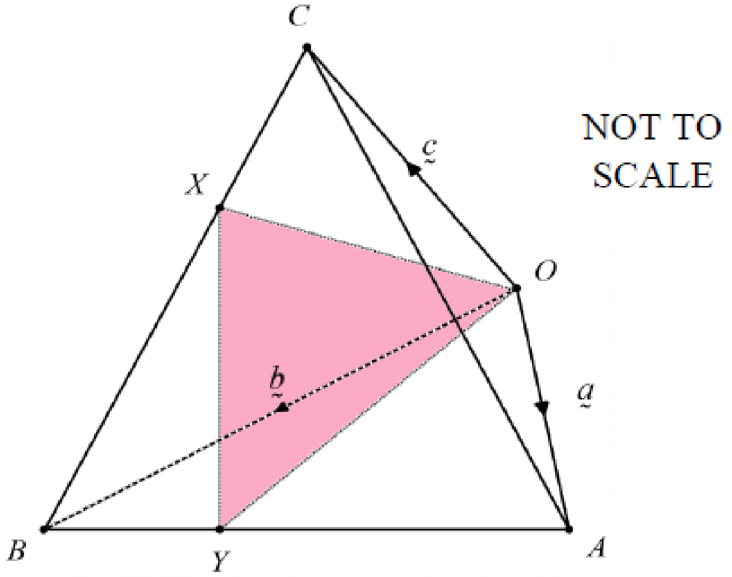
$$\sum_{k=1}^{n} kI_k = -I_1 + I_{n+1} + \frac{1}{e^{\frac{1}{4}}}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots \left(\frac{1}{4}\right)^{n+1}\right]$$

That last term is a GP with n-terms and $a = \left(\frac{1}{4}\right)^2$, $r = \left(\frac{1}{4}\right)$

$$\sum_{k=1}^{n} kI_k = -I_1 + I_{n+1} + \frac{1}{e^{\frac{1}{4}}} \times \left[\frac{\left(\frac{1}{4}\right)^2\left(1 - \left(\frac{1}{4}\right)^n\right)}{1 - \frac{1}{4}}\right] = I_{n+1} - I_1 + \frac{1 - \left(\frac{1}{4}\right)^n}{12e^{\frac{1}{4}}}$$

As required!

(b) The diagram shows a regular tetrahedron $OABC$ with sides of length 1.



Let X and Y be points on the edges such that $|\overrightarrow{BX}| : |\overrightarrow{XC}| = |\overrightarrow{AY}| : |\overrightarrow{YB}| = x : 1 - x$, where $0 < x < 1$.

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, and $\overrightarrow{OC} = \underset{\sim}{c}$.

(i) Show that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = \frac{1}{2}$. **1**

As OABC is a regular tetrahedron, each side is an equilateral triangle. Hence, the angles formed are $\frac{\pi}{3}$ radians.

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \times |\boldsymbol{b}| \times \cos\frac{\pi}{3} = \frac{1}{2}$$

Similarly,

$$\boldsymbol{a} \cdot \boldsymbol{c} = |\boldsymbol{a}| \times |\boldsymbol{c}| \times \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\boldsymbol{b} \cdot \boldsymbol{c} = |\boldsymbol{b}| \times |\boldsymbol{c}| \times \cos\frac{\pi}{3} = \frac{1}{2}$$

(ii) Show that $\left\|\overrightarrow{OY}\right\| = \sqrt{1-x+x^2}$.	**2**

$$\overrightarrow{OY} = \overrightarrow{OB} + \overrightarrow{BY}$$

$$\overrightarrow{OY} = \boldsymbol{b} + (1-x)\overrightarrow{BA}$$

$$\overrightarrow{OY} = \boldsymbol{b} + (1-x)(-\boldsymbol{b} + \boldsymbol{a})$$

$$\overrightarrow{OY} = (1-1+x)\boldsymbol{b} + (1-x)\boldsymbol{a}$$

$$\overrightarrow{OY} = (1-x)\boldsymbol{a} + (x)\boldsymbol{b}$$

$$\left|\overrightarrow{OY}\right|^2 = \overrightarrow{OY} \cdot \overrightarrow{OY} = [(1-x)\boldsymbol{a} + (x)\boldsymbol{b}] \cdot [(1-x)\boldsymbol{a} + (x)\boldsymbol{b}]$$

$$\left|\overrightarrow{OY}\right|^2 = (1-x)^2 \times \boldsymbol{a}\cdot\boldsymbol{a} + 2x(1-x) \times \boldsymbol{a}\cdot\boldsymbol{b} + x^2 \times \boldsymbol{b}\cdot\boldsymbol{b}$$

From part (i) and that the lengths are 1.

$$\left|\overrightarrow{OY}\right|^2 = (1-x)^2 \times 1 + 2x(1-x) \times \frac{1}{2} + x^2 \times 1$$

$$\left|\overrightarrow{OY}\right|^2 = 1 - 2x + x^2 + x - x^2 + x^2$$

$$\left|\overrightarrow{OY}\right|^2 = 1 - x + x^2$$

$$\left|\overrightarrow{OY}\right| = \sqrt{1 - x + x^2}$$

NOTE: Length needs to be found using the dot-product as ‘a’ and ‘b’ are not perpendicular to each other, so you cannot use the formula on the reference sheet.

(iii) Determine the range of values of $\angle YOX$. **4**

$$\overrightarrow{OX}\cdot\overrightarrow{OY}=\left|\overrightarrow{OX}\right|\times\left|\overrightarrow{OY}\right|\times\cos\angle YOX$$

So, we've found $\overrightarrow{OY}$ **earlier, we need to do the same OX.**

$$\overrightarrow{OX}=\overrightarrow{OC}+\overrightarrow{CX}$$

$$\overrightarrow{OX}=c+(1-x)\overrightarrow{CB}$$

$$\overrightarrow{OX}=c+(1-x)(-c+b)$$

$$\overrightarrow{OX}=(x)c+(1-x)b$$

$$\overrightarrow{OX}=(1-x)b+(x)c$$

This is identical to OY, so we can conclude that

$$\left|\overrightarrow{OX}\right|=\sqrt{1-x+x^2}$$

Going back to the dot-product.

$$\overrightarrow{OX}\cdot\overrightarrow{OY}=\left|\overrightarrow{OX}\right|\times\left|\overrightarrow{OY}\right|\times\cos\angle YOX$$

$$\{(1-x)b+(x)c\}\cdot\{(1-x)a+(x)b\}=\sqrt{1-x+x^2}\times\sqrt{1-x+x^2}\times\cos\angle YOX$$

$$\cos\angle YOX=\frac{(1-x)^2\times a\cdot b+(1-x)x\times b\cdot b+x(1-x)\times a\cdot c+x^2\times b\cdot c}{1-x+x^2}$$

As $a\cdot b=a\cdot c=b\cdot c=\frac{1}{2}$ **and** $b\cdot b=1$, **factorizing out** $\frac{1}{2}$

$$\cos\angle YOX=\frac{(1-x)^2+2x(1-x)+x(1-x)+x^2}{2(1-x+x^2)}$$

$$\cos\angle YOX=\frac{1-2x+x^2+2x-2x^2+x-x^2+x^2}{2(1-x+x^2)}=\frac{1+x-x^2}{2(1-x+x^2)}$$

$$\cos \angle YOX = \frac{1 + x - x^2}{2(1 - x + x^2)}$$

The range of values of cosine will determine the range of possible values of the angle.

As $0 < x < 1$ – the max/min will occur at either the ends and/or when the derivative is zero.

$$x = 1, and\ x = 0, \cos \angle YOX = \frac{1}{2}$$

$$\angle YOX = \frac{\pi}{3}$$

Now, for the derivative…

$$\frac{d(\cos \angle YOX))}{dx} = \frac{1}{2} \times \frac{(1 - x + x^2) \times (1 - 2x) - (1 + x - x^2) \times (-1 + 2x)}{(1 - x + x^2)^2}$$

$$\frac{d(\cos \angle YOX))}{dx} = \frac{1}{2} \times \frac{1 - 2x - x - 2x^2 + x^2 - 2x^3 + 1 - 2x + x - 2x^2 - x^2 + 2x^3}{(1 - x - x^2)^2}$$

$$\frac{d(\cos \angle YOX))}{dx} = \frac{1}{2} \times \frac{2 - 4x}{(1 - x + x^2)^2}$$

When this is zero,

$$0 = \frac{1 - 2x}{(1 - x + x^2)^2}$$

$$x = \frac{1}{2}$$

$$x = 0, \frac{d(\cos \angle YOX))}{dx} > 0$$

$$x = 1, \frac{d(\cos \angle YOX))}{dx} < 0$$

Hence, $x = \frac{1}{2}$ is a maximum.

So,

$$\cos \angle YOX = \frac{1 + \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2}{2\left(1 - \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2\right)} = \frac{5}{6}$$

Conclusion is that,

$$\frac{1}{2} < \cos \angle YOX \leq \frac{5}{6}$$

Note, it is strictly greater than $\frac{1}{2}$ as $x \neq 0 \text{ or } 1$.

$$\arccos\left(\frac{5}{6}\right) \leq \angle YOX \leq \frac{\pi}{3}$$

Note, the change in direction as cosine is decreasing in $\left[0, \frac{\pi}{2}\right]$.

ALTERNATIVE method to finding the max:

$$\cos \angle YOX = \frac{1 + x - x^2}{2(1 - x + x^2)}$$

Start completing the square…

$$\cos \angle YOX = \frac{-(x^2 - x) + 1}{2(x^2 - x + 1)} = \frac{-\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) + 1}{2\left(x^2 - x + \frac{1}{4} - \frac{1}{4} + 1\right)}$$

$$= \frac{-\left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right] + 1}{2\left[\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\right]} = \frac{1}{2} \times \frac{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Maximum occurs when denominator is smallest and numerator is largest. Clearly, this is when $x = \frac{1}{2}$ as

$$\frac{5}{4} - \left(x - \frac{1}{2}\right)^2 \leq \frac{5}{4} \text{ and } \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$$