Student Number___________________

ASCHAM SCHOOL

2015

YEAR 12

TRIAL

EXAMINATION

Mathematics

Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 100

- Attempt Sections I and II.
- Section I is worth 10 marks.
- Answer Section I on the multiple choice answer sheet.
- Detach the multiple choice answer sheet from the back of the examination paper.
- Section II contains 6 questions worth 15 marks each.
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher.

Section I

10 marks

Attempt Questions 1-10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

1 Write $\dfrac{30}{1-2i}$ in the form $a+bi$.

(A) $10-20i$ (B) $-10-20i$ (C) $6+12i$ (D) $6-12i$

2 If $2-i$ is a root of the polynomial $x^3-7x^2+17x-15=0$, then the other roots are:

(A) $2-i, 3$ (B) $2+i, 3$ (C) $2+i, -3$ (D) $2-i, -3$

3 The eccentricity of the hyperbola $\dfrac{x^2}{3}-\dfrac{y^2}{4}=1$ is:

(A) $\dfrac{5}{3}$ (B) $\dfrac{\sqrt{21}}{3}$ (C) $1+\dfrac{2}{\sqrt{3}}$ (D) $\sqrt{\dfrac{7}{3}}$

4 If the polynomial equation $P(x)=0$ has roots α, β and γ, then the polynomial equation $P(2x-1)=0$ has roots:

(A) $\dfrac{\alpha}{2}+1,\ \dfrac{\beta}{2}+1,\ \dfrac{\gamma}{2}+1$ (B) $\dfrac{\alpha+1}{2},\ \dfrac{\beta+1}{2},\ \dfrac{\gamma+1}{2}$

(C) $2\alpha-1,\ 2\beta-1,\ 2\gamma-1$ (D) $\alpha-\dfrac{1}{2},\ \beta-\dfrac{1}{2},\ \gamma-\dfrac{1}{2}$

5 The gradient of the tangent to the curve $3x^2-y^2=11$ at $(-2,1)$ is:

(A) -6 (B) 6 (C) $-\dfrac{3}{2}$ (D) $\dfrac{3}{2}$

6 The equation of the conic whose distance from the point $(2,0)$ is half its distance to the line $x=8$ is given by:

(A) $\frac{x^2}{16}+\frac{y^2}{12}=1$ (B) $\frac{x^2}{4}+\frac{y^2}{3}=1$ (C) $\frac{x^2}{16}-\frac{y^2}{12}=1$ (D) $\frac{(x+4)^2}{48}+\frac{y^2}{24}=1$

7 Given the curve $y=f(x)$, the curve $y=f(|x|)$ is represented by:

(A) A reflection of the curve $y=f(x)$ in the x-axis.

(B) A reflection of the curve $y=f(x)$ in the y-axis.

(C) A reflection of the curve $y=f(x)$ for $x\geq 0$ in the y-axis.

(D) A reflection of the curve $y=f(x)$ for $y\geq 0$ in the x-axis.

8 If $x+y=1$ and $x^2+y^2=2$, then the value of x^3+y^3 is:

(A) $1\frac{1}{2}$ (B) 2 (C) $2\frac{1}{2}$ (D) $3\frac{1}{2}$

9 Which of the following gives an expression for $\int \frac{dx}{\sqrt{7-6x-x^2}}$?

(A) $\sin^{-1}\left(\frac{x+3}{2}\right)+C$ (B) $\sin^{-1}\left(\frac{x+3}{4}\right)+C$

(C) $\sin^{-1}\left(\frac{x-3}{2}\right)+C$ (D) $\sin^{-1}\left(\frac{x-3}{4}\right)+C$

10 Which of the following statements is FALSE?

(A) $\int_{-3}^{3} x\cos(x)dx=0$ (B) $\int_{-3}^{3} e^{-x^2}dx=2\int_{0}^{3} e^{-x^2}dx$

(C) $\int_{0}^{\pi}\sin^8 x dx>\int_{0}^{\pi}\sin(8x)dx$ (D) $\int_{0}^{1} x^8 dx<\int_{0}^{1} x^9 dx$

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) $z = -2 + 3i$ and $w = 3 - 2i$ are complex numbers.

(i) Express $z + w$ in modulus –argument form. 2

(ii) Express $z\overline{w}$ in the Cartesian form. 2

(b) Sketch the region of the Argand diagram whose points satisfy:

$$\left|z - \overline{z}\right| \leq 2 \text{ and } -\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{4}.$$ 2

(c) The cube roots of unity are the solutions to $z^3 = 1$

(i) Find the three roots in modulus-argument form and sketch them on an Argand diagram.. 4

(ii) Write the three roots in Cartesian form. 1

(iii) If one of the complex roots is ω, show that $1 + \omega + \omega^2 = 0$. 2

(iv) Evaluate $(1+\omega)(1+\omega^2)(1+\omega^5)(1+\omega^8)(1+\omega^{11})$. 2

Question 12 (15 marks) Use a SEPARATE writing booklet.

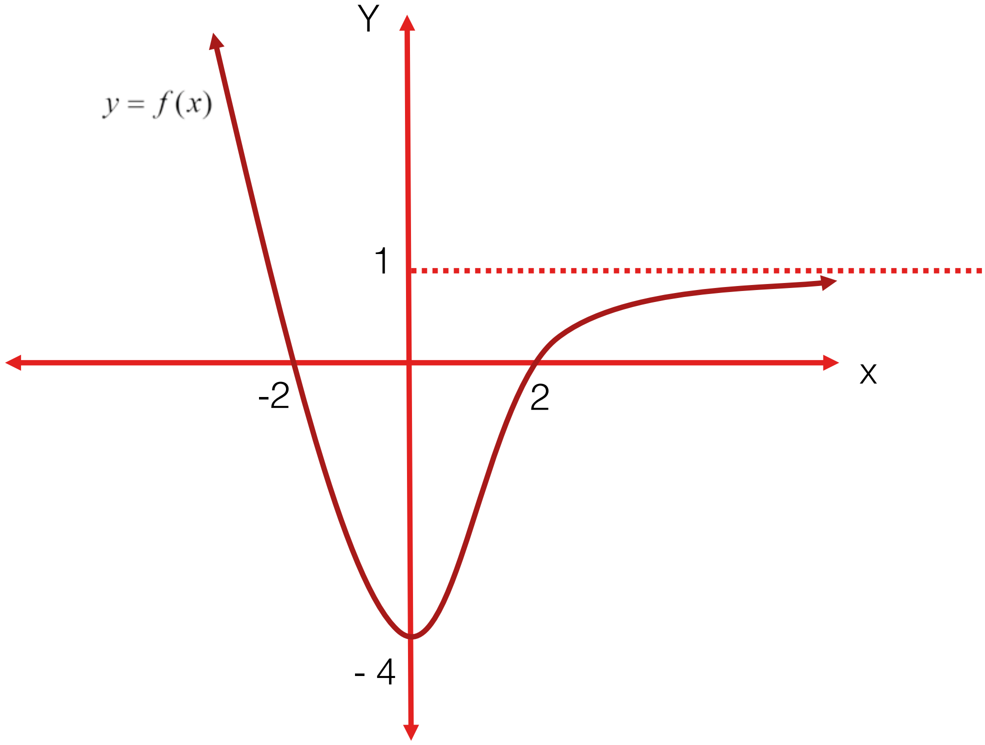
(a) Find $\int \cos^4 x dx$. **2**

(b) Evaluate $\int_0^1 xe^{-x} dx$ **3**

(c) (i) Express $\dfrac{x+7}{x^2(x+2)}$ in the form $\dfrac{A}{x^2}+\dfrac{B}{x}+\dfrac{C}{x+2}$. **2**

(ii) Hence, or otherwise, find $\int \dfrac{x+7}{x^2(x+2)} dx$. **2**

(d) The diagram below shows the graph of $y = f(x)$.



On separate half page graphs, sketch the following:

(i) $y = (f(x))^2$ **2**

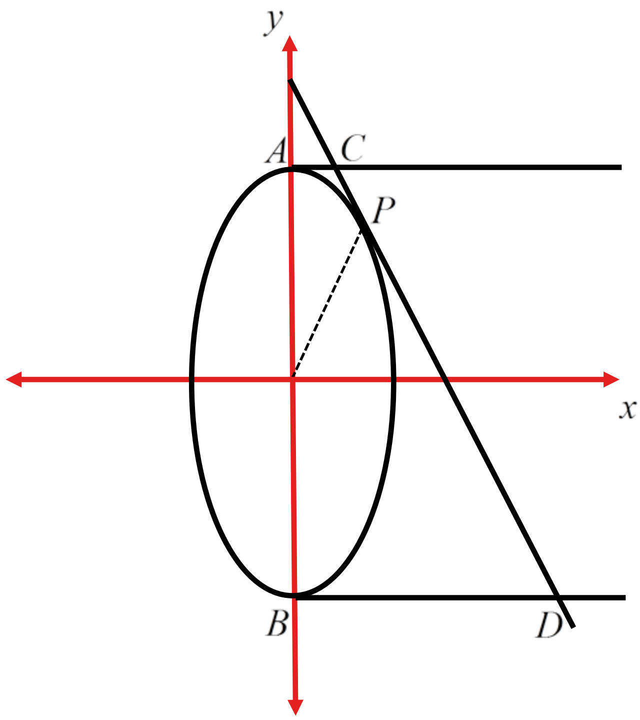
(ii) $y^2 = f(x)$ **2**

(iii) $y = x.f(x)$ **2**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows the ellipse which has equation $\frac{x^2}{9}+\frac{y^2}{16}=1$.

The point $P(3\cos\theta, 4\sin\theta)$ lies on the ellipse. The ellipse meets the y-axis at the points A and B. The tangents to the ellipse at A and B meet the tangent at P at the points C and D respectively.



(i) Find the eccentricity, coordinates of the foci and the equations of the directrices. **4**

(ii) Show that the equation of the tangent to the ellipse at P is $3y\sin\theta+4x\cos\theta=12$. **2**

(iii) Find the numerical value of $AC\times BD$. **3**

(b) (i) Sketch the region of the Cartesian plane bounded by: **2**

$y=x^2$, $y=1$, $x=4$ and the x-axis.

(ii) A volume is generated by rotating this region about the x-axis. Find this volume by using the method of slicing. **4**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that the equation of the normal to the hyperbola $xy = c^2$ at $P\left(cp, \frac{c}{p}\right)$ is $p^3x - py = c(p^4 - 1)$. 2

(ii) The normal at $P\left(cp, \frac{c}{p}\right)$ meets the hyperbola $xy = c^2$ again at $Q\left(cq, \frac{c}{q}\right)$. Prove that $p^3q = -1$. 2

(iii) Hence, show that the locus of the midpoint R of PQ is given by $c^2(x^2 - y^2)^2 + 4x^3y^3 = 0$. 3

(b) A particle of mass 1kg moves in a straight line against a resistance of $v + v^3$, where v is its velocity $\left(m\ddot{x} = -m\left(v + v^3\right)\right)$.

Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$.

(i) Show that v is related to the displacement x by $x = \tan^{-1}\left(\dfrac{Q - v}{1 + Qv}\right)$. 2

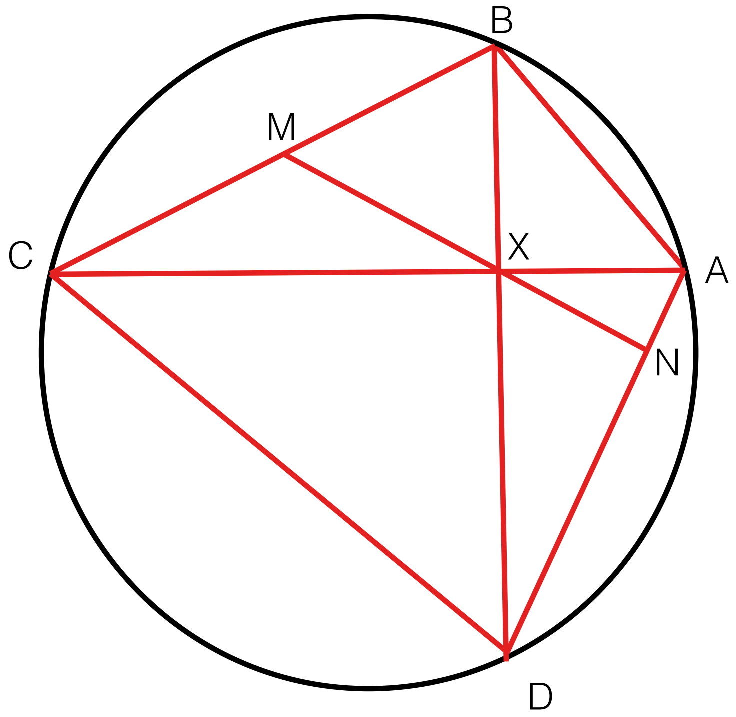
(ii) Show that the time t which has elapsed when the particle is travelling with velocity R is given by $t = \dfrac{1}{2}\log_e\left(\dfrac{Q^2\left(1 + R^2\right)}{R^2\left(1 + Q^2\right)}\right)$. 2

(iii) Find R^2 as a function of t. 2

(iv) Find the limiting values of v and x as $t \to \infty$. 2

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a)



$ABCD$ is a cyclic quadrilateral. The diagonals AC and BD Intersect at right angles at X. M is the midpoint of BC. MX produced meets AD at N.

(i) Copy the diagram showing the above information.

(ii) Prove that $\angle MBX = \angle MXB$. **3**

(iii) Show that MN is perpendicular to AD. **4**

(b) Find the equation with rational coefficients whose roots are the reciprocals of the squares of the roots of $x^3 - x^2 - 14x - 24 = 0$. **3**

(c) (i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. **2**

(ii) Hence or otherwise find all solutions of: **3**

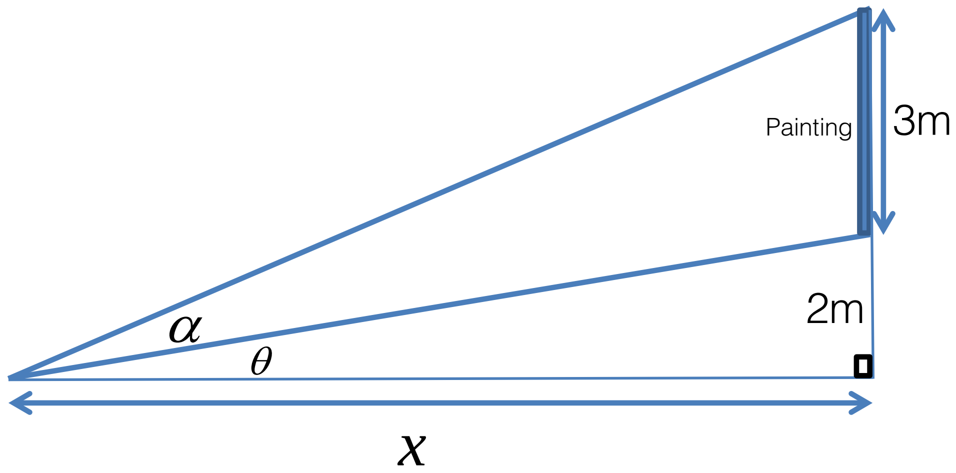
$\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$.

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) If $I_n = \int_1^e (\log_e x)^n \, dx$ for $n \geq 0$, show that $I_n = e - nI_{n-1}$ for $n \geq 1$. 3

(ii) Hence, evaluate I_4. 2

(b) The base of a 3m high painting in the Louvre Museum is 2m above the eye-level of an observer who is x m from the base of the wall on which the painting is hanging. The angle α is the viewing angle at eye-level and is the difference between the angles of elevation of the top and bottom of the painting as seen by the observer.



(i) Show that $\alpha = \tan^{-1}\left(\dfrac{3x}{x^2+10}\right)$. 3

(ii) Hence find how far the observer should stand to maximise the viewing angle. 3

(c) Given that $f(x) = x^6 + 4x^5 - 3x^4 - 8x^3 + 35x^2 - 60x - 225$ has zeroes at $x = \pm\sqrt{5}$ and a double zero, factorise $f(x)$ over:

(i) the real field. 3

(ii) the complex field. 1

Student Number ……………………………………

ASCHAM SCHOOL

YEAR 12 Trial Mathematics Extension 2 Exam

MULTIPLE-CHOICE ANSWER SHEET

1.	A ⬭	B ⬭	C ⬭	D ⬭
2.	A ⬭	B ⬭	C ⬭	D ⬭
3.	A ⬭	B ⬭	C ⬭	D ⬭
4.	A ⬭	B ⬭	C ⬭	D ⬭
5.	A ⬭	B ⬭	C ⬭	D ⬭
6.	A ⬭	B ⬭	C ⬭	D ⬭
7.	A ⬭	B ⬭	C ⬭	D ⬭
8.	A ⬭	B ⬭	C ⬭	D ⬭
9.	A ⬭	B ⬭	C ⬭	D ⬭
10.	A ⬭	B ⬭	C ⬭	D ⬭

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\;dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\;dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\;dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\;dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

SOLUTIONS AND MARKING SCHEME
MATHS EXT 2 HSC TRIAL
EXAM - ASCHAM SCHOOL 2015

Section 1 (Multiple Choice)

1. $\frac{30}{1-2i} \times \frac{1+2i}{1+2i} = \frac{30(1+2i)}{1+4}$

$= \frac{30}{5}(1+2i) = 6+12i$

$\boxed{C}$

2. $(2-i)(2+i) = 4+1 = 5$

$5 \times 3 = 15$

$\therefore$ roots are: $2-i;\ 2+i,\ 3$

$\boxed{B}$

3. $\frac{b^2}{a^2} = e^2 - 1$

$\therefore e^2 = \frac{b^2}{a^2} + 1 = \frac{4}{3} + 1 = \frac{7}{3}$

$e = \sqrt{\frac{7}{3}}$

$\boxed{D}$

4. if $x = \frac{\alpha+1}{2}$

$P(2x-1) = P(2(\frac{\alpha+1}{2})-1)$

$= P(\alpha+1-1) = P(\alpha)$

$= 0$

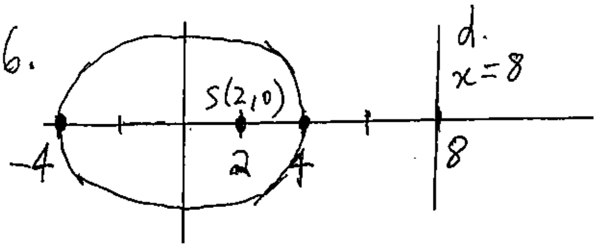
$\boxed{B}$

5. $3x^2 - y^2 = 11$

$\frac{d}{dx}(3x^2 - y^2) = \frac{d}{dx}(11)$

$6x - 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{6x}{2y} = \frac{6 \times 2}{2 \times 1} = -6$ $\boxed{A}$

6. 

$a = 4 \quad \therefore a^2 = 16$

$\boxed{A}$

7. $\boxed{C}$

8. $x^2 + y^2 = (x+y)^2 - 2xy$

$2 = 1^2 - 2xy$

$\therefore xy = -\frac{1}{2}$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$= 1(2 - (-\frac{1}{2}))$

$= 2\frac{1}{2}$

$\boxed{C}$

9. $7 - 6x - x^2 = -(x^2 + 6x + 9) + 7 + 9$

$= 16 - (x+3)^2$

$= 4^2 - (x+3)^2$

$\therefore \int \frac{dx}{\sqrt{4^2 - (x+3)^2}} = \sin^{-1}\left(\frac{x+3}{4}\right) + C$

$\boxed{B}$

10. A) is true since $x.\cos(x)$ is an odd function
(odd function)$\times$(even funct$^{\underline{n}}$)

B) is true since e^{-x^2} is an even function.

C) true since $\int_0^{\pi} \sin 8x\,dx = 0$

D) False due to greater concavity of x^9. $\boxed{D}$

Section II

Question 11

(a) i) $z + w = -2 + 3i + 3 - 2i$

$= 1 + i$

$= \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

ii) $z\bar{w} = (-2+3i)(3+2i)$

$= -6 - 4i + 9i - 6$

$= -12 + 5i$

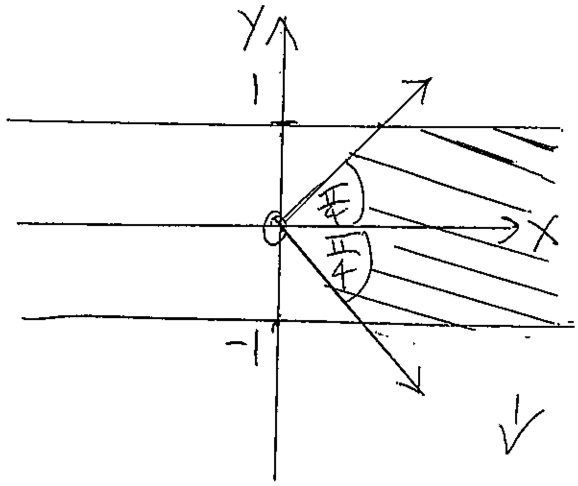
(b) let $z = x + iy$

$\bar{z} = x - iy$

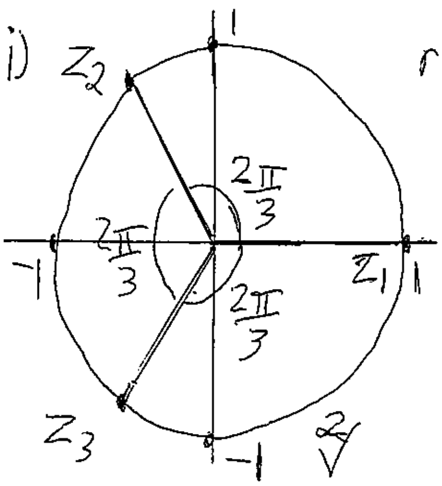
$z - \bar{z} = x + iy - (x - iy)$

$= 2yi$

$|z - \bar{z}| = |2y| \leq 2 \quad \therefore |y| \leq 1$



(c) i)



roots are:

$z_1 = 1\,\text{cis}\,0$

$z_2 = 1\,\text{cis}\left(\frac{2\pi}{3}\right)$

$z_3 = 1\,\text{cis}\left(\frac{4\pi}{3}\right)$

ii) $z_1 = 1\,\text{cis}\,0 = 1$

$z_2 = 1\,\text{cis}\,\frac{2\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= -\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$

$z_3 = 1\,\text{cis}\,\frac{4\pi}{3} = -\frac{1}{2} - i\left(\frac{\sqrt{3}}{2}\right)$

iii) let $\omega = z_2 = -\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$

then $\omega^2 = \left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 i^2 + 2\left(-\frac{1}{2}\right)\left(i\left(\frac{\sqrt{3}}{2}\right)\right)$

$= \frac{1}{4} - \frac{3}{4} - i\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{1}{2} - i\left(\frac{\sqrt{3}}{2}\right) = z_3$

$1 + \omega + \omega^2 = 1 + \left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right) + \left(-\frac{1}{2} - i\left(\frac{\sqrt{3}}{2}\right)\right)$

$= 1 - 1$

$= 0$

iv) from iii) $1 + \omega + \omega^2 = 0$

$\therefore 1 + \omega = -\omega^2$

$1 + \omega^2 = -\omega$

also $\omega^3 = 1$ (since $z^3 = 1$)

$\therefore 1 + \omega^5 = 1 + \omega^3 \times \omega^2$

$= 1 + \omega^2$

$= -\omega$

$1 + \omega^8 = 1 + \omega^3 \times \omega^3 \times \omega^2$

$= 1 + \omega^2 = -\omega$

$1 + \omega^{11} = 1 + (\omega^3)^3 \times \omega^2 = -\omega$

hence

$(1+\omega)(1+\omega^2)(1+\omega^5)(1+\omega^8)(1+\omega^{11})$

$= (-\omega^2)(-\omega)(-\omega)(-\omega)(-\omega)$

$= -\omega^6 = -(\omega^3)^2$

$= -1$

Question 12

(a) $\int \cos^4 x \, dx$

$$\cos^4 x = (\cos^2 x)^2$$
$$= \left(\tfrac{1}{2}(\cos 2x + 1)\right)^2$$
$$= \tfrac{1}{4}(\cos^2 2x + 2\cos 2x + 1)$$
$$= \tfrac{1}{4}\left(\tfrac{1}{2}(\cos 4x + 1) + 2\cos 2x + 1\right)$$
$$= \tfrac{1}{8}\cos 4x + \tfrac{1}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}$$

1 ✓

$$= \tfrac{1}{8}(\cos 4x + 4\cos 2x + 3)$$

$$\int \cos^4 x \, dx = \tfrac{1}{8}\int (\cos 4x + 4\cos 2x + 3)\, dx$$
$$= \tfrac{1}{8}\left[\tfrac{1}{4}\sin 4x + 2\sin 2x + 3x\right] + C$$

1 ✓

$$= \tfrac{1}{32}\sin 4x + \tfrac{1}{4}\sin 2x + \tfrac{3}{8}x + C$$

(b) $\int_0^1 x.e^{-x}\, dx$

Let $\int x.e^{-x}\,dx = \int u.dv$

where $u = x$ $\quad dv = e^{-x}dx$

$du = dx$ $\quad v = -e^{-x}$ 1 ✓

$$\therefore \int x.e^{-x}\,dx = uv - \int v\,du$$
$$= [-x.e^{-x}] + \int e^{-x}dx \quad 1 ✓$$

$$\therefore \int_0^1 x.e^{-x}\,dx = \left[-x.e^{-x}\right]_0^1 - \left[e^{-x}\right]_0^1$$
$$= [-e^{-1} + 0] - [e^{-1} - e^0]$$
$$= 1 - \frac{2}{e} \quad 1 ✓$$

(c) i) $x + 7 = A(x+2) + Bx(x+2) + Cx^2$ 1 ✓

$$= Ax + 2A + Bx^2 + 2Bx + Cx^2$$
$$= x^2(B+C) + x(A+2B) + 2A$$

$\therefore 2A = 7$ and $A = 3\tfrac{1}{2}$

$A + 2B = 1$ or $3\tfrac{1}{2} + 2B = 1$

$2B = -2\tfrac{1}{2}$

$B = -\frac{5}{4}$

$B + C = 0$

$\therefore C = -B = \frac{5}{4}$

$$\frac{x+7}{x^2(x+2)} = \frac{7}{2x^2} - \frac{5}{4x} + \frac{5}{4(x+2)}$$

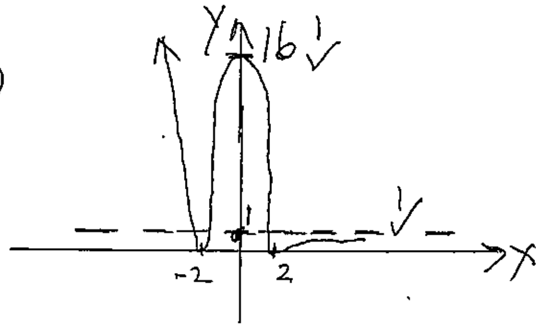
1 ✓

ii) $$\int \frac{x+7}{x^2(x+2)}dx = \frac{7}{2}\int x^{-2}dx - \frac{5}{4}\int \frac{1}{x}dx + \frac{5}{4}\int \frac{1}{x+2}dx \quad 1 ✓$$

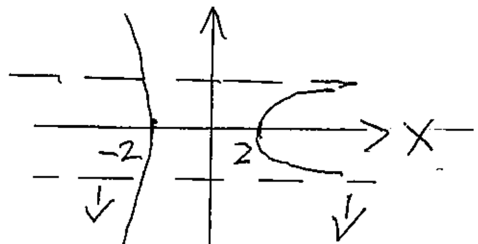
$$= -\frac{7}{2x} - \frac{5}{4}\ln x + \frac{5}{4}\ln(x+2) + C$$

1 ✓

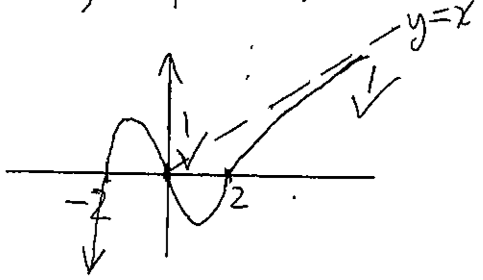
(d) i)



ii)



iii)



Question 13

(a) i) Since ellipse has major axis vertical, it is in the form:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$$\therefore b^2=9 \quad \text{and } a^2=16 \qquad 1 ✓$$

$$\frac{b^2}{a^2}=1-e^2$$

$$e^2=1-\frac{9}{16}$$

$$=\frac{7}{16}$$

$$e=\frac{\sqrt{7}}{4} \qquad 1 ✓$$

coordinates of foci: $(0, \pm ae)$

$$(0, \pm\sqrt{7}) \qquad 1 ✓$$

equations of directrices:

$$y=\pm\frac{a}{e}=\pm 4\div\frac{\sqrt{7}}{4}$$

$$y=\pm\frac{16}{\sqrt{7}} \qquad 1 ✓$$

ii) $$\frac{d}{dx}\left(\frac{x^2}{9}+\frac{y^2}{16}\right)=\frac{d}{dx}(1)$$

$$\frac{2x}{9}+\frac{2y}{16}\cdot\frac{dy}{dx}=0$$

$$\frac{y}{8}\frac{dy}{dx}=-\frac{2x}{9}$$

$$\frac{dy}{dx}=\frac{-16x}{9y}=m \qquad 1 ✓$$

at $P(3\cos\theta, 4\sin\theta)$, $m=\frac{-16\times 3\cos\theta}{9\times 4\sin\theta}$

$$=-\frac{4\cos\theta}{3\sin\theta}$$

tangent at P:

$$y-y_1=m(x-x_1) \qquad 1 ✓$$

$$y-4\sin\theta=-\frac{4\cos\theta}{3\sin\theta}(x-3\cos\theta)$$

$$(3\sin\theta)y-12\sin^2\theta=-(4\cos\theta)x+12\cos^2\theta$$

$$3y\sin\theta+4x\cos\theta=12(\sin^2\theta+\cos^2\theta)$$

$$=12$$

For C: $y=4$ in equation of tangent

$$12\sin\theta+4x\cos\theta=12$$

$$4x\cos\theta=12(1-\sin\theta)$$

$$x=\frac{12(1-\sin\theta)}{4\cos\theta}$$

$$\therefore AC=\frac{3(1-\sin\theta)}{\cos\theta} \qquad 1 ✓$$

For D: $y=-4$ in equation of tangent

$$-12\sin\theta+4x\cos\theta=12$$

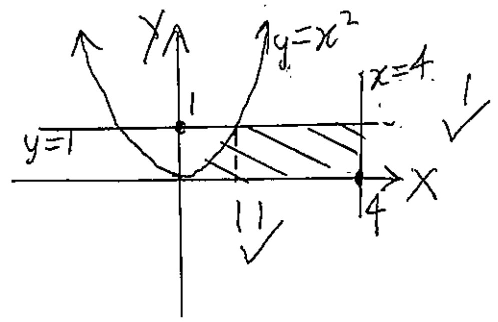
$$x=\frac{12(1+\sin\theta)}{4\cos\theta}$$

$$\therefore BD=\frac{3(1+\sin\theta)}{\cos\theta} \qquad 1 ✓$$

$$AC\times BD=\frac{3(1-\sin\theta)}{\cos\theta}\times\frac{3(1+\sin\theta)}{\cos\theta}$$

$$=\frac{9(1-\sin^2\theta)}{\cos^2\theta}=9 \qquad 1 ✓$$

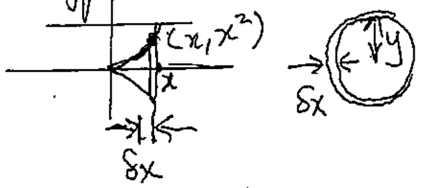
(b) i



1 ✓ 1 ✓

ii) Split volume into V_1 and V_2. V_1 (for $0\le x\le 1$) and V_2 $(1\le x\le 4)$.

typical slice for V_1:



$$\delta V=\pi y^2\delta x \qquad 1 ✓$$

$$=\pi(x^2)^2\delta x$$

$$\delta V=\pi x^4\delta x$$

$$V_1=\lim_{\delta x\to 0}\pi x^4\delta x=\pi\int_0^1 x^4\,dx$$

$$=\pi\left[\frac{x^5}{5}\right]_0^1$$

$$=\frac{\pi}{5}u^3 \qquad 1 ✓$$

$$V_2=\pi\int_1^4 1^2\,dx=\pi\left[x\right]_1^4=4\pi-\pi \qquad 1 ✓$$

$$=3\pi\ u^3$$

$$V=V_1+V_2=\frac{\pi}{5}u^3+3\pi u^3$$

$$=\frac{16}{5}\pi\ u^3 \qquad 1 ✓$$

Question 14

(a) i) $y = c^2 x^{-1}$ $\quad P(cp, \frac{c}{p})$

$\frac{dy}{dx} = -c^2 x^{-2}$

$m_{tangent}$ at $P = -\frac{c^2}{(cp)^2} = -\frac{1}{p^2}$

$\therefore m_{normal} = p^2$ 1 ✓

$y - y_P = m_{normal}(x - x_P)$

$y - \frac{c}{p} = p^2(x - cp)$ 1 ✓

$py - c = p^3(x - cp)$

$py - c = p^3x - cp^4$

$p^3x - py = cp^4 - c$
$= c(p^4 - 1)$ ①

ii) sub $Q(cq, \frac{c}{q})$ into ①:

$p^3(cq) - p(\frac{c}{q}) = c(p^4 - 1)$ 1 ✓

$c(p^3q - \frac{p}{q}) = c(p^4 - 1)$

$p^3q^2 - p = p^4q - q$

$p^3q^2 - p^4q = p - q$

$p^3q(q - p) = p - q$

$p^3q = \frac{p-q}{-(p-q)}$ 1 ✓

$p^3q = -1$

iii) R is MP of PQ

i.e. $R\left(\frac{cp + cq}{2}, \frac{1}{2}\left(\frac{c}{p} + \frac{c}{q}\right)\right)$ 1 ✓

so $x = \frac{c(p+q)}{2}$ ① and

$y = \frac{c(p+q)}{2pq}$ ②

the locus of R is

$c^2(x^2 - y^2)^2 + 4x^3y^3 = 0$ ③

if ① and ② satisfy ③.

Sub ① and ② into ③:

$c^2(x^2-y^2)^2 + 4x^3y^3$

$= c^2\left(\frac{c^2(p+q)^2}{4} - \frac{c^2(p+q)^2}{4p^2q^2}\right)^2$

$+ 4\left(\frac{c^3(p+q)^3}{8}\right)\left(\frac{c^3(p+q)^3}{8p^3q^3}\right)$ 1 ✓

$= c^2\left(\frac{p^2q^2c^2(p+q)^2 - c(p+q)^2}{4p^2q^2}\right)^2$

$+ \frac{c^6(p+q)^6}{16p^3q^3}$

$= c^2\frac{(c^2(p+q)^2(p^2q^2-1))^2}{16p^4q^4} + \frac{c^6(p+q)^6}{16p^3q^3}$

$= \frac{c^6(p+q)^4(p^2q^2-1)^2 + c^6(p+q)^6pq}{16p^4q^4}$

$= \frac{c^6(p+q)^4}{16p^4q^4}\left((p^2q^2-1)^2 + (p+q)^2pq\right)$

Now $(p^2q^2-1)^2 + (p+q)^2pq$

$= p^4q^4 - 2p^2q^2 + 1 + (p^2 + 2pq + q)pq$

$= p^4q^4 - 2p^2q^2 + 1 + p^3q + 2p^2q^2 + pq^3$

$= p^4q^4 + 1 + (-1) + pq^3$

$= pq^3(p^3q + 1)$ 1 ✓

$= pq^3(-1 + 1)$

$= 0$

$\therefore$ ① and ② satisfy ③

and ③ is the locus of R.

Question 14 (continued)

(b) i) $m\ddot{x} = -m(v+v^3)$

$\ddot{x} = -v - v^3$ ①

$v\dfrac{dv}{dx} = -v - v^3$

$\dfrac{dv}{dx} = -1 - v^2$

$\dfrac{dx}{dv} = -\dfrac{1}{1+v^2}$

$x = -\int \dfrac{dv}{1+v^2}$

$x = -\tan^{-1} v + c$ ✓

now $x = 0$ when $v = Q$

$0 = -\tan^{-1} Q + c$

$\therefore c = \tan^{-1} Q$

so $x = -\tan^{-1} v + \tan^{-1} Q$
$= \tan^{-1} Q - \tan^{-1} v$

Now $\tan x = \tan(\tan^{-1} Q - \tan^{-1} v)$

$= \dfrac{\tan(\tan^{-1} Q) - \tan(\tan^{-1} v)}{1 + \tan(\tan^{-1} Q)\tan(\tan^{-1} v)}$

$\tan x = \dfrac{Q - v}{1 + Qv}$ ✓

$\therefore x = \tan^{-1}\left(\dfrac{Q-v}{1+Qv}\right)$ ④

ii) from ①: $\dfrac{dv}{dt} = -v - v^3$

$\therefore \dfrac{dt}{dv} = \dfrac{-1}{v+v^3} = -\dfrac{1}{v(1+v^2)}$

$= -\dfrac{1}{v} + \dfrac{v}{1+v^2}$ ✓

$\therefore t = -\int \dfrac{1}{v}\,dv + \dfrac{1}{2}\int \dfrac{2v}{1+v^2}\,dv$

$\therefore t = -\ln v + \frac{1}{2}\ln(1+v^2) + c$

when $t = 0$, $v = Q$

$0 = -\ln Q + \frac{1}{2}\ln(1+Q^2) + c$

$\therefore c = \ln Q - \frac{1}{2}\ln(1+Q^2)$

and $t = -\ln v + \frac{1}{2}\ln(1+v^2) + \ln Q - \frac{1}{2}\ln(1+Q^2)$

$= \ln\left(\dfrac{Q}{v}\right) + \dfrac{1}{2}\ln\left(\dfrac{1+v^2}{1+Q^2}\right)$

$= \dfrac{1}{2}\left(2\ln\left(\dfrac{Q}{v}\right) + \ln\left(\dfrac{1+v^2}{1+Q^2}\right)\right)$

$= \dfrac{1}{2}\left[\ln\dfrac{Q^2}{v^2} + \ln\left(\dfrac{1+v^2}{1+Q^2}\right)\right]$

$t = \dfrac{1}{2}\ln\left(\dfrac{Q^2(1+v^2)}{v^2(1+Q^2)}\right)$ ✓

and when $v = R$;

$t = \dfrac{1}{2}\ln\left(\dfrac{Q^2(1+R^2)}{R^2(1+Q^2)}\right)$ as required ②

iii) from ②: $2t = \ln\left(\dfrac{Q^2(1+R^2)}{R^2(1+Q^2)}\right)$ ✓

$e^{2t} = \dfrac{Q^2(1+R^2)}{R^2(1+Q^2)}$

$e^{2t}R^2(1+Q^2) = Q^2(1+R^2)$

$e^{2t}R^2(1+Q^2) = Q^2R^2 + Q^2$

$e^{2t}R^2(1+Q^2) - Q^2R^2 = Q^2$

$R^2\left(e^{2t}(1+Q^2) - Q^2\right) = Q^2$

$\therefore R^2 = \dfrac{Q^2}{e^{2t}(1+Q^2) - Q^2}$ ✓ ③

iv) from ③;

$v^2 = \dfrac{Q^2}{e^{2t}(1+Q^2) - Q^2}$

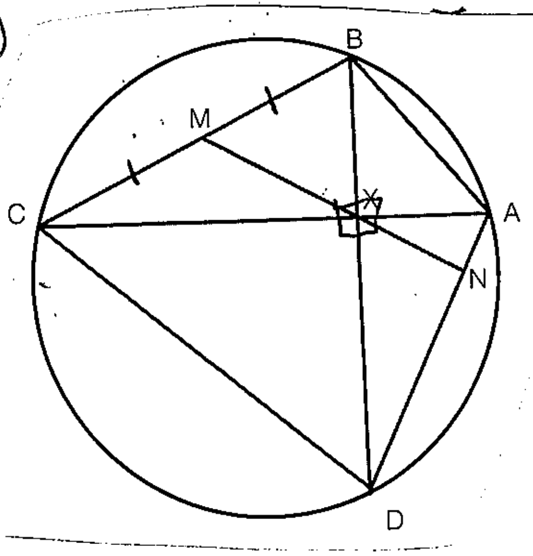
as $t \to \infty$, $v^2 \to 0$ and $v \to 0$ ✓

also from ④;

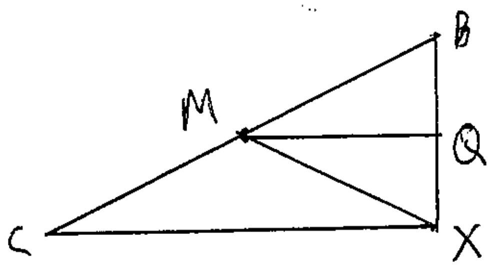
$x \to \tan^{-1}(Q)$ ✓

Question 15

(a) i)



ii) consider ΔBCX



Construct $MQ \parallel CX$.

$\therefore \angle BQM = \angle BXC = 90°$ (Corresponding $\angle$s on $\parallel$ lines are $=$)

In ΔBMQ and ΔBCX;

$\angle MBQ = \angle CBX$

$\angle BQM = \angle BXC = 90°$

$\therefore \Delta BMQ ||| \Delta BCX$ (AA) ✓

$\frac{BM}{BC} = \frac{BQ}{BX}$ (common ratio of sides in similar Δs)

but $BM = MC$ (given)

$\therefore \frac{BQ}{BX} = \frac{BM}{BC} = \frac{1}{2}$ ✓

and $BQ = QX$

In ΔBMQ and ΔXMQ;

$BQ = QX$ (from above)

$\angle BQM = \angle XQM = 90°$

MQ is common

$\therefore \Delta BMQ \equiv \Delta XMQ$ (SAS) ✓

$\therefore \angle MBQ = \angle MXQ$ (matching $\angle$s in $\equiv \Delta$s)

and $\angle MBX = \angle MXB$

iii) In ΔBMX;

$MB = MX$ (equal base $\angle$s of isosceles ΔBMX, $\angle MBX = \angle MXB$)

but $MB = MC$ (given) ✓

$\therefore MX = MC$

and $\angle MCX = \angle MXC$ (base $\angle$s of isosceles ΔMCX)

Now $\angle MXC = \angle AXN$ (vertically opposite $\angle$s)

$\therefore \angle MCX = \angle AXN$ ✓

or $\angle BCX = \angle AXN$ (1)

In the circle $ABCD$;

$\angle CBD = \angle CAD$ ($\angle$s on the same chord) ✓

i.e. $\angle CBX = \angle XAN$ (2)

in ΔCBX and ΔXAN

$\angle BCX = \angle AXN$ (from (1))

$\angle CBX = \angle XAN$ (from (2))

$\therefore \Delta CBX ||| \Delta XAN$ (AA) ✓

hence $\angle BXC = \angle ANX = 90°$ (matching $\angle$s in $|||$ Δs)

$\therefore MN \perp AD$

Question 15 (continued)

(b) $\left(\frac{1}{\sqrt{x}}\right)^3 - \left(\frac{1}{\sqrt{x}}\right)^2 - 14\left(\frac{1}{\sqrt{x}}\right) - 24 = 0$ ✓

$\frac{1}{x\sqrt{x}} - \frac{1}{x} - \frac{14}{\sqrt{x}} - 24 = 0$ ✓

$\left(\frac{1}{\sqrt{x}}\left(\frac{1}{x} - 14\right)\right)^2 = \left(24 + \frac{1}{x}\right)^2$

$\frac{1}{x}\left(\frac{1}{x^2} - \frac{28}{x} + 14^2\right)x^3 = \left(24^2 + \frac{48}{x} + \frac{1}{x^2}\right)x^3$

$1 - 28x + 196x^2 = 576x^3 + 48x^2 + x$

$576x^3 - 148x^2 + 29x - 1 = 0$ ✓

(c) i) $\sin(A+B) = \sin A\cos B + \sin B\cos A$

$\sin(A-B) = \sin A\cos B - \sin B\cos A$

$\sin(A-B) + \sin(A+B) = 2\sin A\cos B$ ✓

when $A = 2x$, $B = x$

$\sin(2x - x) + \sin(2x + x)$ ✓
$= 2\sin 2x\cos x$ ✓

$\therefore \sin x + \sin 3x = 2\sin 2x\cos x$ (1)

ii) $\sin x + \sin 2x + \sin 3x = 0$

$2\sin 2x\cos x + \sin 2x = 0$ ✓
(from (1))

$\sin 2x(2\cos x + 1) = 0$

$\sin 2x = 0$ or $\cos x = -\frac{1}{2}$

$0 \leq 2x \leq 4\pi$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ ✓

rel $x = \frac{\pi}{3}$

$x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$= \frac{2\pi}{3}, \frac{4\pi}{3}$ ✓

Question 16

(a) i) $I_n = \int_1^e (\ln x)^n\,dx$

let $I_n = \int_1^e u\,dv$

where $u = (\ln x)^n$ $dv = dx$

$\frac{du}{dx} = n(\ln x)^{n-1} \times \left(\frac{1}{x}\right)$, $v = x$ ✓

$\int u\,dv = [uv] - \int v\,du$

$\int_1^e (\ln x)^n dx = \left[x.(\ln x)^n\right]_1^e - \int_1^e x\left(\frac{1}{x}\right) n(\ln x)^{n-1} dx$

$= \left[e.(\ln e)^n - 1.(\ln 1)^n\right] - n\int_1^e (\ln x)^{n-1} dx$ ✓

$= e.1^n - nI_{n-1}$

$= e - n.I_{n-1}$

ii) $I_4 = e - 4I_3$

$= e - 4[e - 3I_2]$

$= e - 4[e - 3(e - 2I_1)]$

$= e - 4[e - 3(e - 2\{e - 1I_0\})]$ ✓

now $I_0 = \int_1^e (\ln x)^0 dx$

$= \int_1^e 1\,dx = e - 1$

$\therefore I_4 = e - 4[e - 3(e - 2\{e - (e-1)\})]$

$= e - 4[e - 3(e - 2\times 1)]$

$= e - 4[e - 3e + 6]$

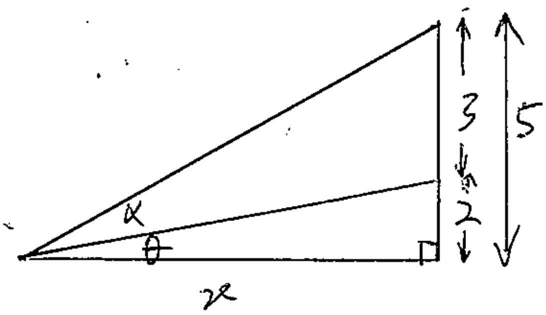
$= e - 4[6 - 2e]$

$= e + 8e - 24$

$= 9e - 24$ ✓

Question 16 (continued)

(b) i)



$$\tan(\alpha+\theta) = \frac{\tan\alpha + \tan\theta}{1-\tan\alpha\tan\theta}$$

$$\frac{5}{x} = \frac{\tan\alpha + \frac{2}{x}}{1-(\tan\alpha)(\frac{2}{x})}$$

$$\frac{5}{x}\left(1-\left(\frac{2}{x}\right)\tan\alpha\right) = \tan\alpha + \frac{2}{x}$$

$$\frac{5}{x} - \frac{10}{x^2}\tan\alpha = \tan\alpha + \frac{2}{x}$$

$$\frac{5}{x} - \frac{2}{x} = \tan\alpha + \frac{10}{x^2}\tan\alpha$$

$$\frac{3}{x} = \left(1+\frac{10}{x^2}\right)\tan\alpha$$

$$\frac{3}{x} = \left(\frac{x^2+10}{x^2}\right)\tan\alpha$$

$$\therefore \tan\alpha = \frac{3x^{\cancel{2}}}{\cancel{x}(x^2+10)}$$

$$\alpha = \tan^{-1}\left(\frac{3x}{x^2+10}\right)$$

ii) for max α; $\frac{d\alpha}{dx} = 0$.

$$\frac{d\alpha}{dx} = \frac{1}{1+\left(\frac{3x}{x^2+10}\right)^2} \times \frac{d}{dx}\left(\frac{3x}{x^2+10}\right)$$

$$= \frac{\frac{3(x^2+10)-(3x)(2x)}{(x^2+10)^2}}{\frac{(x^2+10)^2+(3x)^2}{(x^2+10)^2}}$$

$$= \frac{10-3x^2}{(x^2+10)^2+(3x)^2}$$

$= 0$ when $10 - 3x^2 = 0$

$\therefore x = \pm\sqrt{\frac{10}{3}}$

and $x = \sqrt{\frac{10}{3}}$ since $x > 0$.

$x = \sqrt{\frac{10}{3}}$ produces max α since: $\frac{d\alpha}{dx} > 0$ when $x < \sqrt{\frac{10}{3}}$ and $\frac{d\alpha}{dx} < 0$ when $x > \sqrt{\frac{10}{3}}$.

c) i) $(x-\sqrt{5})(x+\sqrt{5}) = x^2-5$ is a factor of $f(x)$. Hence:

$$\begin{array}{r l}
 & x^4+4x^3+2x^2+12x+45 \\
x^2-5 \,\big) & x^6+4x^5-3x^4-8x^3+35x^2-60x-225 \\
 & x^6 \quad -5x^4 \\
 & 4x^5+2x^4-8x^3+35x^2-60x-225 \\
 & 4x^5 \quad -20x^3 \\
 & 2x^4+12x^3+35x^2-60x-225 \\
 & 2x^4 \quad -10x^2 \\
 & 12x^3+45x^2-60x-225 \\
 & 12x^3 \quad -60x \\
 & 45x^2-225 \\
 & 45x^2-225
\end{array}$$

for double root:

let $g(x) = x^4+4x^3+2x^2+12x+45$

$g'(x) = 4x^3+12x^2+4x+12$

$= x^2(4x+12)+1(4x+12)$

$= 4(x^2+1)(x+3)$

$g(-3) = (-3)^4+4(-3)^3+2(-3)^2+12(-3)+45$

$= 0$ and $g'(-3) = 0$

$\therefore f(x) = (x^2-5)(x+3)^2(x^2+bx+c)$

$(-5)\times(9)\times c = -225$ (equating constant terms)

$\therefore c = 5$

$(-5)\times(6)c + (-5)\times 9\times b = -60$ (equate terms in x)

$-30\times 5 - 45b = -60$

$-45b = -60 \quad 150$

$b = -2$

$f(x) = (x-\sqrt{5})(x+\sqrt{5})(x+3)^2(x^2-2x+5)$

ii) $x^2-2x+5 = x^2-2x+1+4 = (x-1)^2-(2i)^2$

$f(x) = (x-\sqrt{5})(x+\sqrt{5})(x+3)^2(x-1-2i)(x-1+2i)$