
Student Number

Barker College

2025 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Staff Involved:

- GDH – Mr Hanlon
- ALY – Ms Young
- BHC – Mr Carruthers*

PM MONDAY 11TH AUGUST

50 copies

Topics Covered

TOPICS COVERED		
Complex Numbers	Proof	Integration
Vectors	Mechanics	

General Instructions:

- Write your Student Number at the top of this page and on all writing booklets
- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A separate reference booklet is provided
- Marks may not be awarded for careless or poorly arranged working
- Diagrams are not to scale
- For questions in Section II, show relevant mathematical reasoning and / or calculations

Total marks: 100

Section I – 10 marks (pages 2 - 5)

- Attempt Questions 1 – 10
- Allow about 20 minutes for this section

Section II – 90 marks (pages 7 - 14)

- Attempt Questions 11 - 16
- Allow about 2 hours and 40 minutes for this section
- Answer each question in a SEPARATE writing booklet.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 20 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 The argument of a complex number z is $\frac{\pi}{7}$.

What is $\text{Arg}\left(z^{10}\right)$ equal to?

A. $\frac{3\pi}{7}$

B. $\frac{-3\pi}{7}$

C. $\frac{4\pi}{7}$

D. $\frac{-4\pi}{7}$

2 Let $\underset{\sim}{a} = \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -2m \\ 2 \\ 2 \end{pmatrix}$, where m is a real constant. For what value of m will the vector $\underset{\sim}{b} - \underset{\sim}{a}$ be perpendicular to the vector $\underset{\sim}{b}$?

A. 0 only

B. 2 only

C. 0 or 2

D. 0 or –2

3 Consider the following proposition.

"If a positive integer is not a perfect square, then it is not prime."

If a number was found which had the following properties, which would be a counterexample to the proposition?

A. A positive integer that is not a perfect square and is prime.

B. A positive integer that is not a perfect square and is not prime.

C. A positive integer that is a perfect square and is prime.

D. A positive integer that is a perfect square and is not prime.

4 A particle moves along a curve so that its position at time t is given by $\underset{\sim}{x} = \begin{pmatrix} 2t \\ t^2 \\ t^3 \end{pmatrix}$.

The acceleration at $t = 1$ is:

A. $2\underset{\sim}{j} + 6\underset{\sim}{k}$

B. $\underset{\sim}{i} + \underset{\sim}{j} + 6\underset{\sim}{k}$

C. $\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$

D. $\underset{\sim}{j} + 6\underset{\sim}{k}$

5 Which of the following is an expression for $\int \frac{\sin^3 x - \cos^3 x}{\sin x - \cos x}\, dx$?

A. $x - \frac{1}{2}\sin^2 x + c$

B. $x + \frac{1}{2}\sin^2 x + c$

C. $x + \frac{1}{2}\cos 2x + c$

D. $x - \frac{1}{2}\cos 2x + c$

6 Which of the following statements is the negation of the statement

"$\forall x \in \mathbb{R}^+$, $\exists y \in \mathbb{R}^+$ such that $xy = 1$" ?

A. $\exists x,\ y \in \mathbb{R}^+$ such that $xy \neq 1$

B. $\exists x \in \mathbb{R}^+$ such that $\forall y \in \mathbb{R}^+$, $xy \neq 1$

C. $\forall x,\ y \in \mathbb{R}^+$, $xy \neq 1$

D. $\forall x \in \mathbb{R}^+$, there exists a positive real number y such that $xy \neq 1$

7 Using the substitution $x = \pi - u$, the definite integral $\int_0^{\pi} x \sin x \, dx$ will simplify to:

A. 0

B. $\dfrac{\pi^2}{4}\int_0^{\pi} \sin x \, dx$

C. $\dfrac{\pi}{2}\int_0^{\pi} \sin x \, dx$

D. $\int_0^{\pi} \sin x \, dx$

8 Let z be a complex number such that $z^3 = \dfrac{i}{\bar{z}}$.

Which of the following is a possible value for z ?

A. $e^{i\frac{\pi}{4}}$

B. $e^{-i\frac{\pi}{4}}$

C. $e^{i\frac{\pi}{3}}$

D. $e^{-i\frac{\pi}{3}}$

9 Which of the following statements is the contrapositive of the statement "if $x > y > 0$ then $0 < \frac{1}{x} < \frac{1}{y}$" ?

A. If $0 \geq \frac{1}{x}$ or $\frac{1}{x} \geq \frac{1}{y}$ then $x \leq y$ or $y \leq 0$.

B. If $0 \geq \frac{1}{x}$ and $\frac{1}{x} \geq \frac{1}{y}$ then $x \leq y$ and $y \leq 0$.

C. If $0 \geq \frac{1}{x} \geq \frac{1}{y}$ then $x \leq y \leq 0$.

D. If $x \leq y \leq 0$ then $0 \geq \frac{1}{x} \geq \frac{1}{y}$.

10 Which of the following integrals has the largest value?

A. $\int_0^{\frac{\pi}{4}} \sin x \, dx$

B. $\int_0^{\frac{\pi}{4}} \sin^2 x \, dx$

C. $\int_0^{\frac{\pi}{4}} 1 - \sin x \, dx$

D. $\int_0^{\frac{\pi}{4}} 1 - \sin^2 x \, dx$

End of Section I

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Section II

90 marks
Attempt Questions 11 - 16
Allow about 2 hours and 40 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 - 16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **[START THE QUESTION 11 WRITING BOOKLET]**

(a) Let $z = a - 2bi$. Find complex numbers for the following, in the form $x + iy$.

(i) $\dfrac{1}{\overline{z}}$ **2**

(ii) z^2 **1**

(b) (i) Using Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, find an expression for $e^{-i\theta}$ in terms of $\cos\theta$ and $\sin\theta$. **1**

(ii) Hence show that $\sin\theta = \dfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. **1**

(iii) Use the result from part (ii) to show that $\sin^3\theta = \dfrac{3}{4}\sin\theta - \dfrac{1}{4}\sin 3\theta$. **3**

(c) (i) Find real numbers A and B such that $\dfrac{6x^2+3x+17}{(x-3)\left(x^2+2x+5\right)} = \dfrac{A}{x-3} + \dfrac{Bx+1}{x^2+2x+5}$. **2**

(ii) Hence find $\displaystyle\int \frac{6x^2+3x+17}{(x-3)\left(x^2+2x+5\right)}\,dx$. **3**

(d) Prove the following statement for positive n using a proof by contradiction.

"For each irrational number n, $\sqrt{n}$ is also irrational." **2**

Question 12 (15 marks) **[START THE QUESTION 12 WRITING BOOKLET]**

(a) Assume that the tides rise and fall in simple harmonic motion. At low tide the depth of water in a harbour was 6 m deep, and at high tide it was 10 m deep. Low tide was at 8:00 am and the high tide following it was at 3:00 pm.

(i) What is the amplitude of the motion? **1**

(ii) What is the period of the motion? **1**

(iii) What was the depth of the harbour at 11:30 am? **1**

(iv) Starting with $y = a\cos(nt+\alpha)+c$ and by using your answers from parts (i), (ii) and (iii), show that the depth y m of the water in the harbour is given by
$y = 8 - 2\cos\left(\frac{\pi}{7}t\right)$ where t is the number of hours after 8:00 am. **2**

(v) A boat needs at least 9 m of water to pass through the harbour. Between which times after 8:00 am on this day could the ship navigate safely through the harbour? **2**

(b) $P(z) = z^4 - 10z^3 + 117z^2 - 412z + 1460$.

$P(z)$ has zeroes $2a+bi$ and $3a+2bi$, where a and b are integers.

(i) Find the values of a and b and hence find all the zeroes of $P(z)$. **3**

(ii) Factorise $P(z)$ into its quadratic factors with real coefficients. **1**

(c) It is given that $3^x > 2x + \frac{1}{2}\ln x$ for $x > 0$.
By forming a series of n inequalities, show that $3^{n+1} > 2n^2 + 2n + 3 + \ln(n!)$. **4**

Question 13 (15 marks) **[START THE QUESTION 13 WRITING BOOKLET]**

(a) Given that $u_1 = 1$ and $u_{n+1} = 2u_n + (-1)^{n+1}$, prove that $u_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)$, where $n \in \mathbb{Z}^+$, by mathematical induction. **3**

(b) (i) Use a suitable substitution to show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{\sin 2x + \cos 2x}\,dx$. **4**

(c) The acceleration of a particle P moving along the x-axis is given by $\ddot{x} = 4x^3 - 12x$.

(i) If P is initially at the origin with velocity u, show that its velocity is given by $v^2 = 2x^4 - 12x^2 + u^2$. **2**

(ii) If $u = 4$, show that the particle oscillates within the interval $-\sqrt{2} \le x \le \sqrt{2}$. **3**

(iii) Is the motion referred to in part (ii) simple harmonic motion? Give a reason for your answer. **1**

Question 14 (15 marks) **[START THE QUESTION 14 WRITING BOOKLET]**

(a) When an aircraft touches down on a runway, two different retarding forces combine to ultimately bring the aircraft to rest. There is a constant frictional force of $\frac{-1}{5}M$ newtons, where the mass of the aircraft is M kg. There is also a force of $\frac{-1}{125}Mv^2$ newtons, where v is the aircraft's velocity in m/s, due to the reverse thrust of the engines. The reverse thrust of the engines is applied from 10 seconds after touchdown until the plane stops.

(i) The aircraft's speed at touchdown is 80 m/s.

Show that, at the instant the reverse thrust of the engines takes effect, $v = 78$ m/s and the aircraft's displacement from its touchdown point $x = 790$ m. **3**

(ii) Show that for $t > 10$, and until the aircraft stops, $\frac{d^2x}{dt^2} = \frac{-1}{125}\left(25 + v^2\right)$. **1**

(iii) Show that when $t > 10$, $x = 790 + 62.5\left[\ln 6109 - \ln\left(25 + v^2\right)\right]$. **3**

(iv) Calculate how far from the touchdown point the aircraft travels before it comes to rest. Answer to the nearest metre. **1**

(b) Three different particles move in the xy-plane according to trajectories defined parametrically by:

$$\underset{\sim}{R}_1(t) = \cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$$

$$\underset{\sim}{R}_2(t) = \cos(2t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}$$

$$\underset{\sim}{R}_3(t) = \sin(t)\underset{\sim}{i} + \cos(t)\underset{\sim}{j}$$

For each particle, describe:

1. the shape of the path
2. the period of the motion
3. the direction of the motion **3**

(c) A body of mass 1 kg is moving in a horizontal straight line and is acted on by a force $F = 8x^3 + 36x$ newtons, where x is the displacement of the body in metres, measured from the origin.

If the body is initially one metre to the right of the origin with velocity of 11 m/s, find the displacement function $x(t)$. **4**

Question 15 (15 marks) **[START THE QUESTION 15 WRITING BOOKLET]**

(a) There are two lines $\underset{\sim}{R}_1 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\underset{\sim}{R}_2 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

(i) Show that these lines intersect and find the point of intersection. **2**

(ii) Find the shortest distance from the point $P(1,4,0)$ to the line $\underset{\sim}{R}_1$. **3**

(b) Let $I_n = \displaystyle\int_0^1 \frac{1}{\left(1+x^2\right)^n}\,dx$ for $n \geq 1$.

(i) Use integration by parts to show that $I_n = \dfrac{1}{2^n} + 2n \displaystyle\int_0^1 \frac{x^2}{\left(1+x^2\right)^{n+1}}\,dx$. **2**

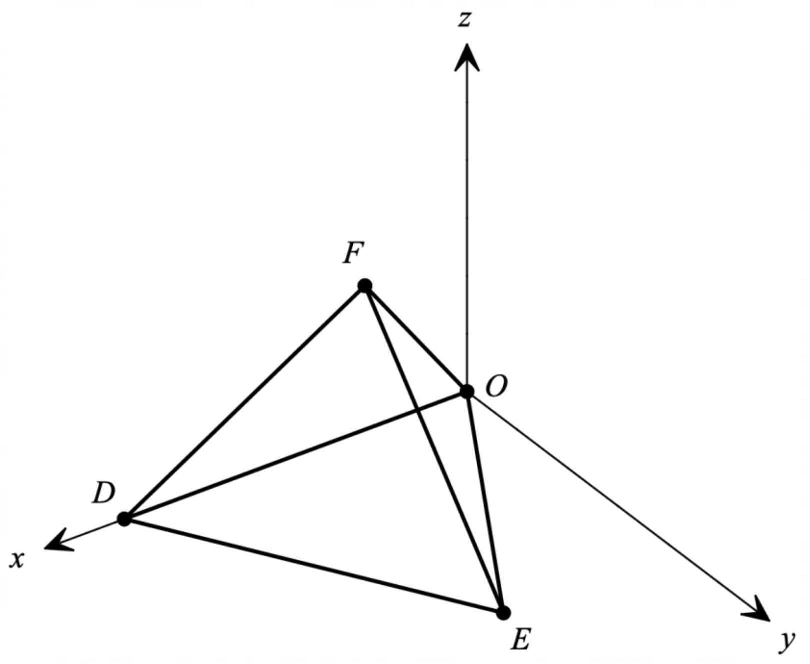
(ii) Hence, or otherwise, show that $I_{n+1} = \dfrac{1}{n2^{n+1}} + \dfrac{2n-1}{2n} I_n$. **3**

(iii) Evaluate $\displaystyle\int_0^1 \frac{1}{\left(1+x^2\right)^2}\,dx$. **1**

Question 15 continues on page 12

Question 15 (continued)

(c) The faces of the tetrahedron *ODEF* are equilateral triangles of side length 1 unit. The base *ODE* lies in the *xy*-plane with one vertex at $O(0,0,0)$ and one vertex at $D(1,0,0)$, with *F* above the *xy*-plane



(i) Find the coordinates of *E*. **1**

(ii) Find the coordinates of *F*. **3**

End of Question 15

Question 16 (15 marks) **[START THE QUESTION 16 WRITING BOOKLET]**

(a) Find $\displaystyle\int \frac{dx}{\sqrt{4x^2-1}}$ by making a suitable substitution. **3**

(b) Given that the product of n consecutive numbers is divisible by $n!$, use mathematical induction to prove that $n^5 - n$ is divisible by 2, 3 and 5 for integers $n \geq 2$. **4**

(c) Consider the line ℓ defined parametrically by $\underset{\sim}{R}(\lambda) = \underset{\sim}{a} + \lambda \underset{\sim}{b}$ and the sphere S defined by $\left|\underset{\sim}{v} - \underset{\sim}{c}\right| = r$.

(i) Suppose ℓ is a tangent to S. Given that the line and the sphere have only one point of intersection, prove that the position vector of the point of contact is given by $\underset{\sim}{R}(\lambda)$ when $\lambda = \dfrac{\underset{\sim}{b} \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right)}{\left|\underset{\sim}{b}\right|^2}$. **3**

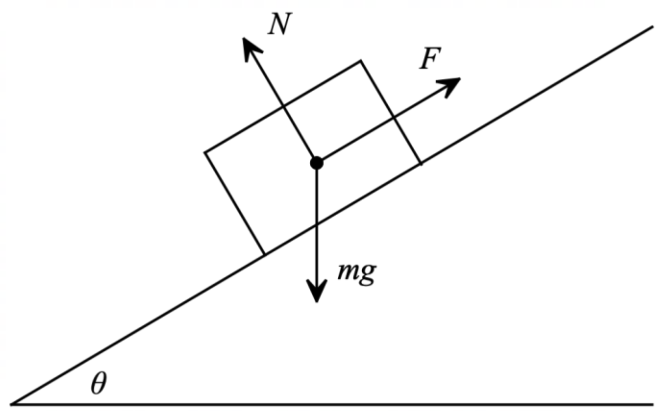
Note: you may ***not*** *use the fact that the tangent is perpendicular to the radius.*

(ii) Hence, show that the tangent to the sphere is perpendicular to the radius at the point of contact. **1**

Question 16 continues on page 14

Question 16 (continued)

(d) An object of mass m rests on a plane inclined at a variable angle θ from the horizontal. It experiences a friction force, F, and a normal force, N. The friction force is $F = \mu N$ where μ is constant. The normal force, N, is equal in magnitude to the force the object exerts on the plane perpendicular to the plane. Let α be the maximal angle that the plane can be tilted before the object begins to slide down the plane.



For some $\theta > \alpha$, a horizontal force H is applied to the object so that if H is sufficiently large, then the object will slide up the plane and if H is sufficiently small, then the object will slide down the plane.

Show that, for the object to remain motionless, H must satisfy
$mg\tan(\theta - \alpha) \leq H \leq mg\tan(\theta + \alpha)$ **4**

End of Paper

2025 Trial Extension 2

(1) $z^{10} = \left(r\,\text{cis}\frac{\pi}{7}\right)^{10}$

$= r^{10}\,\text{cis}\,\frac{10\pi}{7}$

$\therefore \text{Arg}(z^{10}) = \frac{10\pi}{7} = -\frac{4\pi}{7} \quad \therefore \boxed{D}$

(2) $\underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix} -2m+4 \\ 4 \\ -4 \end{pmatrix}$

$(\underline{b} - \underline{a}) \cdot \underline{b} = 0$ for perpendicular

$\therefore \begin{pmatrix} -2m+4 \\ 4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -2m \\ 2 \\ 2 \end{pmatrix} = 0$

$\therefore 4m^2 - 8m + 8 - 8 = 0$

$4m(m-2) = 0 \quad \therefore \boxed{C}$

$m = 0$ or 2

(3) Need a +ve integer that is not a perf. □ but is prime $\therefore \boxed{A}$

(4) $\dot{\underset{\sim}{x}} = \begin{pmatrix} 2 \\ 2t \\ 3t^2 \end{pmatrix} \quad \ddot{x} = \begin{pmatrix} 0 \\ 2 \\ 6t \end{pmatrix}$

At $t = 1$, $\ddot{x} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \quad \therefore \boxed{A}$

(5) $\displaystyle\int \frac{(\sin x - \cos x)(\sin^2 x + \sin x\cos x + \cos^2 x)}{(\sin x - \cos x)}\,dx$

$= \displaystyle\int \sin^2 x + \cos^2 x + \sin x\cos x\,dx$

$= \displaystyle\int 1 + \underbrace{\cos x}_{[f'(x)]}\,\underbrace{\sin x}_{[f(x)]}\,dx$

$= x + \dfrac{\sin^2 x}{2} + C \quad \therefore \boxed{B}$

(6) $\exists x \in R^+, \forall y \in R^+, xy \neq 1 \quad \therefore \boxed{B}$

The original statement basically says:
"All +ve real x have a reciprocal & the reciprocal is also +ve real".
The negation therefore is:
"There exists a +ve real x which does not have a ~~+ve~~ +ve real reciprocal"
Note that the original statement is true & thus the negation should be false.

(7) $x = 0, u = \pi$ $\quad \frac{dx}{du} = -1 \quad \therefore dx = -du$
$x = \pi, u = 0$

$I = \displaystyle\int_{\pi}^{0} (\pi - u)\sin(\pi - u)(-du)$

$= \displaystyle\int_0^{\pi} (\pi - u)\sin u\,du = \pi\int_0^{\pi} \sin u\,du - \int_0^{\pi} u\sin u\,du$

$\therefore I = \displaystyle\int_0^{\pi} x\sin x = \pi\int_0^{\pi} \sin u\,du - \int_0^{\pi} x\sin x\,dx$ via change of variable

$\therefore I = \displaystyle\int_0^{\pi} x\sin x = \pi\int_0^{\pi} \sin x\,dx - I$ via change of variable

$\therefore 2I = \pi\displaystyle\int_0^{\pi} \sin x\,dx \quad \therefore I = \frac{\pi}{2}\int_0^{\pi} \sin x\,dx$

$\therefore \boxed{C}$

(8) $z^3\bar{z} = i$

A: $e^{i\frac{3\pi}{4}} e^{i\left(-\frac{\pi}{4}\right)} = e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \quad \therefore \boxed{A}$

[similar for B, C, D but none work]

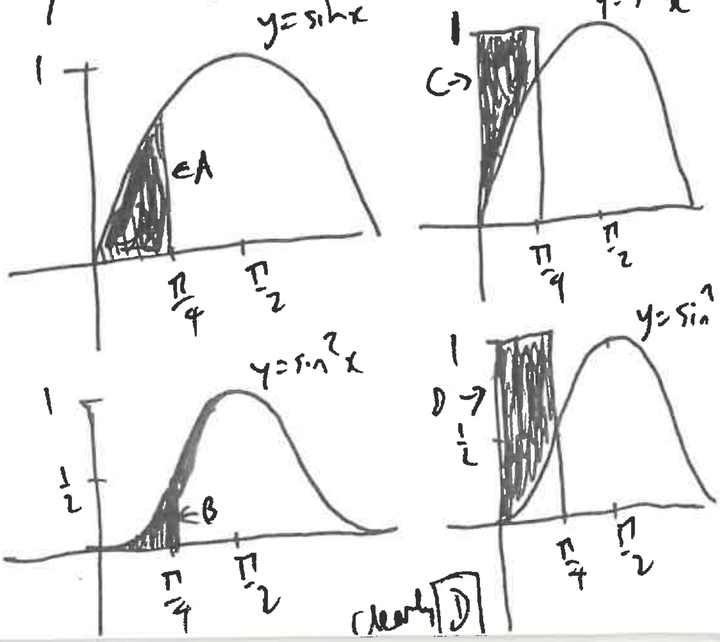
(9) $\sim\left(0 < \frac{1}{x} < \frac{1}{y}\right) \rightarrow \sim(x > y > 0)$

$\therefore$ If $0 \geq \frac{1}{x}$ or $\frac{1}{x} \geq \frac{1}{y}$ then $\quad \therefore \boxed{A}$

$x \leq y$ or $y \leq 0$

Note that to negate ~~an~~ 'AND', we reverse inequalities and change to 'OR'.
A squeezed inequality is an AND situation.

(10)



Clearly $\boxed{D}$

$\therefore$ DCAABBCAAD

11) (a)(i) $\bar{z} = a + 2bi$

$\frac{1}{\bar{z}} = \frac{1}{a+2bi} \times \frac{a-2bi}{a-2bi}$

2

$= \frac{a-2bi}{a^2+4b^2} = \frac{a}{a^2+4b^2} - \frac{2b}{a^2+4b^2} i$

(ii) $z^2 = a^2 - 4abi - 4b^2$

1 $\quad = a^2 - 4b^2 - 4abi$

b) (i) $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

1 $\quad = \cos\theta - i\sin\theta$

(ii) Subtracts ~~Adding~~ vertically:

$\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

1 $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\therefore \sin\theta = \frac{(e^{i\theta} - e^{-i\theta})}{2i}$

(iii) $\sin^3\theta = \frac{e^{3i\theta} - 3e^{2i\theta}e^{-i\theta} + 3e^{i\theta}e^{-2i\theta} - e^{-3i\theta}}{8i^3}$

3

$= \frac{e^{i3\theta} - e^{-i3\theta} - 3(e^{i\theta} - e^{-i\theta})}{-8i}$

$= \frac{2i\sin 3\theta - 3 \times 2i\sin\theta}{-8i}$

$= \frac{2\sin 3\theta - 6\sin\theta}{-8}$

$= \frac{\sin 3\theta - 3\sin\theta}{-4}$

$= \frac{3\sin\theta - \sin 3\theta}{4}$

$= \frac{3}{4}\sin\theta - \frac{\sin 3\theta}{4}$

(c) (i) $6x^2 + 3x + 17 = A(x^2+2x+5) + (Bx+1)(x-3)$

Let $x = 3$: $80 = 20A$ $\quad \therefore A = 4$. 2

Coeff x^2: $6 = 4 + B$ $\quad \therefore B = 2$

(ii) $\int \frac{4}{x-3} + \frac{2x+1}{x^2+2x+5} \, dx$ 3

$= 4\ln|x-3| + \int \frac{2x+2}{x^2+2x+5} - \frac{1}{(x+1)^2+4} \, dx$

$= 4\ln|x-3| + \ln|x^2+2x+5| - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$

(d) Let n be irrational & $\sqrt{n}$ rational

$\therefore \sqrt{n} = \frac{a}{b}$, $\quad a, b \in \mathbb{Z}^+$

$\therefore n = \frac{a^2}{b^2}$ 2

$\therefore n$ rational

But n irrational! $\therefore$ Contradiction

$\therefore$ statement is true.

12) (a) (i) amp $= \frac{10-6}{2} = 2$m

(ii) $\frac{\text{period}}{2} = 7$ hours

$\therefore$ period $= 14$ hours

(iii) $3\frac{1}{2}$ hours is halfway between low & high tide

$\therefore$ Centre of motion $\therefore$ 8m

(iv) $y = 2\cos(nt + \alpha) + 8$

Period $= 14 = \frac{2\pi}{n}$ $\quad \therefore n = \frac{\pi}{7}$

$\therefore y = 2\cos\left(\frac{\pi}{7}t + \alpha\right) + 8$

$y = 2\cos\frac{\pi}{7}t\cos\alpha - 2\sin\frac{\pi}{7}t\sin\alpha + 8$

when $t = 0, y = 6$

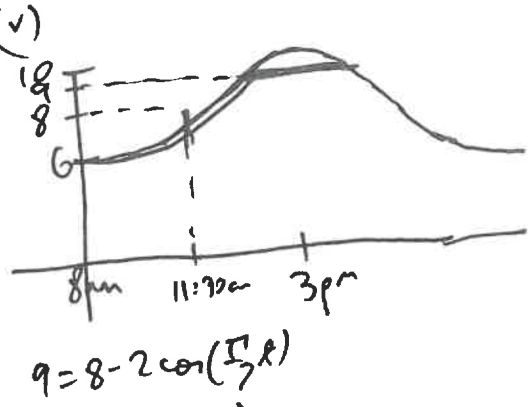
$\therefore 6 = 2\cos\alpha + 8$ $\quad \therefore \cos\alpha = -1$

$\therefore \alpha = \pi$

$\therefore y = 2\cos\frac{\pi}{7}t\cos\pi - 2\sin\frac{\pi}{7}t\sin\pi + 8$

$y = 2\cos\frac{\pi}{7}t(-1) + 8 = 8 - 2\cos\left(\frac{\pi}{7}t\right)$

(v)



$9 = 8 - 2\cos\left(\frac{\pi}{7}t\right)$

$\cos\left(\frac{\pi}{7}t\right) = -\frac{1}{2}$

$\therefore \frac{\pi}{7}t = \frac{2\pi}{3}, \frac{4\pi}{3}$ ...

$t = \frac{14}{3}, \frac{28}{3}$ $\quad \therefore$ 4hr 40min, 9hr 20min

$\therefore$ 12:40pm – 5:20pm

12 (b)(i) Since real coefficients, zeroes: $2a \pm bi$, $3a \pm 2bi$

Sum of zeroes $= 10 = 10a$ $\therefore a = 1$

Prod of zeroes $= 1460 = (4a^2+b^2)(9a^2+4b^2)$

$\therefore (4+b^2)(9+4b^2) = 1460$

$4b^4 + 25b^2 + 36 - 1460 = 0$

$4b^4 + 25b^2 - 1424 = 0$

$b^2 = \dfrac{-25 \pm \sqrt{625 + 16 \times 1424}}{8}$

$= \dfrac{-25 \pm 153}{8} = \dfrac{128}{8}$ or $\dfrac{-178}{8}$

$= 16$ or $-22\frac{1}{4}$

But b integer

$\therefore b = \pm 4$.

$\therefore$ Zeroes: $2 \pm 4i$, $3 \pm 8i$

(ii) sum $= 4$, Prod $= 4 + 16 = 20$; sum $= 6$, Prod $= 9 + 64 = 73$

$\therefore (z^2 - 4z + 20)(z^2 - 6z + 73)$

(c) $3^1 > 2(1) + \frac{1}{2}\ln 1$

$3^2 > 2(2) + \frac{1}{2}\ln 2$

$3^3 > 2(3) + \frac{1}{2}\ln 3$

$\vdots$

$3^n > 2(n) + \frac{1}{2}\ln n$

$\therefore \underbrace{3^1 + 3^2 + 3^3 + \dots + 3^n}_{} > 2(1+2+3+\dots+n) + \frac{1}{2}(\ln 1 + \ln 2 + \ln 3 + \dots + \ln n)$

GS: $a = 3$, $r = 3$, $n = n$

$\therefore \dfrac{3(3^n - 1)}{3-1} > 2\dfrac{n(n+1)}{2} + \frac{1}{2}(\ln(1 \times 2 \times 3 \times \dots \times n))$

$\therefore \frac{3}{2}(3^n - 1) > n(n+1) + \frac{1}{2}\ln(n!)$

$3(3^n - 1) > 2n^2 + 2n + \ln(n!)$

$3^{n+1} - 3 > 2n^2 + 2n + \ln(n!)$

$\therefore 3^{n+1} > 2n^2 + 2n + 3 + \ln(n!)$

(13) (a) $n=1$: $u_1 = \frac{1}{3}(2^2 + (-1)^1) = \frac{1}{3}(4-1) = \frac{3}{3} = 1$

$\therefore$ True for $n = 1$

$n = 2$: $u_2 = \frac{1}{3}(2^3 + (-1)^2) = \frac{1}{3}(8+1) = \frac{9}{3} = 3$

& $u_2 = 2u_1 + (-1)^2 = 2(1) + (1) = 3$

$\therefore$ True for $n = 2$

Let $n = k$ be the integer such that

$u_k = \frac{1}{3}(2^{k+1} + (-1)^k)$

Prove $n = k+1$ is integer such that

$u_{k+1} = \frac{1}{3}(2^{k+2} + (-1)^{k+1})$

Proof: LHS $= 2u_k + (-1)^{k+1}$

$= \frac{2}{3}(2^{k+1} + (-1)^k) + (-1)^{k+1}$ [from let statement]

$= \frac{1}{3}(2^1 2^{k+1} + 2(-1)^k) + (-1)^1(-1)^k$

$= \frac{1}{3}(2^{k+2} + 2(-1)^k + 3(-1)^1(-1)^k)$

$= \frac{1}{3}(2^{k+2} + 2(-1)^k - 3(-1)^k)$

$= \frac{1}{3}(2^{k+2} - (-1)^k)$

$= \frac{1}{3}(2^{k+2} + (-1)(-1)^k)$

$= \frac{1}{3}(2^{k+2} + (-1)^{k+1}) = $ RHS

$\therefore$ Proved by P.M.I

(b)(i) Let $u = a - x$

$\therefore \frac{du}{dx} = -1$

$\therefore dx = -du$

$x = 0$, $u = a$

$x = a$, $u = 0$

$\therefore \int_0^a f(a-x)\,dx = \int_a^0 f(u)(-du)$

$= -\int_a^0 f(u)\,du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$ by change of variable

3 (b) (ii)

$$I = \int_0^{\frac{\pi}{4}} \frac{\sin 2x}{\sin 2x + \cos 2x}\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin\left[2\left(\frac{\pi}{4}-x\right)\right]}{\sin\left[2\left(\frac{\pi}{4}-x\right)\right]+\cos\left[2\left(\frac{\pi}{4}-x\right)\right]}\,dx$$

$$I = \int_0^{\frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{2}-2x\right)}{\sin\left(\frac{\pi}{2}-2x\right)+\cos\left(\frac{\pi}{2}-2x\right)}\,dx$$

$$I = \int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\cos 2x + \sin 2x}\,dx$$

$$\therefore I + I = \int_0^{\frac{\pi}{4}} \frac{\sin 2x + \cos 2x}{\cos 2x + \sin 2x}\,dx$$

$$2I = \int_0^{\frac{\pi}{4}} dx$$

$$I = \frac{1}{2}\left[x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}$$

(c) (i) $\dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 4x^3 - 12x$

$\therefore \frac{1}{2}v^2 = x^4 - 6x^2 + c$

When $x = 0$, $v = u$

$\therefore \frac{u^2}{2} = c$

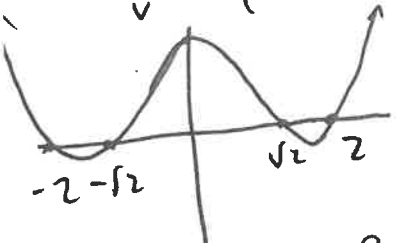
$\therefore \frac{1}{2}v^2 = x^4 - 6x^2 + \frac{u^2}{2}$

$\therefore v^2 = 2x^4 - 12x^2 + u^2$

(ii) $v^2 = 2x^4 - 12x^2 + 16$

$= 2(x^4 - 6x^2 + 8)$

$v^2 = 2(x^2-4)(x^2-2)$



Since initially $x = 0$, particle oscillates between $-\sqrt{2}$ & $\sqrt{2}$ since beyond these points, $v^2 < 0$ which is impossible.

At $x = \sqrt{2}$, $\ddot{x} = -4\sqrt{2}$, $v = 0$ $\therefore$ ~~...~~ heads to the left

At $x = -\sqrt{2}$, $\ddot{x} = 4\sqrt{2}$, $v = 0$

confirming $-\sqrt{2} \le x \le \sqrt{2}$

(iii) $\ddot{x} = -4x(3 + x^2)$

$\neq -n^2(x-a)$

$\therefore$ Not SHM

14 (a) (i) $M\ddot{x} = -\frac{1}{5}M$ at touchdown.

$\therefore \ddot{x} = -0.2$

$\frac{dv}{dt} = -0.2$

$v = -0.2t + c$

When $t = 0$, $v = 80$

$\therefore 80 = c$

$\therefore v = 80 - 0.2t$

At $t = 10$, $v = 80 - 2$
$= 78$ m/s

$\frac{dx}{dt} = 80 - 0.2t$

$\therefore x = 80t - 0.1t^2 + c$

At $t = 0$, $x = 0$ $\therefore c = 0$

$\therefore$ ~~dx~~ $x = 80t - 0.1t^2$

At $t = 10$, $x = 800 - 10$
$= 790$ m

(ii) For $t > 10$,

$M\ddot{x} = -\frac{1}{5}M - \frac{1}{125}Mv^2$

$\therefore \ddot{x} = -\frac{1}{5} - \frac{1}{125}v^2$

$\therefore \frac{d^2x}{dt^2} = -\frac{1}{125}(25 + v^2)$

(iii) $v\frac{dv}{dx} = -\frac{1}{125}(25 + v^2)$

$\therefore \frac{dv}{dx} = -\frac{1}{125}\left(\frac{25+v^2}{v}\right)$

$\therefore \frac{dx}{dv} = -125\left(\frac{v}{25+v^2}\right) = -\frac{125}{2}\left(\frac{2v}{25+v^2}\right)$

$\therefore x = -\frac{125}{2}\ln|25+v^2| + c$

When $x = 790$, $v = 78$

$\therefore 790 = -\frac{125}{2}\ln 6109 + c$ $\therefore c = 790 + \frac{125}{2}\ln 6109$

$\therefore x = 790 + 62.5\ln 6109 - 62.5\ln(25+v^2)$

$x = 790 + 62.5\left[\ln 6109 - \ln(25+v^2)\right]$

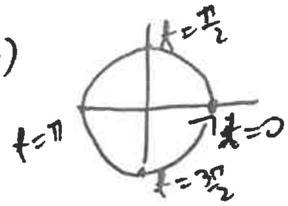
$x = 790 + 62.5\left[\ln 6109 - \ln 25\right]$

(iv) $v = 0$: $x = 790 + 62.5[\ln 6109 - \ln 25]$

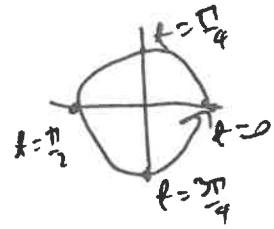
$= 1133.665159\ldots$ m

$\doteqdot 1134$ m (n.m)

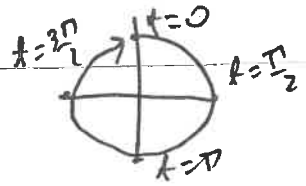
14(b)



$R_1(t)$: circular, 2π, anticlockwise



$R_2 t$: circular, π, anticlockwise



$R_3 t$: circular, 2π, clockwise

(c) ~~[illegible]~~

$\ddot{x} = 8x^3 + 36x$

$\frac{1}{2}v^2 = 2x^4 + 18x^2 + c$

At $x = 1, v = 11$

$\therefore \frac{121}{2} = 2 + 18 + c$

$\therefore c = 40\frac{1}{2}$

$\therefore \frac{1}{2}v^2 = 2x^4 + 18x^2 + 40\frac{1}{2}$

$\therefore v^2 = 4x^4 + 36x^2 + 81$

$v^2 = (2x^2+9)^2$

At $x = 1, v = 11$

$\therefore v = 2x^2 + 9$ not $-(2x^2+9)$

$\frac{dx}{dt} = 2x^2 + 9$

$\frac{dt}{dx} = \frac{1}{2x^2+9} = \frac{1}{2} \times \frac{1}{x^2 + \frac{9}{2}}$

$\therefore t = \frac{1}{2} \frac{\sqrt{2}}{3} \tan^{-1} \frac{\sqrt{2}x}{3} = \frac{1}{3\sqrt{2}} \tan^{-1} \frac{\sqrt{2}x}{3} + c$

At $x = 1, t = 0$ $\therefore 0 = \frac{1}{3\sqrt{2}} \tan^{-1} \frac{\sqrt{2}}{3} + c$

$\therefore c = -\frac{1}{3\sqrt{2}} \tan^{-1} \frac{\sqrt{2}}{3}$

$\therefore t = \frac{1}{3\sqrt{2}} \tan^{-1} \frac{\sqrt{2}x}{3} - \frac{1}{3\sqrt{2}} \tan^{-1} \frac{\sqrt{2}}{3}$

$\frac{1}{3\sqrt{2}} \tan^{-1} \frac{\sqrt{2}x}{3} = t + \frac{1}{3\sqrt{2}} \tan^{-1} \frac{\sqrt{2}}{3}$

$\tan^{-1} \frac{\sqrt{2}x}{3} = 3\sqrt{2}t + \tan^{-1} \frac{\sqrt{2}}{3}$

$\therefore \frac{\sqrt{2}x}{3} = \tan\left[3\sqrt{2}t + \tan^{-1}\frac{\sqrt{2}}{3}\right]$

$\therefore x = \frac{3}{\sqrt{2}} \tan\left[3\sqrt{2}t + \tan^{-1}\frac{\sqrt{2}}{3}\right]$

OR: $t = \frac{1}{3\sqrt{2}}\left(\tan^{-1}\frac{\sqrt{2}x}{3} - \tan^{-1}\frac{\sqrt{2}}{3}\right)$

$\tan(3\sqrt{2}t) = \dfrac{\frac{\sqrt{2}x}{3} - \frac{\sqrt{2}}{3}}{1 + \frac{\sqrt{2}x}{3}\frac{\sqrt{2}}{3}}$

$\left(1 + \frac{2x}{9}\right)\tan(3\sqrt{2}t) = \frac{\sqrt{2}x}{3} - \frac{\sqrt{2}}{3}$

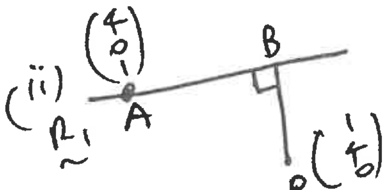
$x\left[\frac{2}{9}\tan(3\sqrt{2}t) - \frac{\sqrt{2}}{3}\right] = -\frac{\sqrt{2}}{3} - \tan(3\sqrt{2}t)$

$x = \dfrac{\frac{\sqrt{2}}{3} + \tan(3\sqrt{2}t)}{\frac{\sqrt{2}}{3} - \frac{2}{9}\tan(3\sqrt{2}t)} = \dfrac{3\sqrt{2} + 9\tan(3\sqrt{2}t)}{3\sqrt{2} - 2\tan(3\sqrt{2}t)}$

(15) (a)(i) $\begin{pmatrix}4\\ \lambda\\ 1+\lambda\end{pmatrix} = \begin{pmatrix}3+\mu\\ 3\mu\\ 2+2\mu\end{pmatrix}$ $\leftarrow \mu = 1$ at P.I. $\leftarrow \lambda = 3$ $\leftarrow$ consistency!

$\therefore$ Intersect at $\begin{pmatrix}4\\3\\4\end{pmatrix}$

(ii)



$\vec{AP} = \begin{pmatrix}-3\\4\\-1\end{pmatrix}$

$\vec{AB} = \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$

$\vec{AB} = \text{Proj}_{\vec{AB}}\vec{AP} = \dfrac{\begin{pmatrix}-3\\4\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}\lambda}{\lambda^2\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}}\lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$

$= \frac{3\lambda}{2\lambda^2}\lambda\begin{pmatrix}0\\1\\1\end{pmatrix} = \frac{3}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}$

$\vec{PB} = \vec{PA} + \vec{AB}$
$= \begin{pmatrix}3\\-4\\1\end{pmatrix} + \frac{3}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}3\\-2.5\\2.5\end{pmatrix}$

$\therefore |\vec{PB}| =$ shortest distance $= \sqrt{9 + 6.25 + 6.25}$

$= \sqrt{21.5}$ units

Note: B is $\begin{pmatrix}4\\1.5\\ \ldots\end{pmatrix}$ & $\lambda = 1.5$ at B & also ...

15 (b) $I_n = \int_0^1 1 \times \frac{1}{(1+x^2)^n}\,dx, \quad n \geq 1$

(i) $I_n = \left[x \times \frac{1}{(1+x^2)^n}\right]_0^1 - \int_0^1 x(-n)(1+x^2)^{-n-1}(2x)\,dx$

$= \left(\frac{1}{2^n} - 0\right) + 2n\int_0^1 x^2(1+x^2)^{-n-1}\,dx$

$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$

(ii) $I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{(x^2+1)-1}{(1+x^2)^{n+1}}\,dx$

$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}}$

$I_n = \frac{1}{2^n} + 2n\left[I_n - I_{n+1}\right]$

$I_n = \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$

$2nI_{n+1} = \frac{1}{2^n} + 2nI_n - I_n$

$2nI_{n+1} = \frac{1}{2^n} + I_n(2n-1)$

$I_{n+1} = \frac{1}{2n\,2^n} + \frac{I_n(2n-1)}{2n}$

$I_{n+1} = \frac{1}{n2^{n+1}} + \frac{2n-1}{2n}I_n$

(iii) Let n=1

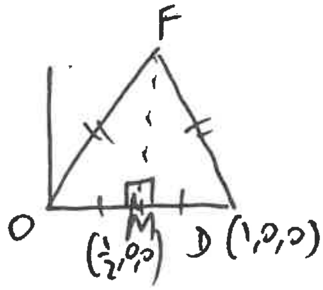
$I_2 = \frac{1}{1\times 2^2} + \frac12 I_1 = \frac14 + \frac12 I_1$

$I_1 = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}$

$\therefore I_2 = \frac14 + \frac12 \times \frac{\pi}{4} = \frac{\pi}{8} + \frac14$

(c)

(i)



$|\vec{OF}| = 1 \quad |\vec{OM}| = \frac12$

$\therefore |\vec{MF}| = \sqrt{1 - \frac14} = \frac{\sqrt3}{2}$

$\therefore E\left(\frac12, \frac{\sqrt3}{2}, 0\right)$

(ii) Let F be (x, y, z)

$\cos\angle DOF = \frac{\vec{OF}\cdot\vec{OD}}{|\vec{OF}||\vec{OB}|}$

$\cos\frac{\pi}{3} = \frac{\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{1\times1}$

$\therefore \frac12 = x$

$\cos\angle EOF = \frac{\vec{OE}\cdot\vec{OF}}{|\vec{OE}||\vec{OF}|}$

$\frac12 = \frac{\begin{pmatrix}\frac12\\\frac{\sqrt3}{2}\\0\end{pmatrix}\cdot\begin{pmatrix}\frac12\\y\\z\end{pmatrix}}{1\times1}$

$\therefore \frac12 = \frac14 + \frac{\sqrt3}{2}y \quad \therefore \frac14 = \frac{\sqrt3}{2}y \quad \therefore y = \frac{1}{2\sqrt3} = \frac{\sqrt3}{6}$

$\therefore F\begin{pmatrix}\frac12\\\frac{\sqrt3}{6}\\z\end{pmatrix}$

Now $|\vec{OF}| = 1 \quad \therefore 1 = \frac14 + \frac{3}{36} + z^2$

$\therefore \frac23 = z^2 \quad \therefore z = \frac{\sqrt2}{\sqrt3} = \frac{\sqrt6}{3}$

$\therefore F\left(\frac12, \frac{\sqrt3}{6}, \frac{\sqrt6}{3}\right)$

OR: $\vec{OF} = \begin{pmatrix}x\\y\\z\end{pmatrix} \quad \vec{DF} = \begin{pmatrix}x-1\\y\\z\end{pmatrix} \quad \vec{EF} = \begin{pmatrix}x-\frac12\\y-\frac{\sqrt3}{2}\\z\end{pmatrix}$

$|\vec{OF}| = |\vec{DF}| = |\vec{EF}| = 1$

$\therefore 1 = x^2+y^2+z^2$

$1 = (x-1)^2+y^2+z^2$

$(x-1)^2 = x^2$

$1 = 2x$

$x = \frac12$

$\therefore \frac34 = y^2+z^2.$

But also $1 = \left(x-\frac12\right)^2 + \left(y-\frac{\sqrt3}{2}\right)^2 + z^2$

$x = \frac12 \therefore 1 = \left(y - \frac{\sqrt3}{2}\right)^2 + z^2$

$\therefore -\sqrt3 y + \frac34 = \frac14$

$\sqrt3 y = \frac12$

$\therefore y = \frac{1}{2\sqrt3} = \frac{\sqrt3}{6}$

$\therefore \frac14 + \frac{3}{36} + z^2 = 1$ ...

(16) Since $\sec^2\theta - 1 = \tan^2\theta$,

(a) let $x = \frac{\sec\theta}{2}$ $\quad \therefore \frac{dx}{d\theta} = \frac{\sec\theta\tan\theta}{2}$

$$\therefore \int \frac{dx}{\sqrt{4x^2-1}} = \frac{1}{2}\int \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{\sec^2\theta-1}}$$

$$= \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\tan\theta}\, d\theta$$

$$= \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}$$

$$= \frac{1}{2}\ln|\sec\theta+\tan\theta|$$

Since $\sec\theta = 2x$,

$$= \frac{1}{2}\ln\left|2x+\sqrt{4x^2-1}\right| + C$$

(b) note that divisibility by primes 2, 3 and 5 means divisible by $2\times3\times5 = 30$.

$n=2$: $2^5-2 = 32-2 = 30 = 30\times 1$

$\therefore$ True for $n=2$.

Let $n=k$ be integer ≥ 2 such that
$k^5-k = 30l$ (l integer)

Prove $n=k+1$ is integer such that
$(k+1)^5-(k+1) = 30m$ (m integer)

Proof: LHS $= (k+1)[(k+1)^4-1]$

$= (k+1)(k^4+4k^3+6k^2+4k)$

$= k^5+5k^4+10k^3+10k^2+4k$

$= 30l+5k^4+10k^3+10k^2+5k$ [from let statement]

$= 30l+5(k^4+2k^3+2k^2+k)$

$= 30l+5k(k^3+2k^2+2k+1)$

$= 30l+5k\underline{(k+1)(k^2+k+1)}$

We'd like $6=3!$ as a factor

$\therefore$ we'd like 3 consecutive numbers

$= 30l+5k(k+1)((k-1)^2+3k)$

$= 30l+5k(k+1)(k-1)^2+15k^2(k+1)$

$= 30l+5(k-1)\underbrace{(k-1)k(k+1)}_{=6p,\ p\in\mathbb{Z}}+15k\underbrace{k(k+1)}_{=2q,\ q\in\mathbb{Z}}$

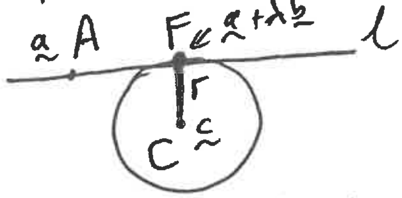
$= 30l+30p(k-1)+30qk$

$= 30(l+p(k-1)+qk) = 30m$

since l, p, k, q integer

$\therefore$ Prove true by P.M.I

(c)(i)



$$|a+\lambda b - c| = r$$

$$\therefore r^2 = [(a-c)+\lambda b]\cdot[(a-c)+\lambda b] = (a-c)\cdot(a-c)+2\lambda\, b\cdot(a-c)+\lambda^2\, b\cdot b$$

$$r^2 = (a-c)\cdot(a-c)$$

$$\lambda^2|b|^2+2\lambda\, b\cdot(a-c)+\underline{|a-c|^2-r^2} = 0$$

$$\Delta = [2b\cdot(a-c)]^2-4|b|^2[|a-c|^2-r^2]$$

$$= 4[b\cdot(a-c)]^2-4|b|^2[|a-c|^2-r^2]$$

Since tangent, λ has only one value $\therefore \Delta = 0$

$$\therefore [b\cdot(a-c)]^2 = |b|^2\underline{[|a-c|^2-r^2]}$$

$$\therefore \lambda^2|b|^2+2\lambda\, b\cdot(a-c)+\frac{[b\cdot(a-c)]^2}{|b|^2} = 0$$

$$\therefore \left[\lambda|b|+\frac{b\cdot(a-c)}{|b|}\right]^2 = 0$$

$$\therefore \lambda = \frac{b\cdot(c-a)}{|b||b|} = \frac{b\cdot(c-a)}{|b|^2}$$

16 (c) (ii)

$\vec{CF} = \underset{\sim}{a} + \lambda \underset{\sim}{b} - \underset{\sim}{c}$

Now $\vec{CF} \cdot \underset{\sim}{b} = (\underset{\sim}{a} + \lambda \underset{\sim}{b} - \underset{\sim}{c}) \cdot \underset{\sim}{b}$

direction vector

$= \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{c} + \lambda\, \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{c} + \lambda |\underset{\sim}{b}|^2$

Using (i)

$= \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{c} + \dfrac{\underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{a})}{|\underset{\sim}{b}|^2} |\underset{\sim}{b}|^2$

$= \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{a} = 0$

$\therefore$ Perpendicular!

(d) Equilibrium (with angle α):

$\therefore F = mg\sin\alpha$

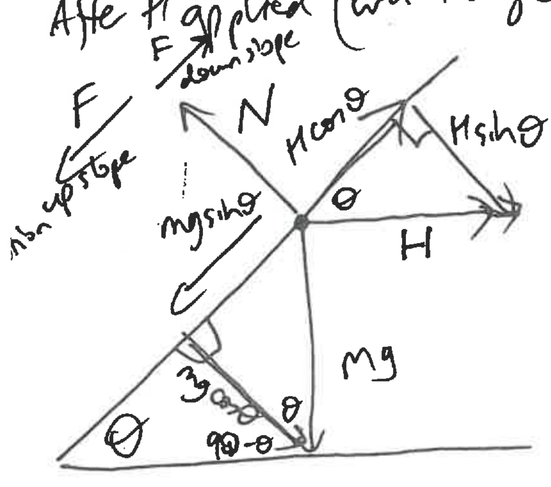
$N = mg\cos\alpha$

$\therefore \dfrac{F}{N} = \tan\alpha$

$\therefore \dfrac{\mu N}{N} = \tan\alpha$

$\therefore \mu = \tan\alpha$

After H applied (with angle θ)



So: $N = H\sin\theta + mg\cos\theta$

Motion up slope: $H\cos\theta > mg\sin\theta + F$

Motion down slope $mg\sin\theta > H\cos\theta + F$

$\therefore$ For motion up slope not to happen,

$H\cos\theta \le mg\sin\theta + F$

& for motion down slope not to happen,

$mg\sin\theta \le H\cos\theta + F$

$\Rightarrow H\cos\theta \le mg\sin\theta + \mu N$

$H\cos\theta \le mg\sin\theta + \tan\alpha(H\sin\theta + mg\cos\theta)$

To create tan, $\div \cos\theta$

$\therefore H \le mg\tan\theta + \tan\alpha(H\tan\theta + mg)$

$H(1 - \tan\theta\tan\alpha) \le mg\tan\theta + mg\tan\alpha$

$H \le \dfrac{mg(\tan\theta + \tan\alpha)}{1 - \tan\theta\tan\alpha}$

$H \le mg\tan(\theta + \alpha)$

$\Rightarrow mg\sin\theta \le H\cos\theta + \tan\alpha(H\sin\theta + mg\cos\theta)$

$\div \cos\theta$

$mg\tan\theta \le H + \tan\alpha(H\tan\theta + mg)$

$mg\tan\theta - mg\tan\alpha \le H + H\tan\theta\tan\alpha$

$mg(\tan\theta - \tan\alpha) \le H(1 + \tan\theta\tan\alpha)$

$\dfrac{mg(\tan\theta - \tan\alpha)}{1 + \tan\theta\tan\alpha} \le H$

$mg\tan(\theta - \alpha) \le H$

$\therefore mg\tan(\theta - \alpha) \le H \le mg\tan(\theta + \alpha)$