
Student Number

Barker College

2024 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Staff Involved:

- ARM - Mr Mallam
- JZT - Mrs Thomas
- BHC - Mr Carruthers*

PM THURSDAY 15TH AUGUST

45 copies

Topics Covered

Complex Numbers	Proof	Integration
Vectors	Mechanics	

General Instructions:

- Write your Student Number at the top of this page and on all writing booklets
- Reading time - 10 minutes
- Working time - 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A separate reference booklet is provided
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale
- For questions in Section II, show relevant mathematical reasoning and / or calculations

Total marks: 100

Section I - 10 marks (pages 2 - 5)

- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II - 90 marks (pages 6 - 12)

- Attempt Questions 11 - 16
- Allow about 2 hours and 45 minutes for this section
- Answer each question in a SEPARATE writing booklet

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 Which expression is equivalent to $\int \sin^3 x \, dx$?

A. $\frac{1}{3}\cos^3 x + \cos x + c$

B. $\frac{1}{3}\cos^3 x - \cos x + c$

C. $\frac{1}{3}\sin^3 x + \sin x + c$

D. $\frac{1}{3}\sin^3 x - \sin x + c$

2 A particle moves along a straight line with an acceleration of $\frac{2}{V}$ m/s^2, where V is the velocity at any instant in m/s.

The initial velocity of the particle is -1 m/s. The particle will move to the:

A. left, increasing in speed

B. left, stop, then move to the right

C. right, increasing in speed

D. right, stop, then move to the left

3 What are the values of the real numbers p and q such that $1-i$ is a root of the equation $z^3 + pz + q = 0$?

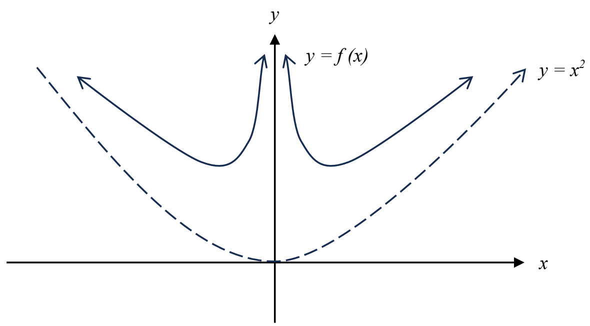
A. $p = 2$ *and* $q = 4$

B. $p = -2$ *and* $q = 4$

C. $p = 2$ *and* $q = -4$

D. $p = -2$ *and* $q = -4$

4 For the graph shown in the diagram below, a possible equation is:



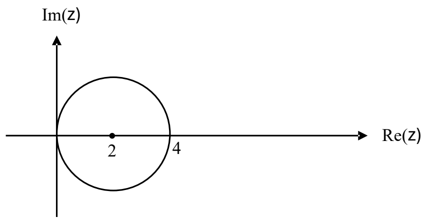
A. $y = \dfrac{x^3 + k}{x}, k > 0$

B. $y = \dfrac{x^3 + k}{x}, k < 0$

C. $y = \dfrac{x^4 + k}{x^2}, k > 0$

D. $y = \dfrac{x^4 + k}{x^2}, k < 0$

5 Which of the following is the equation of the circle below?



A. $(z-2)(\bar{z}-2i) = 4$

B. $(z-2)(\bar{z}-2) = 4$

C. $(z+2)(\bar{z}-2) = 4$

D. $(z+2i)(\bar{z}-2i) = 4$

6 The position of a moving object is given by the cartesian coordinates $(3t, e^t)$, where t is the time in seconds.

Its acceleration is:

A. constant in both magnitude and direction.

B. constant in magnitude only.

C. constant in direction only.

D. constant in neither magnitude or direction.

7 Using an appropriate substitution, $\int e^{2x}\sqrt{e^x - 1}\, dx$ is equivalent to:

A. $\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

B. $\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$

C. $\int \left(u^3 + u\right) du$

D. $\int \left(u^3 - u\right) du$

8 Let $z = \cos\theta + i\sin\theta$, where θ is acute.

The value of $\arg\left(z^2 - z\right)$ is:

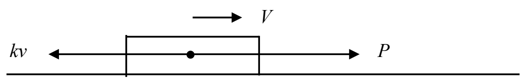
A. $\dfrac{\pi}{2} - \dfrac{\theta}{2}$

B. $\dfrac{\pi}{2} + \dfrac{\theta}{2}$

C. $\dfrac{\pi}{2} - \dfrac{3\theta}{2}$

D. $\dfrac{\pi}{2} + \dfrac{3\theta}{2}$

9 The diagram below shows a box of mass m being pushed by a constant force P across a table.



The box experiences a resistive force due to friction which is proportional to its velocity, that is kv, and it is in the opposite direction.

There are no other forces acting on the box. By considering the forces on the box, its limiting velocity is given by:

A. 0

B. $\dfrac{P}{mk}$

C. $\dfrac{P}{k}$

D. $\sqrt{\dfrac{P}{k}}$

10

$$J = \int_0^1 \sqrt{1-x^4}\, dx$$

$$K = \int_0^1 \sqrt{1+x^4}\, dx$$

$$L = \int_0^1 \sqrt{1-x^8}\, dx$$

Which of the following statements is true for the definite integrals shown above?

A. $J < L < 1 < K$

B. $J < L < K < 1$

C. $L < J < 1 < K$

D. $L < J < K < 1$

End of Section I

Section II

90 marks
Attempt Questions 11 - 16
Allow about 2 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 - 16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **[START A NEW BOOKLET]**

(a) What is the cartesian equation of the line: **2**

$$\underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j} + \lambda\left(-3\underset{\sim}{i} + 6\underset{\sim}{j}\right) \ ?$$

(b) Find r and θ such that: **2**

$$e^{i\pi} - e^{i\frac{\pi}{2}} = re^{i\theta}$$

(c) Find $\int x \sin 2x \, dx$ **3**

(d) For integer m, prove by contrapositive that if m^2 is not divisible by 4, then m is odd. **2**

(e) Prove that a particle with $x = 3\cos^2 t$ is in simple harmonic motion. State its centre of motion and its amplitude. **3**

(f) Three vertices of a parallelogram are $O(0,0,0)$, $A(2,2,1)$ and $B(1,2,2)$. **3**
Find **all** the possible positions of C, the fourth vertex.

Question 12 (15 marks) **[START A NEW BOOKLET]**

(a) (i) Show that $\sqrt{2}-i$ is a root of the equation $z^3-\left(\sqrt{2}-i\right)z^2+8z-8\sqrt{2}+8i=0$ **1**

(ii) Find the other two solutions of the equation. **2**

(b) (i) Use de Moivre's Theorem and the Binomial Theorem to express $\cos 5\theta$ and $\sin 5\theta$ as powers of $\cos\theta$ and $\sin\theta$. **2**

(ii) Hence express $\tan 5\theta$ as a rational function of t, where $t=\tan\theta$. **2**

(iii) By considering the roots of $\tan 5\theta=0$, deduce that: **2**

$$\tan\left(\frac{\pi}{5}\right)\tan\left(\frac{2\pi}{5}\right)\tan\left(\frac{3\pi}{5}\right)\tan\left(\frac{4\pi}{5}\right)=5$$

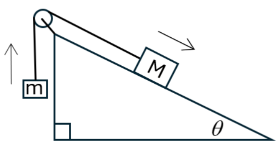
(c) Any three dimensional vector can be expressed as scalar multiples of any three non-parallel vectors.
Find λ, μ, υ such that $\underset{\sim}{d}=\lambda\underset{\sim}{a}+\mu\underset{\sim}{b}+\upsilon\underset{\sim}{c}$ where: **3**

$\underset{\sim}{d}=5\underset{\sim}{i}+5\underset{\sim}{j}+5\underset{\sim}{k}$, $\quad\underset{\sim}{a}=2\underset{\sim}{i}+3\underset{\sim}{j}+\underset{\sim}{k}$, $\quad b=\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}$, $\quad\underset{\sim}{c}=-\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}$

(d) A block of mass M kg is connected to another block of mass m kg (where $M>m$) by a light inextensible string. The string is placed over a smooth pulley on an inclined plane, at an angle of θ to the horizontal, as shown. There is no friction between the surface of the inclined plane and the block with mass M kg.

Initially the system is held in equilibrium, but when released, the block with mass M kg begins to slide down the incline. Neglect the mass of the pulley.

The acceleration due to gravity is g. **3**



Find the simplest expression for the tension, T, in the string.

Question 13 (15 marks) **[START A NEW BOOKLET]**

(a) Let $I_n=\int_0^1\frac{x^n}{(1+x)^2}\,dx$ for $n\geq 0$.

(i) For $n\geq 2$, show that $I_n=\frac{1}{2(n-1)}-\frac{n}{n-1}I_{n-1}$ **4**

(ii) Hence find $\int_0^1\frac{x^3}{(1+x)^2}\,dx$. **2**

(b) Find the shortest distance from the origin to the line through $A(1,3,1)$ and $B(0,1,-1)$. **3**

(c) A body of unit mass is projected vertically upwards, under gravity, from the ground in a medium. This produces a resistance force of kv^2, where v is the velocity and k is a positive constant. The acceleration due to gravity is g.

(i) If the initial velocity is v_0, prove that the maximum height, H, of the body above the ground is:

$$H=\frac{1}{2k}\log_e\left(1+\frac{kv_0^{\,2}}{g}\right)$$ **3**

(ii) In a second projection vertically upwards of the body, it is noticed that the new maximum height reached is $2H$.

Show that the initial velocity of the second projection was $v_0\left(e^{2kH}+1\right)^{\frac{1}{2}}$ **3**

Question 14 (15 marks) **[START A NEW BOOKLET]**

(a) (i) Prove that $\sqrt{10}$ is irrational. **2**

(ii) Hence prove that $\sqrt{2} + \sqrt{5}$ is irrational. **2**

(b) (i) Prove that $\sqrt{ab} \le \dfrac{a+b}{2}$ where $a \ge 0$ and $b \ge 0$. **1**

(ii) Prove that $\sqrt{ab} \le \sqrt{\dfrac{a^2+b^2}{2}}$ **1**

(iii) Prove that $\sqrt[4]{abcd} \le \sqrt{\dfrac{a^2+b^2+c^2+d^2}{4}}$ **2**

(iv) If $1 \le x \le y$, show that $x\,(y-x+1) \ge y$ **2**

(v) Let n and k be positive integers with $1 \le k \le n$.

Prove that $\sqrt{n} \le \sqrt{k(n-k+1)} \le \dfrac{n+1}{2}$. **2**

(vi) For integers $n \ge 1$ prove that $\left(\sqrt{n}\right)^n \le n! \le \left(\dfrac{n+1}{2}\right)^n$. **3**

Question 15 (15 marks) **[START A NEW BOOKLET]**

(a) (i) Expand and simplify $(x+1)\left(x^2-x+1\right)$. **1**

(ii) Hence evaluate $\displaystyle\int_0^\infty \frac{1}{1+x^3}\,dx$. **4**

(b) A stone is projected from a point on the ground and it just clears a fence d metres away. The height of the fence is h metres.

The angle of projection is θ and the speed of projection is v m/s. Air resistance is negligible.
The displacement equations are $x = vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + vt\sin\theta$.

(i) Show that $v^2 = \dfrac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$ **2**

(ii) Show that the maximum height reached is $\dfrac{d^2\tan^2\theta}{4(d\tan\theta - h)}$ **2**

(iii) Show that the stone, when at its maximum height, will just clear the fence if $\tan\theta = \dfrac{2h}{d}$ **2**

(c) (i) Show that $\displaystyle\int_{-a}^{a} \frac{x^4}{1+e^x}\,dx = \int_{-a}^{a} \frac{x^4 e^x}{1+e^x}\,dx$. **2**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_{-2}^{2} \frac{x^4}{1+e^x}\,dx$ **2**

Question 16 (15 marks) **[START A NEW BOOKLET]**

(a) Relative to a fixed origin O, the horizontal unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are pointing due east and north respectively. A particle P, of mass 2kg, is moving under the action of a single constant force of $\underset{\sim}{F}$ newtons.

When $t=0$ seconds, the velocity of P is $\left(3\underset{\sim}{i}-5\underset{\sim}{j}\right)$ m/s and

when $t=4$ seconds, the velocity of P is $\left(11\underset{\sim}{i}+7\underset{\sim}{j}\right)$ m/s.

You may assume that $\underset{\sim}{v}=\underset{\sim}{u}+\underset{\sim}{a}t$, where $\underset{\sim}{v}$ is the velocity at time t with initial velocity $\underset{\sim}{u}$

(i) Calculate the speed of the particle when $t=0$. **1**

(ii) Determine the vector $\underset{\sim}{F}$. **1**

(iii) Find the value of t at the moment the particle is moving due east. **1**

(b) Consider two spheres S_1 and S_2.

$$S_1:(x-1)^2+(y+1)^2+(z-2)^2=64$$

$$S_2:(x+1)^2+(y-3)^2+(z+2)^2=4$$

(i) Show that S_1 and S_2 are tangential (that is, they touch at only one point). **2**

(ii) Find the coordinates of the point of contact. **2**

Question 16 continues on page 12

(c) (i) Show that $e^{in\theta}-e^{-in\theta}=2i\sin(n\theta)$ **1**

(ii) Consider the expression $S_n=1+e^{i\theta}+e^{i2\theta}+e^{i3\theta}+\ldots\ldots+e^{in\theta}$

Show that $S_n=\dfrac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}$ **1**

(iii) Show that $S_n=e^{i\left(\frac{n}{2}\right)\theta}\times\dfrac{\sin\left(\frac{n+1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}$ **3**

(iv) Deduce that:

$$\sin\theta+\sin2\theta+\ldots\ldots\sin n\theta=\frac{\sin\left(\frac{n}{2}\right)\theta\ \sin\left(\frac{n+1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}$$ **3**

End of Paper

Year 12 Extension 2 Mathematics Trial Examination 2024.

Q1 $\int \sin^3 x \, dx$

$= \int (1-\cos^2 x)\sin x \, dx$

$= \int \sin x - \cos^2 x \sin x \, dx$

$= -\cos x + \frac{1}{3}\cos^3 x + c$

$= \frac{1}{3}\cos^3 x - \cos x + c$ (B)

Q2 The initial velocity is -1 m/s. So it is moving to left. Notice that $\ddot{x} = -2$ m/s initially. So acceleration is in the same direction, so the particle will not stop, but move to the left increasing in speed. (A)

Q3 $(1-i)$ is a root of $z^3 + pz + q = 0$, so

$(1-i)^3 + p(1-i) + q = 0$

$-2i(1-i) + p - ip + q = 0$

$-2i - 2 + p - ip + q = 0$

$(p+q-2) + i(-p-2) = 0$

$(p+q-2) - i(p+2) = 0$

$\therefore p = -2$

$q = 4$

(B)

Q4 The curve approaches the parabola as an asymptote from above. Hence $k > 0$. Eliminate (B) and (D).

Notice that $y = f(x)$ is an even function.

(A) is not an even function.

(C) is an even function.

Hence (C).

Q5 The Cartesian equation of the circle is $(x-2)^2 + y^2 = 4$.

Now if $(z-2)(\bar{z}-2) = 4$ then

$((x-2)+iy)((x-2)-iy) = 4$

$(x-2)^2 - iy(x-2) + iy(x-2) + y^2 = 4$

$(x-2)^2 + y^2 = 4$

Hence (B).

Q6 $x = 3t$ $\quad y = e^t$

$\dot{x} = 3$ $\quad \dot{y} = e^t$

$\ddot{x} = 0$ $\quad \ddot{y} = e^t$

So no acceleration in x, only acceleration in y. Acceleration is constant in direction only. Hence (C)

Q7 $\int e^{2x}\sqrt{e^x - 1} \, dx$

Let $u = e^x - 1$

$du = e^x \, dx$

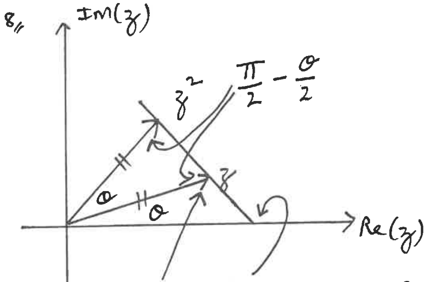
$I = \int e^x \sqrt{e^x - 1} \, (e^x \, dx)$

$= \int (u+1)\sqrt{u} \, du$

$= \int u^{3/2} + u^{1/2} \, du$

(A)

Q8



$\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$ $\quad \arg(z^2 - z) = \theta + \left(\frac{\pi}{2} + \frac{\theta}{2}\right)$

$= \frac{\pi}{2} + \frac{3\theta}{2}$

(D)

Q9 $m\ddot{x} = P - kv$

$\ddot{x} = \frac{P}{m} - \frac{kv}{m}$

Solve $\ddot{x} = 0$

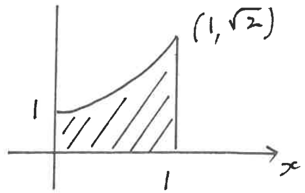
$\frac{P}{m} = \frac{kv}{m}$

$v = \frac{P}{k}$

(C)

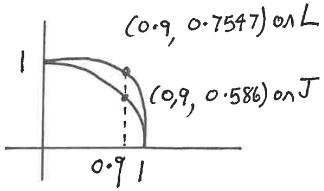
Q10 Is $K < 1$ or $k > 1$?

The graph of $y = \sqrt{1 + x^4}$ between $x = 0$ and $x = 4$ looks like this:



K is the shaded area, so $K > 1$. Eliminate (B) and (D).

which is bigger L or J?

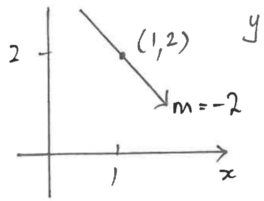


So $L > J$.

Hence (A)

Q11 (a) The Cartesian equation of the line

$\underset{\sim}{R} = \underset{\sim}{i} + 2\underset{\sim}{j} + \lambda(-3\underset{\sim}{i} + 6\underset{\sim}{j})$ is



$y - 2 = -2(x-1)$

$y = -2x + 4$

(b) $e^{i\pi} - e^{i\frac{\pi}{2}}$

$= \cos\pi + i\sin\pi - (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$= -1 - i$

$= \sqrt{2}\, e^{-i\frac{3\pi}{4}}$

Hence $r = \sqrt{2}$ and $\theta = \frac{-3\pi}{4}$

(c) $\int x \sin 2x \, dx$

$= -\frac{1}{2}\cos 2x . x - \int -\frac{1}{2}\cos 2x\,(1)\, dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\, dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

(d) The contrapositive is:

"If m is not odd, then m^2 is divisible by 4."

Suppose that m is even.

Let $m = 2k$ (k an integer).

Then $m^2 = 4k^2$,

which is divisible by 4.

So if m is even, then m^2 is divisible by 4.

Hence, if m^2 is not divisible by 4, then m is odd.

(e) $x = 3\cos^2 t$

$\dot{x} = -6\cos t \sin t$

$= -3\sin 2t$

$\ddot{x} = -6\cos 2t$

But $\cos 2t = 2\cos^2 t - 1$ so

$\ddot{x} = -6(2\cos^2 t - 1)$

$= -12\cos^2 t + 6$

$= -4(3\cos^2 t) + 6$

$= -4x + 6$

$= -4(x - \frac{3}{2})$

which is of the form

$\ddot{x} = -n^2(x - a)$

The centre of motion is $x = \frac{3}{2}$ and the amplitude is $\frac{3}{2}$.

(f) There are 3 possible positions for the fourth vertex, D, to make a parallelogram:

$\vec{OD} = \vec{OA} + \vec{OB}$

$\vec{OD} = \vec{OA} - \vec{OB}$

$\vec{OD} = \vec{OB} - \vec{OA}$

Hence $\vec{OD} = (2,2,1) + (1,2,2) = (3,4,3)$ or

$\vec{OD} = (2,2,1) - (1,2,2) = (1,0,-1)$ or

$\vec{OD} = (1,2,2) - (2,2,1) = (-1,0,1)$.

Q12//

(a) (i) By substitution,

$$(\sqrt{2}-i)^3-(\sqrt{2}-i)(\sqrt{2}-i)^3+8(\sqrt{2}-i)-8\sqrt{2}+8i$$
$$=0+8\sqrt{2}-8i-8\sqrt{2}+8i$$
$$=0$$

Hence $\sqrt{2}-i$ is a root of the equation.

(ii) The sum of the roots is $-\frac{b}{a}=\sqrt{2}-i$.

Hence $\alpha+\beta+\sqrt{2}-i=\sqrt{2}-i$
$$\alpha+\beta=0$$
$$\alpha=-\beta$$

and the product of the roots is $-\frac{d}{a}=8\sqrt{2}-8i$

Hence $\alpha\beta(\sqrt{2}-i)=8\sqrt{2}-8i$
$$\alpha\beta=\frac{8\sqrt{2}-8i}{\sqrt{2}-i}\times\frac{\sqrt{2}+i}{\sqrt{2}+i}$$
$$\alpha(-\alpha)=\frac{16-8\sqrt{2}i+8\sqrt{2}i+8}{3}$$
$$-\alpha^2=8$$
$$\alpha^2=-8$$
$$\alpha=\pm 2\sqrt{2}i$$

If $\alpha=2\sqrt{2}i$ then $\beta=-2\sqrt{2}i$, or vice versa.

(b) Now $(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta$ by de Moivre.

(i) By the Binomial Theorem
$$(\cos\theta+i\sin\theta)^5$$
$$=\cos^5\theta+5\cos^4\theta(i\sin\theta)+10\cos^3\theta(i\sin\theta)^2+10\cos^2\theta(i\sin\theta)^3+5\cos\theta(i\sin\theta)^4+(i\sin\theta)^5$$
$$=\cos^5\theta+5\cos^4\theta(i\sin\theta)-10\cos^3\theta\sin^2\theta-10\cos^2\theta(i\sin^3\theta)+5\cos\theta\sin^4\theta+i\sin^5\theta$$
$$=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta+i(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta)$$

By equating Real parts
$$\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$$
and by equating imaginary parts
$$\sin 5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$$

(ii)
$$\tan 5\theta=\frac{\sin 5\theta}{\cos 5\theta}$$
$$=\frac{5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta}{\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta}$$
(Now divide top and bottom by $\cos^5\theta$)
$$=\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-\tan^2\theta+5\tan^4\theta}$$
$$=\frac{5t-10t^3+t^5}{1-10t^2+5t^4},\text{ where } t=\tan\theta.$$

(iii) Consider solving $\tan 5\theta = 0$.

Then $5t - 10t^3 + t^5 = 0$

$t(5 - 10t^2 + t^4) = 0$

So $t = 0$ OR $t^4 - 10t^2 + 5 = 0$

From $\tan 5\theta = 0$

$5\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

$\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

Hence the four roots of $t^4 - 10t^2 + 5 = 0$

are $\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$, $\tan\frac{3\pi}{5}$, $\tan\frac{4\pi}{5}$

and the product of the roots is $\frac{e}{a} = 5$

Hence $\tan\frac{\pi}{5} \times \tan\frac{2\pi}{5} \times \tan\frac{3\pi}{5} \times \tan\frac{4\pi}{5} = 5$

as required.

(c) $\begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + v \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 2\lambda + \mu - v \\ 3\lambda - \mu + 2v \\ \lambda + 2\mu - v \end{pmatrix}$

$2\lambda + \mu - v = 5$ — ①

$3\lambda - \mu + 2v = 5$ — ②

$\lambda + 2\mu - v = 5$ — ③

① + ② gives $5\lambda + v = 10$ — ④

③ + 2×② gives $7\lambda + 3v = 15$ — ⑤

3×④ gives

$15\lambda + 3v = 30$ — ⑥

⑥ − ⑤ gives

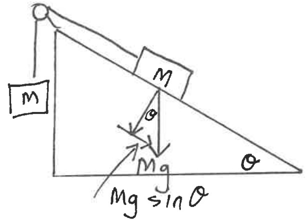
$8\lambda = 15$

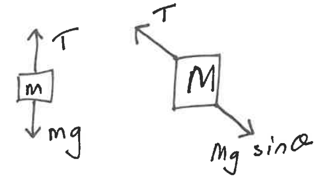
$\therefore \lambda = \frac{15}{8}$

$\therefore v = \frac{5}{8}$

$\therefore \mu = \frac{15}{8}$

(d)





For block M,

$Mg\sin\theta - T = M\ddot{x}$ — ①

For block m,

$T - mg = m\ddot{x}$ — ②

Add ① and ② to get

$Mg\sin\theta - mg = \ddot{x}(m + M)$

$\ddot{x} = \dfrac{Mg\sin\theta - mg}{m + M}$

Substitute this into ② to get

$T = mg + m\left(\dfrac{Mg\sin\theta - mg}{m + M}\right)$

$= \dfrac{mg(m + M) + mMg\sin\theta - m^2 g}{m + M}$

$= \dfrac{mMg + mMg\sin\theta}{m + M}$

$= \dfrac{mMg(1 + \sin\theta)}{m + M}$.

Q13 (a) (i)

$$I_n = \int_0^1 \frac{x^n}{(1+x)^2}\,dx$$

$$= \int_0^1 x^n (1+x)^{-2}\,dx$$

$$= \left[-(1+x)^{-1}x^n\right]_0^1 - \int_0^1 -(1+x)^{-1} n x^{n-1}\,dx$$

$$= -\tfrac{1}{2} + n\int_0^1 \frac{x^{n-1}}{(1+x)}\,dx$$

$$= -\tfrac{1}{2} + n\left[\int_0^1 \frac{x^{n-1}(1+x)}{(1+x)^2}\,dx\right]$$

$$= -\tfrac{1}{2} + n\left[\int_0^1 \frac{x^{n-1}}{(1+x)^2}\,dx + \int_0^1 \frac{x^n}{(1+x)^2}\,dx\right]$$

$$= -\tfrac{1}{2} + n I_{n-1} + n I_n$$

$$I_n - nI_n = -\tfrac{1}{2} + n I_{n-1}$$

$$I_n(1-n) = -\tfrac{1}{2} + n I_{n-1}$$

$$I_n = \frac{-1}{2(1-n)} + \frac{n}{1-n} I_{n-1}$$

$$= \frac{1}{2(n-1)} - \frac{n}{n-1} I_{n-1}, \text{ as required.}$$

(ii) $\int_0^1 \frac{x^3}{(1+x)^2}\,dx = I_3$

$I_3 = \frac{1}{4} - \frac{3}{2} I_2$

$I_2 = \frac{1}{2} - 2I_1$

$I_1 = \int_0^1 \frac{x}{(1+x)^2}\,dx$

Let $u = 1+x$

$du = dx$

when $x = 1$, $u = 2$

when $x = 0$, $u = 1$

$I_1 = \int_1^2 \frac{u-1}{u^2}\,du$

$= \int_1^2 \frac{1}{u} - \frac{1}{u^2}\,du$

$= \left[\ln u + \frac{1}{u}\right]_1^2$

$= \ln 2 - \frac{1}{2}$

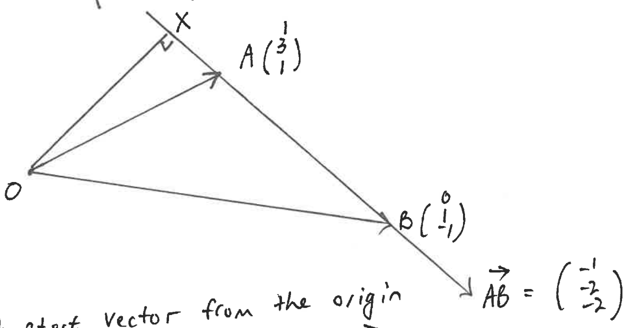
$I_2 = \frac{3}{2} - 2\ln 2$

$I_3 = -2 + 3\ln 2$

Q13 (b) Now $A(1,3,1)$ and $B(0,1,-1)$.

so $\vec{AB} = \vec{OB} - \vec{OA}$

$$= \begin{pmatrix}0\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\3\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$



The shortest vector from the origin to the line AB is the vector $\vec{OX}$ where $\vec{OX} = \vec{OA} + \vec{AX}$ and $\vec{AX}$ is the projection of $\vec{AO}$ onto the line AB.

Now $\text{proj}_{\vec{AB}} \vec{AO}$

$$= \frac{\begin{pmatrix}-1\\-3\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\-2\end{pmatrix}}{1^2+2^2+2^2} \times \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$

$$= \frac{1+6+2}{9} \times \begin{pmatrix}-1\\-2\\-2\end{pmatrix} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$

$\vec{OX} = \vec{OA} + \vec{AX}$

$$= \begin{pmatrix}1\\3\\1\end{pmatrix} + \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$ and $|\vec{OX}| = \sqrt{2}$ units

$$= \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

(c) (i) $v\frac{dv}{dx} = -g - kv^2$

$$\frac{dv}{dx} = \frac{-(g+kv^2)}{v}$$

$$\frac{dx}{dv} = \frac{-v}{g+kv^2}$$

$$\text{Max height} = \int_{V_0}^{0} \frac{-v\,dv}{g+kv^2}$$

$$H = \frac{-1}{2K}\int_{V_0}^{0} \frac{2kv}{g+kv^2}\,dv$$

$$= \frac{1}{2K}\int_{0}^{V_0} \frac{2kv}{g+kv^2}\,dv$$

$$= \frac{1}{2K}\left[\ln(g+kv^2)\right]_0^{V_0}$$

$$= \frac{1}{2K}\ln(g+kv_0^2) - \frac{1}{2K}\ln(g)$$

$$= \frac{1}{2K}\ln\left(\frac{g+kv_0^2}{g}\right) = \frac{1}{2k}\ln\left(1+\frac{kv_0^2}{g}\right)$$

as required.

(ii) Let the new initial velocity be V_1.

$$2H = \frac{1}{2K}\ln\left(1+\frac{kv_1^2}{g}\right)$$

$$4kH = \ln\left(1+\frac{kv_1^2}{g}\right)$$

$$e^{4kH} = 1+\frac{kv_1^2}{g}$$

$$v_1^2 = \frac{g}{k}(e^{4kH}-1)$$

$$\text{So } V_1 = \sqrt{g/k\,(e^{4kH}-1)}$$

$$\text{Now } V_0 = \sqrt{g/k\,(e^{2kH}-1)}$$

$$\text{So } V_1 = \sqrt{g/k\,(e^{2kH}-1)(e^{2kH}+1)}$$

$$= \sqrt{g/k\,(e^{2kH}-1)} \times \sqrt{e^{2kH}+1}$$

$$= V_0 \times \sqrt{e^{2kH}+1}$$

$$= V_0\,(e^{2kH}+1)^{1/2},$$

as required.

Q14 (a)

(i) Proof by contradiction:

Suppose that $\sqrt{10}$ is a rational number and can be written in the form $\sqrt{10} = \frac{p}{q}$ where p and q are integers with no common factors.

Then $10 = \frac{p^2}{q^2}$

$$p^2 = 10q^2 \quad \text{— (1)}$$

so p^2 is even

so p is even

Let $p = 2m$. Sub into (1):

$$(2m)^2 = 10q^2$$

$$4m^2 = 10q^2$$

$$2m^2 = 5q^2$$

$5q^2$ is even

so q^2 is even

so q is even

Contradiction!

p and q cannot both be even as they have no common factors.

Hence $\sqrt{10}$ is not a rational number.

$\sqrt{10}$ is irrational.

(ii) Proof by contradiction:

Suppose that $\sqrt{2}+\sqrt{5}$ is a rational number.

Let $\sqrt{2}+\sqrt{5} = \frac{p}{q}$ where p and q are integers.

Then $(\sqrt{2}+\sqrt{5})^2 = \frac{p^2}{q^2}$

$$7+2\sqrt{10} = \frac{p^2}{q^2}$$

$$2\sqrt{10} = \frac{p^2-7q^2}{q^2}$$

$$\sqrt{10} = \frac{p^2-7q^2}{2q^2}$$

Contradiction!

The RHS is an integer (p^2-7q^2) divided by another integer $(2q^2)$, and therefore rational. But the LHS is $\sqrt{10}$ which we proved in (i) to be irrational.

Hence $\sqrt{2}+\sqrt{5}$ is an irrational number.

Q14 (b)

(i) Now $(a-b)^2 \geqslant 0$

$$a^2 - 2ab + b^2 \geqslant 0$$

$$a^2 + b^2 \geqslant 2ab$$

$$ab \leq \frac{a^2+b^2}{2}$$

Sub $\sqrt{a}$ for a and $\sqrt{b}$ for b

$$\sqrt{a}.\sqrt{b} \leq \frac{(\sqrt{a})^2 + (\sqrt{b})^2}{2}$$

$$\sqrt{ab} \leq \frac{a+b}{2}$$

(ii) Now $ab \leq \frac{a^2+b^2}{2}$ from (i) above

$$\therefore \sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}}$$

(iii) From (ii) above

$$\sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}}$$

So $\sqrt{cd} \leq \sqrt{\frac{c^2+d^2}{2}}$

$$\sqrt{\sqrt{ab} \times \sqrt{cd}} \leq \sqrt{\frac{\left(\sqrt{\frac{a^2+b^2}{2}}\right)^2 + \left(\sqrt{\left(\frac{c^2+d^2}{2}\right)}\right)^2}{2}}$$

$$\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$$

Alternatively:

$$ab \leq \frac{a^2+b^2}{2}$$

and $cd \leq \dfrac{c^2+d^2}{2}$

$$\therefore ab + cd \leq \frac{a^2+b^2}{2} + \frac{c^2+d^2}{2}$$

$$\leq \frac{a^2+b^2+c^2+d^2}{2} \quad ②$$

$$\sqrt{abcd} = \sqrt{(ab)+(cd)}$$

$$\leq \frac{ab+cd}{2} \quad \text{(by given inequality)}$$

$$\leq \frac{a^2+d^2+c^2+d^2}{4} \quad \text{from ②}$$

Taking square roots:

$$\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}.$$

(iv) If $1 \leq x \leq y$, show that

$$x(y-x+1) \geqslant y$$

Now $y \geqslant x$

so $y(x-1) \geqslant x(x-1)$

because $x \geqslant 1$

so $xy - y \geqslant x^2 - x$

$$xy - x^2 + x \geqslant y$$

$$\therefore x(y-x+1) \geqslant y$$

Alternatively:

LHS − RHS

$= xy - x^2 + x - y$

$= y(x-1) - x(x-1)$

$= (y-x)(x-1)$

this is positive or zero

this is positive or zero

$\therefore$ LHS − RHS $\geqslant 0$

$\therefore x(y-x+1) \geqslant y$

(v) Let n and k be positive integers with $1 \leq k \leq n$.

Prove that $\sqrt{n} \leq \sqrt{k(n-k+1)} \leq \frac{n+1}{2}$.

From (iv) $x(y-x+1) \geqslant y$

substitute $x = k$ and $y = n$, then

$$k(n-k+1) \geqslant n$$

$$\therefore \sqrt{k(n-k+1)} \geqslant \sqrt{n}$$

$$\therefore \sqrt{n} \leq \sqrt{k(n-k+1)} \quad — \quad \alpha$$

From (i) $\sqrt{ab} \leq \frac{a+b}{2}$ so

$$\sqrt{k(n-k+1)} \leq \frac{k+(n-k+1)}{2}$$

ie, $\sqrt{k(n-k+1)} \leq \frac{n+1}{2}$

Hence $\sqrt{n} \leq \sqrt{k(n-k+1)} \leq \frac{n+1}{2}$

(vi) For integers $n \geq 1$, to prove that

$$(\sqrt{n})^n \leq n! \leq \left(\frac{n+1}{2}\right)^n$$

Substitute $k=1$ into α above gives $\sqrt{n} \leq \sqrt{1(n)} \leq \frac{n+1}{2}$

" $k=2$ " " " " $\sqrt{n} \leq \sqrt{2(n-1)} \leq \frac{n+1}{2}$

" $k=3$ " " " " $\sqrt{n} \leq \sqrt{3(n-2)} \leq \frac{n+1}{2}$

etc

" $k=n$ " " " " $\sqrt{n} \leq \sqrt{n} \leq \frac{n+1}{2}$

Now multiply all the inequalities above together to get

$$(\sqrt{n})^n \leq \sqrt{(n(n-1)(n-2)\ldots 3\times 2\times 1)^2} \leq \left(\frac{n+1}{2}\right)^n$$

Hence $(\sqrt{n})^n \leq n! \leq \left(\frac{n+1}{2}\right)^n$, as required.

Q15 (a)

(i) $(x+1)(x^2-x+1)$

$= x^3 - x^2 + x + x^2 - x + 1$

$= x^3 + 1$

(ii) $\displaystyle\int_0^\infty \frac{1}{1+x^3}\,dx$

Let $\dfrac{1}{1+x^3} = \dfrac{A}{1+x} + \dfrac{Bx+C}{x^2-x+1}$

$1 = A(x^2-x+1) + (Bx+C)(1+x)$

Let $x=-1$, $1 = 3A$ so $A = \frac{1}{3}$

Let $x = 0$, $1 = A + C$ so $C = \frac{2}{3}$

Let $x = 1$, $1 = A + 2(B+C)$

$1 = \frac{1}{3} + 2B + \frac{4}{3}$ so $B = \frac{-1}{3}$

$$\therefore \int_0^\infty \frac{dx}{1+x^3} = \int_0^\infty \frac{\frac{1}{3}}{1+x} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}\,dx$$

$$= \left[\tfrac{1}{3}\ln|1+x|\right]_0^\infty - \frac{1}{3}\int_0^\infty \frac{x-2}{x^2-x+1}\,dx$$

$$= \left[\tfrac{1}{3}\ln|1+x|\right]_0^\infty - \frac{1}{3}\int_0^\infty \frac{\frac{1}{2}(2x-1) - \frac{3}{2}}{x^2-x+1}\,dx$$

$$= \left[\tfrac{1}{3}\ln|1+x|\right]_0^\infty - \frac{1}{6}\int_0^\infty \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int_0^\infty \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

$$= \left[\tfrac{1}{6}\ln(1+x)^2\right]_0^\infty - \left[\tfrac{1}{6}\ln|x^2-x+1|\right]_0^\infty + \left[\tfrac{1}{2}\cdot\tfrac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right)\right]_0^\infty$$

$$= \left[\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}\right]_0^\infty + \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^\infty$$

This term approaches zero.

$$= 0 + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\infty - 1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} \times \frac{\pi}{2} - \frac{1}{\sqrt{3}}\left(\frac{-\pi}{6}\right)$$

$$= \frac{\pi}{2\sqrt{3}} + \frac{\pi}{6\sqrt{3}}$$

$$= \frac{4\pi}{6\sqrt{3}}$$

$$= \frac{2\pi}{3\sqrt{3}}$$

(b) Now $t = \frac{x}{V\cos\theta}$

(i) $y = -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$

$$y = \frac{-gx^2}{2V^2}\sec^2\theta + x\tan\theta$$

Let $x = d$ and $y = h$, then

$$h = \frac{-gd^2}{2V^2}\sec^2\theta + d\tan\theta$$

$$\frac{gd^2}{2V^2}\sec^2\theta = d\tan\theta - h$$

$$V^2 = \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$$

as required.

(ii) Max height occurs on the axis of symmetry.

$$t = \frac{V\sin\theta}{g}$$

$$y_{max} = -\frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2 + V\left(\frac{V\sin\theta}{g}\right)\sin\theta$$

$$= \frac{-V^2\sin^2\theta}{2g} + \frac{V^2\sin^2\theta}{g}$$

$$= \frac{V^2\sin^2\theta}{2g}$$

$$= V^2\left(\frac{\sin^2\theta}{2g}\right)$$

$$= \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)} \times \frac{\sin^2\theta}{2g}$$

$$= \frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$$

as required.

(iii) $h = \frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$

$$4dh\tan\theta - 4h^2 = d^2\tan^2\theta$$

$$d^2\tan^2\theta - 4dh\tan\theta + 4h^2 = 0$$

$$(d\tan\theta - 2h)^2 = 0$$

$$d\tan\theta - 2h = 0$$

$$\tan\theta = \frac{2h}{d}$$

as required.

Q15 (c)

(i) To show that

$$I = \int_{-a}^{a} \frac{x^4\,dx}{1+e^x} = \int_{-a}^{a} \frac{x^4 e^x\,dx}{1+e^x}$$

Let $x = -u$

$dx = -du$

when $x = a$, $u = -a$

when $x = -a$, $u = a$

$$I = \int_{a}^{-a} \frac{(-u)^4(-du)}{1+e^{-u}}$$

$$= \int_{-a}^{a} \frac{u^4\,du}{1+e^{-u}}$$

$$= \int_{-a}^{a} \frac{u^4 e^u\,du}{1+e^u}$$

$$= \int_{-a}^{a} \frac{x^4 e^x\,dx}{1+e^x}$$

(change of variables)

(ii) $\int_{-2}^{2} \frac{x^4\,dx}{1+e^x} = \int_{-2}^{2} \frac{x^4 e^x\,dx}{1+e^x}$

$$\therefore 2I = \int_{-2}^{2} \frac{x^4(1+e^x)}{(1+e^x)}\,dx$$

$$= \int_{-2}^{2} x^4\,dx$$

$$= \left[\frac{1}{5}x^5\right]_{-2}^{2}$$

$$= 64/5$$

$$\therefore I = \frac{32}{5}$$

Q16 (a) (i) When $t=0$, $\underset{\sim}{v} = 3\underset{\sim}{i} - 5\underset{\sim}{j}$ m/s

Hence speed $= |\underset{\sim}{v}| = |3\underset{\sim}{i} - 5\underset{\sim}{j}|$

$= \sqrt{3^2+5^2}$

$= \sqrt{34}$ m/s

(ii) Using $\underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{a}t$

$11\underset{\sim}{i} + 7\underset{\sim}{j} = (3\underset{\sim}{i} - 5\underset{\sim}{j}) + \underset{\sim}{a} \times 4$

$8\underset{\sim}{i} + 12\underset{\sim}{j} = 4\underset{\sim}{a}$

$\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ m/s^2

Hence $F = ma$

$= 2(2\underset{\sim}{i} + 3\underset{\sim}{j})$

$= 4\underset{\sim}{i} + 6\underset{\sim}{j}$ N

(iii) $\underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{a}t$

$= (3\underset{\sim}{i} - 5\underset{\sim}{j}) + (2\underset{\sim}{i} + 3\underset{\sim}{j})t$

$= (2t+3)\underset{\sim}{i} + (3t-5)\underset{\sim}{j}$

when the particle is moving east, $\underset{\sim}{j} = 0$ so

$3t - 5 = 0$

$t = \frac{5}{3}$ seconds.

(b)

(i) The centre of S_1 is at $C_1(1, -1, 2)$ and its radius is 8.

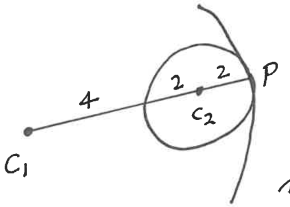
The centre of S_2 is at C_2 $(-1, 3, -2)$ and its radius is 2.

The distance between C_1 and C_2 is

$d = \sqrt{(1--1)^2 + (-1-3)^2 + (2--2)^2} = \sqrt{36} = 6$ units

Notice that S_2 is inside S_1.

The two spheres touch at P.



$C_1C_2 = 6$

$C_2P = \underline{2}$

8

The two spheres are tangential.

(ii) To find the coordinates of the point of contact, we need to extend the direction vector $\overrightarrow{C_1C_2}$ by $\frac{1}{3}$.

Consider the $\underset{\sim}{i}$ component: from 1 to -1 extended by $\frac{1}{3}$ takes us to $-\frac{5}{3}$.

Consider the $\underset{\sim}{j}$ component: from -1 to 3 extended by $\frac{1}{3}$ takes us to $\frac{13}{3}$.

Consider the $\underset{\sim}{k}$ component: from 2 to -2 extended by $\frac{1}{3}$ takes us to $-\frac{10}{3}$.

Hence $P = \left(\frac{-5}{3}, \frac{13}{3}, \frac{-10}{3}\right)$.

(c) (i) $e^{in\theta} - e^{-in\theta}$

$= \cos(n\theta) + i\sin(n\theta) - [\cos(-n\theta) + i\sin(-n\theta)]$

$= \cos(n\theta) + i\sin(n\theta) - [\cos(n\theta) - i\sin(n\theta)]$

$= \cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + i\sin(n\theta)$

$= 2i\sin(n\theta)$

(ii) $S_n = 1 + e^{i\theta} + e^{i2\theta} + \ldots + e^{in\theta}$

$$= \frac{1[(e^{i\theta})^{(n+1)} - 1]}{e^{i\theta} - 1}$$

$$= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$, as required

(iii) Now $S_n = \dfrac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$ from above.

$$S_n = \frac{e^{i(\frac{n+1}{2})\theta}\left[e^{i(\frac{n+1}{2})\theta} - e^{-i(\frac{n+1}{2})\theta}\right]}{e^{i(\theta/2)}\left[e^{i(\theta/2)} - e^{-i(\theta/2)}\right]}$$

$$= \frac{e^{i(\frac{n+1}{2})\theta}}{e^{i(\theta/2)}} \times \frac{2i\sin(\frac{n+1}{2})\theta}{2i\sin(\theta/2)}$$

$$= e^{i(\frac{n}{2})\theta} \times \frac{\sin(\frac{n+1}{2})\theta}{\sin(\frac{\theta}{2})}$$, as required.

(iv) Now $\sin\theta + \sin 2\theta + \ldots + \sin n\theta$

$= \operatorname{Im}\left(1 + e^{i\theta} + e^{i2\theta} + e^{i3\theta} + \ldots + e^{in\theta}\right)$

$$= \operatorname{Im}\left(e^{i(\frac{n}{2})\theta} \times \frac{\sin(\frac{n+1}{2})\theta}{\sin(\frac{\theta}{2})}\right)$$

$$= \operatorname{Im}\left[\left(\cos(\tfrac{n}{2})\theta + i\sin(\tfrac{n}{2})\theta\right) \times \frac{\sin(\frac{n+1}{2})\theta}{\sin(\theta/2)}\right]$$

$$= \operatorname{Im}\left[\frac{\cos(\frac{n}{2})\theta\sin(\frac{n+1}{2})\theta}{\sin(\theta/2)} + i\,\frac{\sin(\frac{n}{2})\theta\sin(\frac{n+1}{2})\theta}{\sin(\theta/2)}\right]$$

$$= \sin(\tfrac{n}{2})\theta \times \frac{\sin(\frac{n+1}{2})\theta}{\sin(\frac{\theta}{2})}$$, as required.

End of Paper