BAULKHAM HILLS HIGH SCHOOL
MARKING COVER SHEET.

YEAR 12 Trial
EXTENSION 2
2011

STUDENT NUMBER:____________________

TEACHER NAME:____________________

QUESTION	MARK
1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	/ 120
PERCENTAGE	%

BAULKHAM HILLS HIGH SCHOOL

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 120
- Attempt Questions 1 – 8
- All questions are of equal value
- Start a separate piece of paper for each question.
- Put your student number and the question number at the top of each sheet.

Question 1 (15 marks) - Start on a new page		**Marks**
a)	Find the following indefinite integrals	
	(i) $\int \cos^3 x \, dx$	**2**
	(ii) $\int \frac{x-2}{x^2+1} \, dx$	**2**
	(iii) $\int x \sin 2x \, dx$	**2**
b)	Evaluate $\int_0^1 \frac{dx}{\sqrt{3-2x-x^2}}$	**3**
c)	Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$	**3**
d)	Find $\int \frac{2x \, dx}{x^3-2x^2+9x-18}$	**3**

Question 2 (15 marks) - Start on a new page		
a)	(i) Express $z_1 = \dfrac{7+4i}{3-2i}$ in the form $a+ib$ where a, b are real	2
	(ii) On an Argand diagram sketch the locus of the point representing the complex number z such that $\lvert z - z_1 \rvert = \sqrt{5}$	1
	(iii) Prove that the locus passes through the origin and find the greatest value of $\lvert z \rvert$	2
b)	Let $z = 2 + 3i$ and $w = 1 + i$ Find zw and $\frac{1}{w}$ in the form $x + iy$	2
c)	(i) Express $(1 - \sqrt{3}i)$ in modulus argument form	2
	(ii) Hence write $\left(1 - \sqrt{3}i\right)^{10}$ in the form $x + iy$	2
d)	The complex number $z = x + iy$ when x and y are real, is such that $\lvert z - i \rvert = \operatorname{Im}(z)$	
	(i) Show that the locus of point P representing z has Cartesian equation $y = \dfrac{1}{2}(x^2+1)$ and sketch the locus	2
	(ii) By finding the gradients of the tangents to this curve which pass through the origin, find the set of possible values of $\arg z$ $(-\pi < \arg z \leq \pi)$	2

Question 3 (15 marks) - Start a new page		**Marks**
a)	Sketch the graph $y = (2x+1)(x+1)$ clearly showing all intercepts on the co-ordinate axes and the co-ordinates of any turning points.	2
b)	Use the graph of part (a) to sketch the graphs below, showing clearly the intercepts on the co-ordinate axes, the co-ordinates of any turning points and the equation of any asymptotes. (i) $y = \log_e[(2x+1)(x+1)]$ (ii) $y = \dfrac{1}{(2x+1)(x+1)}$	2 2
c)	The region bounded by the curve $y = \dfrac{1}{(2x+1)(x+1)}$ the co-ordinate axes and the line $x = 4$ is rotated through one complete revolution about the y axis. (i) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral . (ii) Evaluate the integral in part (i).	 2 4
d)	When $P(x) = x^4 + ax^3 + b$ is divided by $x^2 + 4$, the remainder is $-x + 13$. Find the values of a and b.	3

Question 4 (15 marks) - Start a new page

a) A hyperbola has Cartesian equation

$$3x^2 - y^2 = 12$$

find (i) its eccentricity **1**

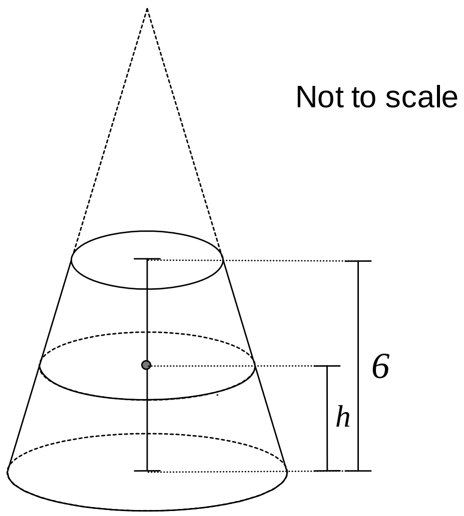
(ii) the co-ordinates of its foci **1**

(iii) the equations of the directrices **1**

(iv) the equations of the asymptotes **1**

hence sketch the hyperbola indicating all the features of your diagram. **1**

b) A right elliptical cone has its top cut off through a plane parallel to its elliptical base. The remaining solid has an ellipse as its base



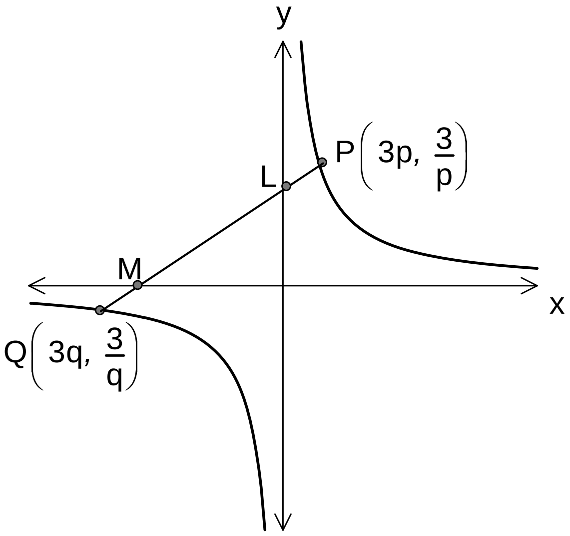
The remaining solid has the ellipse $\frac{x^2}{9} + \frac{4y^2}{9} = 1$ as its base, and another ellipse $x^2 + 4y^2 = 1$ as its top.

The height of the solid is 6 units.

(i) Given that the area of an ellipse with equation $\frac{x^2}{9} + \frac{4y^2}{9} = 1$ is πab, show that the area of the ellipse at height h units above the base is $A = \frac{\pi(h-9)^2}{18}$ **3**

(ii) Hence find the volume of the soild. **3**

c) A plane curve is defined implicitly by $x^2 + 2xy + y^5 = 4$
This curve has a horizontal tangent at $P(x, y)$ show that $x = \alpha$ is a root of the equation $x^5 + x^2 + 4 = 0$ **4**

Question 5 (15 marks) - Start a new page		**Marks**
a)	The equation $x^3 + px - 1 = 0$ has 3 non zero roots α, β, γ	
	(i) Find the values of $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^4 + \beta^4 + \gamma^4$ in terms of p and show that p must be negative.	4
	(ii) Find the monic equation with coefficients in terms of p where roots are $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma}, \frac{\gamma}{\alpha\beta}$	2
b)	A chord PQ of the rectangular hyperbola $xy = 9$ meets the asymptotes at L and M as shown 	
	i) Show that the equation of the chord PQ is $pqy + x = 3(p + q)$	1
	ii) Find the co-ordinates of N the midpoint of PQ	2
	iii) Show that $PL = MQ$	2
	iv) If the chord PQ is a tangent to the parabola $y^2 = 3x$ find the locus of N	2
c)	Solve in terms of a $a^x = e^{2x-1}$ where $a > 0, a \neq \frac{1}{e^2}$	2

Question 6 (15 marks) - Start a new page		
a)	Solve the equation $x^4 - 6x^3 + 9x^2 + 6x - 20 = 0$ given $(2 + i)$ is one of it's zeroes.	3
b)	$P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad where\ a > b > 0.$ The tangent and normal at point P cut y-axis at A and B respectively, and S is a focus of the ellipse	
	i) Show that $\angle ASB = 90°$	2
	ii) Hence show that A, P, S and B are concyclic and state the coordinates of the centre of the circle through A, P, S and B.	3
c)	Prove that $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$	2
	Hence evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$	3
d)	Draw a neat sketch of $y = \frac{1}{\sin^{-1} x}$	2

Question 7 (15 marks) - Start a new page **Marks**

a) Given that $1, \omega$ and ω^2 are the three cube roots of unity

i) Find the value of $(1+2\omega+3\omega^2)(1+2\omega^2+3\omega)$ **3**

ii) If the equations $x^3-1=0$ and $px^5+qx+r=0$ have a common root, evaluate **3**

$$(p+q+r)(p\omega^5+q\omega+r)(p\omega^{10}+q\omega^2+r)$$

b) i) Show that

$$\left(1-\sqrt{x}\right)^{n-1}.\sqrt{x}=\left(1-\sqrt{x}\right)^{n-1}-\left(1-\sqrt{x}\right)^{n}$$ **1**

ii) If $I_n=\int_0^1\left(1-\sqrt{x}\right)^n dx$ for $n\geq 0$ **3**

Show that $I_n=\frac{n}{n+2}\,I_{n-1}\ for\ n\geq 1$

And hence evaluate I_{100}

c) Prove that the volume, V, the area of the curved surface, S, and the radius of the base, r, of a right circular cone are connected by the equation **2**

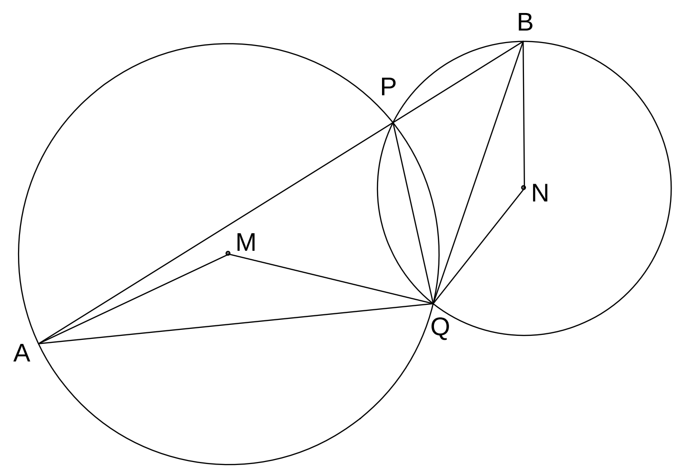
$$9V^2=r^2(S^2-\pi^2r^4)$$

3

Show that the maximum volume for a given curved surface area S, is

$$\frac{2^{\frac{1}{2}}S^{\frac{3}{2}}}{\pi^{\frac{1}{2}}3^{\frac{7}{4}}}$$

Question 8 (15 marks) - Start a new page

a)	Prove by mathematical induction that $7^n + 3n(7^n) - 1$ is divisible by 9.	3
b)	i) Write the general solution of $$\tan 4\theta = 1$$	1
	ii) Use De Moivre's Thereom to find $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$ and hence determine the result $$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$	4
	iii) Find the roots of $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ in the form $x = \tan\theta$ and hence prove that $$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$$	3
c)	 In the given diagram, two circles whose centres are M and N intersect in P and Q. A line drawn through P meets the two circles in A and B. Prove that $\angle MAQ = \angle NBQ$.	4
	End of Exam	

Q1) i) $\int \cos^3 x \, dx$

(1) $= \int \cos x(1-\sin^2 x)\,dx$ 1

$= \int \cos x\,dx - \int \sin^2 x \cos x\,dx$

$= \sin x - \dfrac{\sin^3 x}{3} + c$ 1

ii) $\displaystyle\int \frac{x-2}{x^2+1}\,dx$

$= \displaystyle\int \frac{x\,dx}{x^2+1} - \int \frac{2}{x^2+1}\,dx$

$= \frac{1}{2}\ln(x^2+1) - 2\tan^{-1} x + C$ 1 1

iii) $\int x \sin 2x\,dx$

let $u = x$ $\quad v' = \sin 2x$

$u' = 1$ $\quad v = -\frac{1}{2}\cos 2x$

$\therefore I = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$ 1

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$ 1

b) $\displaystyle\int_0^1 \frac{dx}{\sqrt{-x^2-2x+3}}$

$= \displaystyle\int_0^1 \frac{dx}{\sqrt{-(x^2+2x+1)+4}}$

$= \displaystyle\int_0^1 \frac{dx}{\sqrt{4-(x+1)^2}}$ 1

$= \left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_0^1$ 1

$= \sin^{-1} 1 - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$ 1

c) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\cos x+\sin x}$ (4)

let $t = \tan\frac{x}{2}$ $\quad x=\frac{\pi}{2}$ $t=1$; $x=0$ $t=0$

$\therefore I = \displaystyle\int_0^1 \left(\frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\right)\cdot\frac{2dt}{1+t^2}$ 1

$= \displaystyle\int_0^1 \frac{2\,dt}{1+t^2+1-t^2+2t}$ 1

$= \displaystyle\int_0^1 \frac{dt}{1+t}$

$= \left[\ln(1+t)\right]_0^1$ 1

$= \ln 2 - \ln 1$

$= \ln 2$.

d) $\displaystyle\int \frac{2x\,dx}{x^3-2x^2+9x-18}$

$= \displaystyle\int \frac{2x\,dx}{(x^2+9)(x-2)}$ 1

let $\dfrac{2x}{(x-2)(x^2+9)} \equiv \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+9}$

$A(x^2+9) + (Bx+C)(x-2) \equiv 2x$

$x = 2 \rightarrow$ $13A = 4$ $\quad\therefore A = \frac{4}{13}$

Const term $\quad 9A - 2C = 0$

$2C = 9\times\frac{4}{13}$ $\quad C = \frac{18}{13}$

term in x^2 $\quad A+B = 0$ $\quad B = -\frac{4}{13}$ 1

$\therefore I = \dfrac{1}{13}\left[\displaystyle\int \frac{4}{x-2}\,dx + \int \frac{18-4x}{x^2+9}\,dx\right]$

$= \frac{1}{13}\left[4\ln(x-2) + \frac{18}{3}\tan^{-1}\frac{x}{3} - 2\ln(x^2+9)\right]$

$= \frac{4}{13}\ln(x-2) + \frac{6}{13}\tan^{-1}\frac{x}{3} - \frac{2}{13}\ln(x^2+9) + C$ 1

b) $z = 2+3i \quad w = (1+i)$

$zw = (2+3i)(1+i)$

$= -1+5i$ 1

$\frac{1}{w} = \frac{1}{1+i} \times \frac{1-i}{1-i}$

$= \frac{1-i}{2}$

$= \frac{1}{2} - \frac{1}{2}i.$ 1

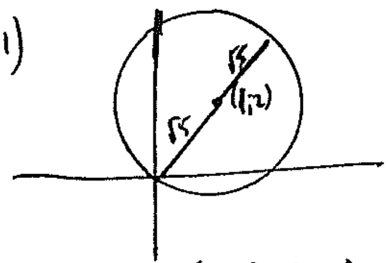
a) i) $z_1 = \frac{7+4i}{3-2i} \times \frac{3+2i}{3+2i}$ 1

$= \frac{21+14i+12i-8}{9+4}$

$= \frac{13+26i}{13}$ 1

$= 1+2i.$

ii)



1

eqn of circle is $(x-1)^2+(y-2)^2 = \sqrt{5}^2$

$\therefore x^2-2x+1+y^2-4y+4 = 5$

$x^2+y^2-2x-4y = 0$ 1

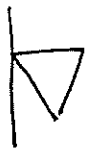
$x=0 \; y=0$ satisfies the eqn.

max $|z|$ = diameter of circle

i.e. $2\sqrt{5}$ 1

c) i) $(1-\sqrt{3}i)$

$= 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ 1



$= 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)$ 1

or $2 \operatorname{cis} -\frac{\pi}{3}$

ii) $(1-\sqrt{3}i)^8$

$= 2^8 \operatorname{cis} -\frac{8\pi}{3}$ 1

$= 2^8 \operatorname{cis} -\frac{2\pi}{3}$

$= 2^8\left(\cos -\frac{2\pi}{3} + i\sin -\frac{2\pi}{3}\right)$

$= 2^8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -128 - \cancel{128i}\; 128\sqrt{3}i$ 1

d) i) $|z-i| = \operatorname{Im} z$

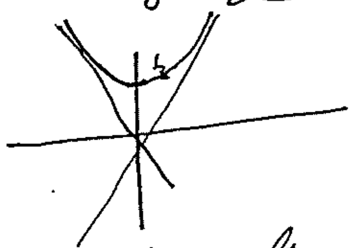
$|x+iy-i| = y$

$x^2+(y-1)^2 = y^2$

$x^2+y^2-2y+1 = y^2$ 1

$\cancel{x^2 = 2y-1}$

$y = \frac{1}{2}(x^2+1)$



1

ii) equation of tangents through O are

$y = mx.$

$mx = \frac{1}{2}(x^2+1)$

$x^2-2mx+1 = 0$

only one soln

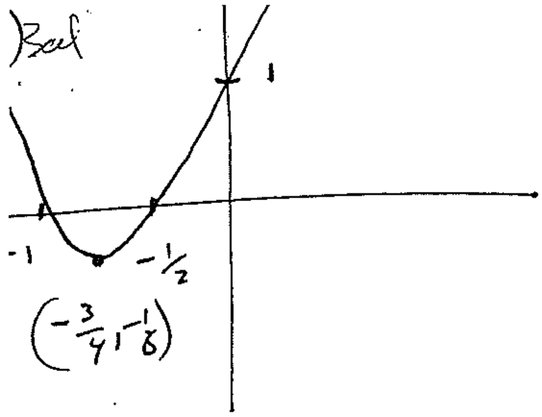
$\therefore \Delta = 0$

$4m^2-4 = 0$ 1

$m = \pm 1$

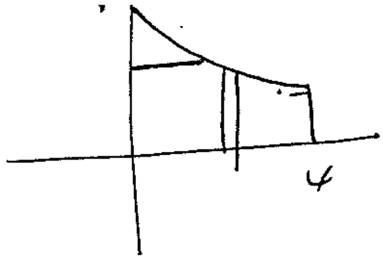
$\therefore \frac{\pi}{4} \le \arg z \le \frac{3\pi}{4}$ 1

)Bcd



2

c)



③

Let the thickness of shell be δx.

$\delta V = 2\pi x \cdot y \cdot \delta x$ 1

$= 2\pi \dfrac{x}{(2x+1)(x+1)} \delta x$

6

$V = \sum_{x=0}^{x=4} \delta V$

$= 2\pi \int_0^4 \dfrac{x\,dx}{(2x+1)(x+1)}$ 1

Let $\dfrac{x}{(2x+1)(x+1)} = \dfrac{a}{2x+1} + \dfrac{b}{x+1}$

$a = -1 \quad b = 1$ 1

$\therefore V = 2\pi \int_0^4 \left(\dfrac{1}{x+1} - \dfrac{1}{2x+1} \right) dx$

$= 2\pi \left[\ln(x+1) - \frac{1}{2}\ln(2x+1) \right]_0^4$ 1

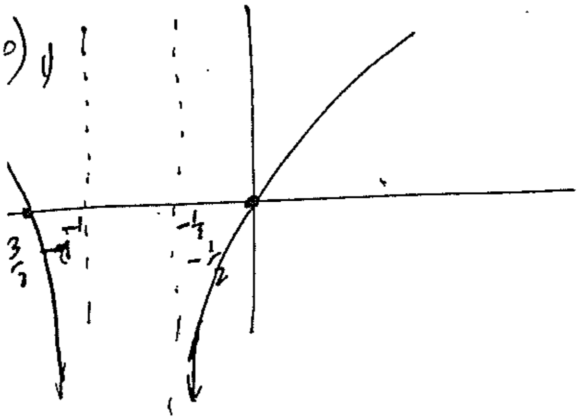
$= 2\pi \left[\ln 5 - \ln 1 - \frac{1}{2}\ln 9 + \frac{1}{2}\ln 1 \right]$

$= 2\pi \left(\ln 5 - \frac{1}{2}\ln 9 \right)$

$= 2\pi \ln \frac{5}{3}$ u³. 1

3

ii) i)



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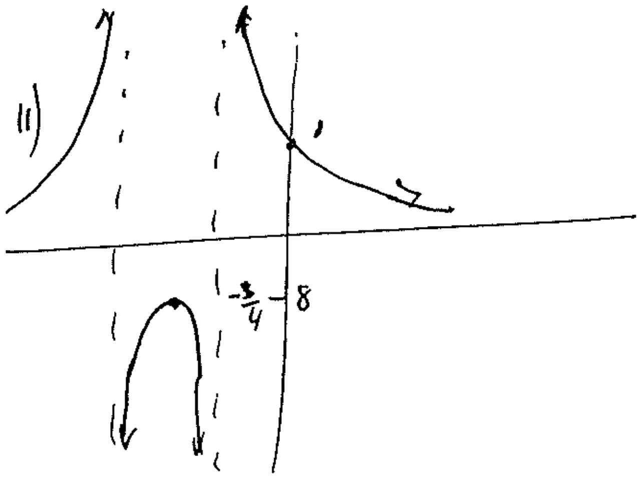
2

ii)



d) $x^4 + ax^3 + b = (x^2+4)Q(x) - x + 13$

Let $x = 2i$ $\quad 16 - 8ai + b = -2i + 13$ 1

$\therefore a = \frac{1}{4}$ 1

$b = -3$ 1

4) $3x^2 - y^2 = 12$

$\frac{x^2}{4} - \frac{y^2}{12} = 1$

$a = 2 \quad b = 2\sqrt{3}$

$b^2 = a^2(e^2 - 1)$

$12 = 4(e^2 - 1)$

$e^2 - 1 = 3$

$e = 2 \quad — (i)$ 1

$S\,(4,0) \quad S'\,(-4,0) \quad —(ii)$ 1

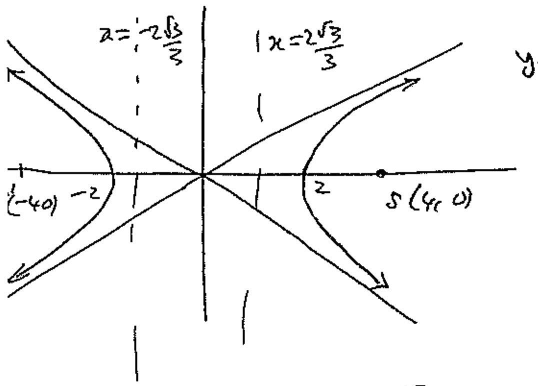
~~$y = \mp\sqrt{3}x$ (iii)~~

rectrices

$\cancel{y=}\ x = \mp \frac{a}{e}$

$x = \mp \frac{2}{\sqrt{3}}$

or $x = \mp \frac{2\sqrt{3}}{3}$ 1



$y = \mp\sqrt{3}x$ 1 (asymptotes)

c) $x^2 + 2xy + y^5 = 4$

$2x + 2y + 2xy' + 5y^4y' = 0$

$y'(2x + 5y^4) = -(2x + 2y)$

$y' = \frac{-2(x+y)}{2x + 5y^4}$ 1

$= 0$ when ~~$x = -y$~~

$y = -x$ 1

subst $y = -x$ in eqn

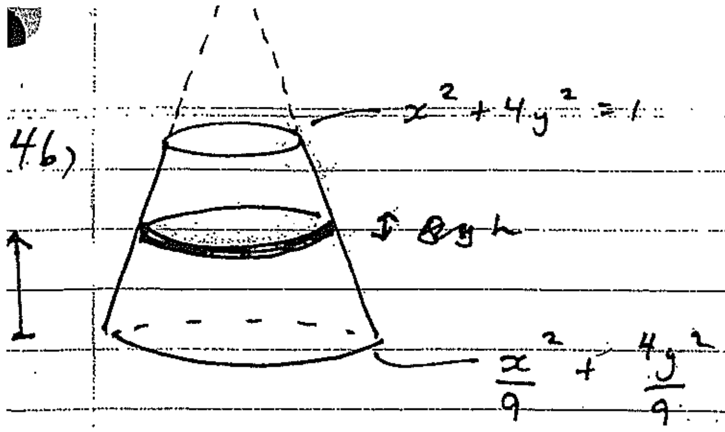
$\therefore x^2 + 2x(-x) + (-x)^5 = 4$

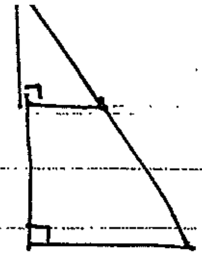
$x^2 - 2x^2 - x^5 = 4$ 1

$-x^2 - x^5 = 4$ 1

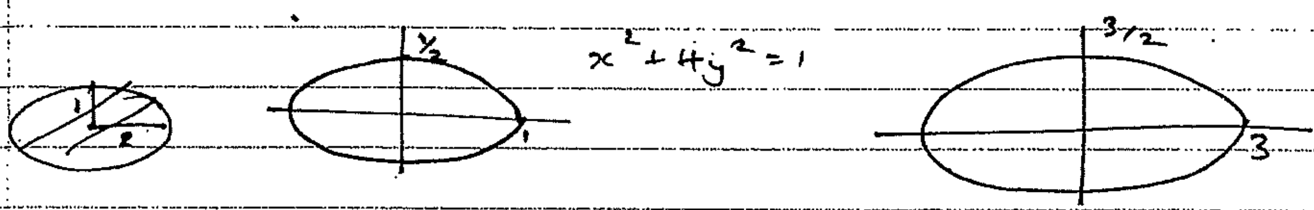
$x^5 + x^2 + 4 = 0$

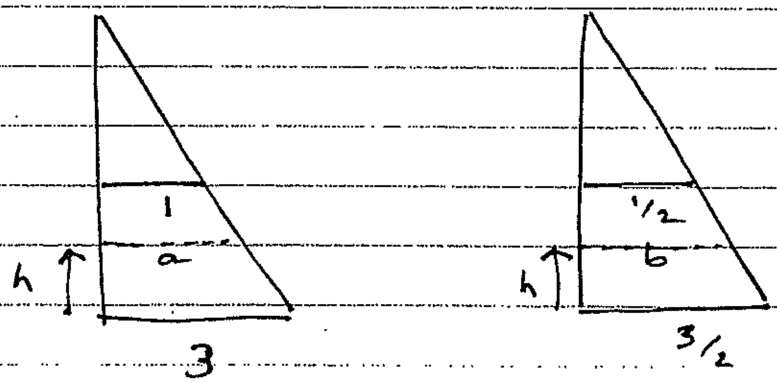
46)





1) ~~Show that the~~ volume ~~of the~~ cylindrical disc at is





$a = mh + b$

If $h=0$, $a=3$

$3 = 0 + b \therefore b = 3$

If $h = 6$, $a = 1$

$1 = 6m + 3$

$-2 = 6m$

$m = -\frac{1}{3}$

$\therefore a = -\frac{1}{3}h + 3$

1

$b = mh + c$

If $h=0$, $b = \frac{3}{2}$

$\frac{3}{2} = 0 + c$

$c = \frac{3}{2}$

If $h=6$, $b = \frac{1}{2}$

$\frac{1}{2} = 6m + \frac{3}{2}$

$-1 = 6m$

$m = -\frac{1}{6}$

$\therefore b = -\frac{1}{6}h + \frac{3}{2}$

1

$\therefore$ Area of ellipse at height h

$= \pi\left(-\frac{1}{3}h+3\right)\left(-\frac{1}{6}h+\frac{3}{2}\right)$

$\therefore$ ~~$\Delta V = \pi$~~

$$= \pi \cdot -\frac{1}{3}(h-9) \cdot -\frac{1}{6}(h-9)$$

$$= \frac{1}{18}\pi\,(h-9)^2 \qquad 1$$

$$\times \boxed{} = \frac{3}{2}$$

$$\boxed{} = \frac{3}{2} \times 6$$

$$\therefore \Delta V = \frac{1}{18}\pi \cdot (h-9)^2 \cdot \Delta h$$

$$V = \frac{\pi}{18}\int_0^6 (h-9)^2 \cdot dh \qquad 1$$

$$= \frac{\pi}{18} \cdot \left[\frac{(h-9)^3}{3}\right]_0^6 \qquad 1$$

$$= \frac{\pi}{18}\left(\frac{(-3)^3}{3} - \frac{(-9)^3}{3}\right) \qquad \frac{729}{3}$$

$$= \frac{\pi}{18}\left(-9 + 243\right)$$

$$= \frac{234\pi}{18}$$

$$= 13\pi \qquad 1$$

~~Given that the area of an ellipse is πab units^2,~~

(i) ~~Given that~~ Show that the area

(5)

a) i) $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

$\sum\alpha = 0 \quad \sum\alpha\beta = p$ 1

$\therefore \sum\alpha^2 = 0 - 2p = -2p$ 1

$x^3 = -px + 1$

$x^4 = -px^2 + x$

$\alpha^4 = -p\alpha^2 + \alpha$ etc 1

$\therefore \sum\alpha^4 = -p\sum\alpha^2 + \sum\alpha$

$= -p \cdot -2p + 0$

$= 2p^2$. 1

$\sum\alpha^2 > 0$ if not one non zero reals

$\therefore -2p > 0 \therefore p < 0$

roots are $\frac{\alpha^2}{\alpha\beta\gamma}, \frac{\beta^2}{\alpha\beta\gamma}, \frac{\gamma^2}{\alpha\beta\gamma}$

$\alpha\beta\gamma = 1$

$\therefore$ roots are $\alpha^2, \beta^2, \gamma^2$ 1

at $y = x^2$

$\therefore x = \sqrt{y}$

$\therefore y^{3/2} + py^{1/2} - 1 = 0$

$y^{1/2}(y+p) = 1$

$y(y^2 + 2py + p^2) = 1$ 1

$\therefore y^3 + 2py^2 + p^2y - 1 = 0$

b) eqn of chord is

$y - \frac{3}{p} = \frac{\frac{3}{p} - \frac{3}{q}}{3p - 3q}(x - 3p)$

$y - \frac{3}{p} = \frac{-1}{pq}(x - 3p)$ 1

$\therefore pqy + x = 3(p+q)$

mid pt of PQ is $x = \frac{3p+3q}{2} = \frac{3}{2}(p+q)$

$y = \frac{\left(\frac{3}{p} + \frac{3}{q}\right)}{2} = \frac{3(p+q)}{2pq}$ 1 1

L is pt $x = 0 \quad y = \frac{3(p+q)}{pq}$

M is pt $y = 0 \quad x = 3(p+q)$

$\therefore$ MP of LM is $~~\frac{3(p+q)}{2pq}~~$ 3P

$\frac{3(p+q)}{2}, \frac{3(p+q)}{2pq}$

$\therefore$ MP of LM is same as MP of QP 2

$\therefore LP = MQ$.

$pqy + x = 3(p+q)$ — (1)

$y^2 = 3x$ — (2)

subst $x = \frac{y^2}{3} \Rightarrow$ (1)

$\therefore pqy + \frac{y^2}{3} = 3(p+q)$

$\therefore y^2 + 3pqy - 9(p+q) = 0$

this only has one soln 1

$\therefore \Delta = 0$

$\therefore 9p^2q^2 = -36(p+q)$

$\therefore (pq)^2 = -4(p+q)$

N is pt $x = \frac{3}{2} \cdot \frac{-(pq)^2}{4} = \frac{-3}{8}(pq)^2$ ×

$y = \frac{3}{2pq}(p+q) = \frac{3}{2pq} \cdot \frac{(pq)^2}{-4}$

$= \frac{-3}{8}(pq)$ ×

$\Rightarrow \quad x = -\frac{8}{3}y^2$

but $pq < 0$ 1

$\therefore y > 0 \quad x < 0$

c) $a^x = e^{2x-1}$

$\ln a^x = \ln e^{2x-1}$ ✓

$x \ln a = (2x-1) \ln e$

$x \ln a = 2x - 1$

$2x - x \ln a = 1$

$x(2 - \ln a) = 1$

$x = \frac{1}{2 - \ln a}$ ✓

$a > 0$ $\ln a \neq 2$

(6)

(6)

a) $x^4 - 6x^3 + 9x^2 + 6x - 20 = 0$

if $(2+i)$ is a zero

$2-i$ is also a zero

$(x-(2+i))(x-(2-i))$

$= x^2 - 4x + 5$

$(x^4 - 6x^3 + 9x^2 + 6x - 20) \div (x^2 - 4x + 5)$

$= x^2 - 2x - 4$

$x^2 - 2x - 4 \Rightarrow x = \frac{2 \mp \sqrt{4+16}}{2}$

$= 1 \mp \frac{\sqrt{20}}{2}$

$= 1 \mp \sqrt{5}$

$\therefore$ roots are $2 \mp i,\ 1 \mp \sqrt{5}$.

b) $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ — T

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ — N

sub $x = 0$ in T $\therefore \frac{y\sin\theta}{b} = 1$

$y = \frac{b}{\sin\theta}$

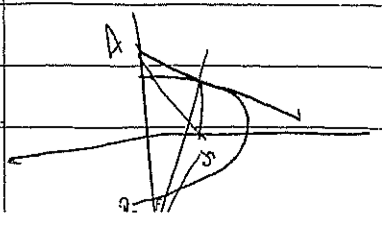
A is pt $\left(0, \frac{b}{\sin\theta}\right)$

sub $x = 0$ in N

$\frac{-by}{\sin\theta} = a^2 - b^2$

$y = \frac{(b^2 - a^2)\sin\theta}{b}$

B is pt $\left(0, \frac{b^2 - a^2\sin\theta}{b}\right)$



$m_{AS} \cdot m_{PS}$

$= \frac{\frac{-b}{\sin\theta}}{ae} \times \frac{\frac{a^2-b^2}{b}\sin\theta}{ae}$

$= \frac{a^2 - b^2}{a^2e^2}$

$= \frac{a^2 - a^2(1-e^2)}{a^2e^2}$

$= \frac{a^2 - a^2 + a^2e^2}{a^2e^2}$

$m_{AS} \cdot m_{PS}$

$= \frac{\frac{-b}{\sin\theta}}{ae} \times \frac{\frac{a^2-b^2}{b}\sin\theta}{ae}$

$= \frac{-(a^2-b^2)}{a^2e^2}$

$= -\left(\frac{a^2 - a^2 + a^2e^2}{a^2e^2}\right)$

$= -1$

$\therefore AS \perp PS$.

$\therefore \angle ASB = 90°$.

now $APB = 90°$ (TANGENT $\perp$ NORMAL)

$\angle ASB = 90°$

$\therefore$ AB subtends same angle at S and P

$\therefore$ APSB are cyclic

AB is diameter.

$\therefore$ centre is $\left(0, \left(\frac{b}{\sin\theta} + \frac{b^2-a^2}{b}\sin\theta\right)/2\right)$

c $\int_0^a f(x)\,dx$

let $u = a - x$

$du = -dx$

when $x = a$ $u = 0$

$x = 0$ $u = a$

$\therefore I = \int_a^0 -f(a-u)\,du$ 1

$= \int_0^a f(a-u)\,du$

let $u = x$

$\therefore I = \int_0^a f(a-x)\,dx$ 1 #

$\int_0^\pi \frac{x \sin x}{1+\cos^2 x}\,dx$

$= \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$ 1

$= \int_0^\pi \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx$

$= \int_0^\pi \frac{\pi \sin x - x\sin x}{1+\cos^2 x}\,dx$

$= \int_0^\pi \frac{\pi \sin x\,dx}{1+\cos^2 x} - \int \frac{x \sin x}{1+\cos^2 x}\,dx$

$I = \int_0^\pi \frac{\pi \sin x\,dx}{1+\cos^2 x} - I$

$2I = \int_0^\pi \frac{\pi \sin x}{1+\cos^2 x}$ 1

let $u = \cos x$ $\quad x = \pi \;\; u = -1$

$du = -\sin x\,dx$ $\quad x = 0 \;\; u = 1$

$\therefore 2I = \pi \int_1^{-1} \frac{-du}{1+u^2}$

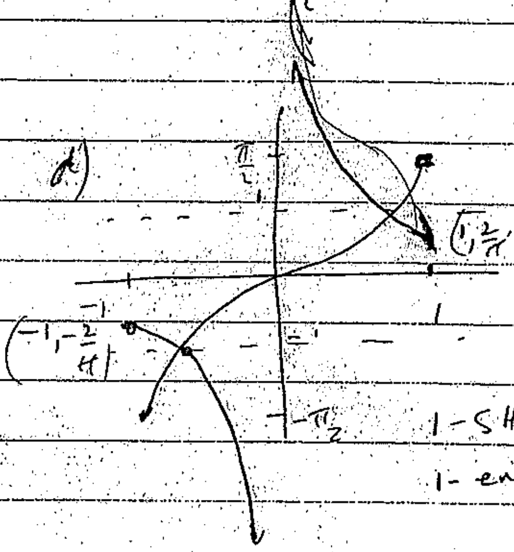
$= \left[\tan^{-1} u\right]_1^{-1}$

$= \left(\frac{\pi}{4} + \frac{\pi}{4}\right)$

$= \frac{\pi^2}{2}$ 1

$I = \frac{\pi^2}{4}$

d)



2

1 - SHADE

1 - end pts

7

7

a) i) $1+\omega+\omega^2=0$ 1

$(1+2\omega+3\omega^2)(1+2\omega^2+3\omega)$

$=(\omega+2\omega^2)(\omega^2+2\omega)$ 1

$=\omega^3+2\omega^2+2\omega^4+4\omega^3$

$=1+2\omega^2+2\omega+4$

$=(5+2(\omega^2+\omega+1)-2)$

$=5-2=3.$ 1

ii) Cannot root must be $1, \omega$ or ω^2 1

if $x=1$ $p+q+r=0$

$x=\omega$ $p\omega^5+q\omega+r=0$ 1

$x=\omega^2$ $p\omega^{10}+q\omega^2+r=0$

$\therefore (p+q+r)(p\omega^5+q\omega+r)(p\omega^{10}+q\omega^2+r)=0$ 1

b) i) $(1-\sqrt{x})^{n-1}-(1-\sqrt{x})^n$

$=(1-\sqrt{x})^{n-1}(1-(1-\sqrt{x}))$

$=(1-\sqrt{x})^{n-1}(1-1+\sqrt{x})$

$=\sqrt{x}(1-\sqrt{x})^{n-1}$ 1

ii) $I_n=\int_0^1(1-\sqrt{x})^n dx$

let $u=(1-\sqrt{x})^n$ $v'=1$

$u'=n(1-\sqrt{x})^{n-1}\cdot\frac{-1}{2\sqrt{x}}$ $v=x$

$\therefore I_n=\left[x(1-\sqrt{x})^n\right]_0^1+\frac{n}{2}\int\frac{(1-\sqrt{x})^{n-1}}{\sqrt{x}}\cdot x\,dx$ 1

$=0+\frac{n}{2}\int(1-\sqrt{x})^{n-1}\sqrt{x}\,dx$

$=\frac{n}{2}\int(1-\sqrt{x})^{n-1}dx-\frac{n}{2}\int(1-\sqrt{x})^n dx$

$=\frac{n}{2}I_{n-1}-\frac{n}{2}I_n$

$\therefore (1+\frac{n}{2})I_n=\frac{n}{2}I_{n-1}$

$\therefore I_n=\frac{n}{n+2}I_{n-1}$ 1

$\cdot I_1=\frac{1}{3}I_0$

but $I_0=\int_0^1 dx=[x]_0^1=1$

$\therefore I_1=\frac{1}{3}$

$I_2=\frac{2}{4}\cdot\frac{1}{3}$

$I_3=\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}$

$\therefore I_{100}=\frac{\cancel{100}}{102}\cdot\frac{99}{101}\cdot\frac{98}{\cancel{100}}\cdot\frac{97}{\cancel{99}}\cdots\frac{\cancel{3}}{\cancel{5}}\cdot\frac{2}{\cancel{4}}\cdot\frac{1}{\cancel{3}}$

$=\frac{2}{102\cdot101}$ 1

7

7c)

$$V = \frac{1}{3}\pi r^2 h$$

$$9V^2 = \pi^2 r^4 h^2$$

now $S = \pi r l$

where $l^2 = h^2 + r^2$

$$\therefore S = \pi r \sqrt{h^2+r^2} \quad 1$$

$$h^2 + r^2 = \frac{S^2}{\pi^2 r^2}$$

$$h^2 = \frac{S^2}{\pi^2 r^2} - r^2$$

$$h^2 = \frac{S^2 - \pi^2 r^4}{\pi^2 r^2}$$

$$\therefore 9V^2 = \pi^2 r^4 \left(\frac{S^2 - \pi^2 r^4}{\pi^2 r^2}\right)$$

$$9V^2 = r^2\left(S^2 - \pi^2 r^4\right) \quad * \quad 1$$

max Volume when $9V^2$ is max

$$\therefore \frac{d\,9V^2}{dr} = 2rS^2 - 6\pi^2 r^5$$

$=0$ when

$$r = 0 \text{ or } r^4 = \frac{S^2}{3\pi^2} \quad 1$$

$$\text{or } r^2 = \frac{S}{3^{1/2}\pi}$$

now $V = \frac{1}{3}\pi r^2 h$ $\quad 1$

$$= \frac{1}{3}\pi \cdot \frac{S}{3^{1/2}\pi} \sqrt{\frac{S^2 - \pi^2 r^4}{\pi^2 r^2}}$$

$$= \frac{S}{3^{3/2}} \sqrt{\frac{S^2 - \pi^2 \frac{S^2}{3\pi^2}}{\pi^2 \cdot \frac{S}{3^{1/2}\pi}}}$$

$$= \frac{S}{3^{3/2}} \sqrt{\frac{2S^2}{3} \cdot \frac{3^{1/2}}{S\pi}}$$

$$= \frac{S}{3^{3/2}} \sqrt{\frac{2S}{3^{1/2}\pi}}$$

$$= \frac{S}{3^{3/2}} \cdot \frac{2^{1/2} S^{1/2}}{3^{1/4}\pi^{1/2}}$$

$$= \frac{2^{1/2} S^{3/2}}{3^{7/4}\pi^{1/2}} \quad 1$$

8 a) when $n=1$

$7^n+3n(7^n)-1$

$=7+3\cdot7-1$

$=7+21-1=27$

which is $\div$ by 9

$\therefore$ true for $n=1$

assume true for some value k

i.e. $7^k+3k(7^k)-1=9N$

where N is an integer

must prove $7^{k+1}+3(k+1)(7^{k+1})-1$

is $\div$ by 9

$=7.7^k+3k.7.7^k+3.7.7^k-1$

$=7(7^k+3k.7^k-1)+7+21.7^k-1$

$=7.9N+6+21.7^k$

$=9(7N)+3(2+7.7^k)$

which is $\div$ by 9 if $2+7.7^k$ is $\div$ by 3

if $k=1$ $\quad 2+7.7^k=2+49=51$ $\div$ by 3

assume $2+7^{k+1}=3P$ (P is an integer)

must prove $2+7^{k+2}$ is $\div$ by 3

$2+7^{k+2}=2+7.7^{k+1}$

$=7(2+7.7^{k+1})-12$

$=21P-12$

$=3(7P-4)$ which is $\div$ by 3

$\therefore 7^{k+1}+3(k+1)7^{k+1}-1$ is $\div$ by 9

$\therefore$ as true for $n=1$ must be true for $n=2$

true for $n=2$ - - $n=3$

and so on for all n

$\therefore 7^n+3n(7^n)-1$ is $\div$ by 9.

b) $\tan 4\theta=1$

$4\theta=\frac{\pi}{4}+2k\pi \quad k=0,\mp1,\ldots$

or $\frac{5\pi}{4}+2k\pi \quad k=0,\mp1,\ldots$

or $4\theta=\frac{\pi}{4}+k\pi \quad k=0\mp1\ldots$

$\therefore \theta=\frac{\pi}{16}+\frac{k}{4}\pi \quad k=0\mp1\ldots$

$(\cos\theta+i\sin\theta)^4=\cos4\theta+i\sin4\theta$

$\therefore \cos4\theta+i\sin4\theta=\cos^4\theta+4i\cos^3\theta\sin\theta+6i^2\cos^2\theta\sin^2\theta+4i^3\cos\theta\sin^3\theta+i^4\sin^4\theta$

$=(\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta)+i(4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta)$

$\therefore \cos4\theta=\cos^4\theta-4\cos^2\theta\sin^2\theta+\sin^4\theta$

$\sin4\theta=4\cos^3\theta\sin\theta+4\cos^3\theta\sin\theta$.

$\tan4\theta=\frac{\sin4\theta}{\cos4\theta}=\frac{4\cos^3\theta\sin\theta-4\cos^3\theta\sin^3\theta}{\cos^4\theta-4\cos^2\theta\sin^2\theta+\sin^4\theta}$

$\div$ top + bottom by $\cos^4\theta$

$\therefore \tan4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-4\tan^2\theta+\tan^4\theta}$

let $\tan4\theta=1$ and $\tan\theta=x$

$\therefore \frac{4x-4x^3}{1-4x^2+x^4}=1$

$x^4+4x^3-6x^2-4x+1=0$

when $x=\tan\theta$.

$\therefore x=\tan\frac{\pi}{16},\tan\frac{5\pi}{16},\tan\frac{9\pi}{16},\tan\frac{13\pi}{16}$

$\Sigma\alpha^2=(\Sigma\alpha)^2-2\Sigma\alpha\beta$

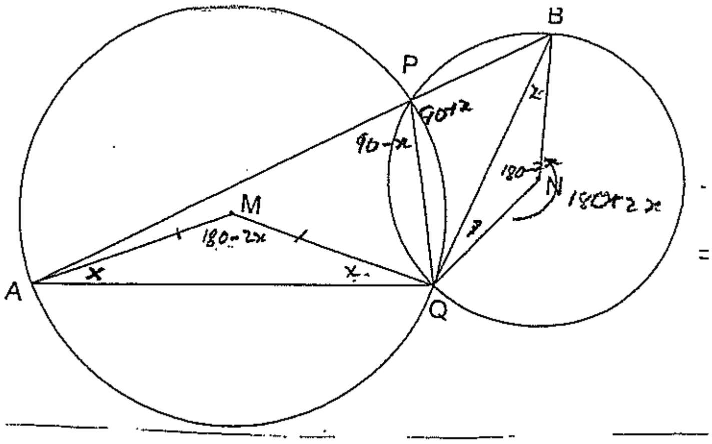
$=(-4)^2-6.-2=16+12=28$

$\therefore \tan^2\frac{\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{9\pi}{16}+\tan^2\frac{13\pi}{16}=28$

but $\tan^2\frac{9\pi}{16}=\tan^2\frac{7\pi}{16}$ and $\tan^2\frac{13\pi}{16}=\tan^2\frac{3\pi}{16}$

$\therefore \tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}=28$

c



let $\angle MAQ = x^\circ$

$AM = MQ$ (equal radii)

$\therefore \triangle AMQ$ is isosceles (1)

$\therefore \angle MAQ = \angle MQA = x$

$\angle AMQ = 180 - 2\angle MAQ$ (angle sum of triangle)

$\angle AMQ = 180 - 2x$.

$\angle APQ = \frac{1}{2}\angle AMQ$ (ANGLE AT CENTRE IS TWICE ANGLE AT CIRCUMFERENCE) (1)

$\therefore \angle APQ = 90 - x$

$APQ + \angle BPQ = 180^\circ$ (st line)

$\therefore \angle BPQ = 90 + x$.

reflex $\angle BNQ = 2\angle BPQ$ (") (1)

OBTUSE $\therefore \angle BNQ = 180 + 2x$.

in $\triangle BNQ$ $\angle BNQ = 180 - 2x$

but $\angle NBQ = \angle NQB$ ($NB = NQ$ equal radii)

$\angle NBQ = \frac{180 - (180 - 2x)}{2}$

$= x$ (1) 4

$= \angle MAQ$

$\therefore \angle NBQ = \angle MAQ$