

Student Number _____

ASCHAM SCHOOL

2011
YEAR 12

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

Total marks – 120

Attempt Questions 1-8

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks)

Use a SEPARATE writing booklet.

(a) Find $\int \sin^3 \theta d\theta$. 2

(b) (i) Express $\frac{3x+1}{(x+1)(x^2+1)}$ in the form $\frac{a}{x+1} + \frac{bx+c}{x^2+1}$. 2

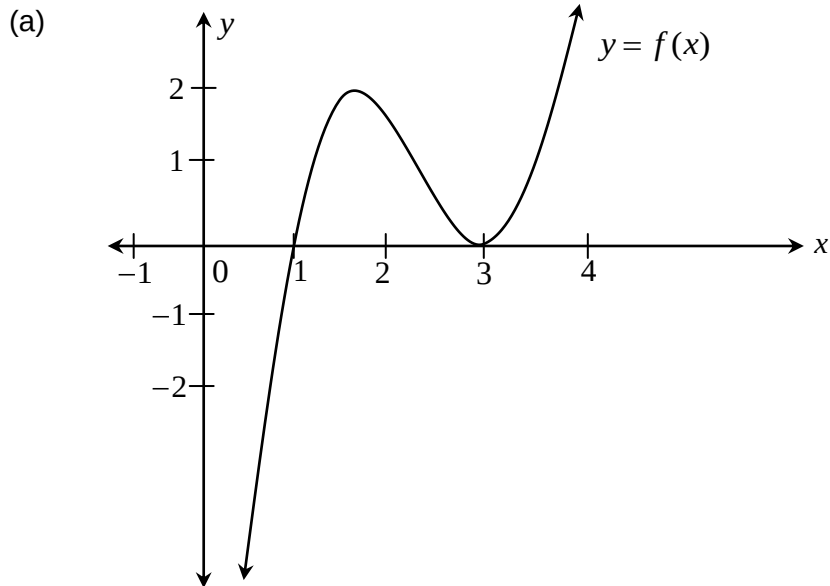
(ii) Hence find $\int \frac{3x+1}{(x+1)(x^2+1)} dx$. 2

(c) Use the substitution $x = 2\sin \theta$, or otherwise, to evaluate $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}} dx$. 3

(d) Find $\int x^2 \sqrt{3-x} dx$. 3

(e) Evaluate $\int_0^1 \tan^{-1} \theta d\theta$. 3

QUESTION 2 (15 marks)
Start a new writing booklet.



The diagram above is a sketch of the function $y = f(x)$.

On separate diagrams sketch:

(i) $y = (f(x))^2$ 2

(ii) $y = \sqrt{f(x)}$ 2

(iii) $y = \ln f(x)$ 2

(iv) $y^2 = f(x)$ 2

(b) (i) If $f'(x) = \frac{2-x}{x^2}$ and $f(1) = 0$, find $f''(x)$ and $f(x)$. 3

(ii) Explain why the graph of $f(x)$ has only one turning point and find the value of the function at that point, stating whether it is a maximum or a minimum value. 2

(iii) Show that $f(4)$ and $f(5)$ have opposite signs and draw a sketch of $f(x)$. 2

QUESTION 3 (15 marks)
Start a new writing booklet.

(a) Express $(\sqrt{3} + i)^8$ in the form $x + iy$. 3

(b) On an Argand diagram, sketch the region where the inequalities 3

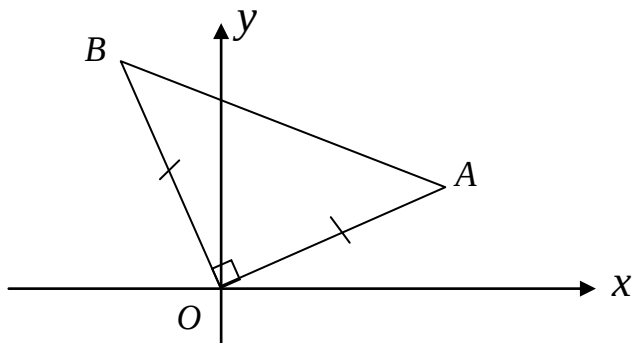
$$|z| \leq 3 \text{ and } -\frac{2\pi}{3} \leq \arg(z + 2) \leq \frac{\pi}{6} \text{ both hold.}$$

(c) Show that $\frac{1 + \sin \theta + i \cos \theta}{1 + \sin \theta - i \cos \theta} = \sin \theta + i \cos \theta$. 3

(d) (i) Express $z = \frac{-1 + i}{\sqrt{3} + i}$ in modulus-argument form. 2

(ii) Hence evaluate $\cos \frac{7\pi}{12}$ in surd form. 2

(e) The Argand diagram below shows the points A and B which represent the complex numbers z_1 and z_2 respectively.



Given that $\triangle BOA$ is a right-angled isosceles triangle, show that $(z_1 + z_2)^2 = 2z_1z_2$. 2

QUESTION 4 (15 marks)
Start a new writing booklet.

- (a) If $z = 1 + i$ is a root of the equation $z^3 + pz^2 + qz + 6 = 0$ where p and q are real, find p and q . **3**

- (b) Show that if the polynomial $f(x) = x^3 + px + q$ has a multiple root, then $4p^3 + 27q^2 = 0$. **3**

- (c) The base of a solid is the region in the first quadrant bounded by the curve $y = \sin x$, the x -axis and the line $x = \frac{\pi}{2}$. **3**

Find the volume of the solid if every cross-section perpendicular to the base and the x -axis is a square.

- (d) (i) Find the five roots of the equation $z^5 = 1$. Give the roots in modulus-argument form. **2**

- (ii) Show that $z^5 - 1$ can be factorised in the form :

$$z^5 - 1 = (z - 1)(z^2 - 2z \cos \frac{2\pi}{5} + 1)(z^2 - 2z \cos \frac{4\pi}{5} + 1) \quad \text{2}$$

- (iii) Hence show that $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$. **2**

QUESTION 5 (15 marks)
Start a new writing booklet.

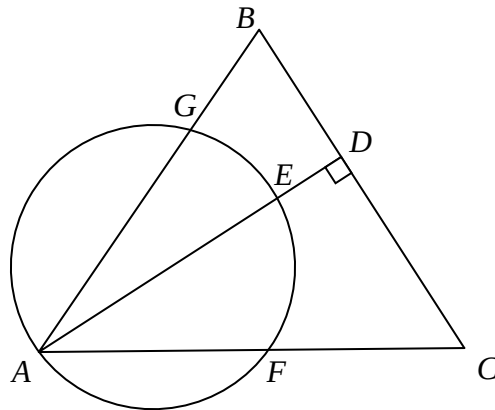
- (a) The ellipse $(x-1)^2 + \frac{y^2}{4} = 1$ is rotated about the y -axis.

Use the method of slicing to find the volume of the solid formed by the rotation.

4

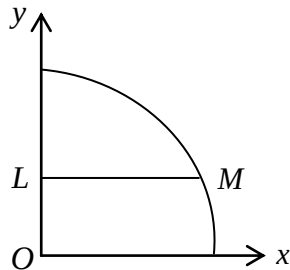
- (b) In the triangle ABC , AD is the perpendicular from A to BC . E is any point on AD and the circle drawn with AE as diameter cuts AC at F and AB at G .

4



Prove B, G, F and C are concyclic.

- (c) The diagram below shows the part of the circle $x^2 + y^2 = a^2$ in the first quadrant.



- (i) If the horizontal line LM through $L(0, b)$, where $0 < b < a$, divides the area between the curve and the coordinates axes into two equal parts, show that

$$\sin^{-1} \frac{b}{a} + \frac{b\sqrt{a^2 - b^2}}{a^2} = \frac{\pi}{4}.$$

3

- (ii) If the radius of the circle is 1 unit, show that b can be found by solving the equation

$$\sin 2\theta = \frac{\pi}{2} - 2\theta, \text{ where } \theta = \sin^{-1} b.$$

3

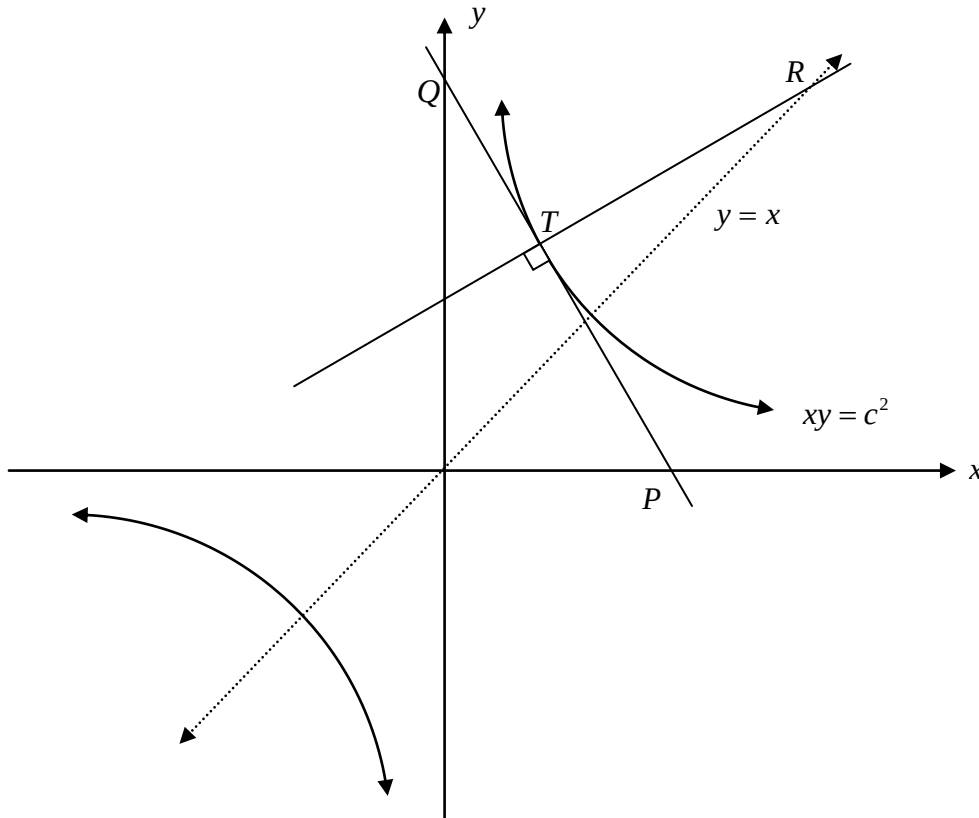
- (iii) Without attempting to solve the equation, how could θ (and hence b) be approximated? 1

QUESTION 6 (15 marks)
Start a new writing booklet.

- (a) An ellipse has equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$ with vertices $A(2,0)$ and $A'(-2,0)$. P is a point (x_1, y_1) on the ellipse.
- (i) Find its eccentricity, coordinates of its foci, S and S' , and the equations of its directrices. **3**
- (ii) Prove that the sum of the distances SP and $S'P$ is independent of the position of P . **3**
- (iii) Show that the equation of the tangent to the ellipse at P is $\frac{xx_1}{4} + \frac{yy_1}{3} = 1$. **3**
- (iv) The tangent at $P(x_1, y_1)$ meets the directrix at T . Prove that angle PST is a right angle. **3**
- (b) If $a + b + c = 1$,
- (i) Prove $a^2 + b^2 \geq 2ab$. **1**
- (ii) Prove $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$. **2**

QUESTION 7 (15 marks)
Start a new writing booklet.

- (a) The point $T(ct, \frac{c}{t})$ lies on the hyperbola $xy = c^2$.
 The tangent at T meets the x -axis at P and the y -axis at Q .
 The normal at T meets the line $y = x$ at R .



*not to
scale*

You may assume that the tangent at T has equation $x + t^2y = 2ct$.

- (i) Find the coordinates of P and Q . **2**
- (ii) Find the equation of the normal at T . **2**
- (iii) Show that the x -coordinate of R is $x = \frac{c}{t}(t^2 + 1)$. **2**
- (iv) Prove that $\triangle PQR$ is isosceles. **3**
- (b) (i) If $I_n = \int \frac{dx}{x^2 + 1}^n$ prove that $I_n = \frac{1}{2(n-1)} \left[\frac{x}{(x^2 + 1)^{n-1}} + (2n-3)I_{n-1} \right]$. **4**
- (ii) Hence evaluate $\int_0^1 \frac{dx}{x^2 + 1}^2$. **2**

QUESTION 8 (15 marks)
Start a new writing booklet.

Marks

- (a) A plane of mass M kg on landing, experiences a variable resistive force due to air resistance of magnitude Bv^2 newtons, where v is the speed of the plane. That is, $M \ddot{x} = -Bv^2$.

- (i) Show that the distance (D_1) travelled in slowing the plane from speed V to speed U under the effect of air resistance only, is given by:

4

$$D_1 = \frac{M}{B} \ln\left(\frac{V}{U}\right)$$

After the brakes are applied, the plane experiences a constant resistive force of A Newtons (due to brakes) as well as a variable resistive force, Bv^2 . That is, $M \ddot{x} = -(A + Bv^2)$.

- (ii) After the brakes are applied when the plane is travelling at speed U , show that the distance D_2 required to come to rest is given by:

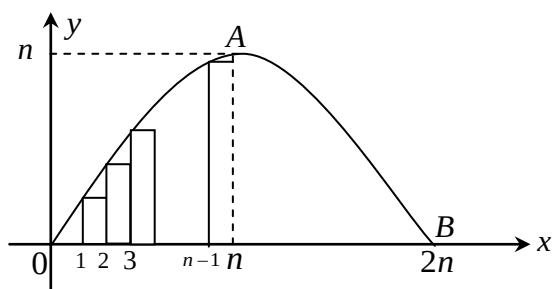
4

$$D_2 = \frac{M}{2B} \ln\left[1 + \frac{B}{A} U^2\right].$$

- (iii) Use the above information to estimate the total stopping distance after landing, for a 100 tonne plane if it slows from 90 m/s to 60 m/s under a resistive force of $125v^2$ Newtons and is finally brought to rest with the assistance of a constant braking force of magnitude 75 000 Newtons. (Note: 1 Newton (N) = 1 kg. m/s²)

2

(b)



The diagram above represents the curve $y = n \sin \frac{\pi x}{2n}$, $0 \leq x \leq 2n$, where n is any integer $n \geq 2$.

The points $O(0,0)$, $A(n,n)$ and $B(2n,0)$ lie on this curve.

- (i) By considering the areas of the lower rectangles of width 1 from $x = 0$ to $x = n$, prove that

$$\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \sin \frac{3\pi}{2n} + \dots + \sin \frac{\pi(n-1)}{2n} < \frac{2n}{\pi}.$$

3

- (ii) Hence or otherwise, explain why $2n \sum_{r=1}^{n-1} \sin \frac{\pi r}{2n} < \frac{\pi n^2}{2}$.

2

END OF PAPER

SOLUTIONS & MARKING SCHEME

Extra TRIAL HSC ASCHAM
2011

Question 1

$$\begin{aligned} a) \int \sin^3 \theta d\theta &= \int \sin \theta (\sin^2 \theta) d\theta \\ &= \int \sin \theta (1 - \cos^2 \theta) d\theta \\ &= \int \sin \theta d\theta + \int \cos^2 \theta (-\sin \theta) d\theta \end{aligned}$$

$$= -\cos \theta + \frac{\cos^3 \theta}{3} + C$$

by substitution: where $u = \cos \theta$
 $\frac{du}{d\theta} = -\sin \theta$

$$\begin{aligned} \int \cos^2 \theta (-\sin \theta) d\theta &= \int u^2 du \\ &= \frac{u^3}{3} = \frac{\cos^3 \theta}{3} \end{aligned}$$

$$\begin{aligned} b) (i) \quad 3x+1 &= a(x^2+1) + (bx+c)(x+1) \\ &= ax^2 + a + bx^2 + (b+c)x + c \\ &= (a+b)x^2 + (b+c)x + a+c \end{aligned}$$

$$\therefore a+b=0 \quad (1)$$

$$b+c=3 \quad (2)$$

$$a+c=1 \quad (3)$$

$$\text{from (1): } a = -b \quad (4)$$

Sub (4) into (3):

$$\begin{cases} -b+c=1 & (5) \\ b+c=3 & (2) \end{cases}$$

$$(2) + (5): 2c = 4$$

$$\therefore c = 2 \quad (6)$$

$$\begin{aligned} \text{Sub (6) into (2): } a+2 &= 1 \Rightarrow a = -1 \\ b &= 1 \end{aligned}$$

$$\therefore \frac{3x+1}{(x+1)(x^2+1)} = \frac{-1}{x+1} + \frac{x+2}{x^2+1}$$

$$\begin{aligned}
 \text{(ii)} \quad \int \frac{3x+1}{(x+1)(x^2+1)} dx &= \int \frac{-1}{x+1} dx + \int \frac{x+2}{x^2+1} dx \\
 &= -\ln(x+1) + \frac{1}{2} \int \frac{2x}{x^2+1} dx + 2 \int \frac{1}{x^2+1} dx \\
 &= -\ln(x+1) + \frac{1}{2} \ln(x^2+1) + 2 \tan^{-1} x + C
 \end{aligned}$$

$$\text{(c)} \quad \int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}} \times 2\cos\theta d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\sin^2\theta d\theta \quad \checkmark$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2(2\sin^2\theta) d\theta$$

$$= 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 - \cos 2\theta) d\theta \quad \text{since } \cos 2\theta = 1 - 2\sin^2\theta$$

$$= 2 \left[\theta - \frac{1}{2} \sin 2\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2 \left[\left(\frac{\pi}{3} - \frac{1}{2} \sin \frac{2\pi}{3} \right) - \left(\frac{\pi}{6} - \frac{1}{2} \sin \frac{\pi}{3} \right) \right]$$

$$= 2 \left(\frac{\pi}{6} \right) = \frac{\pi}{3} \quad \checkmark$$

$$\text{since } x = 2\sin\theta$$

$$dx = 2\cos\theta d\theta$$

$$\text{when } x = 1,$$

$$1 = 2\sin\theta$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

$$\text{when } x = \sqrt{3}$$

$$\sqrt{3} = 2\sin\theta$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$

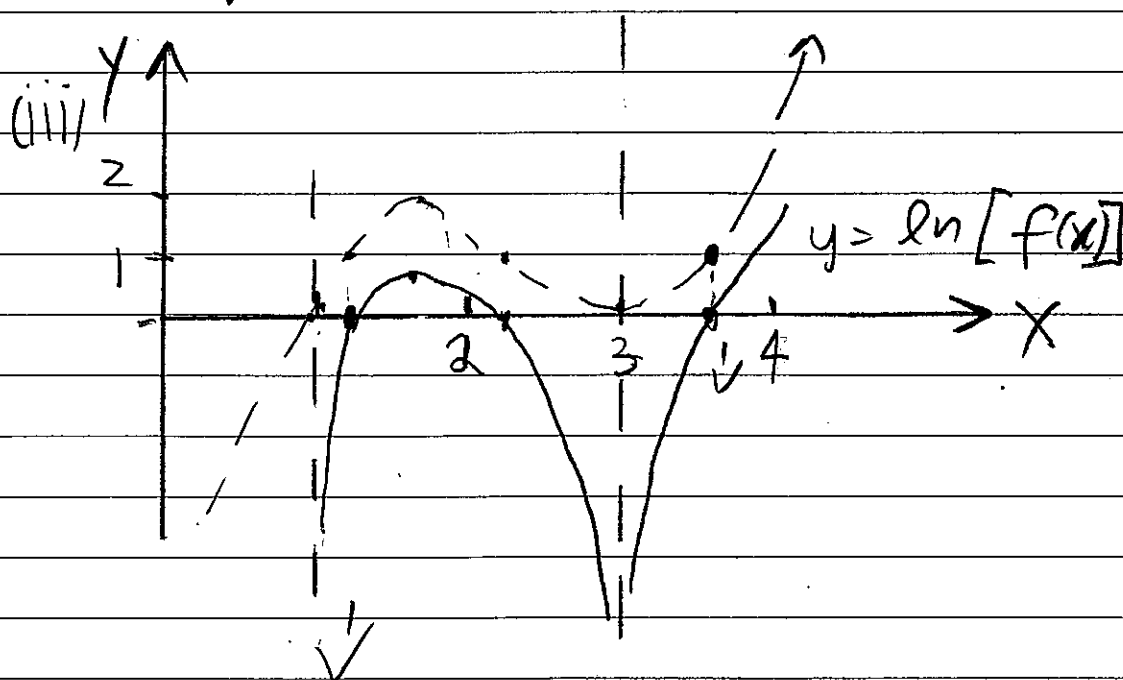
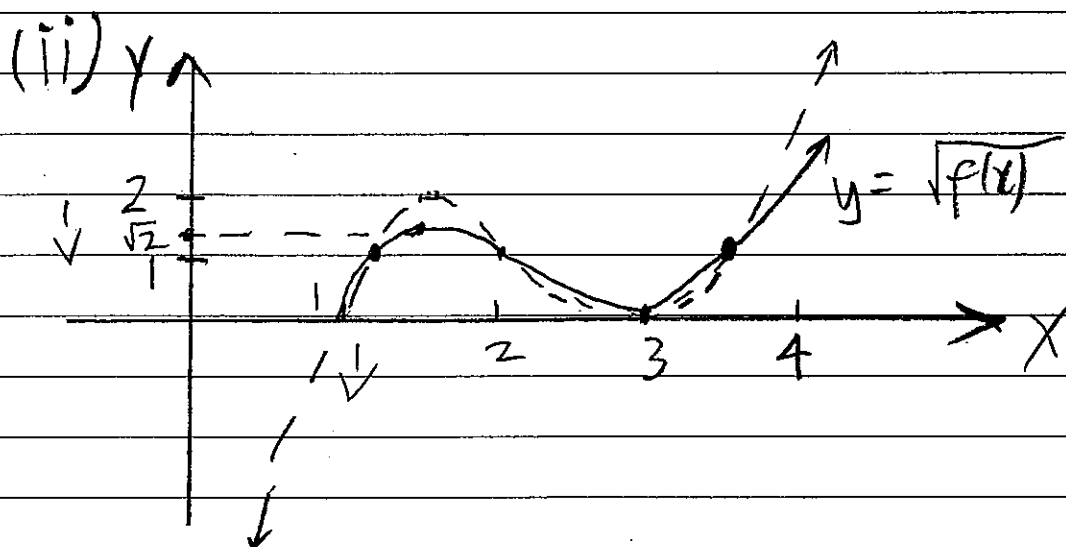
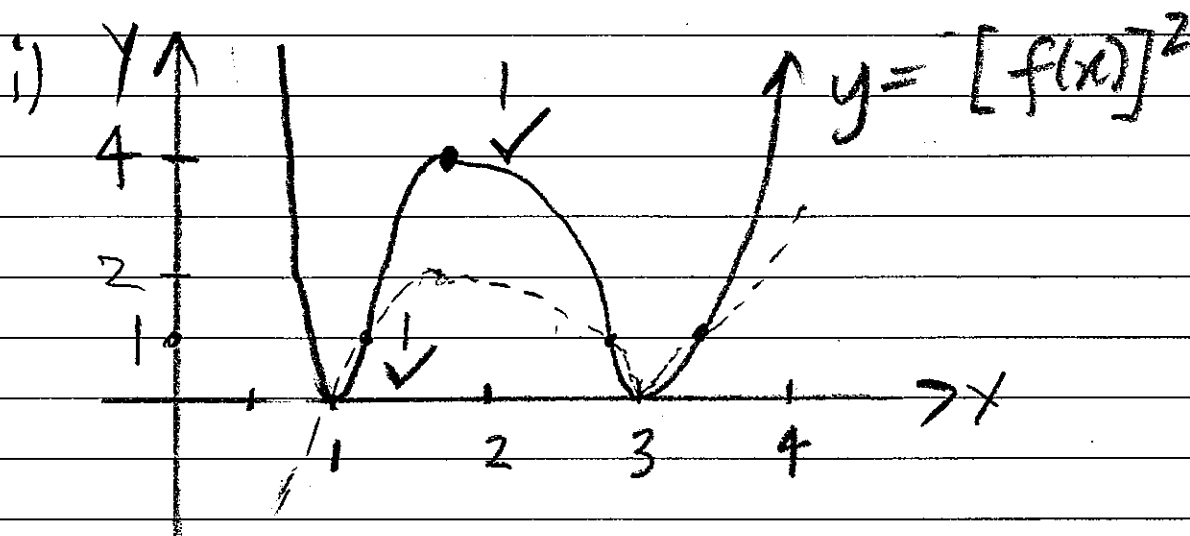
$$\begin{aligned}
 (d) \quad & \int x^2 \sqrt{3-x} \, dx \quad \left| \begin{array}{l} \text{let } u = 3-x \\ \therefore x = 3-u \\ \text{and } dx = -du \end{array} \right. \\
 &= \int (3-u)^2 \sqrt{u} (-du) \\
 &= - \int (9 - 6u + u^2) \sqrt{u} \, du \\
 &= - \int (9u^{1/2} - 6u^{3/2} + u^{5/2}) \, du \\
 &= - \frac{9u^{3/2}}{(3/2)} + \frac{6u^{5/2}}{(5/2)} - \frac{u^{7/2}}{(7/2)} + C \\
 &= -6\sqrt{(3-x)^3} + \frac{12}{5}\sqrt{(3-x)^5} - \frac{2}{7}\sqrt{(3-x)^7} + C
 \end{aligned}$$

$$(e) \text{ let } \int_0^1 1 \cdot \tan^{-1} \theta \, d\theta = \int_0^1 u \, dv$$

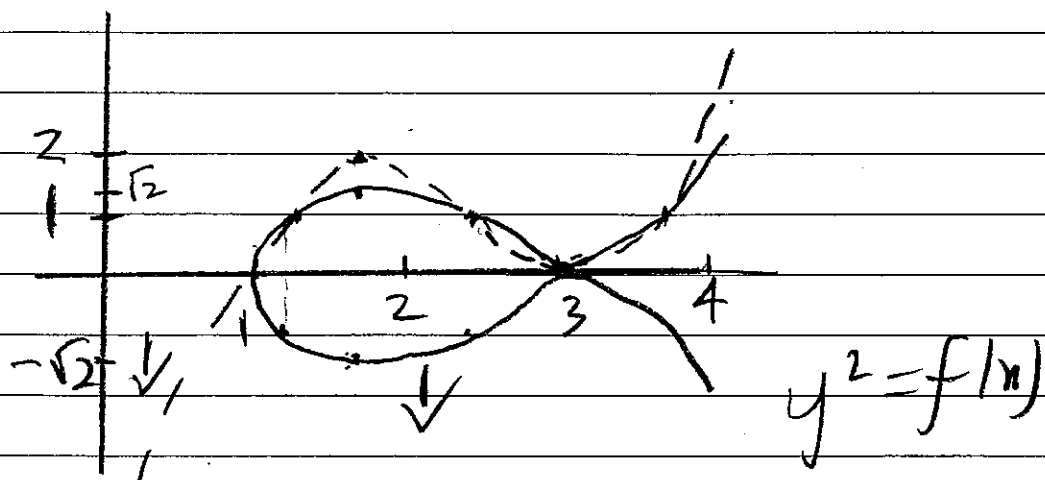
$$\begin{aligned}
 \text{where } u &= \tan^{-1} \theta, \quad v = \theta \\
 du &= \frac{1}{1+\theta^2} d\theta, \quad dv = 1 d\theta
 \end{aligned}$$

$$\begin{aligned}
 \int_0^1 u \, dv &= [uv]_0^1 - \int_0^1 v \, du \\
 &= [\theta \cdot \tan^{-1} \theta]_0^1 - \int_0^1 \frac{\theta}{1+\theta^2} d\theta \quad \checkmark \\
 &= \tan^{-1} 1 - \left[\frac{1}{2} \ln(1+\theta^2) \right]_0^1 \\
 &= \frac{\pi}{4} - \frac{1}{2} \ln 2
 \end{aligned}$$

Question 2



(iv)



$$(b) (i) \quad f'(x) = \frac{2-x}{x^2} = 2x^{-2} - x^{-1}$$

$$\begin{aligned} f''(x) &= -4x^{-3} + x^{-2} \quad \checkmark \\ &= -\frac{4}{x^3} + \frac{x}{x^3} = \frac{x-4}{x^3} \end{aligned}$$

$$f(x) = \int \left(2x^{-2} - \frac{1}{x} \right) dx$$

$$= \frac{2x^{-1}}{-1} - \ln x + C$$

$$= -\frac{2}{x} - \ln x + C \quad \checkmark$$

$$f(1) = -\frac{2}{1} - \ln(1) + C = 0$$

$$= -2 + C = 0 \Rightarrow C = 2 \quad \checkmark$$

$$f(x) = -\frac{2}{x} - \ln x + 2$$

(ii) $f(x)$ has a turning point
when $f'(x) = 0$

$$f'(x) = 0 \text{ when } \frac{2-x}{x^2} = 0$$

i.e. when $x = 2$

this is the only point when $f(x)$ is stationary since it is the only point where the gradient is zero.

This means that there is only one turning point.

$$\text{when } x=2, \quad f(x) = -\frac{2}{2} - \ln 2 + 2$$

$$f(2) = 1 - \ln 2 = 0.30685 \dots$$

$$f''(2) = \frac{2-4}{2^3} = -\frac{1}{4} < 0$$

$\therefore f(x)$ is concave down at $x=2$

hence $(2, 1 - \ln 2)$ is a maximum turning point. ✓

$$(iii) \quad f(4) = -\frac{2}{4} - \ln 4 + 2$$

$$= 1\frac{1}{2} - \ln 4$$

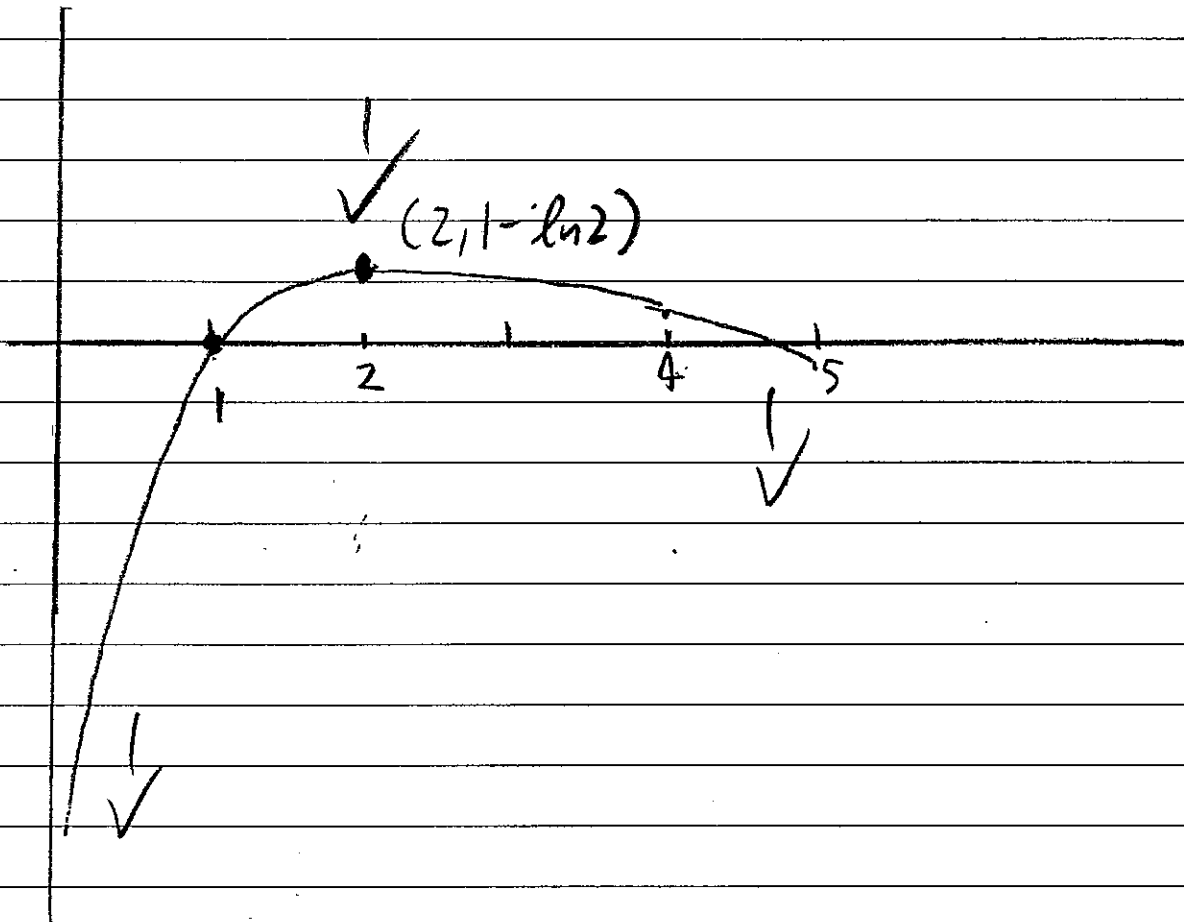
$$= 0.1137 \dots > 0$$

$$f(5) = -\frac{2}{5} - \ln 5 + 2$$

$$= 1\frac{3}{5} - \ln 5$$

$$= -0.009438 \dots < 0 \quad \checkmark$$

$f(x)$ is only defined for $x > 0$
since $\ln(x)$ is only defined for $x > 0$.



Question 3

(a) $(\sqrt{3} + i)^8 =$

let $z = \sqrt{3} + i$
 $|z| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= \sqrt{4} = 2$

$z = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\cos\theta + i2\sin\theta$

$\left. \begin{array}{l} \cos\theta = \frac{\sqrt{3}}{2} \\ \sin\theta = \frac{1}{2} \end{array} \right\} \theta = \frac{\pi}{6} \checkmark$

$\therefore z = 2 \operatorname{cis} \frac{\pi}{6}$

$(\sqrt{3} + i)^8 = (2 \operatorname{cis} \frac{\pi}{6})^8 = 2^8 \operatorname{cis} (8 \times \frac{\pi}{6})$

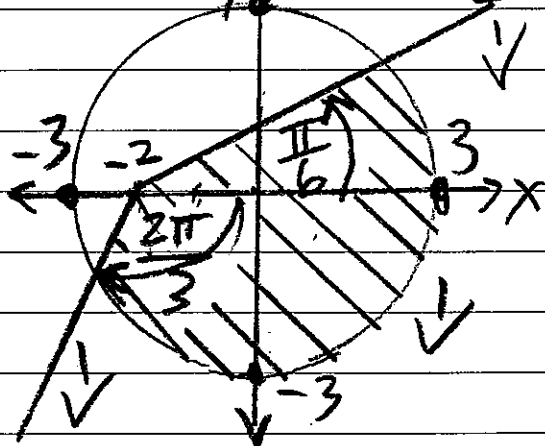
$= 256 \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right)$

$= 256 \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2} \right)$

$= -128 - i128\sqrt{3}$

(b) $|z| \leq 3$

$-\frac{2\pi}{3} \leq \arg(z+2) \leq \frac{\pi}{6}$



$$(c) \frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$$

$$\text{LHS} = \frac{(1 + \sin\theta) + i\cos\theta}{(1 + \sin\theta) - i\cos\theta} \times \frac{(1 + \sin\theta) + i\cos\theta}{(1 + \sin\theta) + i\cos\theta} \quad \checkmark$$

$$= \frac{(1 + \sin\theta)^2 - \cos^2\theta + 2i(1 + \sin\theta)\cos\theta}{(1 + \sin\theta)^2 + \cos^2\theta} \quad \checkmark$$

$$= \frac{(1 + \sin\theta)^2 - (1 - \sin^2\theta) + 2(1 + \sin\theta)i\cos\theta}{(1 + \sin\theta)^2 + (1 - \sin^2\theta)}$$

$$= \frac{(1 + \sin\theta)^2 - (1 + \sin\theta)(1 - \sin\theta) + (1 + \sin\theta)2i\cos\theta}{(1 + \sin\theta)^2 + (1 + \sin\theta)(1 - \sin\theta)}$$

$$= \frac{(1 + \sin\theta)((1 + \sin\theta) - (1 - \sin\theta) + 2i\cos\theta)}{(1 + \sin\theta)((1 + \sin\theta) + (1 - \sin\theta))}$$

$$= \frac{2\sin\theta + 2i\cos\theta}{2} = \sin\theta + i\cos\theta = \text{RHS}$$

↓

$$(d) (i) \quad z = \frac{-1 + i}{\sqrt{3} + i}$$

$$\begin{aligned} \text{let } z_1 &= -1 + i, \quad r_1 = \sqrt{(-1)^2 + 1^2} = \sqrt{2}, \quad \left. \begin{aligned} \cos\theta_1 &= -\frac{1}{\sqrt{2}} \\ \sin\theta_1 &= \frac{1}{\sqrt{2}} \end{aligned} \right\} \theta_1 = \frac{3\pi}{4} \\ &= \sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \\ &= \sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4} \right) \end{aligned}$$

$$\begin{aligned} \text{let } z_2 &= \sqrt{3} + i, \quad r_2 = \sqrt{(\sqrt{3})^2 + 1^2} = 2, \quad \left. \begin{aligned} \cos\theta_2 &= \frac{\sqrt{3}}{2} \\ \sin\theta_2 &= \frac{1}{2} \end{aligned} \right\} \theta_2 = \frac{\pi}{6} \\ &= 2 \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) \\ &= 2 \operatorname{cis} \frac{\pi}{6} \end{aligned}$$

$$z = \frac{z_1}{z_2} = \frac{\sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4} \right)}{2 \operatorname{cis} \left(\frac{\pi}{6} \right)}$$

$$= \frac{\sqrt{2}}{2} \operatorname{cis} \left(\frac{3\pi}{4} - \frac{\pi}{6} \right) = \frac{1}{\sqrt{2}} \operatorname{cis} \left(\frac{9\pi}{12} - \frac{2\pi}{12} \right)$$

$$= \frac{1}{\sqrt{2}} \operatorname{cis} \left(\frac{7\pi}{12} \right)$$

$\downarrow$

$$(ii) \quad z = \frac{-1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{-\sqrt{3}+1 + (\sqrt{3}-1)i}{(3+1)}$$

$$= \left(\frac{1-\sqrt{3}}{4} \right) + \left(\frac{\sqrt{3}-1}{4} \right) i$$

equating real parts from (i)

$$\frac{1}{\sqrt{2}} \cos \left(\frac{7\pi}{12} \right) = \frac{1-\sqrt{3}}{4}$$

$$\therefore \cos \left(\frac{7\pi}{12} \right) = \frac{\sqrt{2}-\sqrt{6}}{4} \quad \checkmark$$

(e) From the diagram, $z_2 = iz_1$ ① $\checkmark$

since multiplying z_1 by i rotates it 90° in an anticlockwise direction.

$$\therefore (z_1 + z_2)^2 = (z_1 + iz_1)^2 \quad (\text{from ①})$$

$$= z_1^2 (1+i)^2 = z_1^2 (1+2i-1)$$

$$= 2iz_1^2 = 2z_1(iz_1) \quad \checkmark$$

$$= 2z_1 z_2 \quad (\text{from ①})$$

as required.

Question 4

(a) If $z_1 = 1+i$ is a root, then
 $z_2 = \bar{z}_1 = 1-i$ is also a root.

$$z_1 z_2 z_3 = -6 \quad (\text{product of roots})$$

$$(1+i)(1-i)z_3 = -6$$
$$2z_3 = -6 \quad z_3 = -3 \quad \checkmark$$

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = q \quad (\text{sum product of roots})$$

$$(1+i)(1-i) + (1+i)(-3) + (1-i)(-3) = q$$

$$q = 2 - 3 - 3 - 3i + 3i$$

$$q = -4 \quad \checkmark$$

$$z_1 + z_2 + z_3 = -p \quad (\text{sum of roots})$$

$$(1+i) + (1-i) + (-3) = -p$$
$$-1 = -p$$

$$\therefore p = 1 \quad \checkmark$$

(b) If $f(x)$ has a multiple root α
then $f'(\alpha)$ has the same root α .

$$\text{i.e. } f(\alpha) = 0 \quad \text{and} \quad f'(\alpha) = 0$$

$$f'(x) = 3x^2 + p \Rightarrow f'(\alpha) = 3\alpha^2 + p = 0$$

$$\therefore p = -3\alpha^2$$
$$\text{and } \alpha^2 = \frac{p}{-3} \quad \textcircled{1}$$
$$\checkmark$$

$$f(x) = x^3 + px + q$$
$$= x(x^2 + p) + q$$

$$f(x) = x(x^2 + p) + q$$

$$= x\left(\frac{p}{3} + p\right) + q \quad (\text{from ①})$$

$$= \left(\frac{2p}{3}\right)x + q = 0$$

$$\therefore 2px + 3q = 0$$

$$x = -\frac{3q}{2p}$$

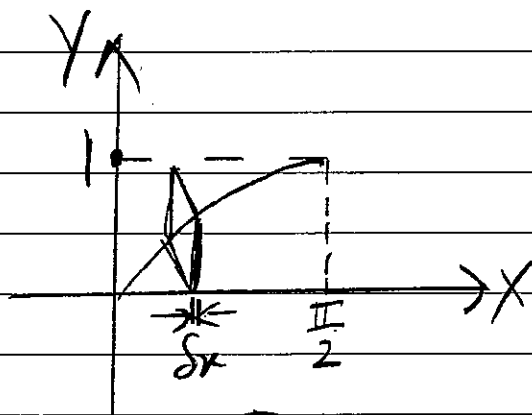
$$x^2 = \frac{9q^2}{4p^2}$$

$$\text{So } \left(-\frac{p}{3}\right) = \frac{9q^2}{4p^2} \quad (\text{from ①})$$

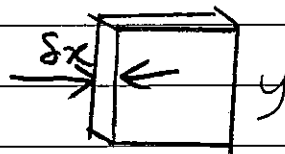
$$-4p^3 = 27q^2 \quad \checkmark \quad \text{or } 4p^3 + 27q^2 = 0$$

as required.

(c)



Cross-section



$$A = y^2 = \sin^2 x$$

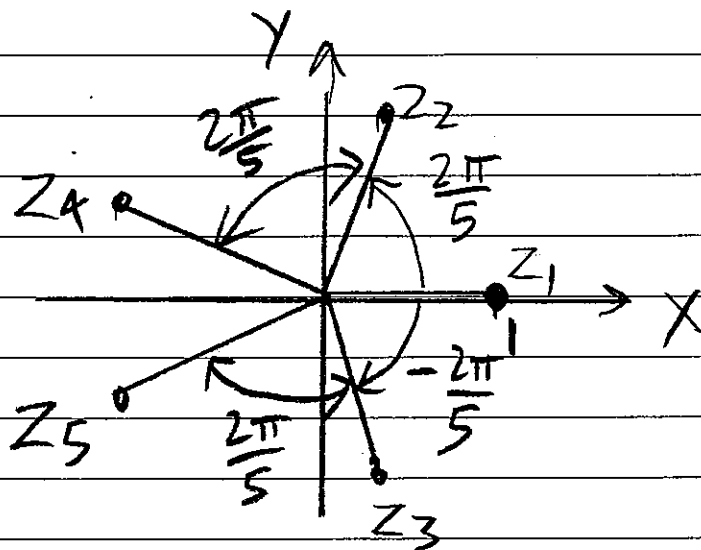
$$V = \lim_{\delta x \rightarrow 0} \sum_{x=0}^{\frac{\pi}{2}} (\sin^2 x) \delta x$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2x) \, dx \quad \left(\begin{array}{l} \text{since} \\ \cos 2x = 1 - 2\sin^2 x \end{array} \right)$$

$$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{2}} = \frac{1}{2} \left[\frac{\pi}{2} - \frac{1}{2} \sin \pi \right]$$

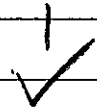
$$= \frac{\pi}{4} \text{ u}^3. \quad \checkmark$$

(d) (i) $z^5 = 1$



roots will be $z_1 = 1$

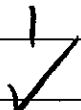
$$z_2 = 1 \operatorname{cis} \frac{2\pi}{5}$$



$$z_3 = 1 \operatorname{cis} \left(-\frac{2\pi}{5}\right)$$

$$z_4 = 1 \operatorname{cis} \left(\frac{4\pi}{5}\right)$$

$$z_5 = 1 \operatorname{cis} \left(-\frac{4\pi}{5}\right)$$



$$\begin{aligned} z^5 - 1 &= (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5) \\ &= (z - 1)(z - \operatorname{cis} \frac{2\pi}{5})(z - \operatorname{cis} (-\frac{2\pi}{5}))(z - \operatorname{cis} (\frac{4\pi}{5})) \\ &\quad \times (z - \operatorname{cis} (-\frac{4\pi}{5})) \end{aligned}$$

now $(z - \operatorname{cis} (\frac{2\pi}{5}))(z - \operatorname{cis} (-\frac{2\pi}{5}))$

$$= (z^2 - z(\operatorname{cis} \frac{2\pi}{5} + \operatorname{cis} (-\frac{2\pi}{5})) + \operatorname{cis} (\frac{2\pi}{5})\operatorname{cis} (-\frac{2\pi}{5}))$$

$$= (z^2 - z(\cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5}) + \cos(\frac{2\pi}{5}) - i\sin(\frac{2\pi}{5}))$$

$$+ (\cos(\frac{2\pi}{5})\cos(-\frac{2\pi}{5}) + i^2(\sin(\frac{2\pi}{5})\sin(-\frac{2\pi}{5})))$$

$$= z^2 - 2z\cos(\frac{2\pi}{5}) + \cos^2(\frac{2\pi}{5}) + \sin^2(\frac{2\pi}{5})$$

$$= z^2 - 2z\cos(\frac{2\pi}{5}) + 1$$

(since $\cos(-\frac{2\pi}{5}) = \cos \frac{2\pi}{5}$ and $\sin(-\frac{2\pi}{5}) = -\sin(\frac{2\pi}{5})$)

Similarly $(z - \text{cis}(\frac{4\pi}{5}))(z - \text{cis}(-\frac{4\pi}{5}))$

$$= z^2 - 2z \text{cis}(\frac{4\pi}{5}) + 1$$

so $z^5 - 1 = (z - 1)(z^2 - 2z \text{cis}(\frac{2\pi}{5}) + 1)(z^2 - 2z \text{cis}(\frac{4\pi}{5}) + 1)$ ①
as required.

(iii) in the expansion of ① above equate coefficients of z from both sides: ↓

$$\text{LHS} = 0z$$

$$\begin{aligned} \text{RHS} &= z + 2z \text{cis}(\frac{2\pi}{5}) + 2z \text{cis}(\frac{4\pi}{5}) \\ &= z(1 + 2 \text{cis}(\frac{2\pi}{5}) + 2 \text{cis}(\frac{4\pi}{5})) \end{aligned}$$

$$\therefore 1 + 2 \text{cis} \frac{2\pi}{5} + 2 \text{cis}(\frac{4\pi}{5}) = 0 \quad \checkmark$$

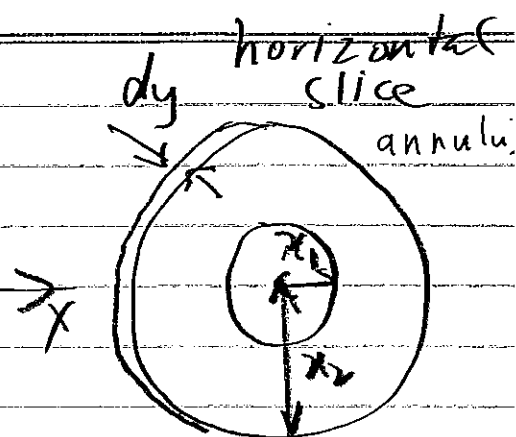
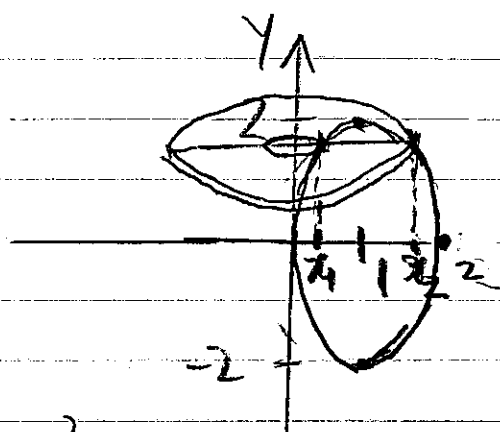
$$2 \text{cis} \frac{2\pi}{5} + 2 \text{cis} \frac{4\pi}{5} = -1$$

$$\text{cis} \frac{2\pi}{5} + \text{cis} \frac{4\pi}{5} = -\frac{1}{2}$$

as required.

Question 5

(a)



$$(x-1)^2 = 1 - \frac{y^2}{4}$$

$$x-1 = \pm \sqrt{1 - \frac{y^2}{4}}$$

$$\delta V = \pi(x_2^2 - x_1^2) \delta y$$

$$\delta V = \pi(x_2 + x_1)(x_2 - x_1) \delta y$$

$$x_1 = 1 - \sqrt{1 - \frac{y^2}{4}}$$

$$x_2 = 1 + \sqrt{1 - \frac{y^2}{4}}$$

$$x_1 + x_2 = 2$$

$$x_2 - x_1 = 2\sqrt{1 - \frac{y^2}{4}}$$

$$\therefore \delta V = \pi(2)(2\sqrt{1 - \frac{y^2}{4}}) \delta y$$

$$V = \lim_{\delta y \rightarrow 0} \sum_{y=-2}^2 \delta V$$

$$= 2\pi \int_0^2 4\sqrt{1 - \frac{y^2}{4}} dy = 4\pi \int_0^2 \sqrt{4 - y^2} dy$$

$$\text{let } y = 2\sin\theta \quad \frac{dy}{d\theta} = 2\cos\theta \quad \therefore dy = 2\cos\theta d\theta$$

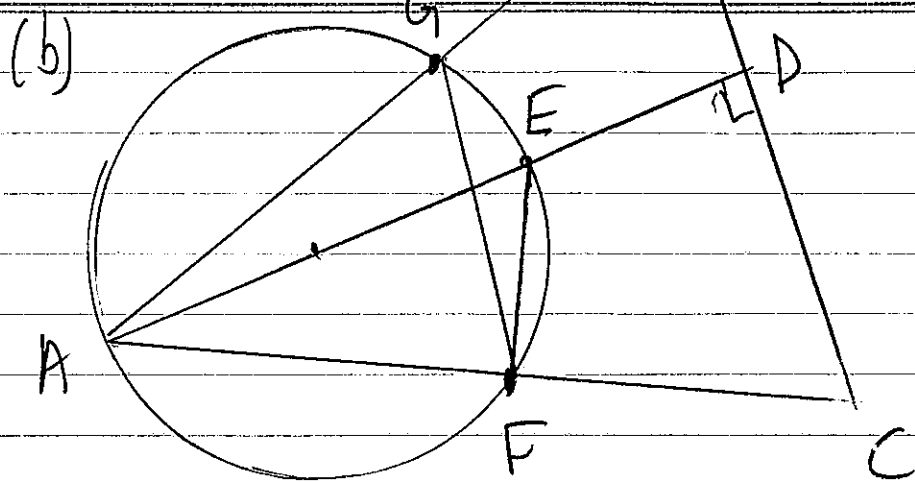
$$\text{When } y=0, \theta=0 \quad \text{when } y=2, \theta = \frac{\pi}{2}$$

$$V = 4\pi \int_0^{\frac{\pi}{2}} \sqrt{4 - (2\sin\theta)^2} (2\cos\theta d\theta)$$

$$= 4\pi \int_0^{\frac{\pi}{2}} (2\cos\theta)^2 d\theta = 8\pi \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$

$$= 8\pi \int_0^{\frac{\pi}{2}} (\cos 2\theta + 1) d\theta = 8\pi \left[\frac{1}{2} \sin 2\theta + \theta \right]_0^{\frac{\pi}{2}}$$

$$= 4\pi^2$$



(ii) $a=1$ ①, $\theta = \sin^{-1}(b)$ ②

hence $b = \sin \theta$ ③
sub ①, ② and ③ into part (i):

$$\sin^{-1}\left(\frac{b}{1}\right) + \frac{b\sqrt{1^2 - b^2}}{1^2} = \frac{\pi}{4} \quad \checkmark$$

$$\theta + \sin \theta \sqrt{1 - \sin^2 \theta} = \frac{\pi}{4} \quad \checkmark$$

$$\theta + \sin \theta \cos \theta = \frac{\pi}{4} \quad \checkmark$$

$$2\theta + 2\sin \theta \cos \theta = \frac{\pi}{2}$$

$$2\theta + \sin 2\theta = \frac{\pi}{2} \text{ as required.}$$

(iii) we use Newton's method to approximate it
(or halving the interval) $\checkmark$

Question 6

(a) (i) $\frac{x^2}{4} + \frac{y^2}{3} = 1 = \frac{x^2}{2^2} + \frac{y^2}{(\sqrt{3})^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$$b^2 = a^2(1 - e^2) \Rightarrow 1 - \frac{b^2}{a^2} = e^2$$

$$1 - \frac{3}{4} = \frac{1}{4} = e^2 = \left(\frac{1}{2}\right)^2 \Rightarrow e = \frac{1}{2} \quad \checkmark$$

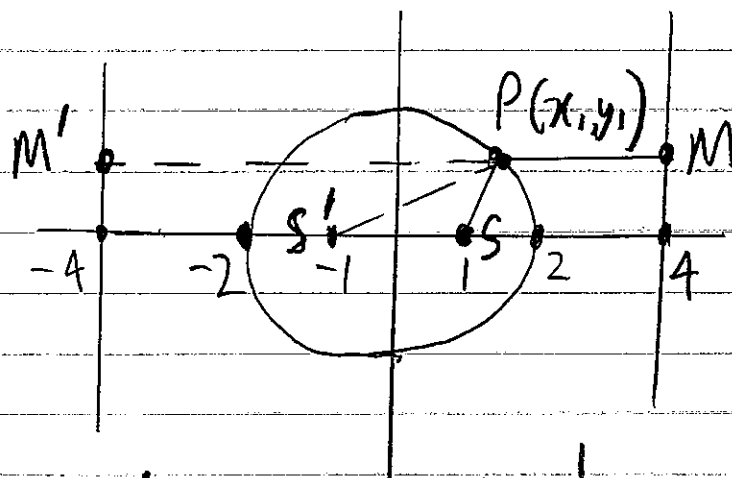
$$S(ae, 0) = S\left(2 \times \frac{1}{2}, 0\right) = S(1, 0) \quad \checkmark$$

$$S'(-ae, 0) = S'(-1, 0)$$

d: $x = \frac{a}{e} = \frac{2}{\frac{1}{2}} = 4$ so $x = 4$

d': $x = -4$

(ii)



$$PS = ePM$$

$$PS' = ePM'$$

$$PS + PS' = ePM + ePM' \quad \checkmark$$

$$= e(PM + PM')$$

$$= e(MM')$$

$$= \frac{1}{2} \times 8 = 4 \quad \checkmark \quad \text{(independent of } P(x, y) \text{)}$$

as required.

$$\frac{x^2}{4} + \frac{y^2}{3} = 1 \quad (1)$$

$$(ii) \quad \frac{d}{dx} \left(\frac{x^2}{4} \right) + \frac{d}{dx} \left(\frac{y^2}{3} \right) = \frac{d}{dx} (1)$$

$$\frac{2x}{4} + \frac{2y}{3} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{4} \times \frac{3}{2y} = -\frac{3x}{4y}$$

$$\text{at } (x_1, y_1) \quad m = -\frac{3x_1}{4y_1}$$

$$\text{so tangent has eqn: } y - y_1 = m(x - x_1)$$

$$y - y_1 = -\frac{3x_1}{4y_1} (x - x_1)$$

$$4y_1 y - 4y_1^2 = -3x_1 x + 3x_1^2$$

$$\frac{3x_1 x}{12} + \frac{4y_1 y}{12} = \frac{3x_1^2}{12} + \frac{4y_1^2}{12}$$

$$\frac{x_1 x}{4} + \frac{y_1 y}{3} = \frac{x_1^2}{4} + \frac{y_1^2}{3} = 1$$

as required. (from (1))

$$(iv) \quad P(x_1, y_1) \quad S(1, 0)$$

for T: sub $x=4$ into eqn. of tangent

$$\frac{4x_1}{4} + \frac{y_1 y}{3} = 1 \Rightarrow y = \frac{3(1-x_1)}{y_1}$$

$$T(4, \frac{3}{y_1}(1-x_1)) \quad \checkmark$$

$$m_{PS} = \frac{y_1}{x_1 - 1} \quad \checkmark \quad m_{ST} = \frac{\frac{3}{y_1}(1-x_1)}{(4-1)} = \frac{1-x_1}{y_1} \quad \checkmark$$

$$m_{PS} \times m_{ST} = \frac{y_1}{-(1-x_1)} \times \frac{(1-x_1)}{y_1} = -1$$

$\therefore \angle PST = 90^\circ$ as required.

(b) (i) for all a, b

$$(a-b)^2 \geq 0$$

$$a^2 - 2ab + b^2 \geq 0$$

$$a^2 + b^2 \geq 2ab \quad \text{as required.}$$

(ii) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ac}{abc}$

$$= \frac{(ab + bc + ac)(a + b + c)}{abc}$$

$$= \frac{(a^2b + ab^2 + abc + abc + b^2c + bc^2 + a^2c + abc + ac^2)}{abc}$$

$$= \frac{(3abc + a(b^2 + c^2) + b(a^2 + c^2) + c(b^2 + a^2))}{abc}$$

$$\geq \frac{3abc + a(2bc) + b(2ac) + c(2ba)}{abc}$$

$$\geq \frac{9abc}{abc}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$$

Question 7

$$(a)(i) x + t^2 y = 2ct \quad \text{or} \quad y = -\frac{1}{t^2}x + \frac{2c}{t}$$

at Q: when $x=0$, $t^2 y = 2ct \quad \therefore Q(0, \frac{2c}{t}) \checkmark$
 $y = \frac{2c}{t}$

at P: when $y=0$, $x = 2ct \quad \therefore P(2ct, 0) \checkmark$

$$(ii) m_{\text{tangent}} = -\frac{1}{t^2} \Rightarrow m_{\text{normal}} = t^2 \checkmark$$

$$T(ct, \frac{c}{t})$$

eqn. of normal: $y - \frac{c}{t} = m_{\text{normal}}(x - ct)$

$$y - \frac{c}{t} = t^2(x - ct)$$

$$ty - c = t^3x - ct^4 \quad \checkmark$$

$$t^3x - ty = ct^4 - c$$

(iii) for R: solve $t^3x - ty = ct^4 - c$ ①
 and $y = x$ ②
 simultaneously

$$t^3x - tx = c(t^4 - 1) \quad \checkmark$$

$$x(t^3 - t) = c(t^2 - 1)(t^2 + 1)$$

$$x = \frac{c(t^2 - 1)(t^2 + 1)}{t(t^2 - 1)} \quad \checkmark$$

$$x = \frac{c}{t}(t^2 + 1) \text{ as required.}$$

$$(b) (i) I_n = \int \frac{1 dx}{(x^2+1)^n} = \int u dv$$

$$\text{where } u = (x^2+1)^{-n} \quad v = x$$

$$\frac{du}{dx} = -2nx(x^2+1)^{-n-1} \quad \frac{dv}{dx} = 1 \quad \checkmark$$

$$I_n = uv - \int v du$$

$$= x(x^2+1)^{-n} - \int x(-2nx)(x^2+1)^{-(n+1)} dx$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{-x^2}{(x^2+1)^{n+1}} dx \quad \checkmark$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{-x^2+1-1}{(x^2+1)^{n+1}} dx$$

$$I_n = \frac{x}{(x^2+1)^n} + 2n \int \frac{dx}{(x^2+1)^n} - 2n \int \frac{dx}{(x^2+1)^{n+1}}$$

$$I_n = \frac{x}{(x^2+1)^n} + 2n I_n - 2n I_{n+1} \quad \checkmark$$

$$2n I_{n+1} = \frac{x}{(x^2+1)^n} + I_n (2n-1)$$

replace n with $n-1$

$$2(n-1) I_n = \frac{x}{(x^2+1)^{n-1}} + I_{n-1} (2(n-1)-1) \quad \checkmark$$

$$I_n = \frac{1}{2(n-1)} \left[\frac{x}{(x^2+1)^{n-1}} + (2n-3) I_{n-1} \right]$$

as required.

$$(iv) \quad R\left(\frac{c}{t}(t^2+1), \frac{c}{t}(t^2+1)\right) \checkmark$$

$$Q\left(0, \frac{2c}{t}\right) \quad P(2ct, 0)$$

$$QR^2 = \left(\frac{c}{t}(t^2+1) - \frac{2c}{t}\right)^2 + \left(\frac{c}{t}(t^2+1)\right)^2$$

$$= \frac{c^2}{t^2} (t^2+1-2)^2 + \frac{c^2}{t^2} (t^2+1)^2$$

$$= \frac{c^2}{t^2} ((t^2-1)^2 + (t^2+1)^2)$$

$$= \frac{c^2}{t^2} (t^4 - 2t^2 + 1 + t^4 + 2t^2 + 1)$$

$$= \frac{2c^2}{t^2} (t^4 + 1) \checkmark$$

$$PR^2 = \left(\frac{c}{t}(t^2+1) - 2ct\right)^2 + \left(\frac{c}{t}(t^2+1)\right)^2$$

$$= \frac{c^2}{t^2} ((t^2+1) - 2t^2)^2 + \frac{c^2}{t^2} (t^2+1)^2$$

$$= \frac{c^2}{t^2} (1 - 2t^2)^2 + \frac{c^2}{t^2} (t^2+1)^2$$

$$= \frac{c^2}{t^2} (t^4 - 2t^2 + 1 + t^4 + 2t^2 + 1)$$

$$= \frac{c^2}{t^2} (2t^4 + 2)$$

$$= \frac{2c^2}{t^2} (t^4 + 1) \checkmark = QR^2$$

$\therefore PR = QR$ and $\triangle PQR$ is isosceles.

$$(ii) \int_0^1 \frac{dx}{(x^2+1)^2} = \frac{1}{2(2-1)} \left[\frac{x}{x^2+1} \right]_0^1 + (2 \times 2 - 3) I_1$$

$$= \frac{1}{2} \left[\frac{1}{2} + \int_0^1 \frac{1}{x^2+1} dx \right] \checkmark$$

$$= \frac{1}{4} + \frac{1}{2} \left[\tan^{-1} x \right]_0^1$$

$$= \frac{1}{4} + \frac{1}{2} \left[\frac{\pi}{4} \right]$$

$$= \frac{1}{4} \left(1 + \frac{\pi}{2} \right) \checkmark$$

Question 8

(a) (i) $M\ddot{x} = -Bv^2$
 $\ddot{x} = -\frac{B}{m}v^2$

$$v \cdot \frac{dv}{dx} = -\frac{B}{m}v^2 \quad \checkmark$$

$$\frac{dx}{dv} = -\frac{m}{Bv} \quad \checkmark$$

$$x = -\frac{m}{B} \int_v^u \frac{1}{v} dv \quad \checkmark$$

$$= -\frac{m}{B} [\ln v]_v^u$$

$$= -\frac{m}{B} [\ln(u) - \ln(v)]$$

$$x = \frac{m}{B} \ln\left(\frac{u}{v}\right)^{-1} \quad \checkmark$$

$\therefore D_1 = \frac{m}{B} \ln \frac{v}{u}$ as required.

(ii) $M\ddot{x} = -(A + Bv^2)$

$$v \cdot \frac{dv}{dx} = -\frac{(A + Bv^2)}{m}$$

$$\frac{dv}{dx} = -\frac{(A + Bv^2)}{mv}$$

$$\frac{dx}{dv} = -\frac{mv}{A + Bv^2} \quad \checkmark$$

$$x = -\frac{M}{2B} \int_V^0 \frac{2BV}{A+BV^2} dV \quad \checkmark$$

$$= -\frac{M}{2B} \left[\ln(A+BV^2) \right]_U^0$$

$$= -\frac{M}{2B} \left[\ln(A) - \ln(A+BU^2) \right] \quad \checkmark$$

$$= \frac{M}{2B} \ln \left(\frac{A}{A+BU^2} \right)^{-1} \quad \checkmark$$

$$= \frac{M}{2B} \ln \left(\frac{A+BU^2}{A} \right) = \frac{M}{2B} \ln \left(1 + \frac{B}{A} U^2 \right)$$

as required

(iii) $M = 100 \text{ tonnes} = 100\,000 \text{ kg}$

$V = 90$

$U = 60$

$BV^2 = 125V^2$ so $B = 125$

$A = 75000 \text{ N}$

$D = D_1 + D_2$

$= \frac{M}{B} \ln \left(\frac{V}{U} \right) + \frac{M}{2B} \ln \left(1 + \frac{B}{A} U^2 \right) \quad \checkmark$

$= \frac{100000}{125} \ln \left(\frac{90}{60} \right) + \frac{100000}{2 \times 125} \ln \left(1 + \frac{125}{75000} \times 60^2 \right)$

$= 800 \ln \left(\frac{3}{2} \right) + 400 \ln(7)$

$= 1102.74 \text{ m (6 sig figs)} \quad \checkmark$

(b)	(i) x	1	2	3	$n-1$
	$y = n \sin \frac{\pi x}{2n}$	$n \sin \frac{\pi}{2n}$	$n \sin \frac{2\pi}{2n}$	$n \sin \frac{3\pi}{2n}$	$n \sin \frac{(n-1)\pi}{2n}$

$$\text{Areas of rectangles} = 1 \times n \sin\left(\frac{\pi}{2n}\right) + 1 \times n \sin\left(\frac{2\pi}{2n}\right) + 1 \times n \sin\left(\frac{3\pi}{2n}\right) + \dots + 1 \times n \sin\left(\frac{(n-1)\pi}{2n}\right) \quad \checkmark$$

$$\text{Area under curve} = \int_0^n n \sin\left(\frac{\pi x}{2n}\right) dx$$

$$= - \left[n \left(\frac{2n}{\pi} \right) \cos\left(\frac{\pi x}{2n}\right) \right]_0^n$$

$$= - \frac{2n^2}{\pi} \left[\cos\left(\frac{\pi n}{2n}\right) - \cos(0) \right]$$

$$= \frac{2n^2}{\pi} \quad \checkmark$$

Areas of rectangles < Area under curve

$$n \sin\left(\frac{\pi}{2n}\right) + n \sin\left(\frac{2\pi}{2n}\right) + n \sin\left(\frac{3\pi}{2n}\right) + \dots + n \sin\left(\frac{(n-1)\pi}{2n}\right) < \frac{2n^2}{\pi}$$

$$n \left(\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \sin \frac{3\pi}{2n} + \dots + \sin \frac{(n-1)\pi}{2n} \right) < n \left(\frac{2n}{\pi} \right) \quad \checkmark$$

$$\therefore \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \sin \frac{3\pi}{2n} + \dots + \sin \frac{(n-1)\pi}{2n} < \frac{2n}{\pi} \quad \textcircled{1}$$

(ii) From ①: $\sum_{r=1}^{n-1} \sin\left(\frac{\pi r}{2n}\right) < \frac{2n}{\pi}$ as required.

$$2n \sum_{r=1}^{n-1} \sin\left(\frac{\pi r}{2n}\right) < \frac{4n^2}{\pi} = \frac{4\pi n^2}{\pi^2} \quad \textcircled{2} \quad \checkmark$$

but $\frac{4}{\pi} < \frac{\pi}{2}$ (since $8 < \pi^2$)

$$\text{so } \frac{4\pi n^2}{\pi^2} < \frac{\pi^2 n^2}{2\pi} = \frac{\pi n^2}{2} \quad \checkmark$$

$$\therefore \sum_{r=1}^{n-1} \sin\left(\frac{\pi r}{2n}\right) < \frac{4\pi n^2}{\pi^2} < \frac{\pi n^2}{2}$$

as required.