BAULKHAM HILLS HS PERSEVERE

2025 **HSC TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 3 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
Total marks: 100	**Section I – 10 marks** (pages 2 – 5) • Attempt Questions 1 – 10 • Allow about 15 minutes for this section. **Section II – 90 marks** (pages 6 – 12) • Attempt Questions 11 – 16 • Allow about 2 hours 45 minutes for this section.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section.

Use the multiple-choice answer page in the writing booklet for Questions 1 – 10.

1 Consider the following statement:

"If my cricket team plays badly, they are not training enough".

Which one of the following statements is the contrapositive of the statement above?

A. If they are not training enough, then my cricket team plays badly.

B. If they are training enough, then my cricket team does not play badly.

C. If my cricket team doesn't play badly, then they are training enough.

D. If my cricket team plays badly, then they need more training.

2 A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $mkv + mkv^2$ newtons when its velocity is $v\ \text{ms}^{-1}$ where k is a positive constant. At time t seconds the particle has displacement x metres from a fixed point O on the line and velocity $v\ \text{ms}^{-1}$. Which of the following is an expression for x in terms of v?

A. $\dfrac{1}{k}\displaystyle\int \frac{1}{1+v}\,dv$

B. $\dfrac{1}{k}\displaystyle\int \frac{1}{v+v^2}\,dv$

C. $-\dfrac{1}{k}\displaystyle\int \frac{1}{v+v^2}\,dv$

D. $-\dfrac{1}{k}\displaystyle\int \frac{1}{1+v}\,dv$

3 What is the principal argument of $\dfrac{-3\sqrt{2}-i\sqrt{6}}{2+2i}$?

A. $-\dfrac{11\pi}{12}$

B. $\dfrac{7\pi}{12}$

C. $\dfrac{11\pi}{12}$

D. $\dfrac{13\pi}{12}$

4 If $i^n = p$, then what is a simplified expression in terms of p for i^{2n+3}?

A. $p^2 - i$

B. $p^2 + i$

C. $-p^2 i$

D. $p^2 i$

5 Which expression is equal to $\int 3\sqrt{x}\ln x dx$?

A. $2x\sqrt{x}\left(\ln x - \dfrac{2}{3}\right) + C$

B. $2x\sqrt{x}\left(\ln x + \dfrac{2}{3}\right) + C$

C. $\dfrac{1}{\sqrt{x}}\left(\dfrac{3}{2}\ln x - 1\right) + C$

D. $\dfrac{1}{\sqrt{x}}\left(\dfrac{3}{2}\ln x + 1\right) + C$

6 If θ is the angle between $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}$, what is the value of $\cos(2\theta)$?

A. $\dfrac{-24}{25}$

B. $-\dfrac{4}{5}$

C. $\dfrac{7}{25}$

D. $\dfrac{14}{25}$

7 With a suitable substitution, which of the following is an equivalent expression to $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x dx}{\sec^2 x - 3\tan x + 1}$?

A. $\displaystyle\int_1^{\sqrt{3}} \left(\frac{1}{u-1}\right) + \left(\frac{1}{u-2}\right) du$

B. $\displaystyle\int_1^{\sqrt{3}} \left(\frac{1}{3(u-3)}\right) - \left(\frac{1}{3u}\right) du$

C. $\displaystyle\int_1^{\sqrt{3}} \left(\frac{1}{u-2}\right) - \left(\frac{1}{u-1}\right) du$

D. $\displaystyle\int_1^{\sqrt{3}} \left(\frac{1}{u-1}\right) - \left(\frac{1}{u-2}\right) du$

8 The acceleration vector of a particle that starts from rest is given by
$\underset{\sim}{a}(t) = -4\sin(2t)\underset{\sim}{i} + 20\cos(2t)\underset{\sim}{j} - 20e^{-2t}\underset{\sim}{k}$, where $t \geq 0$.
What is the velocity vector of the particle?

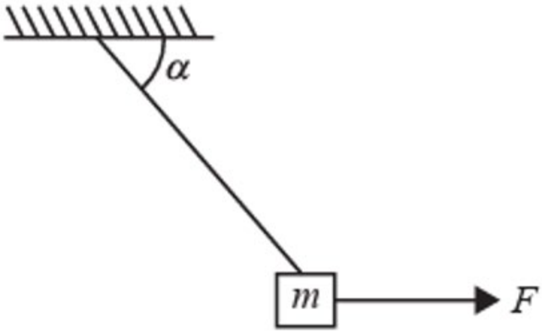
A. $\underset{\sim}{v}(t) = 2\cos(2t)\underset{\sim}{i} + 10\sin(2t)\underset{\sim}{j} + 10e^{-2t}\underset{\sim}{k}$

B. $\underset{\sim}{v}(t) = \left(2\cos(2t) - 2\right)\underset{\sim}{i} + 10\sin(2t)\underset{\sim}{j} + \left(10e^{-2t} - 10\right)\underset{\sim}{k}$

C. $\underset{\sim}{v}(t) = \left(4\cos(2t) - 4\right)\underset{\sim}{i} + 20\sin(2t)\underset{\sim}{j} + \left(20e^{-2t} - 20\right)\underset{\sim}{k}$

D. $\underset{\sim}{v}(t) = \left(8 - 8\cos(2t)\right)\underset{\sim}{i} - 40\sin(2t)\underset{\sim}{j} + \left(40e^{-2t} - 40\right)\underset{\sim}{k}$

9 A particle of mass m kilograms hangs from a string that is attached to a fixed point. The particle is acted on by a horizontal force of magnitude F newtons. The system is in equilibrium when the string makes an angle α to the horizontal, as shown in the diagram below. The tension in the string has magnitude T newtons.



What is the value of $\tan\alpha$?

A. $\dfrac{mg}{T}$

B. $\dfrac{T}{mg}$

C. $\dfrac{F}{mg}$

D. $\dfrac{mg}{F}$

10 If the sum of two unit vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is a unit vector, what is the magnitude of the difference of the two unit vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$?

A. 0

B. $\sqrt{2}$

C. $\sqrt{3}$

D. $\sqrt{5}$

End of Section I

Section II

90 marks

Attempt Questions 11 - 16

Allow about 2 hours 45 minutes for this section.

Instructions

- Answer the questions in the appropriate page of your answer booklet.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing paper is available. If you use extra paper, write your name, and your teacher's name on the paper and clearly indicate which question you are answering.

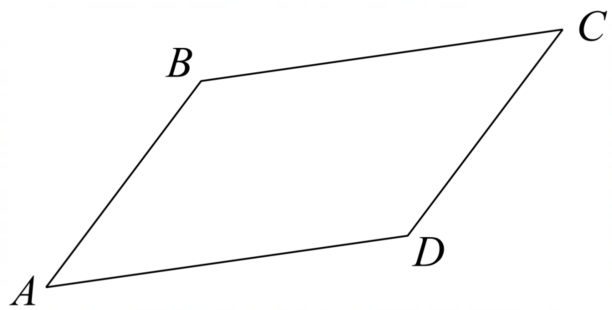
Question 11 (15 marks) Use the Question 11 pages of your answer booklet

(a) Let $z = 1 + i\sqrt{3}$.

(i) Express z in modulus-argument form. **2**

(ii) Hence, simplify $z^{10} + 512z$. **2**

(b) *ABCD* is a parallelogram.

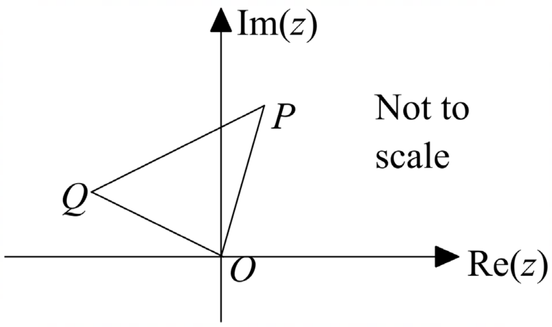


Given that $\overrightarrow{AB} = 6\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\overrightarrow{BC} = 2\underset{\sim}{i} + 5\underset{\sim}{j} + 8\underset{\sim}{k}$, find the size of angle ABC, giving your answer in degrees, to two decimal places. **2**

Question 11 continues on page 7

Question 11 (continued)

(c) Points P and Q represent the complex numbers z and w respectively in the complex plane. Triangle OPQ is an equilateral triangle.



(i) Explain why $wz = z^2\text{cis}\left(\dfrac{\pi}{3}\right)$. **1**

(ii) Prove that $z^2 + w^2 = zw$. **2**

(d) Find $\displaystyle\int \frac{dx}{\sqrt{1+2x-x^2}}$. **2**

(e) The acceleration of a particle undergoing simple harmonic motion along the x – axis is given by $\ddot{x} = -4(x-3)$. Find the maximum speed of the particle if its speed as it passes through $x = 1$ is $4\sqrt{3}$ metres per second. **4**

End of Question 11

Question 12 (15 marks) Use the Question 12 pages of your answer booklet

(a) Prove, by the method of contradiction, that there are no integers a and b which satisfy $a^2 - 8b = 7$. **3**

(b) By integrating one of the two terms in the integrand by parts, or otherwise, find **3**

$$\int \left(2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}} \right) dx.$$

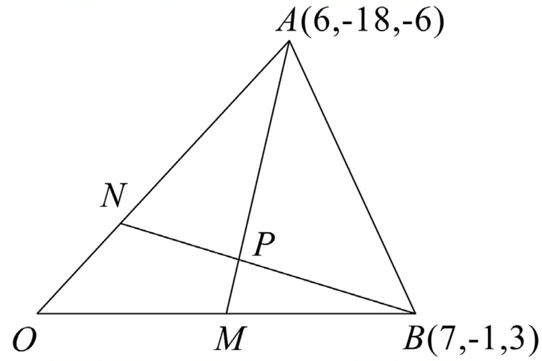
(c) The line l has equation $\dfrac{x-1}{h} = \dfrac{y-1}{3} = \dfrac{3-z}{6}$. Find the value(s) of h for which the angle between l and the x-axis is $120°$. **4**

(d) (i) Show that $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\,dx = 4 - 2\sqrt{3}$. **3**

(ii) Hence find the exact value of $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{1+\sin x}\,dx$. **2**

Question 13 (15 marks) Use the Question 13 pages of your answer booklet

(a) The vertices of triangle OAB have coordinates $A(6,-18,-6), B(7,-1,3)$, where O is a fixed origin. **4**



The point N lies on OA so that $ON : NA = 1 : 2$.

The point M is the midpoint of OB. The point P is the intersection of AM and BN.

By using vector methods, determine the coordinates of P.

(b) Prove, **without** the use of mathematical induction, that $5^{2n} - 1$ is divisible by 8, for all positive integers n. **2**

(c) A particle of mass 1 kg is projected vertically upwards with an initial velocity of $100\ \text{ms}^{-1}$ in a medium in which the resistance force is equal to 0.01 times the square of the particle's velocity, that is $0.01\,v^2$. Use $g = 10\ \text{ms}^{-2}$.

(i) Show that the maximum height reached by the particle is $50\log_e 11$ metres. **3**

(ii) Calculate the downward velocity of the particle on its return to the point of of projection. Give your answer correct to two decimal places. **3**

(d) A polynomial is given by $P(z) = z^4 - 6z^3 + 38z^2 - 94z + 221$. Given that $z = 2 + 3i$ is a root of the equation $P(z) = 0$, find the three other roots of $P(z) = 0$. **3**

Question 14 (16 marks) Use the Question 14 pages of your answer booklet

(a) If $z = r(\cos\theta + i\sin\theta)$, use De Moivre's theorem to solve the complex equation $z^5 = \overline{z}$. **3**
Express your answer(s) in the form $a + ib$.

(b) Let $I_n = \int \operatorname{cosec}^n x\,dx$, for integers $n \geq 0$.

(i) Show that $I_n = \dfrac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$ for integers $n \geq 2$. **3**

(ii) Hence, evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \operatorname{cosec}^4 x\,dx$. **2**

(c) A train of mass m, pulled by a locomotive which exerts a constant (propelling) force P is moving at speed v ms^{-1} along a straight level track against a resistive force mkv, where k is a positive constant. The speed increases from 2 ms^{-1} to 4 ms^{-1} over a time interval of 5 seconds.

(i) Show that $P = 2km\left(\dfrac{2e^{5k} - 1}{e^{5k} - 1}\right)$. **4**

(ii) Find the corresponding distance moved. **3**

(iii) There is an upper bound to the speed that the train can attain. Find the value of this upper bound. **1**

Question 15 (15 marks) Use the Question 15 pages of your answer booklet

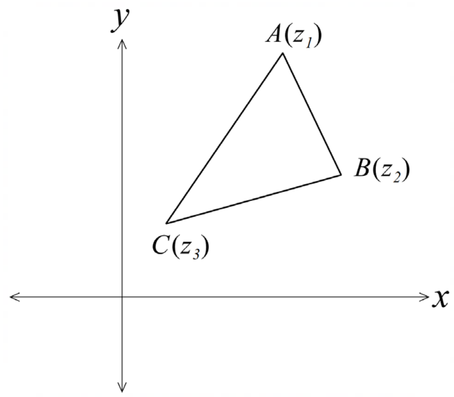
(a) Use the principle of mathematical induction to prove that $3^n > n^3$ for all integers $n > 3$. **3**

(b) Let S be a sphere with the centre at $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and radius $r = 5$. A line l is defined by the parametric equation: $\underset{\sim}{r}(\lambda) = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -1 \\ 0 \end{pmatrix}$, where λ is a scalar and $\lambda \in \mathbb{R}$.

(i) Find the point(s) of intersection between the line and the sphere. **3**

(ii) Find the shortest distance from the centre of the sphere to the line. **2**

(c) The points A, B and C are the points that represent the complex numbers z_1, z_2 and z_3 respectively on the Argand diagram.



Prove that if $\dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{z_1 - z_3}{z_1 - z_2}$, then triangle ABC is an equilateral triangle. **4**

(d) Evaluate $\displaystyle\int \frac{\ln x}{(1+\ln x)^2}\,dx.$ **3**

Question 16 (14 marks) Use the Question 16 pages of your answer booklet

(a) It is given that for any three positive numbers x, y and z it can be shown that $(x+y)(x+z)(y+z) \geq 8xyz$. Do **NOT** prove this. **3**

Use this result to prove that if a, b, and c are any three positive numbers such that each number is less than the sum of the other two, then $(a+b-c)(b+c-a)(c+a-b) \leq abc$.

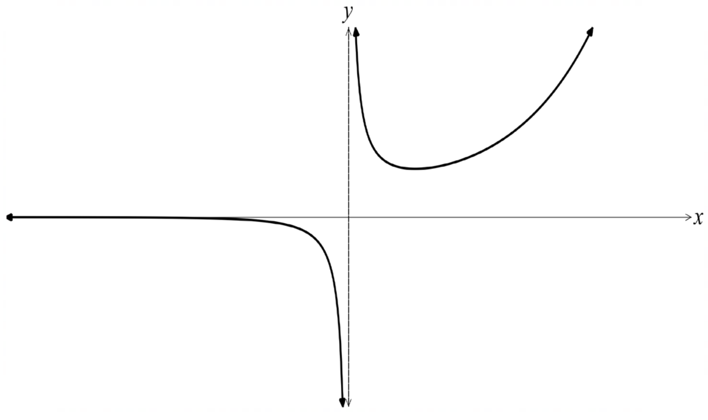
(b) Let $S = (\sqrt{50}+7)^{2000} + (\sqrt{50}-7)^{2000}$.

(i) Prove that S is an even integer. **2**

(ii) Without the use of a calculator, show that $0 < \sqrt{50} - 7 < 0.1$. (You may use the fact that $71^2 = 5041$). **2**

(iii) Hence, or otherwise, prove that the first two thousand digits after the decimal point in the decimal expansion of $(\sqrt{50}+7)^{2000}$ are all the digit 9. **2**

(c) The diagram below is the graph of $y = \dfrac{e^x}{x}$.



(i) Find the coordinates of the stationary point. **1**

(ii) Find a if $\displaystyle\int_a^{2a} \frac{e^x}{x}\,dx = \int_a^{2a} \frac{e^x}{x^2}\,dx$. **2**

(iii) Hence, or otherwise, show that it is not possible to find distinct integers m and n such that $\displaystyle\int_m^{n} \frac{e^x}{x}\,dx = \int_m^{n} \frac{e^x}{x^2}\,dx$. **2**

End of Paper

H4 2025 X2 TRIAL SOLUTIONS

1\. If they are training enough, my cricket team does not play badly.

(B)

2\. $\xleftarrow{mkv+mkv^2}$

$F = ma$

$ma = -mkv(1+v)$

$a = -kv(1+v)$

$v\dfrac{dv}{dx} = -kv(1+v)$

$-\dfrac{1}{k}\displaystyle\int \frac{dv}{1+v} = \int dx$

$x = -\dfrac{1}{k}\displaystyle\int \frac{dv}{1+v}$

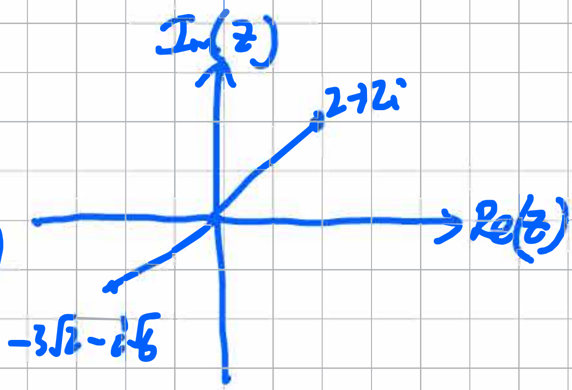
$\therefore$ (D)

3\. $\dfrac{-3\sqrt{2}-i\sqrt{6}}{2+2i} = \dfrac{\sqrt{6}\left(-\sqrt{3}-i\right)}{2\left(1+i\right)}$

$= \dfrac{-\sqrt{6}\times 2\,\text{cis}\left(\pi+\frac{\pi}{6}\right)}{2\times\sqrt{2}\,\text{cis}\,\frac{\pi}{4}}$

$= \sqrt{3}\,\text{cis}\left(\frac{7\pi}{6}-\frac{\pi}{4}\right)$

$= \sqrt{3}\,\text{cis}\left(\frac{11\pi}{12}\right)$



$\therefore$ (C)

4. $i^{2n+3} = i^{2n} \times i^{3}$

$= (i^{n})^{2} \times (-i)$

$= p^{2} \times -i$

$= -p^{2} i$

$\therefore$ C

5. $\int 3\sqrt{x} \ln x \, dx$

$u = \ln x \qquad v' = 3\sqrt{x}$

$u' = \frac{1}{x} \qquad v = \frac{3x^{3/2}}{3/2} = 2x\sqrt{x}$

$= 2x\sqrt{x} \ln x - \int 2\sqrt{x} \, dx$

$= 2x\sqrt{x} \ln x - 2\left(\frac{x^{3/2}}{3/2}\right) + C$

$= 2x\sqrt{x} \ln x - \frac{4}{3} x\sqrt{x} + C$

$= 2x\sqrt{x}\left(\ln x - \frac{2}{3}\right) + C$

$\therefore$ A

6. $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\sqrt{3} - 16 - \sqrt{3} = \sqrt{20}\sqrt{20} \cos\theta$

$\frac{-16}{20} = \cos\theta$

$\cos 2\theta = 2\cos^{2}\theta - 1$

$= 2\left(\frac{-4}{5}\right)^{2} - 1$

$= \frac{7}{25}$

$\therefore$ C

7. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x\,dx}{\sec^2 x - 3\tan x + 1}$

Let $u = \tan x$

$du = \sec^2 x\,dx$

when $x = \frac{\pi}{3}$, $u = \sqrt{3}$

$x = \frac{\pi}{4}$, $u = 1$

$$= \int_1^{\sqrt{3}} \frac{du}{u^2+1-3u+1}$$

$$= \int_1^{\sqrt{3}} \frac{du}{u^2-3u+2}$$

$$= \int_1^{\sqrt{3}} \frac{du}{(u-2)(u-1)}$$

$$= \int_1^{\sqrt{3}} -\frac{1}{u-1} + \frac{1}{u-2}\,du$$

$\therefore$ C

8. $\underset{\sim}{a}(t) = -4\sin(2t)\underset{\sim}{i} + 20\cos(2t)\underset{\sim}{j} - 20e^{-2t}\underset{\sim}{k}$

$\underset{\sim}{v}(t) = 2\cos(2t)\underset{\sim}{i} + 10\sin(2t)\underset{\sim}{j} + 10e^{-2t}\underset{\sim}{k} + \underset{\sim}{c}$

when $t=0$, $\underset{\sim}{v}(0) = 0$ where $\underset{\sim}{c} = c_1\underset{\sim}{i} + c_2\underset{\sim}{j} + c_3\underset{\sim}{k}$

$0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k} = (2\cos(0) + c_1)\underset{\sim}{i} + (10\sin 0 + c_2)\underset{\sim}{j} + (10e^0 + c_3)\underset{\sim}{k}$

Equating components

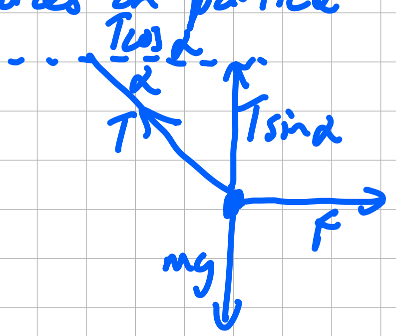
$2 + c_1 = 0$ $\quad$ $c_2 = 0$ $\quad$ $10 + c_3 = 0$

$\therefore c_1 = -2$ $\quad$ $c_2 = 0$ $\quad$ $c_3 = -10$

$\therefore \underset{\sim}{v}(t) = (2\cos(2t) - 2)\underset{\sim}{i} + 10\sin(2t)\underset{\sim}{j} + (10e^{-2t} - 10)\underset{\sim}{k}$

$\therefore$ B

9. Forces on particle



Resolving forces vertically

$F = ma$

$T\sin\alpha - mg = 0$

$T\sin\alpha = mg$ ①

Resolving horizontally

$F = ma$

$T\cos\alpha - F = 0$

$T\cos\alpha = F$ ②

① ÷ ② $\dfrac{T\sin\alpha}{T\cos\alpha} = \dfrac{mg}{F}$

$\therefore T = \dfrac{mg}{F}$

$\therefore$ D

10. Let unit vectors be $\underset{\sim}{u}$ and $\underset{\sim}{v}$

$|\underset{\sim}{u} + \underset{\sim}{v}| = 1$

$(\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} \cdot \underset{\sim}{v}) = 1^2$

$\underset{\sim}{u} \cdot \underset{\sim}{u} + 2\underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{v} = 1$

$|\underset{\sim}{u}|^2 + 2\underset{\sim}{u} \cdot \underset{\sim}{v} + |\underset{\sim}{v}|^2 = 1$

$1 + 2\underset{\sim}{u} \cdot \underset{\sim}{v} + 1 = 1$ since $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are unit vectors

$2\underset{\sim}{u} \cdot \underset{\sim}{v} = -1$

Now $|\underset{\sim}{u} - \underset{\sim}{v}|^2 = (\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v})$

$= \underset{\sim}{u} \cdot \underset{\sim}{u} - 2\underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{v}$

$= |\underset{\sim}{u}|^2 + 1 + |\underset{\sim}{v}|^2$

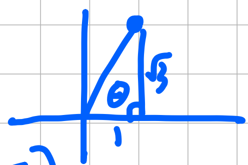
$|\underset{\sim}{u} - \underset{\sim}{v}|^2 = 3$

$|\underset{\sim}{u} - \underset{\sim}{v}| = \sqrt{3}$

$\therefore$ magnitude of $\underset{\sim}{u} - \underset{\sim}{v}$ is $\sqrt{3}$ units

$\therefore$ C

11 a) i) $z = 1 + \sqrt{3} i$



$\tan\theta = \sqrt{3}$

(2) correct solution

(1) correct modulus or argument

$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

ii) $z^{10} + 512z$

(2) correct solution

(1) uses De Moivre's theorem to simplify z^{10}

$= \left(2\operatorname{cis}\frac{\pi}{3}\right)^{10} + 512 \times 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 2^{10}\operatorname{cis}\left(\frac{10\pi}{3}\right) + 1024\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 1024\left(\cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)\right) + 512 + 512\sqrt{3} i$

$= -1024\cos\frac{\pi}{3} - 1024 i\sin\frac{\pi}{3} + 512\sqrt{2} + 512\sqrt{3} i$

$= -1024 \times \frac{1}{2} - 1024\frac{\sqrt{3}}{2} i + 512 + 512\sqrt{3} i$

$= -512 - 512\sqrt{3} i + 512 + 512\sqrt{3} i$

$= 0$

11 b) To find $\angle ABC$

(2) correct solution

(1) uses dot product

$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos(\angle ABC)$

$\begin{pmatrix} -6 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = \sqrt{49}\sqrt{93}\cos(\angle ABC)$

$-12 + 10 - 24 = 7\sqrt{93}\cos(\angle ABC)$

$\frac{-26}{7\sqrt{93}} = \cos\angle ABC$

$\therefore \angle ABC = 112.653\ldots$

$\therefore \angle ABC = 112.65^\circ$ (2dp)

11 c) i) Since ΔPOQ is equilateral $|w| = |z|$ and $\angle POQ = \frac{\pi}{3}$. $\therefore$ w is z rotated $\frac{\pi}{3}$ radians anticlockwise. $\therefore zw = z \; z \text{ cis } \frac{\pi}{3}$

① correct reason

$$= z^2 \text{ cis } \frac{\pi}{3}$$

ii) $z^2 + w^2 = z^2 + \left(z \text{ cis } \frac{\pi}{3}\right)^2$

$$= z^2 + z^2 \text{ cis}\left(\frac{2\pi}{3}\right)$$

$$= z^2 + z^2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$$

$$= z^2\left(1 + -\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$= z^2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$= z^2 \text{ cis } \frac{\pi}{3}$$

$$= zw$$

② correct solution

① expresses w^2 in terms of $z^2 \text{ cis}\left(\frac{2\pi}{3}\right)$

11 d) $\displaystyle\int \frac{dx}{\sqrt{1+2x-x^2}}$

$$= \int \frac{dx}{\sqrt{2-(x^2-2x+1)}}$$

$$= \int \frac{dx}{\sqrt{2-(x-1)^2}}$$

$$= \sin^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C$$

② correct solution

① completes the square

11e) $\ddot{x} = -4(x-3)$

$$v\frac{dv}{dx} = -4(x-3)$$

$$\int_{4\sqrt{3}}^{v} v\,dv = -4\int_{1}^{x}(x-3)\,dx$$

$$\left[\frac{v^2}{2}\right]_{4\sqrt{3}}^{v} = -4\left[\frac{x^2}{2} - 3x\right]_{1}^{x}$$

$$\left[v^2\right]_{4\sqrt{3}}^{v} = -4\left[x^2 - 6x\right]_{1}^{x}$$

$$v^2 - 48 = -4\left[x^2 - 6x - (1-6)\right]$$

$$v^2 = -4x^2 + 24x + 28$$

Centre of motion is $x = 3$.

$\therefore v^2 = -36 + 72 + 28$

$v^2 = 64$

$v = \pm 8$

$\therefore$ maximum speed is 8 m/s.

④ correct solution

③ expresses v^2 in terms of x

② obtains correct integral

① identifies centre at $x=3$ or expresses $\ddot{x}$ in terms of $v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

12a) $a^2 - 8b = 7$

Assume there are integers a and b such that $a^2 - 8b = 7$

$a^2 = 8b + 7$

$\therefore a^2$ is odd (since $8b$ is even, and sum of even and odd is odd)

let $a = 2k+1$ where $k \in \mathbb{Z}$

$\therefore a^2 = 8b + 7$

$(2k+1)^2 = 8b + 7$

$4k^2 + 4k + 1 = 8b + 7$

③ correct solution

② proves a^2 is odd

or ② uses odd or even properties of numbers in a suitable proof

① stating the contradiction and apply number facts

$4k^2+4k-8b=6$

$2k^2+2k-4b=3$

$2(k^2+k-2b)=3$

LHS is even if k and b are integers and RHS is odd

$\therefore$ By contradiction there are no possible integers a and b.

12 b) $\int \left(2\sqrt{1+x^3}+\frac{3x^3}{\sqrt{1+x^3}}\right)dx$

③ correct solution
② integrates either expression by parts
① correctly attempts to integrate by parts

Method 1: integrate $2\sqrt{1+x^3}$ by parts

$\int 2\sqrt{1+x^3}\,dx$

$u=(1+x^3)^{\frac{1}{2}}$ $\quad v'=2$

$u'=\frac{1}{2}(1+x^3)^{-\frac{1}{2}}\times 3x^2$

$u'=\frac{3x^2}{2\sqrt{1+x^3}}$ $\quad v=2x$

$\int 2\sqrt{1+x^3}+\frac{3x^3}{\sqrt{1+x^3}}\,dx$

$=2x\sqrt{1+x^3}-\int\frac{3x^3\,dx}{\sqrt{1+x^3}}+\int\frac{3x^3\,dx}{\sqrt{1+x^3}}+C$

$=2x\sqrt{1+x^3}+C$

Method 2: integrate $\frac{3x^3}{\sqrt{1+x^3}}$ by parts

$\int x\frac{3x^2}{\sqrt{1+x^3}}\,dx$

$u=x$ $\quad v'=3x^2(1+x^3)^{-\frac{1}{2}}$

$u'=1$ $\quad v=2(1+x^3)^{\frac{1}{2}}$

$=2x\sqrt{1+x^3}-\int 2\sqrt{1+x^3}\,dx+C$

$\therefore \int\left(2\sqrt{1+x^3}+\frac{3x^3}{\sqrt{1+x^3}}\right)dx$

$=\int 2\sqrt{1+x^3}\,dx+2x\sqrt{1+x^3}-\int 2\sqrt{1+x^3}\,dx+C$

$=2x\sqrt{1+x^3}+C$

12c) $\dfrac{x-1}{h} = \dfrac{y-1}{3} = \dfrac{3-z}{6}$

④ correct solution
③ obtains equation in h
② finds dot product of vectors
① finds direction vector of the line

$\dfrac{x-1}{h} = \lambda \qquad \dfrac{y-1}{3} = \lambda \qquad \dfrac{3-z}{6} = \lambda$

$x = 1 + \lambda h \qquad y = 1 + 3\lambda \qquad z = 3 - 6\lambda$

Let $\underset{\sim}{r}$ be a position vector for a point on line l

$$\therefore l: \underset{\sim}{r} = \begin{pmatrix}1\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}h\\3\\-6\end{pmatrix}$$

$$\begin{pmatrix}h\\3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} = 1\sqrt{h^2+9+36}\cos 120^\circ$$

$$h = \sqrt{h^2+45}\left(-\frac{1}{2}\right)$$

$$-2h = \sqrt{h^2+45} \qquad (*)$$

$$4h^2 = h^2+45$$

$$3h^2 = 45$$

$$h^2 = 15$$

$$\therefore h = \pm\sqrt{15}$$

$$h = -\sqrt{15} \text{ for } \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Note: if $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is taken as the direction vector of the x axis, $h = -\sqrt{15}$ is the solution

if $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ is taken as the direction vector then $h = \sqrt{15}$ is the solution

Also if $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ is taken as the direction vector then $h = \sqrt{15}$ is another valid solution. $\therefore h = \pm\sqrt{15}$

12d) i) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\,dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1-\sin x}{1-\sin^2 x}\,dx$$

③ correct solution
② correctly integrates
① expresses integrand in appropriate form which can be integrated

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\cos^2 x} - \sin x(\cos x)^{-2}\, dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sec^2 x - \sin x(\cos x)^{-2}\, dx$$

$$= \Big[\tan x - (\cos x)^{-1}\Big]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$= \left(\tan\frac{2\pi}{3} - \frac{1}{\cos\frac{2\pi}{3}}\right) - \left(\tan\frac{\pi}{3} - \frac{1}{\cos\frac{\pi}{3}}\right)$$

$$= -\sqrt{3} + 2 - \sqrt{3} + 2$$

$$= 4 - 2\sqrt{3}$$

<u>OR</u>

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\, dx$$

let $t = \tan\frac{x}{2}$

$dx = \frac{2dt}{1+t^2}$

when $x = \frac{2\pi}{3}$, $t = \tan\frac{\pi}{3}$
$= \sqrt{3}$

when $x = \frac{\pi}{3}$, $t = \tan\frac{\pi}{6}$
$= \frac{1}{\sqrt{3}}$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{2dt}{(1+t^2)\left(1+\frac{2t}{1+t^2}\right)}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{2dt}{1+t^2+2t}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} 2(t+1)^{-2}$$

$$= \Big[-2(t+1)^{-1}\Big]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \left[\frac{-2}{t+1}\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= -2\left(\frac{1}{\sqrt{3}+1} - \frac{1}{\frac{1}{\sqrt{3}}+1}\right)$$

$$= -2\left(\frac{1}{\sqrt{3}+1} - \frac{\sqrt{3}}{\sqrt{3}+1}\right)$$

$$= 2\,\frac{\sqrt{3}-1}{\sqrt{3}+1}\;\frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{\cancel{2}(3+1-2\sqrt{3})}{\cancel{2}}$$

$$= 4-2\sqrt{3}$$

12dii) $$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{1+\sin x}\,dx$$

Let $u = \pi - x$

$du = -dx$

when $x = \frac{2\pi}{3}$, $u = \frac{\pi}{3}$

$x = \frac{\pi}{3}$, $u = \frac{2\pi}{3}$

② correct solution

① correct integrand in terms of u

$$= \int_{\frac{2\pi}{3}}^{\frac{\pi}{3}} \frac{\pi-u}{1+\sin(\pi-x)}\,(-du)$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi-u}{1+\sin u}\,du$$

$$\therefore \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{1+\sin x}\,dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{(\pi-x)\,dx}{1+\sin x}$$

$$2\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x\,dx}{1+\sin x} = \pi\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\,dx$$

$$\therefore \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x\,dx}{1+\sin x} = \frac{\pi}{2}\left(4-2\sqrt{3}\right)$$
$$= \pi\left(2-\sqrt{3}\right)$$

13 a) $\overrightarrow{ON} = \frac{1}{3}(6,-18,6)$ — (4) correct solution

$= (2,-6,-2)$ — (3) obtains constants of "λ" and "μ"

$\overrightarrow{OM} = \frac{1}{2}(7,-1,3)$

$= \left(\frac{7}{2},-\frac{1}{2},\frac{3}{2}\right)$ — (2) obtains coordinates of $\overrightarrow{PA}$

let $\overrightarrow{NP} = k\overrightarrow{NB}$, $0<k<1 \wedge k\in\mathbb{R}$

$\overrightarrow{NP} = k(\underset{\sim}{b}-\underset{\sim}{n})$ — (1) obtains coordinates of $\overrightarrow{NP}$

$= k\left[(7,-1,3)-(2,-6,-2)\right]$

$= k(5,5,5)$

$\overrightarrow{MP} = \overrightarrow{MO}+\overrightarrow{ON}+\overrightarrow{NP}$

$= -\underset{\sim}{m}+\underset{\sim}{n}+(5k,5k,5k)$

$= -\left(\frac{7}{2},-\frac{1}{2},\frac{3}{2}\right)+(2,-6,-2)+(5k,5k,5k)$

$\overrightarrow{MP} = \left(5k-\frac{3}{2},\ 5k-\frac{11}{2},\ 5k-\frac{7}{2}\right)$

Now $\overrightarrow{PA} = \overrightarrow{PN}+\overrightarrow{NA}$

$= -\overrightarrow{NP}+2\overrightarrow{ON}$

$= (-5k,-5k,-5k)+(4,-12,-4)$

$= (-5k+4,-5k-12,-5k-4)$

But $\overrightarrow{MP} = \lambda\overrightarrow{MA}$ (M, P and A are collinear), $\lambda\in\mathbb{R}$

$\left(5k-\frac{3}{2},\ 5k-\frac{11}{2},\ 5k-\frac{7}{2}\right) = \lambda(-5k+4,-5k-12,-5k-4)$

$$5k - \frac{3}{2} = -5\lambda k + 4\lambda \quad (1)$$

$$5k - \frac{11}{2} = -5\lambda k - 12\lambda \quad (2)$$

$$5k - \frac{7}{2} = -5\lambda k - 4\lambda \quad (3)$$

$(1) - (2)$ $\quad 4 = 16\lambda$

$$\lambda = \frac{1}{4}$$

Sub in (1) $\quad 5k - \frac{3}{2} = -\frac{5k}{4} + 1$

$$\frac{25k}{4} = \frac{5}{2}$$

$$25k = 10$$

$$k = \frac{2}{5}$$

$$\therefore \vec{OP} = \vec{ON} + \vec{NP}$$

$$= (2,-6,-2) + (5k, 5k, 5k)$$

$$= (2,-6,-2) + (2, 2, 2)$$

$$\therefore \vec{OP} = (4, -4, 0)$$

$\therefore P$ is the point $(4, -4, 0)$

13b) $\quad 5^{2n} - 1 = (5^n)^2 - 1$

$$= (5^n + 1)(5^n - 1)$$

(2) correct proof

(1) proves result is divisible by 4 (or equivalent merit)

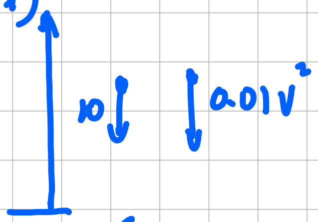
Now 5^n is odd and $5^n + 1$ and $5^n - 1$ are two consecutive even numbers, therefore one of them must be a multiple of 4.

Since one is even and the other it can be written as $2k$ (k is an integer) and the other is a multiple of 4 it can be written as $4m$ (m is in integer.

$$\therefore 5^{2n} - 1 = 2k\,4m$$

$= 8km$ which is divisible by 8

13 c) i)



$$F = ma$$
$$a = -10 - 0.01v^2 \quad \checkmark$$
$$v\frac{dv}{dx} = -10 - \frac{1}{100}v^2$$
$$v\frac{dv}{dx} = \frac{-(1000 + v^2)}{100}$$
$$\int_{100}^{0} \frac{v\,dv}{1000+v^2} = -\frac{1}{100}\int_{0}^{H} dx \quad \checkmark$$
$$\frac{1}{2}\int_{100}^{0} \frac{2v\,dv}{1000+v^2} = -\frac{1}{100}\Big[x\Big]_0^H$$
$$50\int_{0}^{100} \frac{2v\,dv}{1000+v^2} = H - 0$$
$$H = 50\Big[\ln(1000+v^2)\Big]_0^{100}$$
$$= 50 \ln 11000 - \ln 1000$$
$$= 50 \ln \frac{11000}{1000}$$
$\therefore$ Max height $= 50 \ln 11$

(3) correct solution
(2) correct integrand
(1) correct equation of motion

13 c ii)

$$a = 10 - 0.01v^2$$
$$v\frac{dv}{dx} = 10 - \frac{v^2}{100}$$
$$v\frac{dv}{dx} = \frac{1000 - v^2}{100}$$
$$-\frac{1}{2}\int_{0}^{V} \frac{-2v\,dv}{1000-v^2} = \frac{1}{100}\int_{0}^{50\ln 11} dx$$
$$-\frac{1}{2}\Big[\ln|1000-v^2|\Big]_0^V = \frac{1}{100}\Big[x\Big]_0^{50\ln 11}$$
$$-50\left(\ln|1000-v^2| - \ln 1000\right) = 50\ln 11 - 0$$

(3) correct solution
(2) correctly integrates to find an expression involving V
(1) obtains correct integrand

$$-50 \ln\left|\frac{1000-v^2}{1000}\right| = 50\ln 11$$

$$\ln\left|\frac{1000-v^2}{1000}\right| = -\ln 11$$

$$\frac{1000-v^2}{1000} = e^{-\ln 11}$$

$$\frac{1000-v^2}{1000} = \frac{1}{11}$$

$$1000-v^2 = \frac{1000}{11}$$

$$v^2 = 1000 - \frac{1000}{11}$$

$$v^2 = 909.09\ldots$$

$$v = 30.15\ ms^{-1}$$

13d) $P(z) = z^4 - 6z^3 + 38z^2 - 94z + 221$

3 correct solution

Since coefficients are real complex roots occur in conjugate pairs

2 obtains simultaneous equations to give other roots

$\therefore$ $2-3i$ is also a root.

1 identifies $2-3i$ is a root with justification

Let roots be $2+3i, 2-3i, \alpha, \beta$.

$\Sigma\alpha$: $\alpha+\beta+2+3i+2-3i = 6$

$\alpha+\beta+4 = 6$

$\alpha+\beta = 2$ 1

$\Sigma\alpha\beta\gamma\delta$: $\alpha\beta(2+3i)(2-3i) = 221$

$13\alpha\beta = 221$

$\alpha\beta = 17$ 2

$$\alpha + \frac{17}{\alpha} = 2$$

$$\alpha^2 + 17 = 2\alpha$$

$$\alpha^2 - 2\alpha + 17 = 0$$

$$\alpha = \frac{2 \pm \sqrt{4-68}}{2}$$

$$\alpha = \frac{2 \pm \sqrt{-64}}{2}$$

$$\alpha = \frac{2 \pm 8i}{2}$$

$$\alpha = 1 \pm 4i$$

$\therefore$ other roots are $2-3i$, $1 \pm 4i$

14 a) $z^5 = \bar{z}$

(3) correct solution

$r^5(\text{cis}\,\theta)^5 = r\,\text{cis}(-\theta + 2k\pi)$ where k is an integer

(2) Equates moduli and arguments and obtains $r=0$, $r=1$ and $\theta = \frac{k\pi}{3}$

Equating moduli:

$r^5 = r$

$r^5 - r = 0$

$r(r^4 - 1) = 0$

$\therefore r = 0$ or $r^4 = 1$

$r^2 = 1$

$\therefore r = 0$ or $r = 1$ $(r > 0)$

Equating arguments

$\text{cis}(5\theta) = \text{cis}(-\theta + 2k\pi)$

$5\theta = -\theta + 2k\pi$

$6\theta = 2k\pi$

$\theta = \frac{k\pi}{3}$

(1) Equates moduli and obtains $r=0$ and $r=1$

or (1) Equates arguments and obtains $\theta = \frac{k\pi}{3}$

$k=0$, $z = 1\,\text{cis}\,0 = 1$

$k=1$, $z = 1\,\text{cis}\,\frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$

$k=2$, $z = 1\,\text{cis}\,\frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$k=-1$, $z = 1\,\text{cis}\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

$k=-2$, $z = 1\,\text{cis}\left(-\frac{4\pi}{3}\right) = \frac{1}{2} - \frac{i\sqrt{3}}{2}$

$k=3$, $z = 1\,\text{cis}\,\pi = -1$

$\therefore z = 0$, $z = \pm 1$, $z = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$, $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

14b i) $I_n = \int \operatorname{cosec}^n x\,dx$

$u = \operatorname{cosec}^{n-2} x \qquad v' = \operatorname{cosec}^2 x$

$u' = (n-2)\operatorname{cosec}^{n-3}(x) - \sin^{-2} x \cos x \qquad v = -\cot x$

$u' = -(n-2)\operatorname{cosec}^{n-2} x \cot x$

$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\,dx$

$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x(\operatorname{cosec}^2 x - 1)\,dx$

$I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2)\left[I_n - I_{n-2}\right]$

$I_n(1 + n - 2) = -\cot x \operatorname{cosec}^{n-2} x + (n-2)I_{n-2}$

$I_n = \dfrac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

(3) correct proof

(2) correct primitive of $\operatorname{cosec}^n x$

(1) splits $\operatorname{cosec}^n x$ into $\operatorname{cosec}^{n-2} x$ and $\operatorname{cosec}^2 x$, and attempts to integrate by parts

14b ii) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \operatorname{cosec}^4 x\,dx$

(2) correct solution

(1) uses recurrence equation

$= \left[\dfrac{-\cot x \operatorname{cosec}^2 x}{3}\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \dfrac{2}{3}\displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \operatorname{cosec}^2 x\,dx$

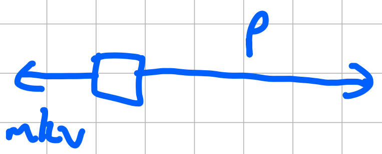
or (1) integrates and evaluates $\operatorname{cosec}^2 x$ with given limits

$= \dfrac{1}{3}\left[\sqrt{3}\times 4 + (\sqrt{3}\times 4)\right] + \dfrac{2}{3}\left[-\cot x\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$

$$= \frac{8\sqrt{3}}{3} + \frac{2}{3}\left(\sqrt{3} + \sqrt{3}\right)$$

$$= 4\sqrt{3}$$

14 c) i)



④ correct solution

③ correct expression in terms of P, m, k

② correct integral

① correct equation of motion

$$F = ma$$

$$ma = P - mkv$$

$$m\frac{dv}{dt} = P - mkv$$

$$\int_2^4 \frac{m\,dv}{P - mkv} = \int_0^5 dt$$

$$[t]_0^5 = \frac{-1}{k}\int_2^4 \frac{-mk\,dv}{P - mkv}$$

$$5 - 0 = \frac{-1}{k}\left[\ln|P - mkv|\right]_2^4$$

$$-5k = \ln|P - 4mk| - \ln|P - 2mk|$$

$$\ln\left|\frac{P - 4mk}{P - 2mk}\right| = -5k$$

$$\frac{P - 4mk}{P - 2mk} = e^{-5k}$$

$$\frac{P - 2mk}{P - 4mk} = e^{5k}$$

$$P - 2mk = Pe^{5k} - 4mke^{5k}$$

$$4mke^{5k} - 2mk = P\left(e^{5k} - 1\right)$$

$$2mk\left(2e^{5k} - 1\right) = P\left(e^{5k} - 1\right)$$

$$\therefore P = 2mk\left(\frac{2e^{5k} - 1}{e^{5k} - 1}\right)$$

14c ii) $mv\frac{dv}{dx} = P - mkv$

$$\int_2^4 \frac{mv\,dv}{P-mkv} = \int_0^x dx$$

$$[x]_0^x = -\int_2^4 \frac{P-mkv-P}{k(P-mkv)}$$

$$= \int_2^4 \frac{-1}{k} + \frac{P}{k}\frac{1}{P-mkv}\,dv$$

$$= \left[\frac{-v}{k}\right]_2^4 - \int_2^4 \frac{P}{mk^2}\left(\frac{-mkv}{P-mkv}\right)dv$$

$$= \frac{-4}{k} + \frac{2}{k} \frac{-P}{mk^2} \ln|P-mkv|\Big|_2^4$$

$$= \frac{-2}{k} - \frac{P}{mk^2}\left[\ln|P-4mk| - \ln|P-2mk|\right]$$

$$\therefore \text{Distance} = \frac{-2}{k} - \frac{P}{mk^2}\ln\left|\frac{P-4mk}{P-2mk}\right|$$

or $\text{Distance} = \frac{P}{mk^2}\ln\left|\frac{P-2mk}{P-4mk}\right| - \frac{2}{k}$

NB: If substituting P from (i),

$$\text{Distance} = \frac{10(2e^{5k}-1)}{e^{5k}-1} - \frac{2}{k}$$

(3) correct solution

(2) integrates to find an expression for x

(1) manipulates integrand to obtain appropriate integrand

c iii) Train will have an upper bound to speed when $\ddot{x} = 0$.

$$m\ddot{x} = P - mkv$$

$$0 = P - mkv$$

$$mkv = P$$

$$v = \frac{P}{mk} \quad \text{or} \quad v = \frac{2(2e^{5k}-1)}{e^{5k}-1}$$

(1) correct solution

15a) Test $n=4$,

$3^4 = 81$ $\quad$ $4^3 = 64$

$3^4 > 4^3$

$\therefore$ True for $n=4$

Assume true for $n=k$, $k \in \mathbb{Z} \wedge k>4$

$3^k > k^3$

(3) correct proof

(2) Uses assumption to make significant progress (ie tries to prove $2k^3-3k^2-3k-1>0$

(1) Proves true for $n=4$

For $n=k+1$, we wish to prove $3^{k+1} > (k+1)^3$

Consider $3^{k+1} - (k+1)^3$

$$= 3 \times 3^k - (k^3+3k^2+3k+1)$$
$$= 3(3^k-k^3) + 2k^3 - 3k^2 - 3k - 1$$
$$= 3(3^k-k^3) + (k^3-3k^2+3k-1) + k^3 - 6k$$
$$= 3(3^k-k^3) + (k-1)^3 + k(k^2-6)$$

>0 since by assumption $3^k > k^3$ and $k-1>0$, $k^2-6>0$ and $k>0$ since $k>3$.

$\therefore$ If true for $n=k$, it is true for $n=k+1$. But it is true for $n=4$, therefore by induction it is true for all positive integers greater than 3.

15 b) i) Let $\underset{\sim}{v}$ be the position vector of a point on the sphere

③ correct solution

② finds $\lambda = 0$ and $\lambda = 1$

$$S: \left| \underset{\sim}{v} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right| = 5$$

Point/s of intersection of $\underset{\sim}{r}(t)$ with the sphere ① obtains quadratic in λ

$$\left| \underset{\sim}{r} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right| = 5$$

$$\left| \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right| = 5$$

$$\sqrt{(3-7\lambda)^2 + (4-\lambda)^2 + (0\lambda+0)^2} = 5$$

$$49\lambda^2 - 42\lambda + 9 + \lambda^2 - 8\lambda + 16 = 25$$

$$50\lambda^2 - 50\lambda = 0$$

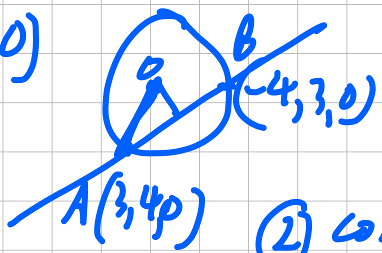
$$\lambda(\lambda - 1) = 0$$

$$\lambda = 0 \text{ or } 1$$

$\therefore$ Points of intersection are

$\lambda = 0$; $(3, 4, 0)$ and $(-4, 3, 0)$

15 b ii) when $\lambda = 0$, point A is $(3, 4, 0)$



Project $\overrightarrow{AB}$ onto line $\underset{\sim}{r}(\lambda)$

$$\left| \operatorname{proj}_{\overrightarrow{AB}} \overrightarrow{AO} \right| = \frac{\begin{pmatrix} 0-3 \\ 0-4 \\ 0-0 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ -1 \\ 0 \end{pmatrix}}{\sqrt{50}}$$

$$= \frac{21 + 4 + 0}{\sqrt{50}}$$

$$= \frac{25}{5\sqrt{2}}$$

② correct solution

① finds length or vector projection onto line l

OR ① finds midpoint of chord and attempts to find distance

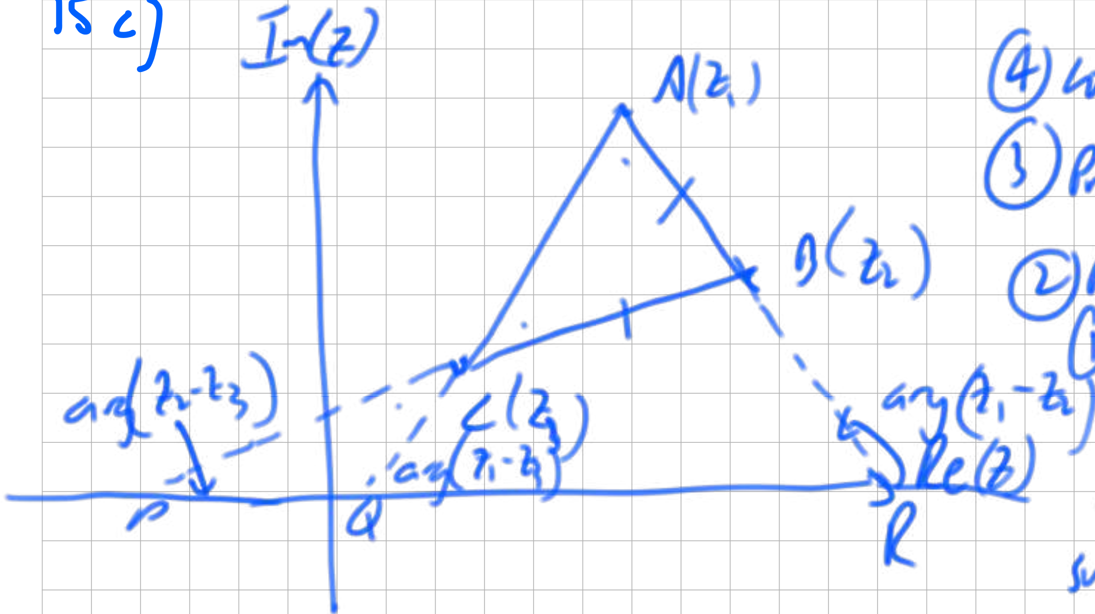
$$= \frac{5\sqrt{2}}{2}$$

$$|\vec{AO}| = \sqrt{3^2+4^2+0^2}$$
$$= 5$$

$$\therefore \text{Perpendicular distance} = \sqrt{5^2 - \left(\frac{5\sqrt{2}}{2}\right)^2}$$
$$= \sqrt{25 - \frac{25}{2}}$$
$$= \sqrt{\frac{25}{2}}$$
$$= \frac{5}{\sqrt{2}}$$

$$\therefore \text{Shortest distance} = \frac{5\sqrt{2}}{2}$$

15 c)



(4) correct solution

(3) Proves BA=BC

(2) Proves $\angle PCQ = \angle CAB$

(1) takes argument of both sides and expresses as subtraction of arguments

Extend AC to meet x axis at Q $\angle CQR = \arg(z_1 - z_3)$

Extend BC to meet x axis at P $\angle CPQ = \arg(z_2 - z_3)$

Extend AB to meet x axis at R $\angle BRx = \arg(z_1 - z_2)$

From $\dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{z_1 - z_3}{z_1 - z_2}$ (*)

$$\arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \arg\left(\frac{z_1 - z_3}{z_1 - z_2}\right)$$

$$\arg(z_2 - z_3) - \arg(z_1 - z_3) = \arg(z_1 - z_3) - \arg(z_1 - z_2)$$

$$\arg(z_1 - z_2) - \arg(z_1 - z_3) = \arg(z_1 - z_3) - \arg(z_2 - z_3)$$

$\angle CAB = \arg(z_1 - z_2) - \arg(z_1 - z_3)$ (Exterior $\angle$ of ΔCAB equals sum of interior opposite angles)

$\angle PCQ = \arg(z_1 - z_3) - \arg(z_2 - z_3)$ (Exterior $\angle$ of ΔPCQ equals sum of interior opposite angles)

$\therefore \angle PCQ = \angle CAB$

$\angle ACB = \angle PCQ$ (vertically opposite angles)

$\therefore \angle ACB = \angle CAB$

$\therefore BA = BC$ (equal $\angle$'s opposite equal sides of ΔABC)

$\therefore |z_1 - z_2| = |z_2 - z_3|$

From (*) $\left|\dfrac{z_2 - z_3}{z_1 - z_3}\right| = \left|\dfrac{z_1 - z_3}{z_1 - z_2}\right|$

$$|z_1 - z_3|^2 = |z_1 - z_2||z_2 - z_3|$$

$$|z_1 - z_3|^2 = |z_1 - z_2|^2$$

$\therefore |z_1 - z_3| = |z_1 - z_2|$

$\therefore AC = AB = BC$

$\therefore \triangle ABC$ is equilateral

15d) $\displaystyle\int \frac{\ln x \, dx}{(1+\ln x)^2} = \int \frac{x \ln x \, dx}{x(1+\ln x)^2}$

$\displaystyle = \frac{-x\ln x}{1+\ln x} + \int dx$

$\displaystyle = x - \frac{x \ln x}{1+\ln x} + C$

$u = x\ln x \qquad v = \frac{-1}{1+\ln x}$

$u' = (1+\ln x) \qquad v' = \frac{1}{x(1+\ln x)^2}$

(3) correct solution

(2) uses integration by parts to transform the integrand into a valid result

(1) transforms integral with a relevant substitution

There were a variety of correct substitutions and approaches, including $u = 1+\ln x$ and $u = \ln x$. Other successful integration by parts include $u = \frac{1}{1+\ln x}, v' = x$

16 a) Since each number is less than sum of other two

$c < a+b \qquad b < a+c \qquad a < b+c$

$a+b-c > 0 \qquad a+c-b > 0 \qquad b+c-a > 0$

$\therefore (a+b-c+a+c-b)(a+b-c+b+c-a)(a+c-b+b+c-a) \geq 8(a+b-c)(a+c-b)(b+c-a)$

$2a \times 2b \times 2c \geq 8(a+b-c)(a+c-b)(b+c-a)$

$abc \geq (a+b-c)(a+c-b)(b+c-a)$

$\therefore (a+b-c)(a+c-b)(b+c-a) \leq abc$

(3) correct solution

(2) substitutes correct expressions into the inequality

(1) obtains $a+b-c>0$ etc

16 b) i) $S = \binom{2000}{0}\sqrt{50}^{2000} + \binom{2000}{1}\sqrt{50}^{1999}7^{1} + \dots + \binom{2000}{1999}\sqrt{50}(7)^{1999} + 7^{2000}$

$+ \binom{2000}{0}\sqrt{50}^{2000} - \binom{2000}{1}\sqrt{50}^{1999}7^{1} + \dots - \binom{2000}{1999}\sqrt{50}\,7^{1999} + 7^{2000}$

$\therefore S = 2\left[\binom{2000}{0}50^{1000} + \binom{2000}{2}50^{999}7^{2} + \dots + 7^{2000}\right]$

$= 2 \times$ integer

$\therefore S$ is even

② correct proof (must justify it is an integer $\sqrt{50}^{2000} = 50^{1000}$)

① expands one binomial correctly

16 b ii) $\sqrt{49} < \sqrt{50} < \sqrt{50.41}$

$7 < \sqrt{50} < 7.1$

$\therefore \quad 0 < \sqrt{50} - 7 < 0.1$

② correct proof

① shows $\sqrt{50} - 7 > 0$

iii) $(\sqrt{50}+7)^{2000} = S - (\sqrt{50}-7)^{2000}$

$= \underset{\text{even integer}}{S} - \underset{\text{at least 1999 zeroes}}{(0.000\dots)}$

② correct proof

① identifies $(\sqrt{50}-7)^{2000} < (10^{-1})^{2000}$ or expresses $(\sqrt{50}+7)^{2000}$ as $S - (\sqrt{50}-7)^{2000}$

since $(\sqrt{50}-7)^{2000} < (10^{-1})^{2000}$

$\therefore$ At least the first 2000 digits after the decimal point are 9s.

(Note: $1-0.1=0.9$, $1-0.01=0.99$, $1-0.001=0.999$. Number of 9's is one more than the number of 0's before the 1)

16 c i) $y = \dfrac{e^x}{x}$

$\dfrac{dy}{dx} = \dfrac{xe^x - e^x}{x^2}$

$\dfrac{dy}{dx} = \dfrac{e^x(x-1)}{x^2}$

Stat points occur when $\dfrac{dy}{dx} = 0$

$\dfrac{e^x(x-1)}{x^2} = 0$

① correct solution

(Note: nature of stationary point is not required to be shown

$\therefore x = 1$ is a stat point

From diagram, only one stationary point occurs

OR $\left[\begin{array}{l} \text{when } x<1, \ \frac{dy}{dx}<0 \\ \text{when } x>1, \ \frac{dy}{dx}>0 \end{array}\right]$

$\therefore$ minimum turning point at $(1, e)$

16 c ii) $\displaystyle\int_a^{2a} \frac{e^x}{x}\,dx = \int_a^{2a} \frac{e^x}{x^2}\,dx$

② correct solution

① finds primitive function using (i)

$$\int_a^{2a} \frac{xe^x}{x^2} - \frac{e^x}{x^2}\,dx = 0$$

$$\int_a^{2a} \frac{xe^x - e^x}{x^2}\,dx = 0$$

$$\left[\frac{e^x}{x}\right]_a^{2a} = 0$$

$$\frac{e^{2a}}{2a} - \frac{e^a}{a} = 0$$

$$e^{2a} - 2e^a = 0$$

$$e^a(e^a - 2) = 0$$

$$e^a - 2 = 0 \qquad (e^a \neq 0)$$

$$e^a = 2$$

$$a = \ln 2$$

16 c ii) $$\int_a^{2a} \frac{e^x}{x^2}\,dx = \left[-\frac{1}{x}e^x\right]_a^{2a} - \int_a^{2a} -\frac{1}{x}e^x\,dx$$

$$\int_a^{2a} \frac{e^x}{x^2}\,dx = -\frac{e^{2a}}{2a} + \frac{1}{a}e^a + \int_a^{2a} \frac{e^x}{x}\,dx$$

since $\int_a^{2a} \frac{e^x}{x^2}\,dx = \int_a^{2a} \frac{e^x}{x}\,dx$,

$$0 = \frac{-e^{2a}}{2a} + \frac{e^a}{a}$$

$$\frac{e^{2a}}{2} = e^a$$

$$\frac{e^a}{2} = 1$$

$$e^a = 2$$

$$\therefore a = \ln 2$$

(2) correct solution

(1) uses integration by parts to express relationship between integrals or using (i)

16 c iii) Using ii) if the integrals are the same,

$$\left[-\frac{1}{x}e^x\right]_m^n = 0$$

$$-\frac{1}{n}e^n + \frac{1}{m}e^m$$

$$\frac{e^m}{m} = \frac{e^n}{n}$$

(2) correct solution

(1) obtains $e^{m-n} = \frac{m}{n}$ and attempts to use proof by contradiction

or (1) obtains $\frac{e^m}{m} = \frac{e^n}{n}$ and attempts to use graph

Method 1: Proof by contradiction. Assume that there exist integers m and n such that $e^{m-n} = \frac{m}{n}$

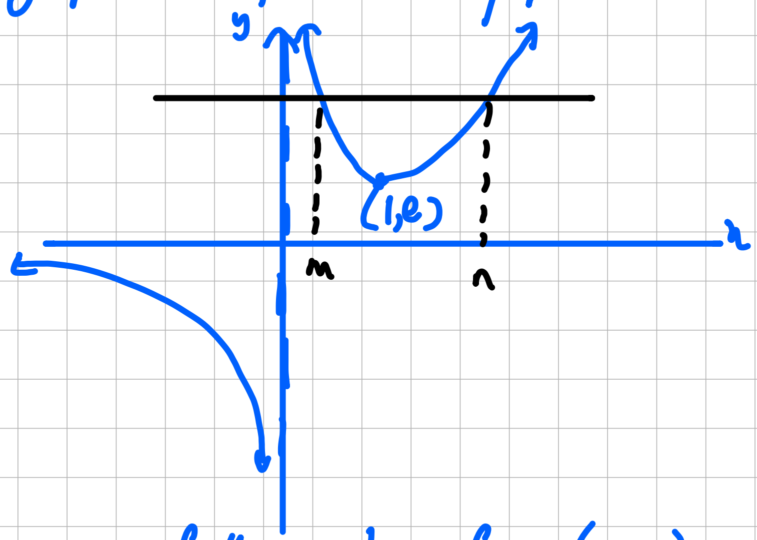
Now $\frac{m}{n}$ is rational (m, n are integers)

e^{m-n} is only rational if $m-n=0$, when $m=n$.

However m and n are distinct so $m-n \neq 0$ and e^{m-n} is irrational.

Method 2:

For this to be true for two distinct values m and n, from the graph in (i), this is only possible when $x > 0$.



However, one of these values of m (or n) must be less than 1. This is impossible as m and n must be distinct integers. Therefore it is not possible to find distinct integers m and n.