Student Number

Barker College

2023 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Staff Involved:
- BHC
- DZP
- GDH*

AM FRIDAY 4TH AUGUST

30 copies

General Instructions:
- Write your Student Number at the top of this page and on all writing booklets
- Reading time - 10 minutes
- Working time - 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A separate reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and / or calculations

Total Marks 100

Section I - 10 marks (pages 2 - 5)
- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 - 14)
- Attempt Questions 11 - 16
- Allow about 2 hours and 45 minutes for this section
- Answer each question in a SEPARATE writing booklet.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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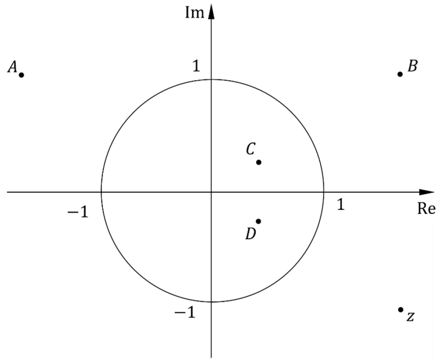
1 The diagram shows the complex number z in the fourth quadrant of the complex plane.

The modulus of z is 2.

Which of the points marked A, B, C or D best shows the position of $\frac{1}{z}$?



A. Point A

B. Point B

C. Point C

D. Point D

2 Which expression is equal to $\int \tan x \sec^{100} x \, dx$?

A. $\tan x \sec^{50} x + C$

B. $\tan^{50} x \sec x + C$

C. $\dfrac{\sec^{100} x}{100} + C$

D. $\dfrac{\tan^{100} x}{100} + C$

3 Which of the following statements is correct?

A. $\forall a, b \in \mathbb{R}$ $\quad \sin a < \sin b \Rightarrow a < b$

B. $\forall a, b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad \sin a < \sin b \Rightarrow a < b$

C. $\forall a, b \in \mathbb{R}$ $\quad \cos a < \cos b \Rightarrow a < b$

D. $\forall a, b \in [0, \pi]$ $\quad \cos a < \cos b \Rightarrow a < b$

4 Each pair of lines given below intersects at $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Which pair of lines are perpendicular?

A. ℓ_1: $\underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix}3\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\0\end{pmatrix}$

B. ℓ_1: $\underset{\sim}{r} = \begin{pmatrix}2\\4\\6\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix}-2\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\3\end{pmatrix}$

C. ℓ_1: $\underset{\sim}{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\-2\\-3\end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix}-1\\-2\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$

D. ℓ_1: $\underset{\sim}{r} = \begin{pmatrix}3\\-2\\8\end{pmatrix} + \lambda\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix}7\\4\\5\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\1\end{pmatrix}$

5 The velocity of a body moving in a straight line is given by $v = f(x)$ where x metres is the distance from origin and v is the velocity in metres per second.
The acceleration of the body in ms^{-2} is given by:

A. $f'(x)$

B. $f'(v)$

C. $xf'(x)$

D. $f(x)f'(x)$

6 Given that $z = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, what is the value of $(\bar{z})^3$?

A. $-27i$

B. $27i$

C. -27

D. 27

7 A unit mass undergoing simple harmonic motion has a period of $\frac{2\pi}{3}$ seconds.

The extremities of the motion are 2 metres apart.

What is the maximum magnitude of the force on the mass?

A. 6 N

B. 9 N

C. 12 N

D. 18 N

8 What are the values of real numbers p and q such that $(2 - i)$ is a root of the equation $z^3 + pz + q = 0$?

A. $p = -11$ and $q = -20$

B. $p = -11$ and $q = 20$

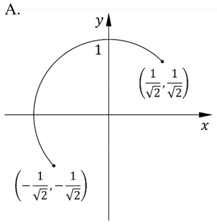
C. $p = 11$ and $q = -20$

D. $p = 11$ and $q = 20$

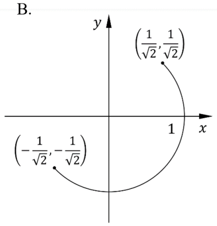
9 Which diagram best shows the curve described by the position vector

$$\underset{\sim}{r}(t) = \sin(t)\,\underset{\sim}{i} - \cos(t)\,\underset{\sim}{j} \quad \text{for} \quad \frac{\pi}{4} \le t \le \frac{5\pi}{4}\,?$$

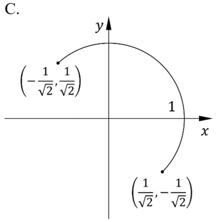
A.



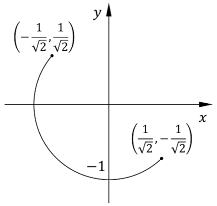
B.



C.



D.



10 The non-zero complex numbers ω and z are linked by the formula $\omega = z + k\bar{z}$, where k is real and $k \neq 0$. z is neither purely real nor purely imaginary.

Which of the following statements is INCORRECT?

A. ω can only be purely imaginary if $k = -1$

B. ω can only be purely real if $k = 1$

C. ω and z must have different moduli

D. ω and z must have different arguments

End of Section I

Section II

90 marks
Attempt Questions 11 - 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 - 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **[USE THE QUESTION 11 WRITING BOOKLET]**

(a) The complex numbers $z = 4e^{\frac{\pi}{3}i}$ and $w = 2e^{\frac{\pi}{6}i}$ are given.

(i) Find $|wz|$. **1**

(ii) Find the value of $\frac{z}{w}$, giving the answer in the form $re^{i\theta}$. **1**

(iii) Hence, or otherwise, find the value of w^2. **1**

(b) (i) Find real numbers A, B, C and D such that **2**

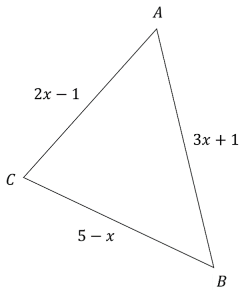
$$\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}.$$

(ii) Hence find $\displaystyle\int \frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}\,dx$. **2**

Question 11 continues on page 7

Question 11 (continued)

(c) Triangle ABC has sides $a = 5 - x$, $b = 2x - 1$ and $c = 3x + 1$ as shown, where $x \in \mathbb{R}$.

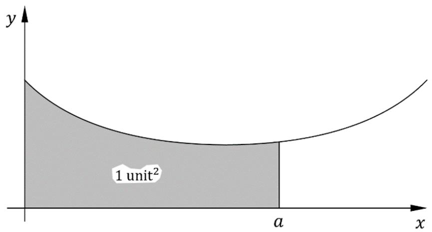


NOT TO SCALE

(i) Prove that $\frac{5}{6} < x < \frac{3}{2}$. **2**

(ii) Given ΔABC is isosceles, prove that $\cos A = \frac{1}{8}$. **2**

(d) The area under the curve $y = \frac{1}{1+\sin x}$ from $x = 0$ to $x = a$ is 1 unit^2.



NOT TO SCALE

Find the value of a. **4**

End of Question 11

Question 12 (15 marks) **[USE THE QUESTION 12 WRITING BOOKLET]**

(a) (i) What is the simplified expansion of $(1+ia)^4$ in ascending powers of a ? **2**

(ii) Hence find the values of a such that $(1+ia)^4$ is real. **1**

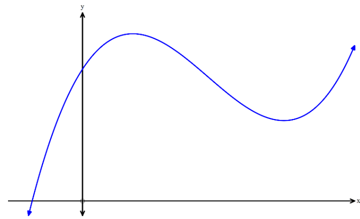
(b) Solve the equation: $z^2 = |z|^2 - 4$ **3**

(c) (i) Prove that the three cube roots of i are $z_1 = e^{\frac{\pi}{6}i}, z_2 = e^{\frac{5\pi}{6}i}$ and $z_3 = -i$. **2**

(ii) Prove that **3**

$$\left(z_3\right)^2 = \left(z_1 i\right)^2 + \left(z_2 i\right)^2.$$

(d) Here is the correct graph of $y = 2x^3 - 15x^2 + 24x + 41$, with labels not shown.



Mr Carruthers and Mr Peattie are trying to resolve a difference.

Mr Carruthers points to the graph and states that the sum of the roots is clearly negative since there is only one real root and it is negative.

Mr Peattie points to the equation and states that the sum of the roots is clearly positive, in fact that it equals 7.5.

(i) Find the coordinates of the stationary points to confirm Mr Carruthers' observation of only one real root. **1**

(ii) By factorising over the complex field, show clearly who is correct. **2**

(iii) Explain the incorrect person's mistake. **1**

End of Question 12

Question 13 (15 marks) **[USE THE QUESTION 13 WRITING BOOKLET]**

(a) Let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be non-zero, non-parallel vectors, where $|\underset{\sim}{u}| = 2|\underset{\sim}{v}|$.

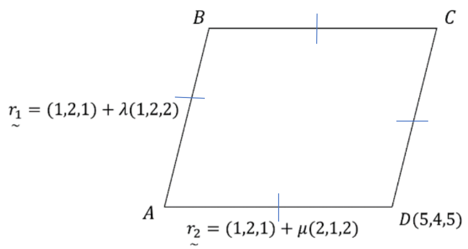
By drawing a diagram, or otherwise, prove that $|\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}| = 2|\text{proj}_{\underset{\sim}{u}}\underset{\sim}{v}|$ **3**

(b) Two sides of the rhombus $ABCD$ are formed by:

$\underset{\sim}{r_1} = (1,2,1) + \lambda(1,2,2)$ passing through A and B and

$\underset{\sim}{r_2} = (1,2,1) + \mu(2,1,2)$ passing through A and D.

The coordinates of D are $(5,4,5)$.



(i) Find the two possible coordinates of B. **3**

(ii) Find the exact area of the rhombus. **3**

Question 13 continues on page 10

Question 13 (continued)

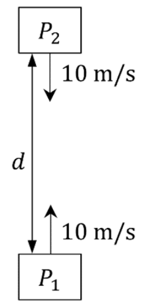
(c) A unit mass is projected vertically in a medium where the resistance to motion is $0.1v^2$ N and $g = 10$ m/s^2.

(i) In the upward direction, show that the equation of motion is: **1**

$$\ddot{x} = -\frac{100 + v^2}{10}.$$

(ii) In the downward direction, show that the terminal velocity is 10 m/s. **1**

(iii) A unit mass, P_2, is projected vertically downward at 10 m/s, and at the same instant a second identical unit mass, P_1, is projected vertically upwards at 10 m/s. Initially they are d metres apart.



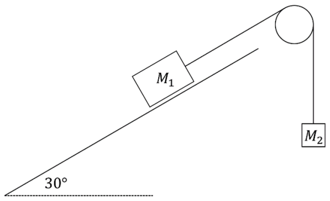
If the two unit masses collide at the instant P_1 reaches its maximum height, show that:

$$d = 5\left(\ln 2 + \frac{\pi}{2}\right) \text{ metres.}$$ **4**

End of Question 13

Question 14 (15 marks) **[USE THE QUESTION 14 WRITING BOOKLET]**

(a) Two masses, M_1 kilograms and M_2 kilograms, are attached by a light, inextensible string. The string is placed over a smooth pulley, and the mass M_1 rests on a smooth frictionless plane inclined at 30 degrees to the horizontal, as shown.
The mass M_2 is suspended vertically by the string.



The mass M_1 accelerates down the inclined plane at 2 m/s^2 and $g = 10$ m/s^2.

Given $M_1 = kM_2$, find the value of k. **3**

(b) Consider the integral

$$I_n = \int_0^1 x^n e^{x^2}\,dx.$$

The recurrence relationship linking I_n and I_{n-2} can be written in the form:

$$I_n = a - f(n) \times I_{n-2},$$

where n is an integer greater than or equal to 2, a is a constant and $f(n)$ is a function of n.

Find a and $f(n)$. **3**

Question 14 continues on page 12

Question 14 (continued)

(c) Consider the expression: $4a^4 + b^4$, where a and b are positive integers.

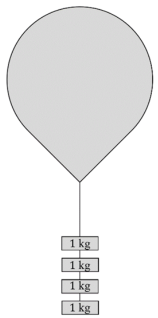
(i) If $a = b$, write down the condition for the expression to be prime. **1**

(ii) By using $p^2 + q^2 = (p+q)^2 - 2pq$,
factorise $4a^4 + b^4$ into the form $(x+y-z)(x+y+z)$. **1**

(iii) Hence prove that $4a^4 + b^4$ is composite if $a \neq b$. **3**

(d) When a large helium balloon is attached to four bricks, each of mass one kilogram, the balloon is neutrally buoyant.

This means that when the balloon and bricks are lifted from the ground and released they are stationary, neither rising nor falling back to the ground.



The string attaching the bricks is cut at some point along its length, and at least one of the bricks falls away, causing the balloon to rise vertically.

The balloon is subject to air resistance of magnitude $0.5v^2$, where v is measured in metres per second, and gravity of 10 ms^{-2}. The balloon and string have negligible mass.

Given that the terminal velocity of the balloon is $2\sqrt{5}$ m/s, how many bricks fell from the balloon when the string was cut? **4**

End of Question 14

Question 15 (15 marks) **[USE THE QUESTION 15 WRITING BOOKLET]**

(a) Sketch the locus defined by $\left|\operatorname{Arg}\left(z+i^3\right)\right| = \dfrac{3\pi}{4}$. **2**

(b) The point A, with coordinates $(0, a, b)$ lies on the line l_1, which has equation:

$$l_1: \quad \underset{\sim}{r} = 6\underset{\sim}{i} + 19\underset{\sim}{j} - \underset{\sim}{k} + \lambda(\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k})$$

(i) Find the values of a and b. **2**

(ii) The point P lies on l_1, such that OP is perpendicular to l_1, where O is the origin. Find the position vector of point P. **4**

(c) A particle is projected to just clear two poles of height h metres at horizontal distances of b and c metres from the point of projection. Air resistance is negligible.
Let v be the velocity of the projection at angle θ to the horizontal.
You may use the Cartesian equation of motion:

$$y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta$$

(i) Show that: $v^2 = \dfrac{(b+c)g\sec^2\theta}{2\tan\theta}$. **2**

(ii) Hence, or otherwise, show that: $\tan\theta = \dfrac{h(b+c)}{bc}$. **3**

(iii) What is the simplest expression in terms of h, b and c, for the greatest height the particle reaches? **2**

End of Question 15

Question 16 (15 marks) **[USE THE QUESTION 16 WRITING BOOKLET]**

(a) (i) If $I_n = \displaystyle\int_0^1 \frac{dx}{\left(1+x^2\right)^n}$ show that for integers $n \geq 1$, $2nI_{n+1} - (2n-1)I_n = \dfrac{1}{2^n}$. **3**

(ii) Hence evaluate $I_3 = \displaystyle\int_0^1 \frac{dx}{\left(1+x^2\right)^3}$. **2**

(b) For a set of POSITIVE, SINGLE-DIGIT INTEGERS,

The MMR (mean-median ratio) = $\dfrac{\text{mean}}{\text{median}}$ (i.e. the arithmetic mean divided by the median).

(i) Show that every set of three positive, single-digit integers has MMR > $\dfrac{2}{3}$. **2**

A set of four positive, single-digit integers has MMR = $\dfrac{4}{5}$.

(ii) Show that, when arranged in ascending order, the middle two integers sum to a multiple of 5. **3**

(iii) Hence find both possible sets of four positive, single-digit integers with MMR = $\dfrac{4}{5}$. **1**

(c) If k, m, n are integers such that $2 \leq k \leq m < n$, show that: **4**

$$\frac{m!}{m^k(m-k)!} < \frac{n!}{n^k(n-k)!}$$

End of Paper

1. $z \approx 2e^{-\frac{\pi}{3}i}$

$\frac{1}{z} = z^{-1} \approx \frac{1}{2}e^{\frac{\pi}{3}i}$

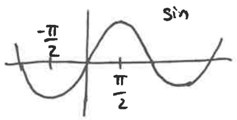
C

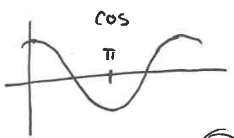
2. $\int (\sec x \tan x)\sec^{99}x\,dx$

$= \frac{\sec^{100}x}{100} + C$

C

3. Is the function increasing in the given interval?





only increasing for B

4. Look for direction vectors whose dot product is 0.

$\begin{pmatrix}-1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\0\end{pmatrix} = -2+2+0 = 0$

A

5. acceleration: $v\frac{dv}{dx} = f(x)f'(x)$

D

6. $z = 3e^{\frac{\pi}{6}i}$

$(\bar{z})^3 = \left(3e^{-\frac{\pi}{6}i}\right)^3 = 27e^{-\frac{\pi}{2}i} = 27(-i) = -27i$

A

7. $\frac{2\pi}{n} = \frac{2\pi}{3}$ $\therefore n = 3$ so $\ddot{x} = -n^2x$

$= -9x$

Extremeties of motion are $x = \pm 1$ and so is max acceleration and force.

Max $\ddot{x} = -9(-1) = 9$ Max $F = m\ddot{x} = 1\times 9 = 9$

B

8. $2-i$ is root, so $2+i$ is too.

Sum of roots is 0, so third root α:

$\alpha + 2 - i + 2 + i = 0 \implies \alpha = -4$

prod. of roots: $(2-i)(2+i)(-4) = \frac{-q}{1}$

$-20 = -q$

$q = 20$

substitute $z = -4$: $(-4)^3 + p(-4) + 20 = 0$

$p = -11$ B

9. $t = \frac{\pi}{4}$: $r(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}\underset{\sim}{i} - \frac{1}{\sqrt{2}}\underset{\sim}{j}$ $\therefore$ C or D

$t = \frac{5\pi}{4}$: $r(\frac{5\pi}{4}) = \frac{-1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j}$ $\therefore$ C or D

$t = \frac{\pi}{2}$: $r(\frac{\pi}{2}) = \underset{\sim}{i}$ $\therefore$ C

10. Test A: $\omega = z - \bar{z} = (x+iy) - (x-iy)$

$= 2iy$ (purely imaginary) OK

Test B: $\omega = z + \bar{z} = x+iy+x-iy$

$= 2x$ (purely real) OK

Test C: Let $z = e^{\frac{\pi}{3}i}$, $k = 1$

$\omega = e^{\frac{\pi}{3}i} + e^{-\frac{\pi}{3}i}$

$= 2\cos\frac{\pi}{3} = 1$

so $|z| = |\omega| = 1$ Incorrect!

C

11. a) i) $|wz| = |w||z|$

$= 2\times 4$

$= 8$

ii) $\frac{z}{w} = \frac{4e^{\frac{\pi}{3}i}}{2e^{\frac{\pi}{6}i}} = \frac{4}{2}e^{(\frac{\pi}{3}-\frac{\pi}{6})i}$

$= 2e^{\frac{\pi}{6}i}$

iii) Since $\frac{z}{w} = w$, $w^2 = z$

$= 4e^{\frac{\pi}{3}i}$

b) i)

$5x^3 - 3x^2 + 2x - 1 \equiv Ax(x^2+1) + B(x^2+1)$
$+ (Cx+D)x^2$

Let $x = 0$:

$-1 = B$

Let $x = i$:

$-5i + 3 + 2i - 1 = -Ci - D$

$C = 3$, $D = -2$

Let $x = 1$:

$5 - 3 + 2 - 1 = 2A + -1(2) + 3 - 2$

$3 = 2A - 1$

$A = 2$

ii) $I = \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{3x-2}{x^2+1}\right)dx$

$= 2\ln|x| + \frac{1}{x} + \frac{3}{2}\int\frac{2x}{x^2+1}dx$

$-2\int\frac{1}{x^2+1}dx$

$I = 2\ln|x| + \frac{1}{x} + \frac{3}{2}\ln(x^2+1)$

$-2\tan^{-1}x + C$

c) i) By the triangle inequality:

$3x + 1 < (2x-1) + (5-x)$

$2x < 3$

$x < \frac{3}{2}$ .... (1)

and

$2x - 1 < (5-x) + (3x+1)$

$-1 < 6$ (true but unhelpful!)

and

$5 - x < (2x-1) + (3x+1)$

$5 < 6x$

$\frac{5}{6} < x$ .... (2)

Combining (1) and (2):

$\frac{5}{6} < x < \frac{3}{2}$

ii) Try $AC = BC$: $2x - 1 = 5 - x$

$3x = 6$

$x = 2$

Doesn't satisfy (i).

Try $AB = AC$: $2x - 1 = 3x + 1$

$-2 = x$

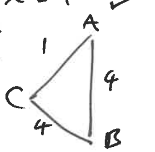
Doesn't satisfy (i).

Try $AB = BC$: $5 - x = 3x + 1$

$x = 1$ ✓

So, $\cos A = \frac{1^2+4^2-4^2}{2\times 1\times 4}$

$= \frac{1}{8}$



(d) Now $\int_0^a \frac{1}{1+\sin x}\,dx = 1$

Let $t = \tan\left(\frac{x}{2}\right)$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2(x/2)$$
$$= \frac{1+t^2}{2}$$
$$dx = \frac{2\,dt}{1+t^2}$$

$$I = \int_0^{\tan(\frac{a}{2})} \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^{\tan(\frac{a}{2})} \frac{2\,dt}{1+t^2+2t}\,dt$$

$$= \int_0^{\tan(\frac{a}{2})} \frac{2}{(1+t)^2}\,dt$$

$$= \left[-2(1+t)^{-1}\right]_0^{\tan(a/2)}$$

$$= \frac{-2}{1+\tan(\frac{a}{2})} - \left(\frac{-2}{1}\right)$$

$$= \frac{-2}{1+\tan(\frac{a}{2})} + 2$$

But $I = 1$ so

$$1 = \frac{-2}{1+\tan(\frac{a}{2})} + 2$$

$$\therefore -1 = \frac{-2}{1+\tan(a/2)}$$

$$\therefore 1+\tan\left(\frac{a}{2}\right) = 2$$

$$\tan\left(\frac{a}{2}\right) = 1$$

$$\therefore \frac{a}{2} = \frac{\pi}{4}$$

$$a = \frac{\pi}{2}$$

Q12 (a)

(i) $(1+ia)^4 = (1)\times 1^4 \times (ia)^0 + (4)\times 1^3 \times (ai)^1 + (6)\times 1^2 \times (ai)^2 + (4)\times 1 \times (ai)^3 + (1)\times 1^0 \times (ai)^4$

$$= 1 + 4ai - 6a^2 - 4a^3 i + a^4$$

(ii) $(1+ai)^4 = (1-6a^2+a^4) + i(4a-4a^3)$

$$= (1-6a^2+a^4) + i(1-a^2)4a$$

Now $(1+ai)^4$ will be purely real if $4a(1-a^2) = 0$.

Hence when $a = 0, 1, -1$

(b) Solve $z^2 = |z|^2 - 4$

Let $z = x+iy$

$$(x+iy)^2 = \left(\sqrt{x^2+y^2}\right)^2 - 4$$

$$x^2 - y^2 + 2xyi = x^2+y^2-4$$

$$2xyi = 2y^2 - 4$$

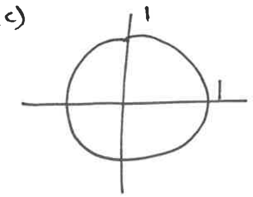
$$xyi = y^2 - 2$$

Now $xy = 0$ and $0 = y^2 - 2$

$$2 = y^2$$
$$y = \pm\sqrt{2}$$
$$x = 0$$

$$\therefore z = \pm\sqrt{2}\,i$$

(c)



$z^3 = i$

(i) $(z_1)^3 = \left(e^{i\frac{\pi}{6}}\right)^3$

$= e^{i\pi/2}$

$= \cos(\pi/2) + i\sin(\pi/2)$

$= 0 + i$

$= i$

$\therefore z_1$ is a cube root of i

$(z_2)^3 = \left(e^{i\frac{5\pi}{6}}\right)^3$

$= e^{i\,5\pi/2}$

$= \cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right)$

$= 0 + i$

$= i$

$\therefore z_2$ is a cube root of i

$(z_3)^3 = (-i)^3$

$= (-i)^2 \times (-i)$

$= -1 \times -i$

$= i$

$\therefore z_3$ is a cube root of i

(ii) RTP that

$(z_3)^2 = (z_1 i)^2 + (z_2 i)^2$

LHS $= (-i)^2$

$= -1$

RHS $= \left(i e^{i\frac{\pi}{6}}\right)^2 + \left(i e^{i\frac{5\pi}{6}}\right)^2$

$= -e^{i\frac{\pi}{3}} - e^{i\frac{5\pi}{3}}$

$= -\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) - \left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i - \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$= -1$

$\therefore$ LHS $=$ RHS

d) i) $y' = 6x^2 - 30x + 24$

$0 = 6(x^2 - 5x + 4)$

$0 = (x-4)(x-1)$

stationary points at $x = 1, 4$

$x = 1$:

$y = 2 - 15 + 24 + 41$

$= 52$

$x = 4$:

$y = 2(4)^3 - 15(4)^2 + 24(4) + 41$

$= 25$

$\therefore$ Turning points at $(1, 52), (4, 25)$

So, both turning points are above the x-axis, the graph is correct and there is only one real root.

ii) $f(-1) = -2 - 15 - 24 + 41$

$= 0$

$\therefore (x+1)$ is a factor

$$
\begin{array}{r}
2x^2 - 17x + 41 \\
x+1 \,\big)\, 2x^3 - 15x^2 + 24x + 41 \\
2x^3 + 2x^2 \\
\hline
-17x^2 + 24x \\
-17x^2 - 17x \\
\hline
41x + 41 \\
41x + 41 \\
\hline
0
\end{array}
$$

$\therefore y = (x+1)(2x^2 - 17x + 41)$

$= 2(x+1)\left(x - \left(\frac{17}{4} + \frac{\sqrt{39}i}{4}\right)\right)\left(x - \left(\frac{17}{4} - \frac{\sqrt{39}i}{4}\right)\right)$

(by quadratic formula)

$\therefore$ roots are:

$-1, \left(\frac{17}{4} + \frac{\sqrt{39}i}{4}\right), \left(\frac{17}{4} - \frac{\sqrt{39}i}{4}\right)$

and sum of roots is:

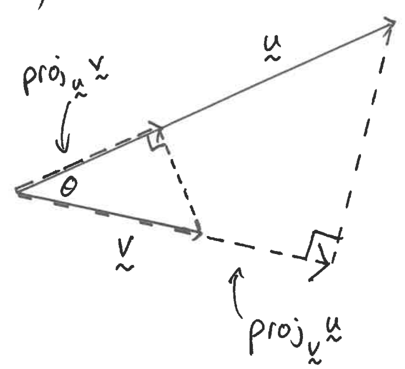
$-1 + \frac{17}{4} + \frac{\sqrt{39}i}{4} + \frac{17}{4} - \frac{\sqrt{39}i}{4}$

$= 7.5$

So Mr Peattie is correct

(iii) Mr Carruthers only saw the real root on the graph, not the two complex roots with positive real parts. They are a conjugate pair, so adding them gives a positive real number (8.5) which, when added to the third root does give 7.5.

13. a)



$|proj_{u} v| = |v| \cos\theta$

$2|proj_{u} v| = 2|v| \cos\theta$ .... (1)

and

$|proj_{v} u| = |u| \cos\theta$

$|proj_{v} u| = 2|v| \cos\theta$ .... (2)

By (1) and (2):

$|proj_{v} u| = 2|proj_{u} v|$

b) i) $A = (1,2,1)$ (when $\mu = \lambda = 0$)

$D = (5,4,5)$

$|AD| = \sqrt{(5-1)^2+(4-2)^2+(5-1)^2}$

$= 6$

For some λ, point B is $(1+\lambda,\ 2+2\lambda,\ 1+2\lambda)$

and $\vec{AB}$ is

$$\begin{pmatrix}1+\lambda\\2+2\lambda\\1+2\lambda\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}\lambda\\2\lambda\\2\lambda\end{pmatrix}$$

so $|\vec{AB}| = \sqrt{\lambda^2+(2\lambda)^2+(2\lambda)^2} = 6$

$\lambda^2 + 4\lambda^2 + 4\lambda^2 = 36$

$9\lambda^2 = 36$

$\lambda = \pm 2$

So, B could be:

$(1+2,\ 2+2(2),\ 1+2(2)) = (3,6,5)$

or

$(1-2,\ 2+2(-2),\ 1+2(-2)) = (-1,-2,-3)$

ii) Choose $\lambda = 2$:

$\vec{AB} = \begin{pmatrix}2\\4\\4\end{pmatrix}$ $\quad \vec{AD} = \begin{pmatrix}5\\4\\5\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}4\\2\\4\end{pmatrix}$

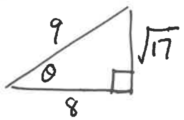
Let $\angle BAD = \theta$

Now, $\vec{AB} \cdot \vec{AD} = |\vec{AB}||\vec{AD}| \cos\theta$

$\begin{pmatrix}2\\4\\4\end{pmatrix} \cdot \begin{pmatrix}4\\2\\4\end{pmatrix} = 6 \times 6 \times \cos\theta$

$32 = 36 \cos\theta$

$\cos\theta = \frac{8}{9}$



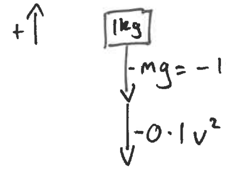
$\sin\theta = \frac{\sqrt{17}}{9}$

Area $ABCD = 2 \times \frac{1}{2} |\vec{AB}||\vec{AD}| \sin\theta$

$= 6 \times 6 \times \frac{\sqrt{17}}{9}$

$= 4\sqrt{17}$

c) i)



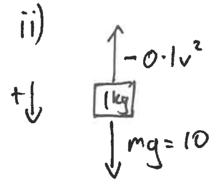
$F = m\ddot{x}$

$-10 - 0.1v^2 = 1 \times \ddot{x}$

$\ddot{x} = \frac{-100 - v^2}{10}$

$= -\frac{100+v^2}{10}$

ii)



Terminal velocity reached when $F = 0$ (and $a = \ddot{x} = 0$).

$F = 10 - 0.1v^2$

$0 = 10 - 0.1v_T^2$

$v_T^2 = 100$

$v_T = 10\,m/s$

iii) For P_1, use (i).

$\ddot{x}_1 = -\frac{100+v^2}{10}$

$v\frac{dv}{dx_1} = -\frac{100+v^2}{10}$

$\frac{-10}{100+v^2} \cdot v\,dv = dx_1$

$-5\int_{10}^{0} \frac{2v}{100+v^2}\,dv = \int_0^{x_1} dx_1$

$-5\left[\ln(100+v^2)\right]_{10}^{0} = x_1$

$x_1 = -5\left[\ln 100 - \ln(100+10^2)\right]$

$= -5 \ln\frac{100}{200}$

$= 5\ln 2$

How long did P_1 take to stop?

$\frac{dv}{dt} = -\frac{100+v^2}{10}$

$\int_{10}^{0} \frac{-10\,dv}{100+v^2} = \int_0^t dt$

$-10\int_{10}^{0} \frac{dv}{10^2+v^2} = t$

$t = -10 \cdot \frac{1}{10}\left[\tan^{-1}\frac{v}{10}\right]_{10}^{0}$

$= \left[\tan^{-1}\frac{v}{10}\right]_0^{10}$

$= \tan^{-1}1 - \tan^{-1}0$

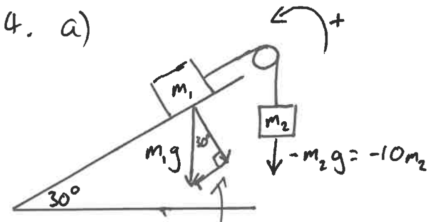
$= \frac{\pi}{4}$

Since P_2 starts at terminal velocity, its velocity is a constant 10m/s for $\frac{\pi}{4}$ s.

$x_2 = 10 \times \frac{\pi}{4} = \frac{5\pi}{2}$, so

$x_1 + x_2 = 5\ln 2 + \frac{5\pi}{2} = 5\left(\ln 2 + \frac{\pi}{2}\right)$

14. a)



$m_1 g \sin 30°$

$= m_1 \times 10 \times \frac{1}{2}$

$= 5m_1$

$F = ma$

$5m_1 - 10m_2 = (m_1 + m_2) \times 2$

$5M_1 - 10M_2 = 2M_1 + 2M_2$

$3M_1 = 12M_2$

$M_1 = 4M_2$

$\therefore k = 4$

b) $I_n = \frac{1}{2}\int_0^1 x^{n-1} . 2xe^{x^2}\,dx$

$= \frac{1}{2}\left[x^{n-1} . e^{x^2}\right]_0^1 - \frac{1}{2}\int_0^1 (n-1)x^{n-2}e^{x^2}\,dx$

$= \frac{e}{2} - \frac{n-1}{2}\int_0^1 x^{n-2}e^{x^2}\,dx$

$= \frac{e}{2} - \frac{n-1}{2} I_{n-2}$

$\therefore a = \frac{e}{2}, \quad f(n) = \frac{n-1}{2}$

c) i) If $a = b$,

$4a^4 + b^4 = 5a^4$

Only prime if $a^4 = 1$,

i.e $a = 1$

ii) $4a^4 + b^4$

$= (2a^2)^2 + (b^2)^2$

$= (2a^2 + b^2)^2 - 2(2a^2)(b^2)$

$= (2a^2 + b^2)^2 - (2ab)^2$

$= (2a^2 + b^2 - 2ab)(2a^2 + b^2 + 2ab)$

iii) So, $4a^4 + b^4$ can be expressed as the product of two factors which will certainly be composite if both factors are greater than 1.

$2a^2 + b^2 + 2ab$ is the sum of 3 positive integers so must be greater than 1.

Now, consider:

$2a^2 + b^2 - 2ab - 1$

$= a^2 - 2ab + b^2 + a^2 - 1$

$= (a-b)^2 + a^2 - 1$

> 0 since, given $a \neq b$, $(a-b)^2 > 0$

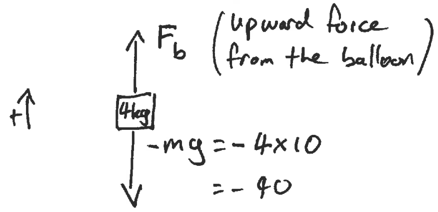
and $a^2 - 1 \geq 0$

$\therefore 2a^2 + b^2 - 2ab - 1 > 0$

and $2a^2 + b^2 - 2ab > 1$

Thus $4a^4 + b^4$ is the product of two integers greater than 1 and must be composite.

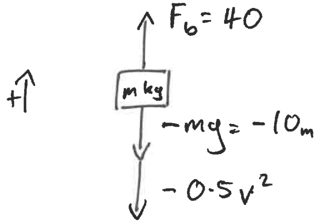
d) In its neutrally buoyant state, net force must be 0.



$F_b + (-40) = 0$

$F_b = 40$

Now, once the string is cut:



$F = ma$

$40 - 10m - \frac{1}{2}v^2 = ma$

But, when terminal velocity is reached, $a = 0$ and $v = 2\sqrt{5}$.

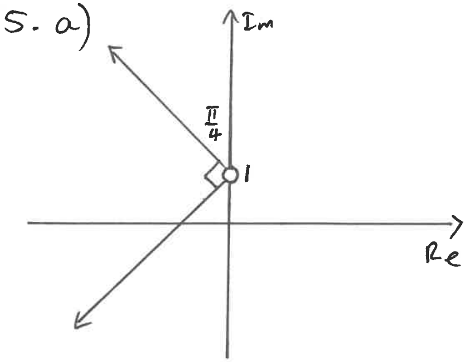
so, $40 - 10m - \frac{1}{2}(2\sqrt{5})^2 = 0$

$40 - 10 = 10m$

$m = 3$

So 1 brick must have fallen.

15. a)



$|\text{Arg}(z - i)| = \frac{3\pi}{4}$

$\text{Arg}(z - i) = \pm\frac{3\pi}{4}$

b) i) $(0, a, b)$ satisfies, so

$\begin{pmatrix}0\\a\\b\end{pmatrix} = \begin{pmatrix}6\\19\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\4\\-2\end{pmatrix}$

$0 = 6 + \lambda(1) \implies \lambda = -6$

$a = 19 + (-6)(4)$

$= -5$

$b = -1 + (-6)(-2)$

$= 11$

ii) $\vec{OP} \cdot \begin{pmatrix}1\\4\\-2\end{pmatrix} = 0$

$\begin{pmatrix}6+\lambda\\19+4\lambda\\-1-2\lambda\end{pmatrix} \cdot \begin{pmatrix}1\\4\\-2\end{pmatrix} = 0$ (since P lies on ℓ_1)

$6 + \lambda + 76 + 16\lambda + 2 + 4\lambda = 0$

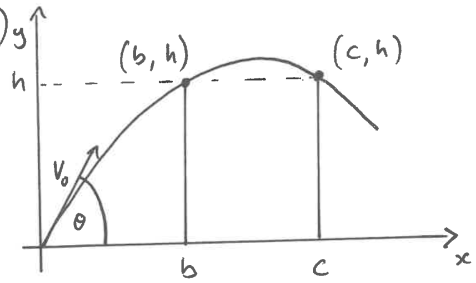
$21\lambda = -84$

$\lambda = -4$

so $\vec{OP} = \begin{pmatrix}6+(-4)\\19+4(-4)\\-1-2(-4)\end{pmatrix} = \begin{pmatrix}2\\3\\7\end{pmatrix}$

$\therefore$ P is $(2, 3, 7)$

c)



i) (b, h) and (c, h) must satisfy the given equation:

$$h = b\tan\theta - \frac{gb^2}{2v^2}\sec^2\theta \quad \ldots (1)$$

$$h = c\tan\theta - \frac{gc^2}{2v^2}\sec^2\theta \quad \ldots (2)$$

Equating (1) and (2):

$$b\tan\theta - \frac{gb^2}{2v^2}\sec^2\theta = c\tan\theta - \frac{gc^2}{2v^2}\sec^2\theta$$

$$b\tan\theta - c\tan\theta = \frac{gb^2}{2v^2}\sec^2\theta - \frac{gc^2}{2v^2}\sec^2\theta$$

$$(b-c)\tan\theta = \frac{g(b^2-c^2)}{2v^2}\sec^2\theta$$

$$\cancel{(b-c)}\tan\theta = \frac{g\cancel{(b-c)}(b+c)}{2v^2}\sec^2\theta$$

$$v^2\tan\theta = \frac{(b+c)g\sec^2\theta}{2}$$

$$v^2 = \frac{(b+c)g\sec^2\theta}{2\tan\theta} \quad \ldots (3)$$

ii) Substitute (3) into (1):

$$h = b\tan\theta - \frac{gb^2\sec^2\theta}{2\,\frac{(b+c)g\sec^2\theta}{2\tan\theta}}$$

$$h = b\tan\theta - \frac{b^2\tan\theta}{(b+c)}$$

$$h = \frac{b^2\tan\theta + bc\tan\theta}{b+c} - \frac{b^2\tan\theta}{b+c}$$

$$h = \frac{bc\tan\theta}{b+c}$$

$$h(b+c) = bc\tan\theta$$

$$\tan\theta = \frac{h(b+c)}{bc}$$

iii) Since neglecting air resistance, max height is reached half way between b and c, i.e. $x = \frac{b+c}{2}$, so

$$y = \frac{b+c}{2}\tan\theta - \frac{(b+c)^2}{4}\cdot\frac{g}{2v^2}\sec^2\theta$$

$$= \frac{b+c}{2}\cdot\frac{h(b+c)}{bc} - \frac{(b+c)^2 g\sec^2\theta}{8\,\frac{(b+c)g\sec^2\theta}{2\tan\theta}}$$

$$= \frac{(b+c)^2h}{2bc} - \frac{(b+c)\tan\theta}{4}$$

$$= \frac{(b+c)^2h}{2bc} - \frac{(b+c)}{4}\cdot\frac{h(b+c)}{bc}$$

$$= \frac{2h(b+c)^2 - h(b+c)^2}{4bc} = \frac{h(b+c)^2}{4bc}$$

Q16 (a)

(i) Let $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$ for integers $n \geq 1$.

RTS that $2n I_{n+1} - (2n-1)I_n = \frac{1}{2^n}$

Now $I_n = \int_0^1 1\times\frac{1}{(1+x^2)^n}\,dx$

$$= \int_0^1 1\times(1+x^2)^{-n}\,dx$$

$$= \left[x(1+x^2)^{-n}\right]_0^1 - \int_0^1 x.(-n)(1+x^2)^{-n-1}(2x)\,dx$$

$$= \left(\frac{1}{2^n} - 0\right) + n\int_0^1 \frac{2x^2}{(1+x^2)^{n+1}}$$

$$= \frac{1}{2^n} + 2n\int_0^1\frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{(1+x^2)-1}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1\frac{dx}{(1+x^2)^n} - 2n\int_0^1\frac{dx}{(1+x^2)^{n+1}}$$

$$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$$

$$I_n - 2nI_n = \frac{1}{2^n} - 2nI_{n+1}$$

$$2nI_{n+1} + (1-2n)I_n = \frac{1}{2^n}$$

$\therefore 2nI_{n+1} - (2n-1)I_n = \frac{1}{2^n}$, as required.

ii) $I_1 = \int_0^1 \frac{1}{1+x^2}\,dx$

$= \left[\tan^{-1}x\right]_0^1$

$= \frac{\pi}{4}$

For $n=1$,

$2(1)I_2 - (2(1)-1)I_1 = \frac{1}{2}$

$2I_2 - \frac{\pi}{4} = \frac{1}{2}$

$2I_2 = \frac{2+\pi}{4}$

$I_2 = \frac{2+\pi}{8}$

For $n=2$,

$2(2)I_3 - (2(2)-1)I_2 = \frac{1}{4}$

$4I_3 - 3\left(\frac{2+\pi}{8}\right) = \frac{1}{4}$

$4I_3 - \left(\frac{6+3\pi}{8}\right) = \frac{2}{8}$

$4I_3 = \frac{8+3\pi}{8}$

$I_3 = \frac{8+3\pi}{32}$

b) i) Let the integers be:

$1 \leq a \leq b \leq c \leq 9$

And, let's assume

$MMR \leq \frac{2}{3}$

$\frac{\frac{a+b+c}{3}}{b} \leq \frac{2}{3}$

$\frac{a+b+c}{3b} \leq \frac{2}{3}$

$3a+3b+3c \leq 6b$

$a+b+c \leq 2b$

$a-b+c \leq 0$

$a+(c-b) \leq 0$

which is a contradiction!
because, since $c \geq b$, $c-b \geq 0$
and the LHS must be positive

$\therefore MMR > \frac{2}{3}$

ii) Let the integers be:

$1 \leq a \leq b \leq c \leq d \leq 9$

$MMR = \frac{\frac{a+b+c+d}{4}}{\frac{b+c}{2}}$

$\frac{4}{5} = \frac{a+b+c+d}{2(b+c)}$

$8b+8c = 5a+5b+5c+5d$

$3b+3c = 5a+5d$

$3(b+c) = 5(a+d)$

Since RHS is divisible by 5, the LHS must be too.
But 3 and 5 are coprime so for LHS to be a multiple of 5, $b+c$ must be a multiple of 5.

iii) $b+c$ must be a multiple of 5. Possible combinations are:

1, 4   1, 9   2, 3   2, 8

3, 7   4, 6   5, 5   6, 9

7, 8

If $b=1$, $c=4$ then

$3(1+4) = 5(a+d)$

$3 = a+d$

which is impossible if
$a \leq b \leq c \leq d$.

But for $b=c=5$,

$3(5+5) = 5(a+d)$

$a+d = 6$

possible: $a=1$, $d=5$

So we have
1, 5, 5, 5
and similarly
1, 7, 8, 8

c) Consider the LHS and expand the factorials:

$$LHS = \frac{\overbrace{m(m-1)(m-2)\ldots(m-k+1)}^{k \text{ terms}}}{m^k} \cdot \frac{(m-k)\ldots(2)(1)}{(m-k)\ldots(2)(1)}$$

$= \frac{\cancel{m}}{\cancel{m}} \cdot \frac{m-1}{m} \cdot \frac{m-2}{m} \cdots \frac{m-(k-1)}{m}$

Similarly,

$RHS = \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-(k-1)}{n}$

each with $k-1$ factors.

Let's compare corresponding factors.
(ie $\frac{m-1}{m}$ and $\frac{n-1}{n}$ etc)

Consider:

$\frac{m-r}{m} - \frac{n-r}{n}$ for some $r \in \mathbb{Z}^+$

$= \frac{mn - nr - mn + mr}{mn}$

$= \frac{mr - nr}{mn}$

$= \frac{r(m-n)}{mn} < 0$ since $m<n$

$\therefore \frac{m-r}{m} < \frac{n-r}{n}$

Thus $\frac{m-1}{m} < \frac{n-1}{n}$, $\frac{m-2}{m} < \frac{n-2}{n}$ etc.

Since each corresponding (positive) factor on LHS is less than its counterpart on RHS, LHS < RHS.