Student Number

Barker College

2021 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Section I – Multiple Choice

Write your student number at the top of this page.

Select the alternative A, B C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = A 2 B 6 C 8 D 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

correct

A ● B ~~●~~ C ○ D ○

Start Here →

1.	A ○	B ○	C ○	D ○	6.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○	7.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○	8.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○	9.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○	10.	A ○	B ○	C ○	D ○

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Barker College

Student Number

2021 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Staff Involved:

BHC ARM*

40 copies

8:30 AM
FRIDAY 20 AUGUST

TOPICS COVERED
- Complex Numbers I & II
- Proof
- Integration
- Vectors
- Mechanics

General Instructions:
- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale unless specifically stated

Total marks: 100

Section I – 10 marks (pages 4-8)
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 9-14)
- Attempt Questions 11-16
- Allow about 2 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 What is the negation of the following statement?

If today is Friday, then I have a maths exam.

A. If I have a maths exam, then today is Friday.

B. Today is not Friday and I do not have a maths exam.

C. If I do not have a maths exam, then today is not Friday.

D. Today is Friday and I do not have a maths exam.

2 Given the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, what is the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$?

A. $-\dfrac{14}{3}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$

B. $-\dfrac{14}{9}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$

C. $-\dfrac{14}{13}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)$

D. $-\dfrac{14}{169}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)$

3 A particle has a displacement of $3\underset{\sim}{i} + \underset{\sim}{j}$ metres at a particular time. The particle moves with constant velocity and two seconds later its displacement is $-\underset{\sim}{i} + 5\underset{\sim}{j}$ metres.
What is the velocity of the particle in m.s^{-1}?

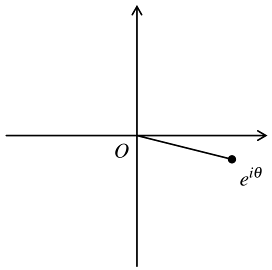
A. $2\underset{\sim}{i} + 6\underset{\sim}{j}$

B. $-4\underset{\sim}{i} + 4\underset{\sim}{j}$

C. $4\underset{\sim}{i} - 4\underset{\sim}{j}$

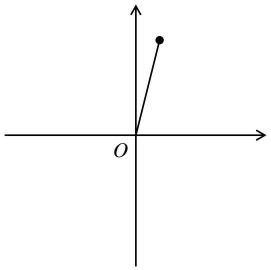
D. $-2\underset{\sim}{i} + 2\underset{\sim}{j}$

4 The Argand diagram shows the complex number $e^{i\theta}$.

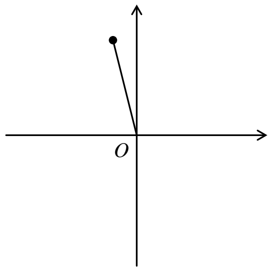


Which of the following diagrams best shows the complex number $ie^{-i\theta}$?

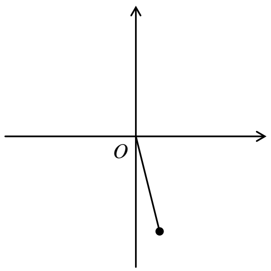
A.



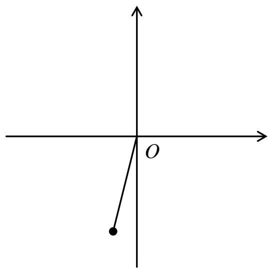
B.



C.



D.



5 Given that a and b are integers, which one of the following statements has a converse that is always true?

A. If a and b are even, then $a + b$ is even

B. If a is divisible by 8, then a is even

C. If a and b are odd, then $a \times b$ is odd

D. If a and b are odd, then $a - b$ is even

6 Which complex number is a cube root of $-i$?

A. $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

B. $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

C. $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

D. $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

7 How many complex numbers satisfy $|z - 2 - i| = 3$ and $|z - 6 + 2i| = 2$ simultaneously?

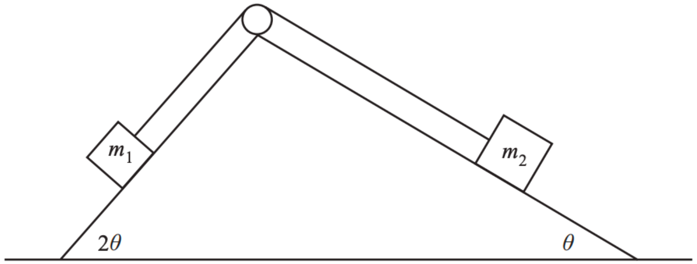
A. 0

B. 1

C. 2

D. Infinitely many

8 Two particles with mass m_1 kg and m_2 kg are connected by a taut light string that passes over a smooth pulley. The particles sit on smooth inclined planes as shown in the diagram below.



If the system is in equilibrium, what is the value of $\frac{m_1}{m_2}$?

A. $\frac{\sec\theta}{2}$

B. $2\sec\theta$

C. $2\cos\theta$

D. $\frac{1}{2}$

9 The integral $\int \frac{dx}{2\sin x + 3\cos x}$ is equivalent to which one of the following expressions?

A. $-2\int \frac{dt}{3t^2 - 4t - 3}$

B. $-\int \frac{dt}{3t^2 - 4t - 3}$

C. $-2\int \frac{dt}{3t^2 - 2t - 3}$

D. $-\int \frac{dt}{3t^2 - 2t - 3}$

10 Given that $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx > 0$, which one of the following must be true?

A. $\forall x \in [a,b]\,(f(x) > g(x))$

B. $\forall x \in [a,b]\,(g(x) > 0)$

C. $\exists x \in [a,b]\,(f(x) > g(x))$

D. $\exists x \in [a,b]\,(g(x) > f(x))$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE Writing Booklet

(a) Consider the complex numbers $w = 1 - 3i$ and $z = -2 + 5i$.

(i) Evaluate $z\bar{z}$. **1**

(ii) Express $\dfrac{w}{z}$ in the form $x + iy$ where x and y are real numbers. **1**

(b) Evaluate $\displaystyle\int_1^e \frac{1}{x}\cos(\ln x)\,dx$. **2**

Give your answer in simplest exact form.

(c) Find a vector perpendicular to both $3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $4\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}$. **2**

(d) (i) Find the square root of $5 - 12i$. **2**

(ii) Hence, solve $z^2 + 3z + 1 + 3i = 0$. **2**

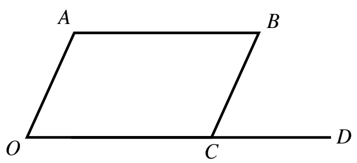
(e) Find $\displaystyle\int \frac{dx}{\sqrt{3 - 2x - x^2}}$. **2**

(f) $OABC$ is a parallelogram. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. **3**

X is the midpoint of AC.

D lies on OC produced such that $\dfrac{OC}{CD} = k$.

If $\overrightarrow{XD} = 3\underset{\sim}{c} - \dfrac{1}{2}\underset{\sim}{a}$, find the value of k.



Question 12 (15 marks) Use a SEPARATE Writing Booklet

(a) Find the equation, in vector form, of the tangent to the circle $|\underset{\sim}{v}| = 5$ at the point $(-3, 4)$. **2**

(b) Using the substitution $x = 4\sin^2\theta$, evaluate $\displaystyle\int_0^2 \sqrt{\frac{x}{4-x}}\,dx$. **3**

Give your answer in simplest exact form.

(c) For positive real numbers a and b, prove that $\dfrac{1}{a} + \dfrac{1}{b} \geq \dfrac{4}{a+b}$. **2**

(d) Consider the following statement, P, for integers x and y:

P: If $x^2 + y^2$ is divisible by 4 then x and y are not both odd.

(i) State the contrapositive of P. **1**

(ii) Hence, prove P by using the contrapositive. **2**

(iii) Without providing a proof, state the conditions on x and y for $x^2 + y^2$ to be divisible by 4. **1**

(e) Use a suitable substitution to evaluate $\displaystyle\int_1^9 \frac{dx}{x(2\sqrt{x} - 1)}$. **4**

Give your answer in simplest exact form.

Question 13 (14 marks) Use a SEPARATE Writing Booklet

(a) On an Argand diagram, the point P representing the complex number z moves such that $|z-(1+i)|=1$.

(i) Sketch the locus of P. **1**

(ii) Find the greatest value of $|z|$. **1**

(iii) Shade the region where $|z-(1+i)|\leq 1$ and $0\leq \operatorname{Arg}(z-1)\leq \frac{\pi}{4}$. **2**

(iv) Find the area of the region in (iii). **1**

(b) A ball of mass m kg is thrown vertically downwards off a very high cliff with speed 20 m.s^{-1}. It experiences acceleration due to gravity of 10 m.s^{-2} and a force due to air resistance of $\frac{1}{100}mv^2$ where v is the velocity of the ball.

(i) Find the terminal velocity that the ball would reach in m.s^{-1}. **1**

(ii) Find the distance through which the ball falls to reach a speed of 30 m.s^{-1}. Give your answer in metres in simplest exact form. **3**

(c) A particle moves in a straight line. Its displacement, x metres, after t seconds is given by

$$x=2\cos\left(2t-\frac{2\pi}{3}\right)+3.$$

(i) Prove the particle is moving in simple harmonic motion about $x = 3$ by showing that $\ddot{x} = -4(x-3)$. **2**

(ii) What is the period of the motion? **1**

(iii) Find all times within the first π seconds when the particle is moving at 2 metres per second in either direction. **2**

Question 14 (15 marks) Use a SEPARATE Writing Booklet

(a) A particle of mass m kg is projected vertically upwards with a velocity of $\sqrt{\frac{g}{k}}$ m.s^{-1} where g is the magnitude of the acceleration due to gravity. It experiences air resistance of magnitude mkv^2. Show that the time taken to reach its maximum height is $\frac{\pi}{4\sqrt{gk}}$. **4**

(b) The point A has a position vector $\begin{pmatrix}-2\\4\\7\end{pmatrix}$ and the point B has position vector $\begin{pmatrix}-1\\3\\8\end{pmatrix}$.

(i) Find a vector equation for ℓ_1, the line passing through A and B. **2**

(ii) The point P has position vector $\begin{pmatrix}0\\2\\3\end{pmatrix}$. **2**

If $\angle PBA=\theta$, show that $\cos\theta=\frac{1}{3}$.

(iii) The line ℓ_2 passes through the point P and is parallel to the line ℓ_1. Find a vector equation for ℓ_2. **1**

(iv) The points C and D both lie on ℓ_2. Given that $AB = PC = DP$ and that the x-coordinate of C is positive, find the coordinates of C and D. **3**

(v) Find the exact area of trapezium $ABCD$, giving your answer as a simplified surd. **3**

Question 15 (17 marks) Use a SEPARATE Writing Booklet

(a) A particle is moving such that $\ddot{x} = 18x^3 + 27x^2 + 9x$.
Initially, $x = -2$ and the velocity, $v = -6$.

(i) Show that $v^2 = 9x^2(1 + x)^2$. **2**

(ii) Hence, or otherwise, show that $\int \frac{1}{x(1+x)} dx = -3t$. **2**

(iii) Hence, find x as a function of t. **4**

(b) Find $\int \left(1+2x^2\right) e^{x^2}\, dx$. **3**

(c) (i) Use de Moivre's Theorem to show that **3**

$$\tan 7\theta = \frac{7t - 35t^3 + 21t^5 - t^7}{1 - 21t^2 + 35t^4 - 7t^6} \quad \text{where } t = \tan\theta$$

(ii) Hence, show that the roots of the equation $x^3 - 21x^2 + 35x - 7 = 0$ are **3**

$$\tan^2\left(\frac{\pi}{7}\right),\ \tan^2\left(\frac{2\pi}{7}\right) \text{ and } \tan^2\left(\frac{3\pi}{7}\right)$$

Question 16 (14 marks) Use a SEPARATE Writing Booklet

(a) (i) For a function $g(x)$, continuous on $[a, b]$, prove that **2**

$$\int_a^b g(x)\,dx = \int_a^b g(a+b-x)\,dx$$

(ii) Hence, or otherwise, for an **even** function $f(x)$, continuous on $[-a, a]$, show that **3**

$$\int_{-a}^{a} \frac{f(x)}{1+e^x} dx = \int_0^a f(x)\,dx$$

(b) For integers $n \geq 0$, let $I_n(x) = \int_0^x t^n e^{-t} dt$.

(i) Prove by induction that **4**

$$I_n(x) = n!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)\right]$$

(ii) Show that **2**

$$0 \leq \int_0^1 t^n e^{-t} dt \leq \frac{1}{n+1}$$

(iii) Hence, show that **1**

$$0 \leq 1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) \leq \frac{1}{(n+1)!}$$

(iv) Hence, find the limiting value of $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ as $n \to \infty$. **2**

Show working to justify your answer.

End of paper

Year 12 Extension 2 Trial Exam

Q1, "If today is Friday, then I have a maths exam" is of the form

$$F \Rightarrow E$$

which is logically equivalent to

$$\sim F \vee E.$$

The negation of $F \Rightarrow E$ is

$$\sim(\sim F \vee E)$$

which is equivalent to

$$F \wedge \sim E,$$

namely,

"It is Friday and I don't have a maths exam."

(d)

Q2, Now $\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a}$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \times \underset{\sim}{b}$$

$$= \frac{6-8-12}{(\sqrt{9})^2} \times (2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$$

$$= -\frac{14}{9}(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$$

(b)

Q3, The displacement in 2 sec is

$$(\underset{\sim}{i} + 5\underset{\sim}{j}) - (3\underset{\sim}{i} + \underset{\sim}{j}) = -4\underset{\sim}{i} + 4\underset{\sim}{j}$$

Hence the velocity in m/s is

$$\frac{-4\underset{\sim}{i} + 4\underset{\sim}{j}}{2} = -2\underset{\sim}{i} + 2\underset{\sim}{j}$$

(d)

Q4, Let $\theta = -\frac{\pi}{6}$ such that

$$e^{i\theta} = e^{i(-\pi/6)}. \text{ Then}$$

$$ie^{-i\theta} = ie^{i\pi/6}$$

$$= i\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

(b)

Q5, The converse of $p \Rightarrow q$ is

$$q \Rightarrow p.$$

The converse of (c) is

"If $a \times b$ is odd, then a and b are odd."

This is always true.

Q6, $z^3 = -i$

$$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

$$\therefore z = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$$

$$z = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

(a)

Q7, $|z - (2+i)| = 3$

is a circle centre $(2,1)$ radius 3 units.

$|z - (6-2i)| = 2$

is a circle centre $(6,-2)$ Radius 2 units.

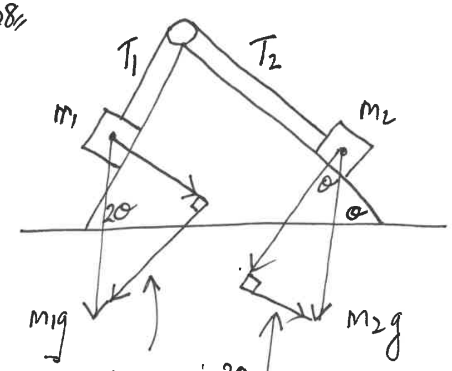
Consider the distance between the two centres:

$(2,1)$ $\quad d = \sqrt{16+9} = 5$ $\quad (6,-2)$

But $3 + 2 = 5$

$\Rightarrow$ The two circles touch at one point (b)

Q8,



$T_1 = m_1 g \sin 2\theta$

$T_2 = m_2 g \sin\theta$

$$T_1 = T_2$$

$$\therefore m_1 g \sin 2\theta = m_2 g \sin\theta$$

$$\therefore \frac{m_1}{m_2} = \frac{\sin\theta}{\sin 2\theta} = \frac{1}{2\cos\theta}$$

(a)

Q9, $\displaystyle\int \frac{dx}{2\sin x + 3\cos x}\, dx$

Let $t = \tan\left(\frac{x}{2}\right)$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$$

$$dx = \frac{2\,dt}{1+t^2}$$

$$I = \int \frac{2\,dt/_{1+t^2}}{2\left(\frac{2t}{1+t^2}\right) + 3\left(\frac{1-t^2}{1+t^2}\right)}$$

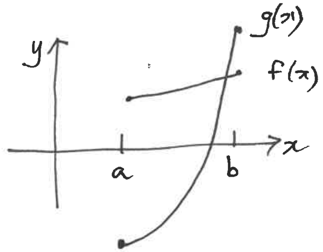
$$= \int \frac{2\,dt}{4t + 3 - 3t^2}$$

$$= \int \frac{-2\,dt}{(3t^2 - 4t - 3)}$$

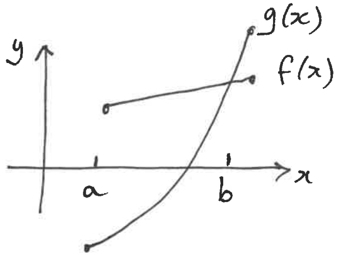
(a)

Q10/ Given that $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx > 0$

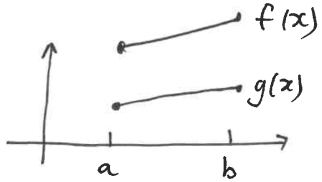
ⓐ is false in this case :



ⓑ is false in this case :



ⓓ is false in this case :



Therefore ⓒ

because if there was not one single x-value for which $f(x) > g(x)$ then $f(x)$ lies entirely underneath $g(x)$, which would make it impossible that

$$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx > 0.$$

Q11/

(a) $w = 1-3i$

$z = -2+5i$

(i) $z\bar{z} = (-2+5i)(-2-5i)$

$= 29$

(ii) $\frac{w}{z} = \frac{1-3i}{-2+5i} \times \frac{-2-5i}{-2-5i}$

$= \frac{-2-5i+6i-15}{29}$

$= \frac{-17}{29} + \frac{1}{29}i$

(b) $\int_1^e \frac{1}{x}\cos(\ln x)\,dx$

$= \sin(\ln x)\Big]_1^e$

$= \sin(1)$

≈ 0.8415

(c) Let the required vector be expressed as $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. We need

$3a - 2b + c = 0$ and

$4a + 2b - 5c = 0$

$7a \quad - 4c = 0$

$a = \frac{4c}{7}$

If $c=1$, $a = \frac{4}{7}$ and $b = \frac{19}{14}$

Hence $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4/7 \\ 19/14 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 19 \\ 14 \end{pmatrix}$

(d) (i)

Let $(x+iy)^2 = 5-12i$

$x^2 - y^2 = 5 \qquad 2xyi = -12i$

$y = \frac{-6}{x}$

$x^2 - \left(\frac{-6}{x}\right)^2 = 5$

$x^4 - 5x^2 - 36 = 0$

$(x^2-9)(x^2+4) = 0$

$x = \pm 3$

when $x = 3$, $y = -2$

when $x = -3$, $y = 2$

Hence $\pm(3-2i)$

(ii) $z^2 + 3z + (1+3i) = 0$

$z = \left[-3 \pm \sqrt{5-12i}\right]/2$

$z = \left[-3 \pm (3-2i)\right]/2$

$z = -i, \quad -3+i$

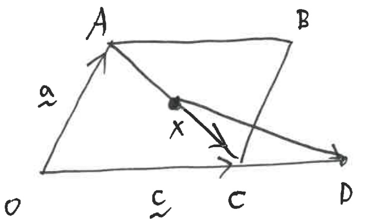
(e) $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$

$\displaystyle= \int \frac{dx}{\sqrt{3-(x^2+2x)}}$

$\displaystyle= \int \frac{dx}{\sqrt{4-(x+1)^2}}$

$\displaystyle= \sin^{-1}\left(\frac{x+1}{2}\right)+c$

(f)



$\dfrac{OC}{CD} = k \Rightarrow CD = \dfrac{\underset{\sim}{c}}{k}$

$\overrightarrow{XD} = 3\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}$

Now $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

$\overrightarrow{OX} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$

$= \underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{a})$

$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c})$

$\overrightarrow{OD} = \overrightarrow{OX} + \overrightarrow{XD}$

$= \dfrac{\underset{\sim}{a}}{2} + \dfrac{\underset{\sim}{c}}{2} + 3\underset{\sim}{c} - \dfrac{\underset{\sim}{a}}{2}$

$= \dfrac{7\underset{\sim}{c}}{2}$

But $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$= \underset{\sim}{c} + \dfrac{\underset{\sim}{c}}{k}$

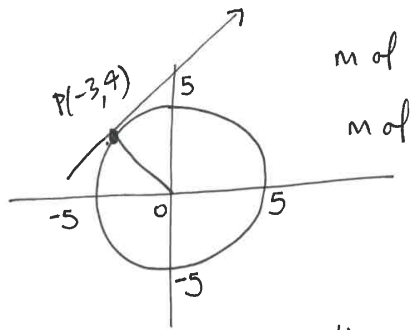
$\therefore \dfrac{7\underset{\sim}{c}}{2} = \underset{\sim}{c} + \dfrac{\underset{\sim}{c}}{k}$

$\dfrac{5\underset{\sim}{c}}{2} = \dfrac{\underset{\sim}{c}}{k}$

$\therefore k = \dfrac{2}{5}$

Q12

(a)



m of OP $= -\frac{4}{3}$

m of tangent $= \frac{3}{4}$

The tangent has a vector equation of the form

$\underset{\sim}{R} = (-3\underset{\sim}{i} + 4\underset{\sim}{j}) + k(4\underset{\sim}{i} + 3\underset{\sim}{j})$

(b) $\displaystyle\int_0^2 \sqrt{\frac{x}{4-x}}\,dx$

Let $x = 4\sin^2\theta$

$dx = 8\sin\theta\cos\theta\,d\theta$

when $x=2$, $\theta = \frac{\pi}{4}$

when $x=0$, $\theta = 0$

$\displaystyle I = \int_0^{\frac{\pi}{4}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}} \times 8\sin\theta\cos\theta\,d\theta$

$\displaystyle= 8\int_0^{\pi/4} \sin^2\theta\,d\theta$

$\displaystyle= 4\int_0^{\pi/4} 1-\cos 2\theta\,d\theta$

$= 4\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$

$= \pi - 2$

(c) Prove that $\frac{1}{a} + \frac{1}{b} \geqslant \frac{4}{a+b}$

Consider LHS - RHS

$$= \frac{1}{a} + \frac{1}{b} - \frac{4}{a+b}$$

$$= \frac{b(a+b) + a(a+b) - 4ab}{ab(a+b)}$$

$$= \frac{a^2 - 2ab + b^2}{ab(a+b)}$$

$$= \frac{(a-b)^2}{ab(a+b)}$$

Now the numerator is always $\geqslant 0$ and the denominator is always > 0.

Hence LHS - RHS $\geqslant 0$.

Therefore

$$\frac{1}{a} + \frac{1}{b} \geqslant \frac{4}{a+b}.$$

(d) P: If x^2+y^2 is divisible by 4 then x and y are not both odd.

(i) The contrapositive of P is:

"If x and y are both odd then x^2+y^2 is not divisible by 4."

(ii) Let $x = 2k+1$

$y = 2j+1$

then $x^2+y^2 = (2k+1)^2 + (2j+1)^2$

$= 4k^2 + 4k + 1 + 4j^2 + 4j + 1$

$= 4(k^2 + k + j^2 + j) + 2$

which is not divisible by 4.

Hence, if x and y are both odd then x^2+y^2 is not divisible by 4.

Therefore, if x^2+y^2 is divisible by 4, then x and y are not both odd.

(iii) For x^2+y^2 to be divisible by 4, the condition on x and y is that they are both even.

(e) $\displaystyle\int_1^9 \frac{dx}{x(2\sqrt{x}-1)}$

Let $x = u^2$

$dx = 2u\,du$

when $x=9$, $u=3$

when $x=1$, $u=1$

$$I = \int_1^3 \frac{2u\,du}{u^2(2u-1)}$$

$$= \int_1^3 \frac{2\,du}{u(2u-1)}$$

Let $\dfrac{2}{u(2u-1)} = \dfrac{A}{u} + \dfrac{B}{2u-1}$

$2 = A(2u-1) + Bu$

$A = -2$, $B = 4$

$$I = \int_1^3 \frac{-2}{u} + \frac{4}{2u-1}\,du$$

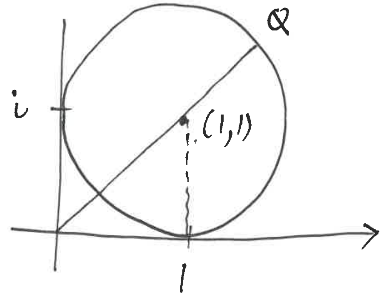
$$= \Big[-2\ln u + 2\ln(2u-1)\Big]_1^3$$

$$= -2\ln 3 + 2\ln 5 - 0$$

$$= \ln\left(\frac{25}{9}\right)$$

Q13. (a) $|z-(1+i)| = 1$

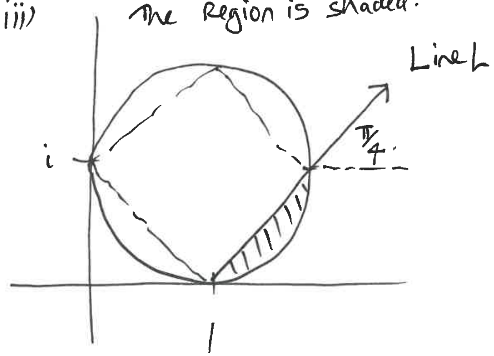
(i)



(ii) The greatest value of $|z|$ occurs when P is at Q.

The greatest value of $|z|$ is $1+\sqrt{2}$

(iii) The Region is shaded.



(iv) Line L has equation $y = x-1$.

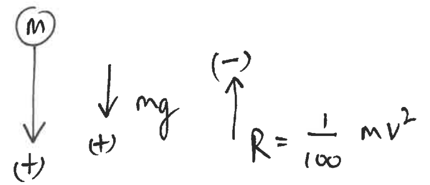
L intersects the circle at $(2,1)$.

Shaded area

$= \frac{1}{4}\left[\pi R^2 - (\sqrt{2})^2\right]$

$= \frac{1}{4}(\pi - 2)$

(b)



$V = 20\ m/s$

(i) $m\ddot{x} = 10m - \frac{1}{100}mv^2$

$\ddot{x} = 10 - \frac{1}{100}v^2$

Put $\ddot{x} = 0$,

$0 = 10 - \frac{1}{100}v^2$

$v^2 = 1000$

$v = 10\sqrt{10}$ m/s.

Terminal velocity is $10\sqrt{10}$ m/s.

(ii) $v\frac{dv}{dx} = 10 - \frac{1}{100}v^2$

$$\frac{dv}{dx} = \frac{10 - \frac{1}{100}v^2}{v}$$

$$\frac{dv}{dx} = \frac{1000 - v^2}{100v}$$

$$\frac{dx}{dv} = \frac{100v}{1000 - v^2}$$

$$x = -50\int \frac{-2v}{1000 - v^2}\,dv$$

$$x = -50\ln(1000 - v^2) + C$$

But when $x = 0$, $v = 20$ so

$C = 50\ln 600$

$\therefore\ x = 50\ln\left(\frac{600}{1000 - v^2}\right)$

When $v = 30$ m/s,

$x = 50\ln\left(\frac{600}{100}\right)$

$x = 50\ln(6)$

(c) $x = 2\cos\left(2t - \frac{2\pi}{3}\right) + 3$

(i) $\dot{x} = -4\sin\left(2t - \frac{2\pi}{3}\right)$

$\ddot{x} = -8\cos\left(2t - \frac{2\pi}{3}\right)$

$\ddot{x} = -4\left(2\cos\left(2t - \frac{2\pi}{3}\right)\right)$

$\ddot{x} = -4(x - 3)$

Hence, the particle is moving in SHM about $x = 3$.

(ii) The period is $\frac{2\pi}{2} = \pi$ seconds.

(iii) To find all the times in the first π seconds when the particle is travelling at 2 m/s left or right,

solve $2 = -4\sin\left(2t - \frac{2\pi}{3}\right)$ and $-2 = -4\sin\left(2t - \frac{2\pi}{3}\right)$

$-\frac{1}{2} = \sin\left(2t - \frac{2\pi}{3}\right)$ $\qquad$ $\frac{1}{2} = \sin\left(2t - \frac{2\pi}{3}\right)$

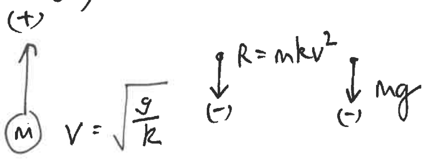
$\therefore\ 2t - \frac{2\pi}{3} = \left(\pi + \frac{\pi}{6}\right), \frac{11\pi}{6}, -\frac{\pi}{6}$ $\qquad$ $2t - \frac{2\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}$

$2t = \frac{11\pi}{6}, \frac{3\pi}{6}$ $\qquad$ $2t = \frac{5\pi}{6}, \frac{9\pi}{6}$

$t = \frac{11\pi}{12}, \frac{\pi}{4}$ $\qquad$ $t = \frac{5\pi}{12}, \frac{3\pi}{4}$

$\therefore\ t = \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{11\pi}{12}$

Q14 (a)



$$m\ddot{x} = -mg - mkv^2$$
$$\ddot{x} = -g - kv^2$$
$$\frac{dv}{dt} = -g - kv^2$$
$$\frac{dt}{dv} = \frac{-1}{g+kv^2}$$
$$t = \frac{-1}{k}\int \frac{dv}{\left(\frac{g}{k}\right)+v^2}$$
$$t = -\frac{1}{k}\sqrt{\frac{k}{g}}\tan^{-1}\left(\frac{v}{\sqrt{g/k}}\right) + c$$

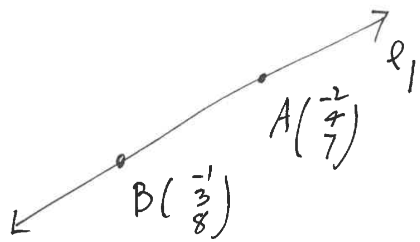
But when $t=0$, $v = \sqrt{g/k}$ so

$$c = \frac{1}{k}\sqrt{\frac{k}{g}}\tan^{-1}(1)$$
$$c = \frac{1}{\sqrt{gk}} \times \frac{\pi}{4}$$
$$\therefore\ t = \frac{\pi}{4\sqrt{gk}} - \frac{1}{\sqrt{gk}}\tan^{-1}\left(\frac{v}{\sqrt{g/k}}\right)$$

when $v=0$,

$$t = \frac{\pi}{4\sqrt{gk}}$$ as required.

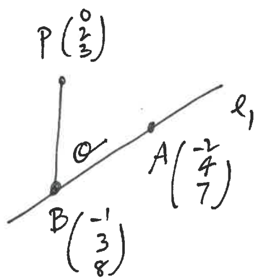
(b) (i)



ℓ_1 is $\vec{OA} + \vec{AB}$

$$= \begin{pmatrix}-2\\4\\7\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

(ii)



Now $\vec{BP} = \begin{pmatrix}1\\-1\\-5\end{pmatrix}$

and $\vec{BA} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$

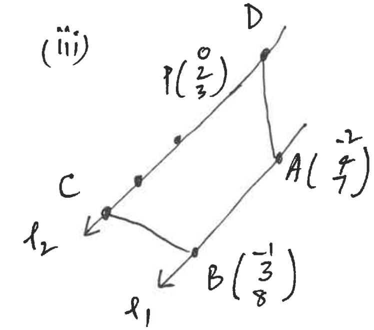
$\therefore\ \vec{BP}\cdot\vec{BA} = -1-1+5 = 3$

$\therefore\ 3 = \sqrt{27} \times \sqrt{3} \times \cos\theta$

$\therefore\ \cos\theta = \frac{3}{3\sqrt{3}\times\sqrt{3}}$

$\cos\theta = \frac{1}{3}$ as required.

(iii)



A vector equation for ℓ_2 is

$$= \begin{pmatrix}0\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

(iv) Now $AB = \sqrt{3}$ so

$C = \begin{pmatrix}1\\1\\4\end{pmatrix}$ and

$D = \begin{pmatrix}-1\\3\\2\end{pmatrix}$

(v) $|CD| = \sqrt{12}$

$|AB| = \sqrt{3}$

$h = \sqrt{24}$

Area $= 9\sqrt{2}$

Q15 (a)

$\ddot{x} = 18x^3 + 27x^2 + 9x$

when $t=0$, $x=-2$ and $v=-6$

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 18x^3 + 27x^2 + 9x$

$\frac{1}{2}v^2 = \frac{9}{2}x^4 + 9x^3 + \frac{9}{2}x^2 + c$

When $x=-2$, $v=-6$ so

$18 = 72 - 72 + 18 + c$

$c = 0$

$\therefore v^2 = 9x^4 + 18x^3 + 9x^2$

$v^2 = 9x^2(x^2 + 2x + 1)$

$v^2 = 9x^2(x+1)^2$ as required.

(ii) From $v^2 = 9x^2(x+1)^2$

$v = \pm 3x(x+1)$

But when $x=-2$, $v=-6$ so

$-6 = -3(-2)(-1)$

$\therefore v = -3x(x+1)$

$\therefore \frac{dx}{dt} = -3x(x+1)$

$\frac{dt}{dx} = \frac{1}{-3x(x+1)}$

$\int -3\,dt = \int \frac{dx}{x(x+1)}$

$\therefore \int \frac{dx}{x(x+1)} = -3t$

as required.

(iii)

$\frac{dx}{dt} = -3x(x+1)$

$t = \int \frac{-1}{3x(x+1)}\,dx$

$t = -\frac{1}{3}\int \frac{1}{x} - \frac{1}{x+1}\,dx$

$t = -\frac{1}{3}[\ln x - \ln(x+1)] + c$

But when $t=0$, $x=-2$ so

$0 = -\frac{1}{3}\ln 2 + c$

$c = \frac{1}{3}\ln 2$

$\therefore t = -\frac{1}{3}\ln\left(\frac{x}{x+1}\right) + \frac{1}{3}\ln 2$

$t = +\frac{1}{3}\ln\left(\frac{2(x+1)}{x}\right)$

$e^{3t} = \frac{2x+2}{x}$

$xe^{3t} = 2x+2$

$-2 = x(2-e^{3t})$

$\therefore x = \frac{-2}{2-e^{3t}}$

(b) $\int (1+2x^2)e^{x^2}\,dx$

$= \int e^{x^2} + 2x^2e^{x^2}\,dx$

$= \int e^{x^2}\,dx + \int x(2xe^{x^2})\,dx$

$= \int e^{x^2}\,dx + \left[e^{x^2}.x - \int e^{x^2}.(1)\,dx\right]$

$= \int e^{x^2}\,dx + xe^{x^2} - \int e^{x^2}\,dx$

$= xe^{x^2}$

(c) (i)

$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$ by de Moivre's Theorem

Also $(\cos\theta + i\sin\theta) = \cos^7\theta + 7\cos^6\theta(i\sin\theta)^1 + 21\cos^5\theta(i\sin\theta)^2 + 35\cos^4\theta(i\sin\theta)^3 + 35\cos^3\theta(i\sin\theta)^4 + 21\cos^2\theta(i\sin\theta)^5 + 7\cos\theta(i\sin\theta)^6 + (i\sin\theta)^7$

by Binomial Theorem

$= \{\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta\} +$

$i\{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \cancel{7}\sin^7\theta\}$

$$\tan 7\theta = \frac{\sin 7\theta}{\cos 7\theta}$$

$$= \frac{7\cos^6\theta \sin\theta - 35\cos^4\theta \sin^3\theta + 21\cos^2\theta \sin^5\theta - \sin^7\theta}{\cos^7\theta - 21\cos^5\theta \sin^2\theta + 35\cos^3\theta \sin^4\theta - 7\cos\theta \sin^6\theta}$$

Divide numerator and denominator by $\cos^7\theta$ to get

$$= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

$$= \frac{7t - 35t^3 + 21t^5 - t^7}{1 - 21t^2 + 35t^4 - 7t^6} \quad \text{if } t = \tan\theta.$$

(ii) $\tan 7\theta = 0$ if

$$7t - 35t^3 + 21t^5 - t^7 = 0$$

$$-t(t^6 - 21t^4 + 35t^2 - 7) = 0$$

$$\therefore t = 0 \text{ or } t^6 - 21t^4 + 35t^2 - 7 = 0 \quad \text{— (1)}$$

Let $t^2 = x$, then (1) becomes

$$x^3 - 21x^2 + 35x - 7 = 0 \quad \text{— (2)}$$

The solutions to (2) lie among the solutions to

$$\tan 7\theta = 0$$

$$7\theta = 0, \pi, 2\pi, 3\pi, \ldots 6\pi$$

$$\theta = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \ldots \frac{6\pi}{7}$$

We let $x = t^2$, so $x = \tan^2\theta$

$\therefore$ The solutions to

$x^3 - 21x^2 + 35x - 7 = 0$ are

$x = \tan^2\left(\frac{\pi}{7}\right)$, $\tan^2\left(\frac{2\pi}{7}\right)$, and $\tan^2\left(\frac{3\pi}{7}\right)$.

Q16, (a) (i)

To prove that $\int_a^b g(x)\,dx = \int_a^b g(a+b-x)\,dx$,

consider $I = \int_a^b g(a+b-x)\,dx$.

Let $u = a+b-x$

$\therefore du = -dx$

when $x = b$, $u = a$

when $x = a$, $u = b$

$$\therefore I = \int_b^a g(u)\,(-du)$$

$$= \int_a^b g(u)\,du$$

$$= \int_a^b g(x)\,dx$$

(u is a dummy variable)

$$\therefore \int_a^b g(x)\,dx = \int_a^b g(a+b-x)\,dx$$

as required.

(b) (ii)

To show that $\int_{-a}^{a} \frac{f(x)}{1+e^x}\,dx = \int_0^a f(x)\,dx$.

where $f(x)$ is an even function.

Now $\int_{-a}^{a} \frac{f(x)}{1+e^x}\,dx$

$$= \int_{-a}^{a} \frac{f(a+(-a)-x)}{1+e^{(a-a-x)}}\,dx$$

$$= \int_{-a}^{a} \frac{f(-x)}{1+e^{-x}}\,dx$$

$$= \int_{-a}^{a} \frac{e^x f(-x)}{1+e^x}\,dx$$

$$= \int_{-a}^{a} \frac{(1+e^x)f(-x)}{1+e^x} - \frac{f(-x)}{1+e^x}\,dx$$

$$= \int_{-a}^{a} f(-x) - \frac{f(-x)}{1+e^x}\,dx$$

$$= \int_{-a}^{a} f(x) - \frac{f(x)}{1+e^x}\,dx, \quad \text{since } f(x) = f(-x)$$

$$= \int_{-a}^{a} f(x)\,dx - \int_{-a}^{a} \frac{f(x)}{1+e^x}\,dx$$

$$\therefore 2\int_{-a}^{a} \frac{f(x)}{1+e^x}\,dx = \int_{-a}^{a} f(x)\,dx$$

$$\therefore 2\int_{-a}^{a} \frac{f(x)}{1+e^x}\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\therefore \int_{-a}^{a} \frac{f(x)}{1+e^x}\,dx = \int_{0}^{a} f(x)\,dx \text{, as required.}$$

(b) Let $I_n = \int_0^x t^n e^{-t}\,dt$, $n \geq 0$ (integers)

(i) To prove that $I_n = n!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}\right)\right]$

when $n=0$, $I_n = \int_0^x t^0 e^{-t}\,dt$

$$= \int_0^x e^{-t}\,dt$$

$$= \left[-e^{-t}\right]_0^x$$

$$= -e^{-x} - (-e^0) = 1 - e^{-x}$$

Same!

and also $I_n = 0!\left[1 - e^{-x}(1)\right]$

$$= 1(1 - e^{-x}) = 1 - e^{-x}$$

$\therefore$ The statement is true when $n = 0$.

Assume that the statement is true when $n = k$, i.e. assume

$$\int_0^x t^k e^{-t}\,dt = k!\left[1 - e^{-x}\left(1 + x + \dots + \frac{x^k}{k!}\right)\right]$$

and then try to prove that

$$\int_0^x t^{k+1} e^{-t}\,dt = (k+1)!\left[1 - e^{-x}\left(1 + x + \dots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}\right)\right]$$

Now $\int_0^x t^{k+1} e^{-t}\, dt$

$= \left[-e^{-t}.\, t^{k+1}\right]_0^x - \int_0^x -e^{-t}(k+1)t^k\, dt$

$= -x^{k+1}e^{-x} + (k+1)\int_0^t t^k e^{-t}\, dt$

$= -x^{k+1}e^{-x} + (k+1)\left[k!\left[1-e^{-x}\left(1+x+\cdots+\frac{x^k}{k!}\right)\right]\right]$

by substituting the assumption.

$= -x^{k+1}e^{-x} + (k+1)!\left[1-e^{-x}\left(1+x+\cdots+\frac{x^k}{k!}\right)\right]$

$= \frac{-x^{k+1}e^{-x}(k+1)!}{(k+1)!} + (k+1)!\left[1-e^{-x}\left(1+x+\cdots+\frac{x^k}{k!}\right)\right]$

$= (k+1)!\left[1-e^{-x}\left(1+x+\cdots+\frac{x^k}{k!}+\frac{x^{k+1}}{(k+1)!}\right)\right]$

$=$ RHS

$\therefore$ The statement is true by mathematical induction.

(ii) Show that

$$0 \le \int_0^1 t^n e^{-t}\, dt \lesssim \frac{1}{n+1}$$

Now $0 \le e^{-t} \le 1$ for $0 \le t \le 1$

$0 \le t^n e^{-t} \le t^n$

$0 \le \int_0^1 t^n e^{-t}\, dt \le \int_0^1 t^n\, dt$

But $\int_0^1 t^n\, dt = \left[\frac{t^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$

$\therefore\ 0 \le \int_0^1 t^n e^{-t} dt \le \frac{1}{n+1}$ as required.

(iii) sub $x=1$ into the result in (i) and then use (ii) to get

$0 \le n!\left[1-e^{-1}\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)\right] \le \frac{1}{n+1}$

$0 \le 1-e^{-1}\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right) \le \frac{1}{(n+1)!}$

(iv) As $n \to \infty$,

$0 \le \lim_{n\to\infty}\left[1-e^{-1}\left(1+\cdots+\frac{1}{n!}\right)\right] \le \lim_{n\to\infty}\frac{1}{(n+1)!}$

But $\lim_{n\to\infty}\frac{1}{(n+1)!} = 0$

$\therefore$

$$\therefore \lim_{n \to \infty} \left[1 - e^{-1} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \right) \right] = 0$$

$\therefore$ As $n \to \infty$,

$$\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \right) \to e$$

because $1 - e^{-1}(e) = 0$.