
Student Number

Barker College

2022 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Section I – Multiple Choice

Write your student number at the top of this page.

Select the alternative A, B C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = A 2 B 6 C 8 D 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

correct

A ● B ~~●~~ C ○ D ○

Start Here →

1.	A ○	B ○	C ○	D ○	6.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○	7.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○	8.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○	9.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○	10.	A ○	B ○	C ○	D ○

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Barker College

Student Number

2022 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Staff Involved:

8:30 AM
FRIDAY 5 AUGUST

BHC*
GDH
KJL

35 copies

TOPICS COVERED
- **Complex Numbers I & II**
- **Proof**
- **Integration**
- **Vectors**
- **Mechanics**

General Instructions:
- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided separately
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale unless specifically stated

Total marks: 100

Section I – 10 marks (pages 4-8)
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 9-14)
- Attempt Questions 11-16
- Allow about 2 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Given that x and y are **positive integers**, which one of the following statements is false?

A. $\forall x\ \exists y \left(\dfrac{2x}{y}=1\right)$

B. $\forall y\ \exists x \left(\dfrac{x}{y}=2\right)$

C. $\forall x\ \exists y \left(\dfrac{x}{2y}=1\right)$

D. $\forall y\ \exists x \left(\dfrac{y}{x}=\dfrac{1}{2}\right)$

2 Let $z=\dfrac{\sqrt{3}-i}{1-i}$.

What are the modulus and argument, respectively, of z?

A. $\sqrt{2}$ and $\dfrac{\pi}{12}$

B. $\sqrt{2}$ and $-\dfrac{\pi}{12}$

C. $\dfrac{1}{\sqrt{2}}$ and $\dfrac{\pi}{12}$

D. $\dfrac{1}{\sqrt{2}}$ and $-\dfrac{\pi}{12}$

3 What is the Cartesian equation of the line with vector equation

$\underset{\sim}{r} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \end{pmatrix}$?

A. $2x + 3y = 36$

B. $3x + 2y = 36$

C. $2x - 3y = 36$

D. $3y - 2x = 36$

4 In the complex plane, $2 + i$ and $6 - 9i$ represent the endpoints of the diameter of a circle.
What is the equation of the circle?

A. $|z - 4 - 4i| = \sqrt{29}$

B. $|z - 4 - 4i| = 29$

C. $|z - 4 + 4i| = \sqrt{29}$

D. $|z - 4 + 4i| = 29$

5 A particle is moving in simple harmonic motion on a straight line.
At time t seconds, it has displacement x metres from a fixed point on the line and velocity v m/s given by $v^2 = 4 - \frac{1}{3}x^2 + \frac{4}{3}x$.
What is the period of the motion?

A. 2π seconds

B. $\frac{2\pi}{3}$ seconds

C. $\frac{2\pi}{\sqrt{3}}$ seconds

D. $2\sqrt{3}\pi$ seconds

6 Which one of the following is an expression for $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} dx$?

A. $x + \frac{1}{4}\cos 2x + C$

B. $x - \frac{1}{4}\cos 2x + C$

C. $x + \frac{1}{2}\cos 2x + C$

D. $x - \frac{1}{2}\cos 2x + C$

7 A particle of mass m falls vertically from rest under gravity in a medium in which the resistance to motion has a magnitude $\frac{1}{30}mv^2$ where v m/s is the speed of the particle and $g = 9.8$ m/s^2 is the acceleration due to gravity.
What is the terminal velocity of the particle?

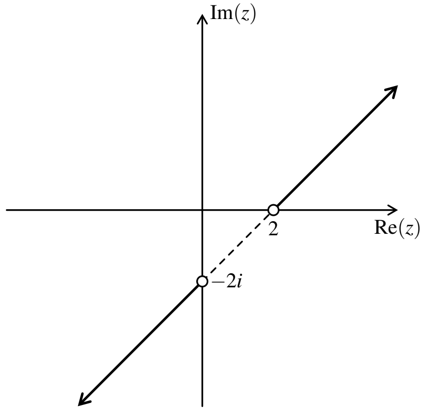
A. 294 m/s

B. 9.8 m/s

C. 30 m/s

D. 17.1 m/s

8 The path traced out by a complex number z is shown by the solid rays in the Argand diagram below.



Which one of the following is the equation of the path traced by z?

A. $\operatorname{Arg}(z-2)-\operatorname{Arg}(z+2i)=\pi$

B. $\operatorname{Arg}(z+2i)-\operatorname{Arg}(z-2)=\dfrac{\pi}{4}$

C. $\operatorname{Arg}(z-2)-\operatorname{Arg}(z+2i)=0$

D. $\operatorname{Arg}(z+2i)-\operatorname{Arg}(z-2)=\pi$

9 Without evaluating the integral, which one of the following must be negative?

A. $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1+\cos x}{x}\,dx$

B. $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\tan x}{x^3}\,dx$

C. $\displaystyle\int_{-1}^{1} \left(x^2-1\right)e^{-x^2}\,dx$

D. $\displaystyle\int_{-1}^{1} \sin\left(x^3\right)\,dx$

10 Let P be a polynomial with real coefficients. If $P(\bar{z})=3+i$, what is the value of $P(z)$?

A. $3+i$

B. $3-i$

C. $-1+3i$

D. $-1-3i$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE Writing Booklet

(a) The quadratic equation $z^2 + pz + q = 0$, where p and q are real, has a root of $\sqrt{2}+i$. Find the values of p and q. **2**

(b) Evaluate $\int_0^1 \tan^{-1} x \, dx$. **2**

(c) Find, correct to the nearest degree, the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$, given that **3**

$$\underset{\sim}{u}+\underset{\sim}{v} = 3\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k} \quad \text{and} \quad \underset{\sim}{u}-\underset{\sim}{v} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}.$$

(d) A subset of the complex plane is described by the relation $|z-(2+4i)| = 1$.

(i) Draw a neat sketch of this relation. **1**

(ii) Given that z is a complex number that satisfies the relation, find the minimum and maximum values of $|z|$. Give your answers in exact form. **2**

(iii) Given that z is a complex number that satisfies the relation, find the minimum and maximum values of $\text{Arg}(z)$, correct to the nearest degree. **2**

(e) The equations of lines ℓ_1 and ℓ_2 are given with respect to a fixed origin. **2**

$$\ell_1 : \underset{\sim}{r_1} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda \begin{pmatrix}-2\\1\\4\end{pmatrix} \quad \text{and} \quad \ell_2 : \underset{\sim}{r_2} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \mu \begin{pmatrix}p\\4\\3\end{pmatrix}$$

where λ and μ are parameters and p is a constant.

What is the value of p if ℓ_1 is perpendicular to ℓ_2?

Question 12 (16 marks) Use a SEPARATE Writing Booklet

(a) At time t, a particle has the position:

$x = 3t^2, \qquad y = t^2 - 2t, \qquad z = t^3$

(i) Find the magnitude of the velocity of the particle at time $t = 1$. **2**

(ii) Find the vector that is the component of the velocity at time $t = 1$ in the direction of $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$. **2**

(b) Consider the conditional statement S, where n is a positive integer:

S: "If n is not divisible by 12, then n is not divisible by 3."

(i) Write, and clearly label, the converse, the contrapositive and the negation of S. **3**

(ii) Determine whether the converse and the contrapositive of S are true or false, briefly justifying each answer. **2**

(c) Let the point A in the complex plane represent $z = 4 + 6i$, where O is the origin.

Let C be the new position of $\overrightarrow{OA}$ after it has been rotated 120° clockwise and its magnitude increased by a factor of 4.

(i) Find the point C in the form $\left(a+b\sqrt{c}\right)+i\left(d+e\sqrt{f}\right)$. **2**

(ii) Find the point B in the form $x + iy$, such that $OABC$ is a parallelogram. **1**

(d) A particle is undergoing simple harmonic motion about the origin with period $\frac{\pi}{4}$ seconds. **4**

Initially, the particle is at rest 5 metres to the right of the origin.

Find the first two times when the particle's speed is half its maximum speed.

Question 13 (15 marks) Use a SEPARATE Writing Booklet

(a) Note that each of parts (α), (β) and (γ) use the following definition of z^2, but they can be solved independently of each other.

Let $z^2 = 5 + 5i$.

(α) Find z and express it in the form $x + iy$, where x and y are real numbers. **3**

(β) If $z^2 w = z^2 + \overline{z^2}\sqrt{2}$, find w and express it in the form $x + iy$. **3**

(γ) If iz is a solution to $x^m + q = 0$, where m and q are positive integers, find a possible value of m and q. **3**

(b) A particle moves in a straight line. At time t seconds, its displacement from a fixed origin is x metres and its velocity is v m/s.

The acceleration of the particle is given by $\ddot{x} = 2x - 4\sqrt{2}$.

At time $t = 0$, the particle is at the origin with $v = 4$ m/s.

(i) Show that $v = 4 - x\sqrt{2}$. **3**

(ii) Find an expression for x in terms of t. **3**

Question 14 (16 marks) Use a SEPARATE Writing Booklet

(a) A particle is projected from a fixed origin, O.

There are two forces acting on the particle: gravity and air resistance.

The particle's displacement is measured in metres, and time is measured in seconds.

(i) The initial horizontal component of the velocity is u. Its subsequent horizontal motion is modelled by $\dfrac{d\dot{x}}{dt} = -k\dot{x}$. **3**

Show that the particle's horizontal displacement is given by $x = \dfrac{u}{k}\left(1 - e^{-kt}\right)$.

(ii) The initial vertical component of the velocity is w. Its subsequent vertical motion is modelled by $\dfrac{d\dot{y}}{dt} = -k\dot{y} - g$ where g is the acceleration due to gravity. **3**

Show that the particle's vertical displacement is given by

$$y = \frac{kw+g}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}.$$

(iii) Show that the Cartesian equation of motion is given by **2**

$$y = \left(\frac{kw+g}{ku}\right)x + \frac{g}{k^2}\ln\left(1 - \frac{kx}{u}\right).$$

(iv) Suppose that $u = 9$ m/s, $w = 6$ m/s, $k = 0.2$ and $g = 10$ m/s^2. Determine whether the particle will pass over a wall of height 0.15m at a horizontal distance of 9 m from the origin. **1**

(b) (i) Prove that $\forall k \in \mathbb{Z}^+$, if k^4 is even, then k is even. **2**

(ii) Prove that $\sqrt[4]{8}$ is irrational. **3**

(c) If $a, b \in \mathbb{R}^+$ and $a + b = 8$, show that $\dfrac{1}{a} + \dfrac{1}{b} \geq \dfrac{1}{2}$. **2**

Question 15 (15 marks) Use a SEPARATE Writing Booklet

(a) Use an appropriate substitution to find $\displaystyle\int \frac{dx}{x\sqrt{1-x}}$. **3**

(b) Let $z = e^{i\theta}$.

(i) Show that $z^n + \dfrac{1}{z^n} = 2\cos(n\theta)$. **1**

(ii) Show that $\left(z+\dfrac{1}{z}\right)^5 = \left(z^5+\dfrac{1}{z^5}\right)+5\left(z^3+\dfrac{1}{z^3}\right)+10\left(z+\dfrac{1}{z}\right)$. **1**

(iii) Hence, find $\displaystyle\int \cos^5\theta \, d\theta$. **2**

(c) Prove by mathematical induction that **4**

$$\frac{d^n}{dx^n}\left(\sqrt{x}\right) = \left[\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!}\right] x^{\frac{-(2n-1)}{2}} \quad \text{for integers } n \geq 2,$$

where $\dfrac{d^n}{dx^n}\left(\sqrt{x}\right)$ is the nth derivative of $\sqrt{x}$.

(d) Let m and n be positive integers. **4**

Use proof by contradiction to show that there is no positive prime number p such that $25m^2 + 30m + 9 = n^2 + p$.

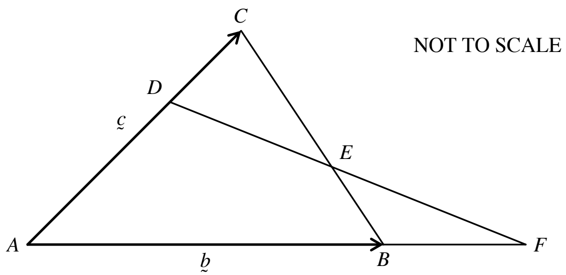
Question 16 (14 marks) Use a SEPARATE Writing Booklet

(a) In $\triangle ABC$ below, D is the point on AC such that $AD:DC = 3:1$.

E is the point on BC such that $BE:EC = 1:3$.

When DE is extended, it meets the extension of AB at F.

Let $\overrightarrow{AB} = \underset{\sim}{b}$ and $\overrightarrow{AC} = \underset{\sim}{c}$.



(i) Show that $\overrightarrow{DE} = \dfrac{3}{4}\underset{\sim}{b} - \dfrac{1}{2}\underset{\sim}{c}$. **2**

(ii) Find the ratio of $AF:BF$. **3**

(b) The point $A(-2, 4, 9)$ is reflected to the point B in the line defined by the equation **3**

$$\underset{\sim}{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix} \text{ where } \lambda \in \mathbb{R}.$$

Find the coordinates of B.

(c) (i) Find $\displaystyle\int \sin^4 x \cos^3 x \, dx$. **2**

(ii) Use integration by parts to prove that **3**

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x \, dx.$$

(iii) Use the reduction formula in part (ii) to confirm your answer for part (i). **1**

End of paper

Barker Mathematics Extension 2 Solutions 5.8.22

Multiple Choice Questions

Q1 C is false when $x = 1$

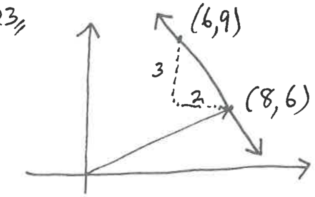
Q2 $z = \dfrac{\sqrt{3} - i}{1 - i} = \dfrac{2\,\text{cis}(-\frac{\pi}{6})}{\sqrt{2}\,\text{cis}(-\frac{\pi}{4})}$

So $|z| = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

and $\arg(z) = -\dfrac{\pi}{6} - \left(-\dfrac{\pi}{4}\right) = \dfrac{\pi}{12}$

Hence A

Q3



OR $8 - 2\lambda = x$
$6 + 3\lambda = y$
$\therefore 24 - 6\lambda = 3x$
$12 + 6\lambda = 2y$
$36 = 3x + 2y$
$\therefore B$

The equation of the line passing through $(8,6)$ and $(6,9)$ is $y - 9 = -\frac{3}{2}(x - 6)$

$2y - 18 = -3x + 18$

$3x + 2y = 36$

Hence B

Q4 The midpoint of the interval joining $(2+i)$ to $(6-9i)$ is $(4-4i)$, the centre of the circle. The distance from $(2+i)$ to $(4-4i)$ is $\sqrt{29}$, the radius of the circle. The equation is

$|z - (4-4i)| = \sqrt{29}$

Hence C.

Q5 Period $= \dfrac{2\pi}{n}$

$\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \dfrac{d}{dx}\left(4 - \tfrac{1}{6}x^2 + \tfrac{2}{3}x\right)$

$= -\dfrac{2}{6}x + \dfrac{2}{3}$

$= -\dfrac{1}{3}(x - 2) \quad \therefore n^2 = \dfrac{1}{3}$

$\therefore n = \dfrac{1}{\sqrt{3}}$

$\therefore$ Period $= \dfrac{2\pi}{\left(\frac{1}{\sqrt{3}}\right)}$

$= 2\sqrt{3}\,\pi$ seconds

Hence D.

Q6 $\cos^3 x + \sin^3 x$

$= (\cos x + \sin x) \times (\cos^2 x - \cos x \sin x + \sin^2 x)$

$\therefore I = \int \cos^2 x - \cos x \sin x + \sin^2 x \, dx$

$I = \int 1 - \cos x \sin x \, dx$

$I = x + \frac{1}{2}(\cos x)^2 + C$

$I = x + \frac{1}{2}\left(\frac{1 + \cos 2x}{2}\right) + C$

$I = x + \frac{1}{4}\cos 2x + C$

Hence A.

OR $I = \int 1 - \dfrac{\sin 2x}{2} \, dx$

$= x + \dfrac{\cos 2x}{4} + C$

Q7 $\downarrow mg$ (+) $\qquad \uparrow \frac{1}{30}mv^2$ (−)

$m\ddot{x} = mg - \frac{1}{30}mv^2$

$\ddot{x} = g - \frac{1}{30}v^2$

when terminal velocity is achieved $\ddot{x} = 0$ so

solve $0 = 9.8 - \frac{1}{30}v^2$

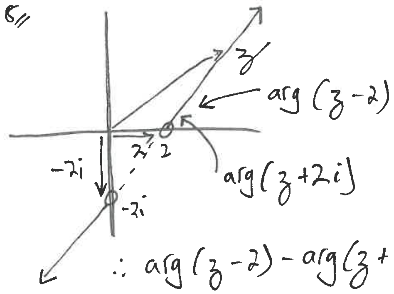
$v^2 = 9.8 \times 30$

$v^2 = 294$

$v = 17.1$ m/sec

Hence D.

Q8



$\therefore \arg(z-2) - \arg(z+2i) = \theta - \theta$
$= 0$

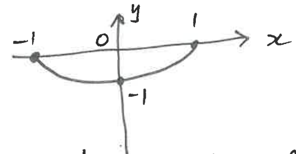
Hence C.

Q9 Which integral must be a negative definite integral?

The integral in A is an odd function and $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$ odd function $= 0$

The integral in C is an even function and the graph of $y = (x^2 - 1)e^{-x^2}$ lies below the x axis between 1 and −1. In fact:

$y = (x^2 - 1)e^{-x^2}$



Hence $\int_{-1}^{1}$ even function = negative answer

Hence C.

D is $\int_{-a}^{a}$ odd function $= 0$

B is $\int_{-a}^{a}$ even function dx whose graph is above the x axis.

In addition, both B & A have asymptotes at $x = 0$, so we can't integrate across the asymptote anyway.

Q10 let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x^1 + a_0 x^0$ where $a_0, \dots, a_n$ real coefficients

$\therefore P(\bar{z}) = a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + \dots + a_1(\bar{z})^1 + a_0(\bar{z})^0$

Now $(\bar{z})^n = (\overline{z^n})$

$\therefore P(\bar{z}) = a_n(\overline{z^n}) + a_{n-1}(\overline{z^{n-1}}) + \dots + a_1(\overline{z^1}) + a_0(\overline{z^0})$

$= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \dots + \overline{a_1 z^1} + \overline{a_0 z^0}$ since $a_0, \dots, a_n$ real

$= \overline{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z^1 + a_0 z^0}$ since $\bar{a} + \bar{b} = \overline{a+b}$

$= \overline{P(z)}$

$\therefore 3+i = \overline{P(z)}$

$\therefore 3-i = P(z)$ $\therefore B$

Q11

(a) Since the coefficients of all terms are Real, then roots occur in conjugate pairs. Hence $\alpha = \sqrt{2} + i$ and $\beta = \sqrt{2} - i$, so

$\alpha + \beta = 2\sqrt{2} = -\frac{b}{a} = -p$

Hence $p = -2\sqrt{2}$

$\alpha\beta = (\sqrt{2}+i)(\sqrt{2}-i)$

$= 2+1$

$= \frac{c}{a}$

$= q$ Hence $q = 3$

(b) $\int_0^1 \tan^{-1} x \, dx$

$= x \tan^{-1} x \Big]_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2} \, dx$

$= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2} \, dx$

$= \frac{\pi}{4} - \frac{1}{2}\left[\ln(1+x^2)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

(c) $\underset{\sim}{u} + \underset{\sim}{v} = 3\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{u} - \underset{\sim}{v} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$\therefore 2\underset{\sim}{u} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$

$\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

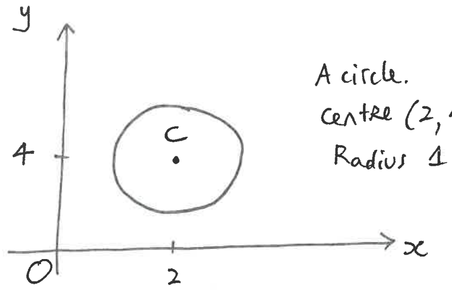
Hence $\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

then $\cos\theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}| \times |\underset{\sim}{v}|}$

$= \frac{2+2+6}{\sqrt{14} \times 3}$

$= \frac{10}{3\sqrt{14}}$

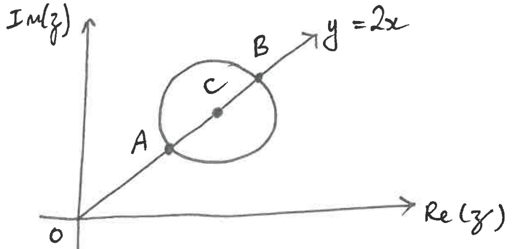
$\therefore \theta = 27^\circ$ (nearest degree) (d)(i) $|z-(2+4i)|=1$



A circle.
Centre $(2,4)$.
Radius 1 unit.

(ii) The centre of the circle is C

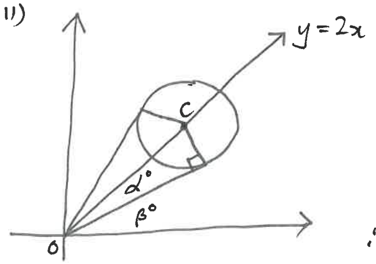
The length of OC is $\sqrt{4^2+2^2}=\sqrt{20}$.

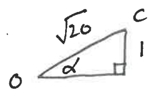


The minimum value of $|z|$ is $OA=\sqrt{20}-1=2\sqrt{5}-1$

The maximum value of $|z|$ is $OB=\sqrt{20}+1=2\sqrt{5}+1$

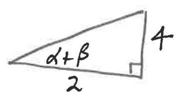
(iii)





$\sin\alpha=\frac{1}{\sqrt{20}}$

$\alpha\approx 12^\circ 55'$



$\tan(\alpha+\beta)=2$

$\alpha+\beta \doteqdot 63^\circ 26'$

$\therefore \beta \doteqdot 50^\circ 31'$

$\therefore$ The minimum value of $Arg(z)\approx 51^\circ$

The maximum value of $Arg(z)=\beta+2\alpha=$

$50^\circ 31'+2\times 12^\circ 55' \doteqdot 76^\circ 21' \doteqdot 76^\circ$

(e) Now l_1 is perpendicular to l_2 if

$$\begin{pmatrix}-2\\1\\4\end{pmatrix}\cdot\begin{pmatrix}p\\4\\3\end{pmatrix}=0$$

$\therefore -2p+4+12=0$

$-2p=-16$

$p=8$

Q12, (a) (i)

$$\left.\begin{aligned}\frac{dx}{dt}&=6t\\ \frac{dy}{dt}&=2t-2\\ \frac{dz}{dt}&=3t^2\end{aligned}\right\}$$

When $t=1$,

$\frac{dx}{dt}=6$

$\frac{dy}{dt}=0$

$\frac{dz}{dt}=3$

$\therefore |\underset{\sim}{v}|=\sqrt{6^2+0^2+3^2}$

$=\sqrt{45}$

$=3\sqrt{5}$ m/s

(ii) Now $\underset{\sim}{v}=\begin{pmatrix}6\\0\\3\end{pmatrix}$ and $\underset{\sim}{u}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$ then

$$\text{proj}_{\underset{\sim}{u}}\underset{\sim}{v}=\frac{12+0+6}{(\sqrt{4+1+4})^2}\underset{\sim}{u}$$

$=2\underset{\sim}{u}$

$=2\begin{pmatrix}2\\-1\\2\end{pmatrix}$

$=\begin{pmatrix}4\\-2\\4\end{pmatrix}$

Q12// (b) S: "If n is not divisible by 12, then n is not divisible by 3."

(i) The converse is:

"If n is not divisible by 3, then n is not divisible by 12."

The contrapositive is:

"If n is divisible by 3, then n is divisible by 12."

the negation is:

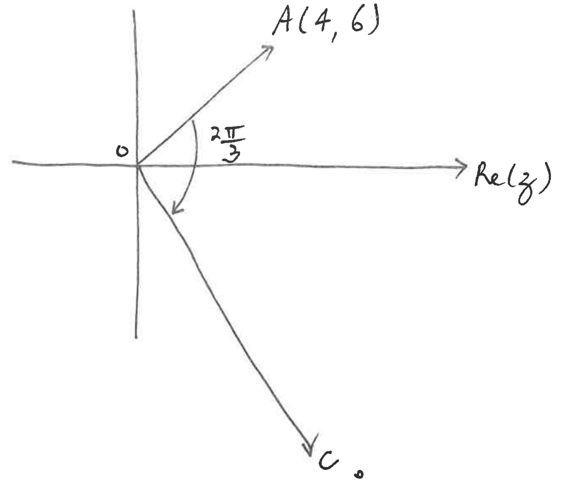
"There exists an integer n with the property that n is not divisible by 12, but n is divisible by 3."

(ii) The contrapositive is false. That is because the contrapositive has the same truth value as S, and S is false. [S is false because 6 is not divisible by 12, but 6 is divisible by 3.]

Alternatively; the contrapositive is false by this counterexample: 6 is divisible by 3, but 6 is not divisible by 12.

The converse is true. That is because if the prime factorisation of n does not have a 3 in it, then n will not be divisible by 12.

Q12// (c)



(i) $\vec{OC} = z \times 4e^{-2\pi/3\, i}$

$= (4+6i) \times 4\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

$= (4+6i) \times 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= (4+6i)(-2-2\sqrt{3}i)$

$= -8 + 12\sqrt{3} - 12i - 8\sqrt{3}i$

$= (12\sqrt{3} - 8) + i(-12 - 8\sqrt{3}) = (-8 + 12\sqrt{3}) + i(-12 - 8\sqrt{3})$

(ii) $\vec{OB} = \vec{OA} + \vec{OC}$

$= (4+6i) + (12\sqrt{3} - 8) + i(-12 - 8\sqrt{3})$

$= (12\sqrt{3} - 4) + i(-6 - 8\sqrt{3})$

(d) Period $= \frac{\pi}{4}$ seconds

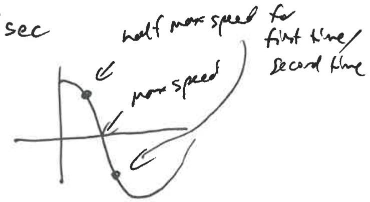
So $\frac{2\pi}{n} = \frac{\pi}{4} \quad \therefore n = 8$

The equation of motion is

$x = 5\cos 8t$

Hence $v = -40 \sin 8t$

So the maximum speed is 40 m/sec



Need to find t when $v = -20$

$-20 = -40 \sin 8t$

$\sin 8t = \frac{1}{2}$

$8t = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore t = \frac{\pi}{48}, \frac{5\pi}{48}$ seconds

Q13 (a) Let $z^2 = 5+5i$

(α) $(x+iy)^2 = 5+5i$

$x^2 - y^2 = 5$ ① $2xyi = 5i$

$2xy = 5$ ②

Now $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= 5^2 + 5^2$

$= 50$

$\therefore x^2+y^2 = \pm 5\sqrt{2}$

But $x^2 \geq 0$ and $y^2 \geq 0$ so

$x^2+y^2 = 5\sqrt{2}$ ③

Now ① + ③ gives

$2x^2 = 5+5\sqrt{2}$

$x = \pm\sqrt{\frac{5+5\sqrt{2}}{2}}$

③ − ① gives

$2y^2 = 5\sqrt{2} - 5$

$y = \pm\sqrt{\frac{5\sqrt{2}-5}{2}}$

From $2xy = 5$ we know x and y are the same sign. Hence

$x+iy = \pm\left(\sqrt{\frac{5+5\sqrt{2}}{2}} + i\sqrt{\frac{5\sqrt{2}-5}{2}}\right)$

(β) Given that $z^2 w = z^2 + \bar{z}^2\sqrt{2}$, to find w.

$(5+5i)w = (5+5i) + (5-5i)\sqrt{2}$

$(1+i)w = (1+i) + (1-i)\sqrt{2}$

$(1+i)w = (1+\sqrt{2}) + (1-\sqrt{2})i$

$w = \frac{(1+\sqrt{2}) + (1-\sqrt{2})i}{(1+i)} \times \frac{(1-i)}{(1-i)}$

$w = \frac{(1+\sqrt{2}) + (1-\sqrt{2})i - (1+\sqrt{2})i + (1-\sqrt{2})}{2}$

$w = \frac{2 - 2\sqrt{2}i}{2}$

$w = 1 - \sqrt{2}i$

(γ) $z^2 = 5+5i = 5\sqrt{2}e^{i\frac{\pi}{4}}$

$\therefore z = \pm\sqrt{5\sqrt{2}}\, e^{i\left(\frac{\pi}{8}\right)}$

$\therefore iz = \pm\sqrt{5\sqrt{2}}\, e^{i\left(\frac{\pi}{8}+\frac{\pi}{2}\right)}$

$iz = \pm\sqrt{5\sqrt{2}}\, e^{i\left(\frac{5\pi}{8}\right)}$

$(iz)^8 = (5\sqrt{2})^4 e^{i5\pi}$

$= 5^4 \times 2^2 \times (\cos 5\pi + i\sin 5\pi)$

$= -2500$

$\therefore$ Possible values of m and q are

$m = 8$ and $q = 2500$

Q13 (b)

(i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - 4\sqrt{2}$

$\frac{1}{2}v^2 = x^2 - 4\sqrt{2}x + C$

But when $x = 0$, $v = 4$ so $C = 8$.

$v^2 = 2x^2 - 8\sqrt{2}x + 16$

$v^2 = 2(x^2 - 4\sqrt{2}x + 8)$

$v^2 = 2(x - 2\sqrt{2})^2$

$v = \pm\sqrt{2}(x - 2\sqrt{2})$

But when $x = 0$, $v = 4$ so

$v = -\sqrt{2}(x - 2\sqrt{2})$

Hence $v = 4 - x\sqrt{2}$,

as required.

(ii) Now $v = \frac{dx}{dt} = 4 - x\sqrt{2}$

$\int \frac{dx}{4 - \sqrt{2}x} = \int dt$

$-\frac{1}{\sqrt{2}} \ln|4 - \sqrt{2}x| + c = t$

But when $t = 0$, $x = 0$ so

$C = \frac{1}{\sqrt{2}} \ln(4)$

$\therefore t = \frac{1}{\sqrt{2}} \ln(4) - \frac{1}{\sqrt{2}} \ln|4 - \sqrt{2}x|$

$t = \frac{1}{\sqrt{2}} \ln\left|\frac{4}{4 - \sqrt{2}x}\right|$

$e^{\sqrt{2}t} = \frac{4}{4 - \sqrt{2}x}$

$4 - \sqrt{2}x = 4e^{-\sqrt{2}t}$

$4 - 4e^{-\sqrt{2}t} = \sqrt{2}x$

$x = \frac{4 - 4e^{-\sqrt{2}t}}{\sqrt{2}}$

$x = 2\sqrt{2} - 2\sqrt{2}e^{-\sqrt{2}t}$

$x = 2\sqrt{2}(1 - e^{-\sqrt{2}t})$

Q14 (a)

(i) $\frac{d\dot{x}}{dt} = -k\dot{x}$

$\int \frac{1}{\dot{x}} \, d\dot{x} = \int -k \, dt$

$\ln(\dot{x}) = -kt + c$

But $\dot{x} = u$ when $t = 0$

so $C = \ln u$

$\therefore \ln \dot{x} = -kt + \ln u$

$\ln\left(\frac{\dot{x}}{u}\right) = -kt$

$\frac{\dot{x}}{u} = e^{-kt}$

$\dot{x} = ue^{-kt}$

$\therefore x = \int ue^{-kt} \, dt$

$x = -\frac{u}{k}e^{-kt} + H$

when $t = 0$, $x = 0$ so

$H = \frac{u}{k}$

$\therefore x = \frac{u}{k} - \frac{u}{k}e^{-kt}$

$x = \frac{u}{k}(1 - e^{-kt})$,

as required.

(ii) $\frac{d\dot{y}}{dt} = -k\dot{y} - g$

$= -k\left(\dot{y} + \frac{g}{k}\right)$

$\int \frac{d\dot{y}}{\dot{y} + \frac{g}{k}} = \int -k \, dt$

$\ln\left(\dot{y} + \frac{g}{k}\right) = -kt + C$

when $t = 0$, $\dot{y} = w$ so

$C = \ln\left(w + \frac{g}{k}\right)$

$\therefore -kt = \ln\left(\frac{\dot{y} + \frac{g}{k}}{w + g/k}\right)$

$\left(w + \frac{g}{k}\right)e^{-kt} = \dot{y} + \frac{g}{k}$

$\dot{y} = \left(w + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}$

$\therefore y = \int\left(\left(w + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}\right) dt$

$y = -\frac{1}{k}\left(w + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}t + F$

But when $t = 0$, $y = 0$ so

$F = \frac{1}{k}\left(w + \frac{g}{k}\right)$

$\therefore y = -\frac{1}{k}\left(w + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}t + \frac{1}{k}\left(w + \frac{g}{k}\right)$

$y = \frac{w}{k}\left(1 - e^{-kt}\right) + \frac{g}{k^2}\left(1 - e^{-kt}\right) - \frac{g}{k}t$

$y = \frac{wk}{k^2}(1 - e^{-kt}) + \frac{g}{k^2}(1 - e^{-kt}) - \frac{g}{k}t$

$y = \frac{kw + g}{k^2}\left(1 - e^{-kt}\right) - \frac{g}{k}t$,

as required.

(iii) From part (i),

$$x = \frac{u}{k}\left(1 - e^{-kt}\right)$$

So $\frac{kx}{u} = 1 - e^{-kt}$

Make t the subject:

$$e^{-kt} = 1 - \frac{kx}{u}$$

$$t = -\frac{1}{k}\ln\left(1 - \frac{kx}{u}\right)$$

Substitute this into y to get

$$y = \frac{kw+g}{k^2}\left(1 - e^{\ln\left(1-\frac{kx}{u}\right)}\right) - \frac{g}{k}\left(\frac{-1}{k}\ln\left(1-\frac{kx}{u}\right)\right)$$

$$y = \frac{kw+g}{k^2}\left(1-\left(1-\frac{kx}{u}\right)\right) + \frac{g}{k^2}\ln\left(1-\frac{kx}{u}\right)$$

$$y = \frac{kw+g}{k^2}\left(\frac{kx}{u}\right) + \frac{g}{k^2}\ln\left(1-\frac{kx}{u}\right)$$

$$y = \left(\frac{kw+g}{ku}\right)x + \frac{g}{k^2}\ln\left(1-\frac{kx}{u}\right),$$ as required.

(iv) Substitution of the numerical values gives

$$y = \left(\frac{0.2\times 6+10}{0.2\times 9}\right)\times 9 + \frac{10}{(0.2)^2}\ln\left(1 - \frac{0.2\times 9}{9}\right)$$

$$y = 56 + (-55.7859)$$

$y \approx 0.214$ m and $0.214 > 0.15$ so the particle will pass over the wall.

Q14 (b)

(i) Proof by contrapositive:

Suppose k is odd.

Let $k = 2a+1$, where a is an integer.

Then $k^4 = (2a+1)^4$

$$= (4a^2+4a+1)(4a^2+4a+1)$$

$$= 16a^4 + 16a^3 + 4a^2 + 16a^3 + 16a^2 + 4a + 4a^2 + 4a + 1$$

$$= 16a^4 + 32a^3 + 8a^2 + 16a^2 + 8a + 1$$

$$= 2\left[8a^4 + 16a^3 + 12a^2 + 4a\right] + 1$$

$= 2j + 1$, which is odd.

So if k is odd, then k^4 will be odd.

Therefore if k^4 is even, then k is even.

(ii) Proof by contradiction:

Suppose that $\sqrt[4]{8}$ is rational.

Let $\sqrt[4]{8} = \frac{p}{q}$ where p and q are integers with no common factors.

Then $8 = \frac{p^4}{q^4}$

$p^4 = 8q^4$

So p^4 is even

$\therefore p$ is even (from (i))

Let $p = 2m$, m an integer

$(2m)^4 = 8q^4$

$16m^4 = 8q^4$

$\therefore q^4 = 2m^4$

$\therefore q^4$ is even

$\therefore q$ is even (from (i))

Contradiction!

Both p and q cannot be even as they have no common factors.

Hence $\sqrt[4]{8}$ is irrational.

(c)

Consider the sign of LHS − RHS.

$$\text{LHS} - \text{RHS} = \frac{1}{a} + \frac{1}{b} - \frac{1}{2}$$

$$= \frac{a+b}{ab} - \frac{1}{2}$$

$$= \frac{2(a+b) - ab}{2ab}$$

but $b = 8 - a$, so

$$\text{LHS} - \text{RHS} = \frac{2(a+8-a) - a(8-a)}{2a(8-a)}$$

$$= \frac{16 - 8a + a^2}{2a(8-a)}$$

$$= \frac{(a-4)^2}{2a(8-a)}$$

The numerator is always ≥ 0.

The denominator is $2ab > 0$ because $a, b \in \mathbb{R}^+$.

Hence LHS − RHS ≥ 0

Hence $\frac{1}{a} + \frac{1}{b} \geq \frac{1}{2}$ as required.

OR

Let $\frac{1}{a} + \frac{1}{b} < \frac{1}{2}$

$\therefore \frac{b+a}{ab} < \frac{1}{2}$

$\therefore \frac{8}{ab} < \frac{1}{2}$

$\therefore 16 < ab$ since $2, a, b$ all positive

$\therefore a(8-a) > 16$

$\therefore 8a - a^2 > 16$

$\therefore a^2 - 8a + 16 < 0$

$\therefore (a-4)^2 < 0$

Contradiction!

$\therefore \frac{1}{a} + \frac{1}{b} \geq \frac{1}{2}$

Q15, (a) $\displaystyle\int \frac{dx}{x\sqrt{1-x}}$

Let $1 - x = u^2$

$-dx = 2u\,du$

$dx = -2u\,du$

$$I = \int \frac{-2u\,du}{(1-u^2)\sqrt{u^2}}$$

$$= -2\int \frac{du}{1-u^2}$$

Let $\frac{1}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$

$1 = A(1+u) + B(1-u)$

$A = \frac{1}{2}$ and $B = \frac{1}{2}$

$$I = -2\int \frac{\frac{1}{2}}{1-u} + \frac{\frac{1}{2}}{1+u}\,du$$

$$I = -\int \frac{1}{1-u} + \frac{1}{1+u}\,du$$

$$I = \int \frac{-1}{1-u} - \frac{1}{1+u}\,du$$

$$I = \ln|1-u| - \ln|1+u|$$

$$I = \ln\left|\frac{1-u}{1+u}\right|$$

$$I = \ln\left|\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right| + c$$

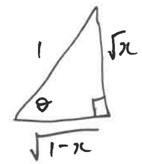
OR

Let $x = \sin^2\theta$

$\therefore dx = 2\sin\theta\cos\theta\,d\theta$

$$\therefore I = \int \frac{2\sin\theta\cos\theta\,d\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}}$$

$$= 2\int \frac{\sin\theta\cos\theta\,d\theta}{\sin^2\theta\cos\theta}$$



$$= 2\int \operatorname{cosec}\theta\,d\theta$$

$$= 2\int \frac{\operatorname{cosec}\theta(\operatorname{cosec}\theta - \cot\theta)\,d\theta}{\operatorname{cosec}\theta - \cot\theta}$$

$$= 2\ln|\operatorname{cosec}\theta - \cot\theta| + c$$

$$= 2\ln\left|\frac{1}{\sqrt{x}} - \frac{\sqrt{1-x}}{\sqrt{x}}\right| + c$$

$$= 2\ln\left|\frac{1-\sqrt{1-x}}{\sqrt{x}}\right| + c$$

(b) $z = e^{i\theta}$

(i) $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$ — ①

$= \cos(n\theta) + i\sin(n\theta)$ by de Moivre

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta)$ — ②

① + ② gives

$z^n + z^{-n} = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$

$= 2\cos(n\theta)$, as required.

(ii) By the binomial theorem,

$(z+\frac{1}{z})^5 = z^5 + 5z^4(\frac{1}{z}) + 10z^3(\frac{1}{z})^2 + 10z^2(\frac{1}{z})^3 + 5z(\frac{1}{z})^4 + (\frac{1}{z})^5$

$= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$

$= (z^5 + \frac{1}{z^5}) + 5(z^3 + \frac{1}{z^3}) + 10(z + \frac{1}{z})$, as required.

(iii) To find $\int \cos^5\theta \, d\theta$.

Now $z + \frac{1}{z} = 2\cos\theta$

$(z+\frac{1}{z})^5 = (2\cos\theta)^5 = 32\cos^5\theta$

$\therefore 32\cos^5\theta = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$

$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$

$\therefore \int \cos^5\theta \, d\theta = \int \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta \, d\theta$

$= \frac{1}{80}\sin 5\theta + \frac{5}{48}\sin 3\theta + \frac{5}{8}\sin\theta + C$

(c) when $n=2$,

$LHS = \frac{d^2}{dx^2}(x^{\frac{1}{2}})$

$= \frac{d}{dx}(\frac{1}{2}x^{-\frac{1}{2}})$

$= -\frac{1}{4}x^{-\frac{3}{2}}$

$RHS = \frac{(-1)^1 \times 2!}{2^3 \times 1!} x^{-\frac{3}{2}}$

$= \frac{-2}{8}x^{-\frac{3}{2}}$

$= -\frac{1}{4}x^{-\frac{3}{2}}$

$LHS = RHS$ when $n=2$.

The statement is true when $n=2$.

Next, assume that the statement is true when $n=k$, ie, assume that $\frac{d^k}{dx^k}(x^{\frac{1}{2}}) = \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}(k-1)!} x^{-\frac{(2k-1)}{2}}$

and then try to prove that

$\frac{d^{k+1}}{dx^{k+1}}(x^{\frac{1}{2}}) = \frac{(-1)^k(2k)!}{2^{2k+1}(k)!} x^{-\frac{(2k+1)}{2}}$

Now $LHS = \frac{d^{k+1}}{dx^{k+1}}(x^{\frac{1}{2}})$

$= \frac{d}{dx}\left(\frac{d^k}{dx^k}(x^{\frac{1}{2}})\right)$

$= \frac{d}{dx}\left(\frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}(k-1)!} x^{-\frac{(2k-1)}{2}}\right)$

$= \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}(k-1)!} \times \frac{-(2k-1)}{2} \times x^{-\frac{(2k-1)}{2}-1}$

$$LHS = \frac{(-1)^{k-1}(2k-2)!\,(-1)(2k-1)}{2^{2k-1}(k-1)! \times 2} x^{\frac{-2k-1}{2}}$$

$$= \frac{(-1)^k (2k-1)!}{2^{2k}(k-1)!} x^{\frac{-(2k+1)}{2}}$$

Now multiply the numerator and denominator of the coefficient by $2k$

$$= \frac{(-1)^k (2k-1)!\,2k}{2^{2k}(k-1)!\,2k} x^{\frac{-(2k+1)}{2}}$$

$$= \frac{(-1)^k (2k)!}{2^{2k+1}(k)!} x^{-\frac{(2k+1)}{2}}$$

$= RHS$

Hence the statement is true by mathematical induction.

(d) Proof by contradiction:

Suppose that there is a positive prime number p such that

$$25m^2 + 30m + 9 = n^2 + p$$

Then $p = 25m^2 + 30m + 9 - n^2$

$p = (5m+3)^2 - n^2$

$p = (5m+3-n)(5m+3+n)$

Notice that $5m+3-n < 5m+3+n$.

Since p is prime then

$5m+3-n = 1$

and $5m+3+n = p$

Then $p+1 = 10m+6$

$p = 10m+5$

$p = 5(2m+1)$

Now $2m+1 \geqslant 3$

$\therefore$ $p = 5(2m+1)$ is a contradiction!

because p cannot be prime and a product of 5 and an integer greater than or equal to 3.

Hence, there is no prime number p such that

$$25m^2 + 30m + 9 = n^2 + p.$$

Q16 (a) (i)

Now $\overrightarrow{CB} = \underset{\sim}{b} - \underset{\sim}{c}$

$\overrightarrow{CE} = \frac{3}{4}(\overrightarrow{CB})$

$= \frac{3}{4}(\underset{\sim}{b} - \underset{\sim}{c})$

$\overrightarrow{CD} = \frac{1}{4}(-\underset{\sim}{c})$

$= -\frac{1}{4}\underset{\sim}{c}$

$\overrightarrow{DE} = \overrightarrow{CE} - \overrightarrow{CD}$

$= \frac{3}{4}(\underset{\sim}{b} - \underset{\sim}{c}) - (-\frac{1}{4}\underset{\sim}{c})$

$= \frac{3}{4}\underset{\sim}{b} - \frac{3}{4}\underset{\sim}{c} + \frac{1}{4}\underset{\sim}{c}$

$= \frac{3}{4}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{c}$, as required.

(ii) To find the ratio of AF : BF

Let $\overrightarrow{AF} = \lambda\overrightarrow{AB} = \lambda\underset{\sim}{b}$

Let $\overrightarrow{DF} = \mu\overrightarrow{DE} = \mu(\frac{3}{4}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{c})$.

Now $\overrightarrow{AF} = \overrightarrow{AD} + \overrightarrow{DF}$

$= \frac{3}{4}\underset{\sim}{c} + \mu(\frac{3}{4}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{c})$

$= \frac{3}{4}\underset{\sim}{c} + \frac{3}{4}\mu\underset{\sim}{b} - \frac{\mu}{2}\underset{\sim}{c}$

$= \frac{3-2\mu}{4}\underset{\sim}{c} + \frac{3\mu}{4}\underset{\sim}{b}$

but $\overrightarrow{AF} = \lambda\underset{\sim}{b}$ so

$\frac{3-2\mu}{4} = 0$ and $\frac{3\mu}{4} = \lambda$

$\therefore \mu = \frac{3}{2}$ $\therefore \lambda = \frac{9}{8}$

Sub $\lambda = \frac{9}{8}$ into

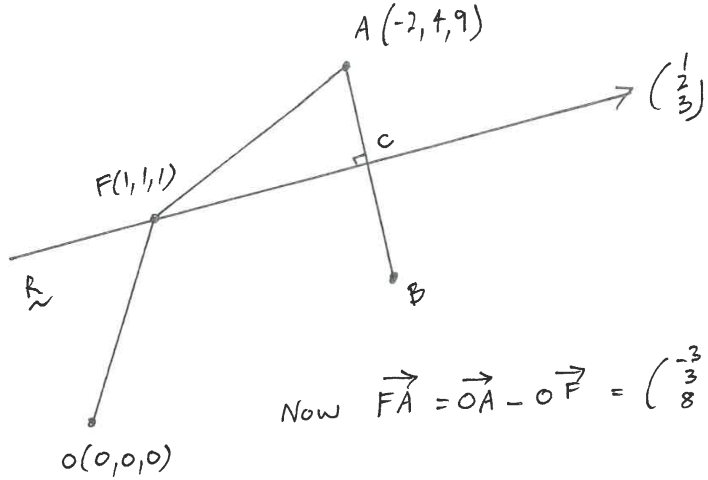
$\overrightarrow{AF} = \lambda\overrightarrow{AB}$

$8\overrightarrow{AF} = 9\overrightarrow{AB}$

$8\overrightarrow{AF} = 9(\overrightarrow{AF} - \overrightarrow{BF})$

$\therefore 9\overrightarrow{BF} = \overrightarrow{AF}$ $\Rightarrow \therefore$ AF : BF = 9 : 1

(b)



Now $\overrightarrow{FA} = \overrightarrow{OA} - \overrightarrow{OF} = \begin{pmatrix}-3\\3\\8\end{pmatrix}$

$\overrightarrow{FC}$ is the projection of $\overrightarrow{FA}$ onto $\underset{\sim}{R}$.

Hence $\overrightarrow{FC} = \frac{\begin{pmatrix}-3\\3\\8\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}}{(\sqrt{14})^2}\underset{\sim}{R}$

$= \frac{-3+6+24}{14}\begin{pmatrix}1\\2\\3\end{pmatrix}$

$= \frac{27}{14}\begin{pmatrix}1\\2\\3\end{pmatrix}$

Now $\vec{AC} = \vec{FC} - \vec{FA}$

$$= \frac{27}{14}\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-3\\3\\8\end{pmatrix}$$

$$= \begin{pmatrix}\frac{69}{14}\\ \frac{6}{7}\\ \frac{-31}{14}\end{pmatrix}$$

And $\vec{OB} = \vec{OA} + 2 \times \vec{AC}$

$$= \begin{pmatrix}-2\\4\\9\end{pmatrix} + 2\begin{pmatrix}\frac{69}{14}\\ 6/7\\ -31/14\end{pmatrix}$$

$$= \begin{pmatrix}55/7\\ 40/7\\ 32/7\end{pmatrix}$$

$\therefore B = \left(\frac{55}{7}, \frac{40}{7}, \frac{32}{7}\right)$

(c)

(i) $\int \sin^4 x \cos^3 x \, dx$

$= \int \sin^4 x (1-\sin^2 x)\cos x \, dx$

$= \int \sin^4 x \cos x - \sin^6 x \cos x \, dx$

$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

(ii) $\int \sin^m x \cos^n x \, dx = I$

x Note: breaking up $\cos^n x = \cos x \cos^{n-1} x$ is necessary. Breaking up $\sin^m x$ was unnecessary though! Might save a few lines not doing it.

$I = \int (\sin^{m-1} x \cos x)(\cos^{n-1} x \sin x)\, dx$

$= (\frac{1}{m}\sin^m x)(\cos^{n-1} x \sin x) - \int \frac{1}{m}\sin^m x \times$

$[\cos^{n-1} x . \cos x + \sin x (n-1)\cos^{n-2} x (-\sin x)]$

$= \frac{1}{m}\sin^{m+1} x \cos^{n-1} x - \int \frac{1}{m}\sin^m x [\cos^n x - (n-1)\cos^{n-2} x \sin^2 x]\, dx$

$= \frac{1}{m}\sin^{m+1} x \cos^{n-1} x - \int \frac{1}{m}\sin^m x \cos^n x - \frac{1}{m}(n-1)\sin^{m+2} x \cos^{n-2} x \, dx$

$= \frac{1}{m}\sin^{m+1} x \cos^{n-1} x - \frac{1}{m} I + \frac{n-1}{m}\int \sin^{m+2} x \cos^{n-2} x \, dx$

$\therefore I + \frac{1}{m} I = \frac{1}{m}\sin^{m+1} x \cos^{n-1} x + \frac{n-1}{m}\int \sin^m x \cos^{n-2} x (1-\cos^2 x)\, dx$

$\therefore \left(\frac{m+1}{m}\right) I = \frac{1}{m}\sin^{m+1} x \cos^{n-1} x + \frac{n-1}{m}\int \sin^m x \cos^{n-2} x - \sin^m x \cos^n x \, dx$

$\therefore \left(\frac{m+1}{m}\right) I = \frac{1}{m}\sin^{m+1} x \cos^{n-1} x + \frac{n-1}{m}\int \sin^m x \cos^{n-2} x \, dx - \frac{n-1}{m} I$

$\therefore \left(\frac{m+1}{m}\right) I + \left(\frac{n-1}{m}\right) I = \frac{1}{m}\sin^{m+1} x \cos^{n-1} x + \frac{n-1}{m}\int \sin^m x \cos^{n-2} x \, dx$

$$\left(\frac{m+n}{m}\right) I = \frac{1}{m}\sin^{m+1}x\cos^{n-1}x + \frac{n-1}{m}\int \sin^m x\cos^{n-2}x\,dx$$

$$\therefore I = \frac{1}{m+n}\sin^{m+1}x\cos^{n-1}x + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

$$\therefore I = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

as required.

(iii) $\int \sin^4 x\cos^3 x\,dx$

$$= \frac{\sin^5 x\cos^2 x}{7} + \frac{2}{7}\int \sin^4 x\cos x\,dx$$

$$= \frac{\sin^5 x\cos^2 x}{7} + \frac{2}{7}\cdot\frac{1}{5}\sin^5 x + C$$

$$= \frac{\sin^5 x(1-\sin^2 x)}{7} + \frac{2}{35}\sin^5 x + C$$

$$= \frac{\sin^5 x}{7} - \frac{\sin^7 x}{7} + \frac{2}{35}\sin^5 x + C$$

$$= \frac{7}{35}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

$$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

which confirms the answer from part (i).

END OF SOLUTIONS.