ASCHAM SCHOOL

2025 YEAR 12
MATHEMATICS EXTENSION 2
TRIAL EXAM

GENERAL INSTRUCTIONS

- Reading time – 10 minutes
- Working time – 180 minutes
- Use black pen, non-erasable
- NESA-approved calculators may be used
- Reference Sheet is provided

Total Marks - 100

Section A – Multiple Choice (1 mark each)

- Attempt Questions 1 to 10.
- Select answers on the separate multiple-choice sheet provided.
- Write your NESA number on the multiple-choice sheet.

Section B – Questions 11 – 16 (15 marks each)

- Start each question in a new booklet.
- If you use a second booklet for a question, place it inside the first.
- Label extra booklets for the same question as, for example, Q11-2 etc.
- Write your NESA number and question number on each booklet.

Section A - Multiple choice (10 marks)
(Mark the correct answer on the sheet provided.)

1. Given $0 < \theta < \dfrac{\pi}{2}$, what is the magnitude of the vector
$\left(\cos\theta\right)\underset{\sim}{i} + \left(\sin\theta\right)\underset{\sim}{j} + \left(\tan\theta\right)\underset{\sim}{k}$?

A) 1

B) $\operatorname{cosec}\theta$

C) $\cot\theta$

D) $\sec\theta$

2. Which is the **negation** of the following statement?

"If snow is falling, then I will not be at the park."

A) If snow is falling, then I will be at the park.

B) If snow is not falling, then I will be at the park.

C) Snow is falling and I will be at the park.

D) Snow is not falling and I will be at the park.

3. A particle whose displacement is x moves in simple harmonic motion such that $\ddot{x} = -4x$.
What is x as a function of t if initially $x = 3$ and $\dot{x} = -6\sqrt{3}$?

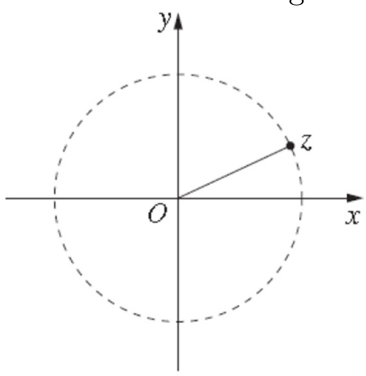
A) $x = 3\cos 2t - 3\sqrt{3}\sin 2t$

B) $x = 3\sin 2t - 3\sqrt{3}\cos 2t$

C) $x = 3\cos 2t + 3\sqrt{3}\sin 2t$

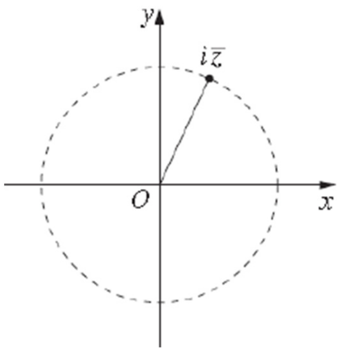
D) $x = 3\sin 2t + 3\sqrt{3}\cos 2t$

4. The complex number z is shown on the Argand diagram below.

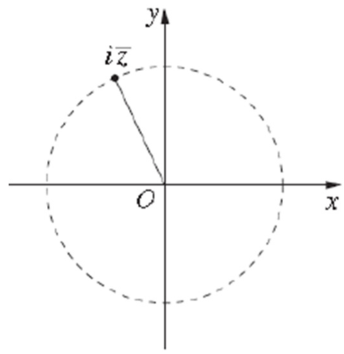


Which of the following best represents $i\overline{z}$?

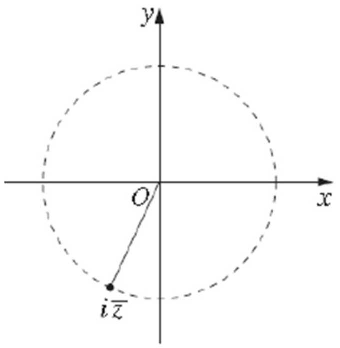
A)



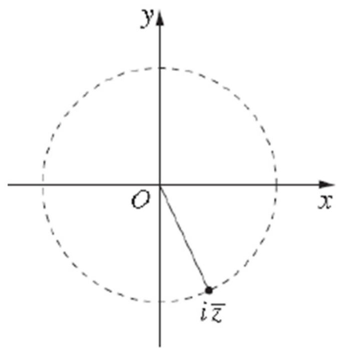
B)



B)



D)



5. P , Q and R are three collinear points with position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively. Q lies between P and R such that $PQ:QR=2:1$.

The position vector of R is given by:

A) $\underset{\sim}{r}=\dfrac{1}{2}\underset{\sim}{q}+\dfrac{3}{2}\underset{\sim}{p}$

B) $\underset{\sim}{r}=\dfrac{3}{2}\underset{\sim}{p}-\dfrac{1}{2}\underset{\sim}{q}$

C) $\underset{\sim}{r}=\dfrac{3}{2}\underset{\sim}{q}-\dfrac{1}{2}\underset{\sim}{p}$

D) $\underset{\sim}{r}=\dfrac{1}{2}\underset{\sim}{p}+\dfrac{3}{2}\underset{\sim}{q}$

6. For which of these statements, for positive integers x and y, is the **converse** necessarily true?

A) If x and y are odd, then $x+y$ is even.

B) If x and y are odd, then x^2+y^2 is even.

C) If x and y are odd, then xy is odd.

D) If x and y are odd, then x^y is odd.

7. Which of the following lines passes through $(0,1,-1)$?

A) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

B) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

C) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

D) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

8. Which of these inequalities is **false**?

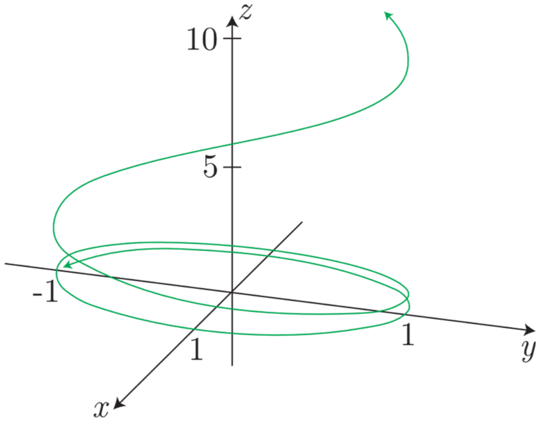
A) $\displaystyle\int_1^2 \frac{1}{1+x}\,dx < \int_1^2 \frac{1}{x}\,dx$

B) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin x}{x}\,dx < \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{x}\,dx$

C) $\displaystyle\int_1^2 e^{-x^2}\,dx < \int_0^1 e^{-x^2}\,dx$

D) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx < \int_0^{\frac{\pi}{4}} \tan^3 x\,dx$

9. Which parametric form best describes the curve below?



A) $\underset{\sim}{r}(t) = \left(\cos t\right)\underset{\sim}{i} + \left(\sin t\right)\underset{\sim}{j} + \left(t\right)\underset{\sim}{k}$

B) $\underset{\sim}{r}(t) = \left(\cos t\right)\underset{\sim}{i} + \left(\sin t\right)\underset{\sim}{j} + \left(e^{-0.5t}\right)\underset{\sim}{k}$

C) $\underset{\sim}{r}(t) = \left(\cos t\right)\underset{\sim}{i} + \left(\sin t\right)\underset{\sim}{j} + \left(\ln t\right)\underset{\sim}{k}$

D) $\underset{\sim}{r}(t) = \left(\cos t\right)\underset{\sim}{i} + \left(\sin t\right)\underset{\sim}{j} + \left(\cos 2t\right)\underset{\sim}{k}$

10. Let $w = e^{i\frac{2\pi}{5}}$.
What is the value of $(1-w)(1-w^2)(1-w^3)(1-w^4)$?

A) -5

B) -4

C) 4

D) 5

(End of Section A. Question 11 begins on the next page.)

Section B (90 marks)
Question 11 (Begin and label a new booklet.) **(15 marks)**

a) Use integration by parts to find $\int \ln(x^2+1)\,dx$. [3]

b) Use the substitution $u = e^x - 1$ to find $\int \frac{e^{2x}}{(e^x-1)^2}\,dx$. [3]

c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{3\cos x - 4\sin x + 5}\,dx$ [4]

d) Convert the line $y = 3x - 5$ to the form $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$. [2]

e) It is known that $5 + 6i$ is a zero of the polynomial $P(z) = 2z^3 - 19z^2 + 112z + k$, where k is real.

i) Find the other two zeroes of $P(z)$. [2]

ii) Find the value of k. [1]

(End of Question 11.)

Question 12 (Begin and label a new booklet.) **(15 marks)**

a) Prove that $\sqrt[3]{2}$ is irrational. [3]

b) Given that x is a real number and $(x+i)^4$ is purely imaginary, find all possible values of x in exact form. [3]

c) Find all the solutions of $z^5 = 4+4i$, giving your answers in modulus-argument form. [3]

d) i) On an Argand diagram sketch the locus of point P representing the complex number z such that $\left|z-\left(\sqrt{3}+i\right)\right| = 1$. [1]

ii) Find the maximum possible values of $|z|$ and $\text{Arg}(z)$. [2]

e) Prove that for all $x, y \in \mathbb{R}$,

$$\frac{(x-1)^2+(y+2)^2}{2} \geq x+y \quad .$$ [3]

(End of Question 12.)

Question 13 (Begin and label a new booklet.) **(14 marks)**

a) A particle is undergoing simple harmonic motion centred about the origin.

Its equation of motion is given by $\ddot{x} = -16x$.

i) Show that $v^2 = 16\left(a^2 - x^2\right)$, where a is the amplitude. [2]

The particle's position at time t seconds is given by: $x = \sqrt{3}\sin 4t - \cos 4t$.

ii) Express x in the form $R\sin\left(4t - \alpha\right)$, where α is in radians. [2]

iii) When does the particle first reach its maximum speed after $t = 0$? [2]

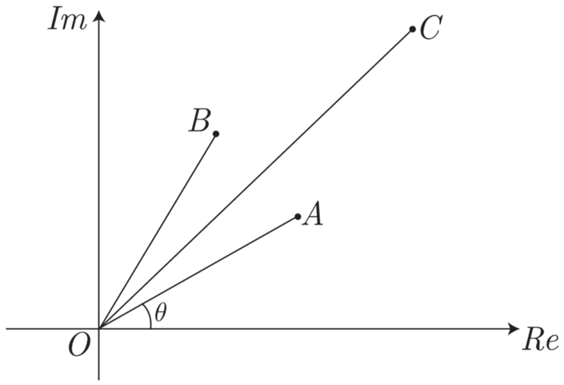
b) i) If $z = \cos\theta + i\sin\theta$, show that $z - \dfrac{1}{z} = 2i\sin\theta$ and $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$. [2]

ii) Hence show that $\sin^5\theta = \dfrac{1}{16}\sin 5\theta - \dfrac{5}{16}\sin 3\theta + \dfrac{5}{8}\sin\theta$. [2]

(Question 13 continues on the next page...)

c) Let $z = e^{i\theta}$ where $0 < \theta < \dfrac{\pi}{2}$.

On the Argand diagram the point A represents z, the point B represents z^2 and the point C represents $z + z^2$.



Copy the diagram into your writing booklet.

i) What type of quadrilateral best describes $OACB$? [1]

ii) Find $\arg(z + z^2)$ in terms of θ. [1]

iii) Find $\left|z + z^2\right|$ in terms of θ. [2]

(End of Question 13.)

Question 14 (Begin and label a new booklet.) **(14 marks)**

a) i) Prove that $x + y \geq 2\sqrt{xy}$ for positive real numbers x, y. [1]

ii) Hence for positive real numbers a, b, c, and d, prove that: [2]

$$a^2 + 4b^2 + 4c^2 + d^2 \geq 8\sqrt{abcd}$$

b) Consider the line with equation $\underset{\sim}{r} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Let A be the point corresponding to $\lambda = 1$ on the line.
Let $P\left(6, 0, 2\right)$ be a point not on the line.

i) Find the projection of $\overrightarrow{AP}$ onto the line. [2]

ii) Hence find the coordinates of point P', the reflection of P about the line. [2]

c) i) Use De Moivre's theorem to show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$. [2]

ii) Deduce that $\tan 3\theta = \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$. [1]

iii) By letting $x = \tan\theta$, find the solutions of $x^3 - 3x^2 - 3x + 1 = 0$, expressing them in exact trigonometric form. [3]

iv) Hence show that $\tan\dfrac{\pi}{12} + \tan\dfrac{5\pi}{12} = 4$. [1]

(End of Question 14.)

Question 15 (Begin and label a new booklet.) **(16 marks)**

a) Evaluate $\int_{-1}^{2} \sqrt{8-2x-x^2}\, dx$ using a suitable substitution. [3]

b) Prove the following statement for all integers $n \geq 3$.

"If $3^n - 4$ is a prime, then n is odd." [3]

c) A particle of mass m kilograms is dropped from rest in a medium where the resistance to motion has magnitude $\frac{1}{10}mv^2$ Newtons when the velocity of the particle is v m/s. After t seconds, the particle has fallen x metres. Take $g = 10$ m/s^2.

i) With a diagram, show that the acceleration is $\ddot{x} = \frac{1}{10}(100 - v^2)$. [1]

The particle hits the ground $\log_e\left(\sqrt{5}\right)$ seconds after it is dropped.

ii) Find its distance fallen in simplest exact form. [5]

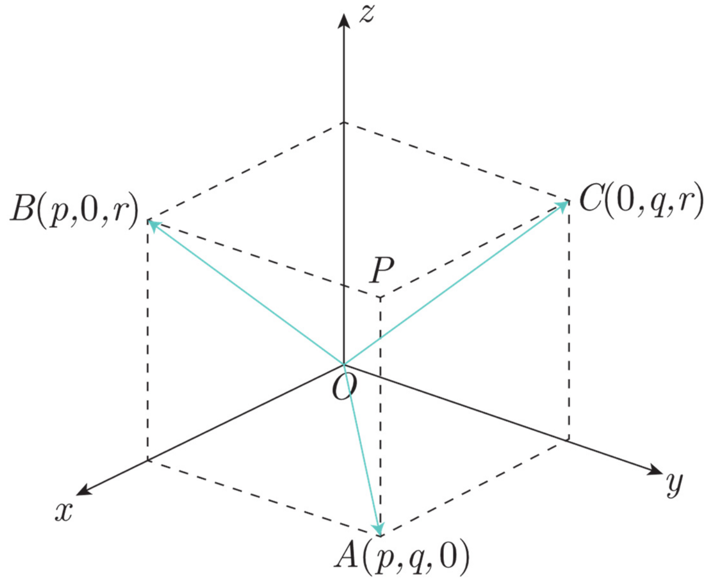
d) Let $I_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\, dx$ for integers $n \geq 0$.

Show that $I_n = \frac{\sqrt{2} - (n-1)I_{n-2}}{n}$ for all integers $n \geq 2$ [4]

(End of Question 15.)

Question 16 (Begin and label a new booklet.) **(16 marks)**

a) The diagram below shows a rectangular prism in the first octant created from $P(p,q,r)$ with all edges parallel to the axes and with OA, OB and OC as face diagonals.



It is known that $\angle AOB = \angle AOC = \angle BOC = \dfrac{\pi}{3}$.

Prove that the rectangular prism is a cube. [4]

(Question 16 continues on the next page...)

b) i) Show that for $k \geq 0$, $2k+3 > 2\sqrt{(k+1)(k+2)}$. [2]

ii) Hence show that for $n \geq 1$,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right).$$ [4]

iii) Is the statement $\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10}$ true for all positive integers N? Justify your answer. [1]

c) Let z_1 and z_2 be two non-zero complex numbers.
It is known that $z_1 + z_2 + 1 = 0$.

i) Show that $|z_1| = |z_2| = 1 \Rightarrow \frac{1}{z_1} + \frac{1}{z_2} + 1 = 0$. [2]

ii) Also show that $\frac{1}{z_1} + \frac{1}{z_2} + 1 = 0 \Rightarrow |z_1| = |z_2| = 1$. [2]

iii) Hence or otherwise show that $|z_1| = |z_2| = 1 \Rightarrow z_1^2 + z_2^2 + 1 = 0$. [1]

(End of Question 16.)

End of exam.

Ascham 2025 Ext 2 Trial Solutions

MC

Q1. $\sqrt{\cos^2\theta + \sin^2\theta + \tan^2\theta}$

$= \sqrt{1 + \tan^2\theta}$

$= \sqrt{\sec^2\theta}$

$= \sec\theta$ as θ in 1st quad. $\boxed{D}$

Q2. $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Snow is falling and I will be ... $\boxed{C}$

Q3. $\ddot{x} = -4x$

$n = 2$ centre: $x = 0$

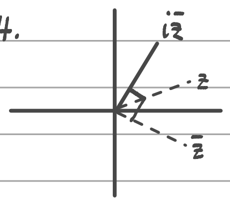
$t=0$: $x = 3$ A or B

$t=0$: $\dot{x} = -6\sqrt{3}$

A: $\dot{x} = -6\sin 2t - 6\sqrt{3}\cos 2t$

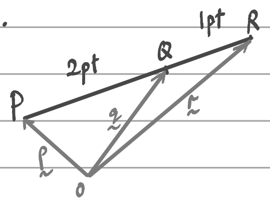
at $t=0$: $\dot{x} = -6\sqrt{3}$ ✓ $\boxed{A}$

Q4.



$\boxed{A}$

Q5.



$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

$= \underset{\sim}{p} + \frac{3}{2}\overrightarrow{PQ}$

$= \underset{\sim}{p} + \frac{3}{2}(\underset{\sim}{q} - \underset{\sim}{p})$

$= \underset{\sim}{p} + \frac{3}{2}\underset{\sim}{q} - \frac{3}{2}\underset{\sim}{p}$

$= -\frac{1}{2}\underset{\sim}{p} + \frac{3}{2}\underset{\sim}{q}$ $\boxed{C}$

Alternatively: $\overrightarrow{QR} = \frac{1}{2}\overrightarrow{PQ}$

$\underset{\sim}{r} - \underset{\sim}{q} = \frac{1}{2}(\underset{\sim}{q} - \underset{\sim}{p})$

$\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

Q6. Checking each converse:

$x+y$ even : $4+6$ X

x^2+y^2 even : 2^2+4^2 X

xy odd : can't think of counterexample ...

x^y odd : 3^4 is odd but y even X

$\boxed{C}$

Q7. A and C out

due to $x=1$ always.

B: try $\lambda = -1$:

$$\underset{\sim}{r} = \begin{pmatrix} 1-1 \\ 1-0 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \checkmark$$

D: try $\lambda = -1$:

$$\underset{\sim}{r} = \begin{pmatrix} 1-1 \\ 0-1 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \times \qquad \boxed{B}$$

Q8. A: $\frac{1}{1+x} < \frac{1}{x}$ in $[1,2]$ ✓

B: $\sin x < 1$ in $[\frac{\pi}{6}, \frac{\pi}{4}]$ ✓

C: e^{-x^2} decreases

from $[0,1]$ to $[1,2]$ ✓

D: as $\tan x \leq 1$ in $[0, \frac{\pi}{4}]$

$\tan^2 x > \tan^3 x$ in $[0, \frac{\pi}{4}]$ FALSE!

$\boxed{D}$

Q9. All radius 1.

As t goes through $(0, 2\pi)$

A) X not linear increase in z-value.

C) X $\ln(t)$ would be large negative.

D) X z-value does not go up and down.

as $t \to \infty$, $z \to 0$ $\therefore$ B $\boxed{B}$

Q10. $w^5 = e^{i2\pi} = 1$

and $1 + w + w^2 + w^3 + w^4 = 0$, $(w \neq 1)$

$= (1 - w^2 - w + w^3)(1 - w^3)(1 - w^4)$

$= (1 - \cancel{w^3} - w^2 + \overset{1}{\cancel{w^5}} - w + w^4 + \cancel{w^3} - \overset{1}{\cancel{w^6}})(1 - w^4)$

$= (2 - w^2 - 2w + w^4)(1 - w^4)$

$= 2 - 2w^4 - w^2 + \overset{1}{\cancel{w^6}} - 2w + 2\overset{1}{\cancel{w^5}} + w^4 - \overset{3}{\cancel{w^8}}$

$= 4 - w - w^2 - w^3 - w^4$

$= 4 - (-1)$

$= 5$ $\boxed{D}$

Alternatively: $w^4 = e^{i\frac{8\pi}{5}} = e^{i(\frac{-2\pi}{5})} = \bar{w}$

$w^3 = e^{i\frac{6\pi}{5}} = e^{i(-\frac{4\pi}{5})} = \overline{w^2}$

$(1-w)(1-\bar{w})(1-w^2)(1-\overline{w^2})$

$= (1 - (w + \bar{w}) + w\bar{w})(1 - (w^2 + \overline{w^2}) + w^2\overline{w^2})$

$= (1 - 2\cos\frac{2\pi}{5} + 1)(1 - 2\cos\frac{4\pi}{5} + 1)$

$= 5$ (with calculator...)

Q11 a)

$$\int \ln(x^2+1)\,dx \qquad u=\ln(x^2+1) \quad v=x$$
$$u'=\frac{2x}{x^2+1} \quad v'=1$$

$$= x\ln(x^2+1) - \int x \times \frac{2x}{x^2+1}\,dx$$

$$= x\ln(x^2+1) - 2\int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\,dx$$

$$= x\ln(x^2+1) - 2\left[x - \tan^{-1}x\right] + C$$

$$= x\ln(x^2+1) - 2x + 2\tan^{-1}x + C$$

b)

$$\int \frac{e^{2x}}{(e^x-1)^2}\,dx \qquad u=e^x-1$$
$$e^x = u+1$$
$$x=\ln(u+1)$$
$$dx = \frac{1}{u+1}\,du$$

$$= \int \frac{(u+1)^2}{u^2}\cdot\frac{1}{u+1}\,du$$

$$= \int \frac{u+1}{u^2}\,du$$

$$= \int \frac{1}{u} + \frac{1}{u^2}\,du$$

$$= \ln|u| - \frac{1}{u} + C$$

$$= \ln|e^x-1| - \frac{1}{e^x-1} + C$$

c)

$$\int_0^{\frac{\pi}{2}} \frac{1}{3\cos x - 4\sin x + 5}\,dx \qquad t=\tan\frac{x}{2}$$
$$x=0,\ t=0$$
$$x=\frac{\pi}{2},\ t=1$$
$$x = 2\tan^{-1}t$$
$$dx = \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{1}{\frac{3(1-t^2)}{1+t^2} - \frac{4(2t)}{1+t^2} + 5}\cdot\frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{3-3t^2-8t+5+5t^2}\,dt$$

$$= \int_0^1 \frac{2}{8+2t^2-8t}\,dt$$

$$= \int_0^1 \frac{1}{t^2-4t+4}\,dt$$

$$= \int_0^1 (t-2)^{-2}\,dt$$

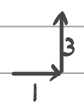
$$= \left[-(t-2)^{-1}\right]_0^1$$

$$= -\frac{1}{-1} + \frac{1}{-2}$$

$$= \frac{1}{2}$$

d)

$$y = 3x-5$$
$$m = \frac{3}{1}$$



$$\underset{\sim}{b} = \begin{pmatrix}1\\3\end{pmatrix}$$

y-int: $(0,-5)$ $\qquad \underset{\sim}{a} = \begin{pmatrix}0\\-5\end{pmatrix}$ or equiv.

$$\therefore\ \underset{\sim}{r} = \begin{pmatrix}0\\-5\end{pmatrix} + \lambda\begin{pmatrix}1\\3\end{pmatrix}$$

e) i) $5-6i$ also a zero as coeff are real by conjugate root theorem.

Σroots : $\quad 5+6i+5-6i+\alpha = \frac{19}{2}$

$$10+\alpha = \frac{19}{2}$$
$$\alpha = \frac{-1}{2}$$

ii) Prod roots: $\quad (5+6i)(5-6i)\times\frac{-1}{2} = \frac{-k}{2}$

$$25+36 = k$$
$$k = 61$$

Q12

a) Assume $\sqrt[3]{2} = \frac{p}{q}$, $p, q \in \mathbb{Z}$ and $\gcd(p,q) = 1$.

$$\sqrt[3]{2}\, q = p$$
$$2q^3 = p^3$$

$\therefore$ p^3 is divisible by 2

$\therefore$ p is divisible by 2 by considering the contrapositive

$\therefore$ Let $p = 2m$, $m \in \mathbb{Z}$

$$2q^3 = (2m)^3$$
$$2q^3 = 8m^3$$
$$q^3 = 4m^3$$

$\therefore$ q^3 is divisible by 2

$\therefore$ q is divisible by 2 by contrapositive

But $\gcd(p,q)$ is 1 from assumption

$\therefore$ $\sqrt[3]{2}$ is irrational by contradiction.

b) $(x+i)^4 = {}^4C_0 x^4 + {}^4C_1 x^3 i + {}^4C_2 x^2 i^2 + {}^4C_3 x i^3 + {}^4C_4 i^4$

$\text{Re}(x+i)^4 = 0$:

$$x^4 - 6x^2 + 1 = 0$$
$$x^2 = \frac{6 \pm \sqrt{36-4}}{2}$$
$$= \frac{6 \pm 4\sqrt{2}}{2}$$
$$= 3 \pm 2\sqrt{2}$$
$$\therefore x = \pm\sqrt{3 \pm 2\sqrt{2}}$$

c) $4 + 4i = 4\sqrt{2}\,\text{cis}\frac{\pi}{4}$

$z = r\,\text{cis}\,\theta$

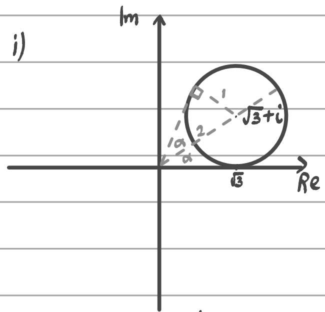
$z^5 = r^5\,\text{cis}\,5\theta$

$r^5 = 4\sqrt{2}$ $\qquad$ $5\theta = \frac{\pi}{4} + 2k\pi$, $k = 0, 1, 2, 3, 4$

$r = \sqrt{2}$

$\therefore$ $z = \sqrt{2}\,\text{cis}\left(\frac{\pi}{20} + \frac{2k\pi}{5}\right)$, $k = 0, 1, 2, 3, 4$

d) i)



ii) $\text{Max } |z| = |\sqrt{3} + i| + 1$

$= 2 + 1$

$= 3$

$\text{Max Arg}(z) = 2\alpha$

$= 2\sin^{-1}\frac{1}{2}$

$= \frac{\pi}{3}$

e) RTP: $\dfrac{x^2 - 2x + 1 + y^2 + 4y + 4}{2} - x - y \geq 0$

$$\text{LHS} = \frac{x^2 - 4x + 4 + y^2 + 2y + 1}{2}$$
$$= \frac{1}{2}(x-2)^2 + \frac{1}{2}(y+1)^2$$

≥ 0 as perfect squares ≥ 0

Q13

a) i) $\ddot{x} = -16x$

$v\dfrac{dv}{dx} = -16x$

$\int v\,dv = \int -16x\,dx$

$\frac{1}{2}v^2 = -8x^2 + c$

$v^2 = -16x^2 + c_1$

$v = 0,\ x = a$

$0 = -16a^2 + c_1$

$16a = c_1$

$\therefore\ v^2 = -16x^2 + 16a^2$

$v^2 = 16(a^2 - x^2)$

ii) $x = \sqrt{3}\sin 4t - \cos 4t$

$x = R\sin(4t - \alpha)$

$= R(\sin 4t\cos\alpha - \cos 4t\sin\alpha)$

$\therefore\ \sqrt{3} = R\cos\alpha \qquad 1 = R\sin\alpha$

$R^2 = \sqrt{3}^2 + 1^2 \qquad \therefore\ \frac{\sqrt{3}}{2} = \cos\alpha$

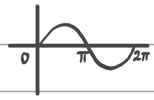
$= 4 \qquad$ rel $\angle = \frac{\pi}{6}$

$R = 2 \qquad \alpha = \frac{\pi}{6}$

$\therefore\ x = 2\sin(4t - \frac{\pi}{6})$

iii) Max speed at centre $x = 0$

$0 = 2\sin(4t - \frac{\pi}{6})$



$4t - \frac{\pi}{6} = 0, \pi, 2\pi \ldots$

$4t = \frac{\pi}{6}$

$t = \frac{\pi}{24}$ sec

b) i) $z = \cos\theta + i\sin\theta \qquad |z| = 1 \Rightarrow \frac{1}{z} = \bar{z}$

$z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = \overline{z^n} = \cos n\theta - i\sin n\theta$

$\therefore\ z^n - \frac{1}{z^n} = 2i\sin n\theta$

ii) $\sin^5\theta = \left(\frac{1}{2i}\left(z - \frac{1}{z}\right)\right)^5$

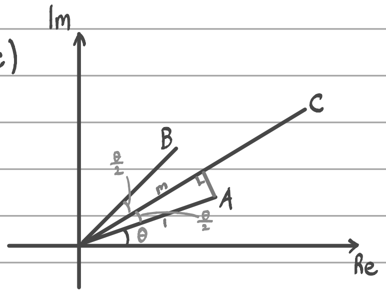
$= \frac{1}{32i}\left(z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}\right)$

$= \frac{1}{32i}\left(z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)\right)$

$= \frac{1}{32i}\left(2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta\right)$

$= \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$

c)



i) Rhombus since $|z| = 1$

so $|z^2| = 1$ and parallelogram has equal sides.

ii) $\arg(z + z^2) = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$

iii) On diagram, $\frac{m}{1} = \cos\frac{\theta}{2}$

$\therefore\ |z + z^2| = |\vec{OC}| = 2\cos\frac{\theta}{2}$

Q14 a) i) $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab$

$a \to \sqrt{x}\ ,\ b \to \sqrt{y}$

$\therefore x + y \geq 2\sqrt{xy}$

ii) using $AM \geq GM$,

$a^2 + 4b^2 \geq 2\sqrt{4a^2b^2} = 4ab$

$4c^2 + d^2 \geq 2\sqrt{4c^2d^2} = 4cd$

$\therefore\ a^2 + 4b^2 + 4c^2 + d^2 \geq 4ab + 4cd$

$\geq 2\sqrt{16abcd}$

$= 8\sqrt{abcd}$

b) i) $A: \begin{pmatrix} -5+2 \\ 1-2 \\ -1+1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$

$\overrightarrow{AP} = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$

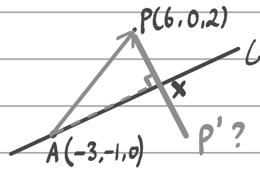
$= \begin{pmatrix} 9 \\ 1 \\ 2 \end{pmatrix}$

$\text{proj}_{D.V.} \overrightarrow{AP} = \dfrac{\begin{pmatrix} 9 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$= \dfrac{18 - 2 + 2}{2^2 + 2^2 + 1^2} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$= 2\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}$

ii)



Let $\overrightarrow{AX}$ be $\text{proj}_{D.V.} \overrightarrow{AP}$

$\therefore\ \overrightarrow{PX} = \overrightarrow{AX} - \overrightarrow{AP}$

$= \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 9 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -5 \\ -5 \\ 0 \end{pmatrix}$

$\therefore\ \overrightarrow{PP'} = 2\overrightarrow{PX}$

$= \begin{pmatrix} -10 \\ -10 \\ 0 \end{pmatrix}$

$\therefore\ \overrightarrow{OP'} = \overrightarrow{OP} + \overrightarrow{PP'}$

$= \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -10 \\ -10 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -4 \\ -10 \\ 2 \end{pmatrix}$ $\therefore P'(-4, -10, 2)$

c) i) $(\cos\theta + i\sin\theta)^3 = c^3 + 3c^2si + 3cs^2i^2 + s^3i^3$

where $c = \cos\theta$ and $s = \sin\theta$

Using De Moivre's : LHS $= \cos 3\theta + i\sin 3\theta$

Equating Re: $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

Equating Im: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

ii) $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta}$

$= \dfrac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$ $\begin{matrix} \div\cos^3\theta \\ \div\cos^3\theta \end{matrix}$

$= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$= \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$

iii) $x = \tan\theta$: $x^3 - 3x^2 - 3x + 1 = 0$

$\tan^3\theta - 3\tan^2\theta - 3\tan\theta + 1 = 0$

$\tan^3\theta - 3\tan\theta = 3\tan^2\theta - 1$

$\dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1} = 1$

$\tan 3\theta = 1$ rel $\angle = \frac{\pi}{4}$

$3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$

$\therefore x = \tan\frac{\pi}{12}, \tan\frac{5\pi}{12}, \cancel{\tan\frac{3\pi}{4}}\ -1.$

iv) $\sum$ roots : $\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} - 1 = \frac{--3}{1}$

$\therefore \tan\frac{\pi}{12} + \tan\frac{5\pi}{12} = 4$

Q15 a) $\int_{-1}^{2} \sqrt{8-2x-x^2}\,dx$

$= \int_{-1}^{2} \sqrt{8-(x^2+2x+1-1)}\,dx$

$= \int_{-1}^{2} \sqrt{9-(x+1)^2}\,dx$ $\qquad x+1 = 3\sin\theta$ $\qquad x=2:\ 3=3\sin\theta,\ \theta=\frac{\pi}{2}$

$\qquad x = 3\sin\theta - 1$

$= \int_{0}^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta$ $\qquad dx = 3\cos\theta\,d\theta$ $\qquad x=-1:\ 0=3\sin\theta,\ \theta=0$

$= \int_0^{\frac{\pi}{2}} 9\cos^2\theta\,d\theta$ as $|\cos\theta| = \cos\theta$ for $\theta\in[0,\frac{\pi}{2}]$

$= 9\int_0^{\frac{\pi}{2}} \frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta$

$= \frac{9}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{9}{2}\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right) - \frac{9}{2}(0+0)$

$= \frac{9\pi}{4}$

b) Contrapositive: if n is even, then 3^n-4 is not prime.

Let $n=2m$, $m\in\mathbb{Z}^+$

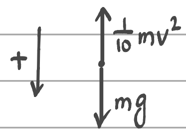
$3^{2m}-4 = (3^m)^2$

$= (3^m-2)(3^m+2)$

since $m\geq 2$, 3^m-2 and 3^m+2 are integers ≥ 2

$\therefore 3^n-4$ is not prime as it has more factors other than 1 and itself.

c) i)



$\therefore F_{net} = mg - \frac{1}{10}mv^2$

$ma = 10m - \frac{1}{10}mv^2$

$a = 10 - \frac{1}{10}v^2$

$= \frac{1}{10}(100-v^2)$

ii) $v\frac{dv}{dx} = \frac{1}{10}(100-v^2)$

$-\frac{1}{2}\int \frac{-2v}{100-v^2}\,dv = \frac{1}{10}\,dx$

$-\frac{1}{2}\ln|100-v^2| = \frac{x}{10}+c$

$x=0,\ v=0:$

$-\frac{1}{2}\ln 100 = c$

$-\frac{1}{2}\ln|100-v^2| = \frac{x}{10} - \frac{1}{2}\ln 100$

$-\frac{1}{2}\ln\left|\frac{100-v^2}{100}\right| = \frac{x}{10}$

$-5\ln\left|\frac{100-v^2}{100}\right| = x$

needs v... see right.

$\therefore$ at $v=\frac{20}{3}$

$x = -5\ln\left|\frac{100-(\frac{20}{3})^2}{100}\right|$

$= -5\ln\left(\frac{5}{9}\right)$

$= 5\ln\frac{9}{5}$ metres

$\frac{dv}{dt} = \frac{1}{10}(100-v^2)$

$\int \frac{1}{(10-v)(10+v)}\,dv = \int\frac{1}{10}\,dt$

$\frac{1}{20}\int -\frac{-1}{10-v}+\frac{1}{10+v}\,dv = \int\frac{1}{10}\,dt$

$\frac{1}{20}\left(-\ln|10-v|+\ln|10+v|\right) = \frac{t}{10}+c$

$\frac{1}{20}\ln\left|\frac{10+v}{10-v}\right| = \frac{t}{10}+c$

$\frac{A}{10-v}+\frac{B}{10+v} = \frac{1}{(10-v)(10+v)}$

$A(10+v)+B(10-v) = 1$

$v=10:\ 20A=1,\ A=\frac{1}{20}$

$v=-10:\ 20B=1,\ B=\frac{1}{20}$

$t=0,\ v=0$

$\frac{1}{20}\ln 1 = 0+c \quad \therefore c=0$

$\therefore \frac{1}{2}\ln\left|\frac{10+v}{10-v}\right| = t$

at $t=\ln(\sqrt{5})$:

$\frac{1}{2}\ln\left|\frac{10+v}{10-v}\right| = \ln\sqrt{5} = \frac{1}{2}\ln 5$

$\left|\frac{10+v}{10-v}\right| = 5$

as $\ddot{x}=0 \Rightarrow 100-v^2=0$

$v=10$ is terminal vel.

$0\leq v<10 \Rightarrow \left|\frac{10+v}{10-v}\right| = \frac{10+v}{10-v} = 5$

$10+v = 50-5v$

$6v = 40$

$v = \frac{20}{3}$

15d) $I_n = \int_0^1 x^{n-1} \times x(1+x^2)^{-\frac{1}{2}}\,dx$

$u = x^{n-1}$ $\quad$ $v = \sqrt{1+x^2}$

$u' = (n-1)x^{n-2}$ $\quad$ $v' = x(1+x^2)^{-\frac{1}{2}}$

$v = \frac{1}{2}\int 2x(1+x^2)^{-\frac{1}{2}}dx$

$= \frac{1}{2}\times 2(1+x^2)^{\frac{1}{2}}$

$= \left[x^{n-1}\sqrt{1+x^2}\right]_0^1 - \int_0^1 \sqrt{1+x^2}\,(n-1)x^{n-2}\,dx$

$= \sqrt{2} - (n-1)\int_0^1 \frac{x^{n-2}(1+x^2)}{\sqrt{1+x^2}}dx$

$I_n = \sqrt{2} - (n-1)\left[I_{n-2} + I_n\right]$

$I_n = \sqrt{2} - (n-1)I_n - (n-1)I_{n-2}$

$I_n(1+n-1) = \sqrt{2} - (n-1)I_{n-2}$

$I = \frac{\sqrt{2} - (n-1)I_{n-2}}{n}$ as required.

Q16 a) $\angle AOB = \frac{\pi}{3}$

$\Rightarrow \cos\frac{\pi}{3} = \frac{\vec{OA}\cdot\vec{OB}}{|\vec{OA}||\vec{OB}|}$

$= \frac{\begin{pmatrix}p\\q\\0\end{pmatrix}\cdot\begin{pmatrix}p\\0\\r\end{pmatrix}}{\sqrt{p^2+q^2}\sqrt{p^2+r^2}}$

$\frac{1}{2} = \frac{p^2}{\sqrt{p^2+q^2}\sqrt{p^2+r^2}}$ ...①

Similarly: $\frac{1}{2} = \frac{q^2}{\sqrt{q^2+p^2}\sqrt{q^2+r^2}}$ ...②

and $\frac{1}{2} = \frac{r^2}{\sqrt{r^2+p^2}\sqrt{r^2+q^2}}$ ..③

RTP (hopefully) $p=q=r$

① = ② : $\frac{p^2}{\cancel{\sqrt{p^2+q^2}}\sqrt{p^2+r^2}} = \frac{q^2}{\cancel{\sqrt{q^2+p^2}}\sqrt{q^2+r^2}}$

$p^2\sqrt{q^2+r^2} = q^2\sqrt{p^2+r^2}$

$p^4(q^2+r^2) = q^4(p^2+r^2)$

$p^4q^2 - q^4p^2 + p^4r^2 - q^4r^2 = 0$

$p^2q^2(p^2-q^2) + r^2(p^2-q^2)(p^2+q^2) = 0$

$(p^2-q^2)\left[p^2q^2 + r^2(p^2+q^2)\right] = 0$

since $[\ldots] > 0$ due to lengths,

$p^2 = q^2 \Rightarrow p = q.$

Similarly for $p=r$ and $q=r$.

$\therefore$ the prism is a cube.

16b) i) RTP: $2k+3 > 2\sqrt{(k+1)(k+2)}$

(Backtrack)

$4k^2+12k+9 \geq 4(k+1)(k+2)$

$= 4(k^2+3k+2)$

$= 4k^2+12k+8$

$LHS^2 = (2k+3)^2$
$= 4k^2+12k+9$

$RHS^2 = 4(k+1)(k+2)$
$= 4(k^2+3k+2)$
$= 4k^2+12k+8$

$\therefore LHS^2 \geq RHS^2$

$\therefore LHS \geq RHS$ as $LHS, RHS > 0$ for $k \geq 0$

ii) Proof by M.I.:

Test $P(1)$: $LHS = \frac{1}{\sqrt{1}} = 1$

$RHS = 2(\sqrt{2}-1)$

≈ 0.8

$\therefore LHS > RHS$ True for $n=1$

Assume $P(k)$: $1 + \frac{1}{\sqrt{2}} + .. + \frac{1}{\sqrt{k}} > 2(\sqrt{k+1}-1)$

RTP $P(k+1)$: $1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1+1}-1) = 2(\sqrt{k+2}-1)$

By assumption:

$$1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1}-1) + \frac{1}{\sqrt{k+1}}$$

$$= 2\sqrt{k+1} + \frac{1}{\sqrt{k+1}} - 2$$

$$= \frac{2(k+1)+1}{\sqrt{k+1}} - 2$$

$$= \frac{2k+3}{\sqrt{k+1}} - 2$$

$$> \frac{2\sqrt{(k+1)(k+2)}}{\sqrt{k+1}} - 2 \quad \text{from i)}$$

$$= 2\sqrt{k+2} - 2$$

$$= 2(\sqrt{k+2} - 1)$$

$\therefore$ Statement true by M.I.

iii) From ii) we know $\sum_{k=1}^{N} \frac{1}{\sqrt{k}} > 2(\sqrt{N+1}-1)$

So statement is <u>false</u> if we choose

$$2(\sqrt{N+1}-1) > 10^{10}$$

$$\sqrt{N+1} > \frac{10^{10}}{2} + 1$$

$$N > \left(\frac{10^{10}}{2}+1\right)^2 - 1$$

as a counter example.

16 c) i) RTP: $|z_1| = |z_2| = 1 \Rightarrow \frac{1}{z_1} + \frac{1}{z_2} + 1 = 0$

$|z|^2 = z\bar{z} = 1 \Rightarrow \frac{1}{z_1} = \bar{z}_1$ and $\frac{1}{z_2} = \bar{z}_2$

$\therefore$ LHS $= \frac{1}{z_1} + \frac{1}{z_2} + 1$

$= \bar{z}_1 + \bar{z}_2 + 1$ using

$= \overline{z_1 + z_2 + 1}$

$= 0$ using given.

ii) RTP: $\frac{1}{z_1} + \frac{1}{z_2} + 1 = 0 \Rightarrow |z_1| = |z_2| = 1$

$\frac{z_2 + z_1}{z_1 z_2} = -1$

$z_2 + z_1 = -z_1 z_2$

$\therefore -1 = -z_1(-1 - z_1)$ using $z_1 + z_2 + 1 = 0$

$-1 = +z_1 + z_1^2$

$0 = z_1^2 + z_1 + 1$

$\therefore z_1$ is a non-real cube root of unity.

as $0 = \frac{z_1^3 - 1}{z_1 - 1}$

$\therefore |z_1| = 1$ and similarly for $|z_2| = 1$

iii) RTP $|z_1| = |z_2| = 1 \Rightarrow z_1^2 + z_2^2 + 1 = 0$

from i) $|z_1| = |z_2| = 1 \Leftrightarrow \frac{1}{z_1} + \frac{1}{z_2} + 1 = 0$

LHS $= z_1^2 + z_2^2 + 1$

$= z_1^2 + (-1 - z_1)^2 + 1$

$= z_1^2 + 1 + 2z_1 + z_1^2 + 1$

$= 2z_1^2 + 2z_1 + 2$

$= 2(z_1^2 + z_1 + 1)$

But $\frac{1}{z_1} + \frac{1}{z_2} + 1 = 0 \Rightarrow z_1^2 + z_1 + 1 = 0$ shown in ii)

$\therefore$ LHS $= 2 \times 0$

$= 0$ as required.