Caringbah High School

Year 12 2025

Mathematics Extension 2

HSC Course

Assessment Task 4 – Trial HSC Examination

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 100

Section I **10 marks**

Attempt Questions 1-10

Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **90 marks**

Attempt Questions 11-16

Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ______________________________ **Class:** ____________

Marker's Use Only							
Section I	**Section II**						**Total**
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**	**Q15**	**Q16**	
/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

---

1. If $\dfrac{1}{x^2+x} = \dfrac{A}{x} + \dfrac{B}{x+1}$, what are the values of A and B?

(A) $A = -1, B = -1$

(B) $A = 1, B = -1$

(C) $A = -1, B = 1$

(D) $A = 1, B = 1$

2. If $z = \dfrac{3+4i}{1+2i}$, the imaginary component of z is:

(A) -2

(B) $\dfrac{2}{5}$

(C) $-\dfrac{2}{5}$

(D) 2

3. Given the complex number $z = a + ib$, where a , $b \in \mathbb{R}$, $a \neq 0$,

$\dfrac{4z\,\overline{z}}{(z+\overline{z})^2}$ is equivalent to:

(A) $4[Re(z) \times Im(z)]$

(B) $1 + \left(\dfrac{Im(z)}{Re(z)}\right)^2$

(C) $4\left([Re(z)]^2 + [Im(z)]^2\right)$

(D) $\dfrac{2 \times Im(z)}{[Re(z)]^2}$

4. Consider the following statement.

"If I live in Berlin then I live in Germany."

What is the negation of this statement?

(A) If I don't live in Berlin then I don't live in Germany.

(B) If I don't live in Germany then I don't live in Berlin.

(C) If I live in Germany then I live in Berlin.

(D) I live in Berlin and I don't live in Germany.

5. The value of $i + i^2 + i^3 + ... + i^{2025}$ is:

(A) i

(B) -1

(C) 1

(D) $-i$

6. Which of the following is the vector equation of a line that passes through the points (8, –2, 8) and (0, 7, 0)?

(A) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 5 \\ 8 \end{pmatrix}$

(B) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 5 \\ 8 \end{pmatrix}$

(C) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -9 \\ 8 \end{pmatrix}$

(D) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -8 \\ -9 \\ -8 \end{pmatrix}$

7. A variable force acts on a particle, causing it to move in a straight line. At time t seconds, where $t \geq 0$, its velocity v metres per second and position x metres from the origin are such that $v = e^x \sin x$.

The acceleration of the particle, in ms^{-2}, can be expressed as:

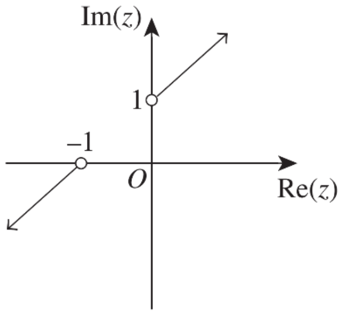
(A) $e^x(\sin x + \cos x)$

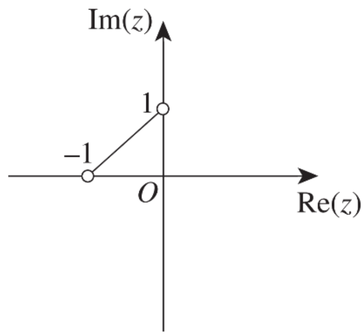
(B) $e^x \sin x(\sin x + \cos x)$

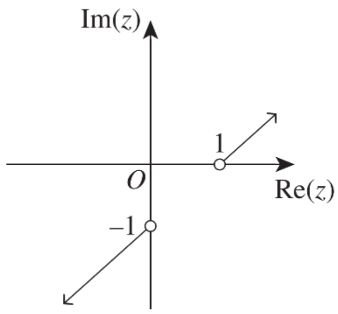
(C) $\frac{1}{2} e^{2x} \sin^2 x$

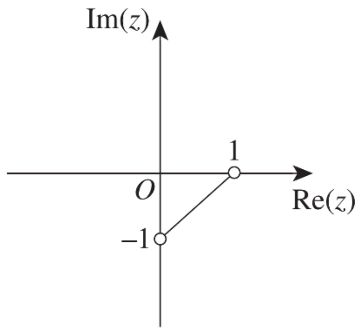
(D) $e^{2x}\left(\sin^2 x + \frac{1}{2} \sin 2x \right)$

8. Which of the following Argand diagrams represents the solutions to the equation $\operatorname{Arg}(z - i) = \operatorname{Arg}(z + 1)$?

(A) 

(B) 

(C) 

(D) 

9. Let $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$.

If $\underset{\sim}{n}$ is a unit vector such that $\underset{\sim}{a} \cdot \underset{\sim}{n} = 0$ and $\underset{\sim}{b} \cdot \underset{\sim}{n} = 0$, then $|\underset{\sim}{c} \cdot \underset{\sim}{n}|$ is equal to:

(A) 2

(B) 3

(C) 4

(D) 5

10. Which expression is equal to

$$\int_0^a [f(a-x)+f(a+x)]\,dx \quad ?$$

(A) $\displaystyle\int_0^a f(x)\,dx$

(B) $\displaystyle\int_0^{2a} f(x)\,dx$

(C) $\displaystyle 2\int_0^a f(x)\,dx$

(D) $\displaystyle\int_{-a}^a f(x)\,dx$

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) If $z = \cos\theta + i\sin\theta$, simplify $z + \dfrac{1}{z}$. **1**

(b) Given that $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to the nearest degree. **2**

(c) Sketch the region defined by $|z + 1| \leq 2$ and $\operatorname{Im}(z) \geq 0$. **2**

(d) Find $\displaystyle\int (x^2 + 6x + 13)^{-1}\ dx$ **2**

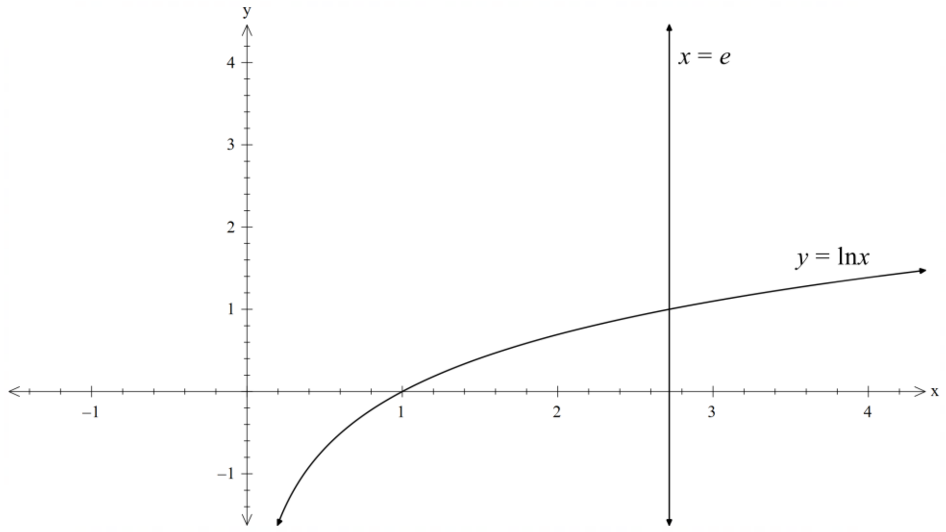
(e) **(i)** Find the square roots of $3 + 4i$. **2**

(ii) Hence, or otherwise, solve the equation **2**

$$z^2 + iz - 1 - i = 0$$

Question 11 continues on page 9

(f) The region R is bounded by the graph of $y = \ln x$, the line $x = e$ and the x-axis. **4**



Find the **volume** of the solid of revolution formed when the region R is rotated about the x-axis.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) By using a suitable substitution, or otherwise, find **2**

$$\int \frac{\sin^3 x}{\cos^2 x}\ dx$$

(b) A body moves in a straight line so that when its displacement from a fixed origin O is x metres, its acceleration, a, is $-4x$ ms^{-2}. **3**

The body accelerates from rest and its velocity, v, is equal to -2 ms^{-1} as it passes through the origin. The body then comes to rest again.

Find v in terms of x for this interval.

(c) Prove by contradiction that $\sqrt{11}-\sqrt{5}<\sqrt{2}$. **2**

(d) Evaluate $\displaystyle\int_0^6 \sqrt{36-x^2}\ dx$ **2**

(e) The point P represents the complex number z in the complex plane. **3**

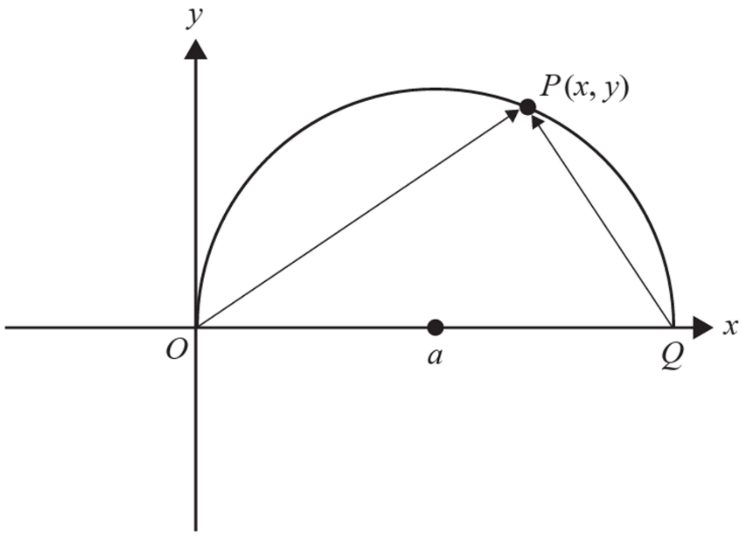
If $z = x + iy$, prove that $\text{Re}\left(\dfrac{z-ib}{z-a}\right)=0$ is the equation of a circle,

where a and b are real, and state its centre and radius in terms of a and b.

Question 12 continues on page 11

(f) OPQ is a semi-circle of radius a with equation $y=\sqrt{a^2-(x-a)^2}$.

$P(x,y)$ is a point on the semi-circle OPQ, as shown below.



(i) Express $\overrightarrow{OP}$ and $\overrightarrow{QP}$ in terms of a, x, y, $\underset{\sim}{i}$ and $\underset{\sim}{j}$. **1**

(ii) Hence determine whether $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{QP}$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving in a straight line and its position at time t is given by

$$x = 2 + \sin 2t + \sqrt{3} \cos 2t$$

(i) Prove that the particle is undergoing simple harmonic motion about $x = 2$. **2**

(ii) Find the amplitude of motion. **1**

(iii) When does the particle first reach its maximum speed after time $t = 0$? **2**

(b) Show that $\dfrac{a^4 + a^2 + 1}{a^2} \geq 3$ for all a. **1**

(c) **(i)** Show that $\left(z - \dfrac{1}{z}\right)^5 = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$ **2**

(ii) By letting $z = \cos\theta + i\sin\theta$, or otherwise, show that **2**

$$32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$$

(iii) Hence, find the exact value of $\displaystyle\int_o^{\frac{\pi}{2}} \sin^5\theta \, d\theta$ **2**

(d) Prove that the product of any three consecutive numbers in even. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The line l has equation $\underset{\sim}{v} = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ b \\ 2 \end{pmatrix}$,

where μ is a parameter and b is a constant.

The points P and Q have coordinates $(-1, 2, 3)$ and $(-2, 4, 2)$ respectively.

(i) Find a vector equation of the line PQ. **1**

Leave your answer in the form of $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$.

(ii) Find the value of b for which the acute angle between the line l and the line PQ is $\cos^{-1}\left(\frac{1}{6}\right)$. **2**

(b) Consider the following proposition:

Suppose $x, y \in \mathbb{R}$. "If $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$."

(i) State the contrapositive of this proposition. **1**

(ii) Hence, prove the above proposition is true. **2**

(c) Two lines $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ have equations

$$\underset{\sim}{r}_1 = \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \text{ and } \underset{\sim}{r}_2 = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \text{ where } \lambda, \mu \in \mathbb{R}$$

The point A lies on $\underset{\sim}{r}_1$ with parameter $\lambda = p$, and the point B lies on $\underset{\sim}{r}_2$ with parameter $\mu = q$.

(i) Show that $\overrightarrow{AB}$ can be expressed as the column vector **1**

$$\begin{pmatrix} -2 - q + p \\ -1 + 2q - 4p \\ -3 + 2q - 3p \end{pmatrix}$$

(ii) Find the value of $\left|\overrightarrow{AB}\right|$ when $\overrightarrow{AB}$ is perpendicular to both $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$. **3**

Question 14 continues on page 14

(d) Consider the spheres S_1 and S_2. **2**

$S_1 : x^2 - 2x + y^2 + 4y + z^2 - 4z = 0$

$S_2 : x^2 + y^2 + z^2 + 10x + 2z + 10 = 0$

Show that S_1 and S_2 intersect.

(e) Find **3**

$$\int \frac{e^x}{e^{2x} + 3e^x + 2}\, dx$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Find **2**

$$\int (1 + 2x^2)e^{x^2}\, dx$$

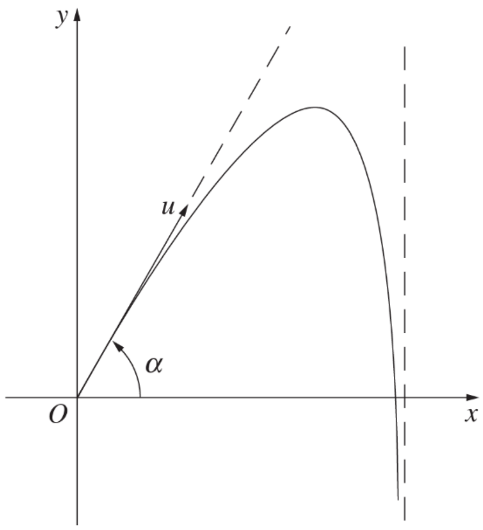
(b) **(i)** Show that $|\sin(\alpha + \beta)| \leq |\sin\alpha| + |\sin\beta|$ **1**

(ii) Under what conditions will equality hold in part (i)? **1**

(c) A particle of mass m is thrown from the top, O, of a very tall building with an initial velocity u at an angle α to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion in the horizontal and vertical directions are given respectively by

$\ddot{x} = -k\dot{x}$ and $\ddot{y} = -k\dot{y} - g$,

where k is a constant and the acceleration due to gravity is g .
(You are not required to show these.)



(i) Derive the result $\dot{x} = ue^{-kt}\cos\alpha$ from the relevant equation of motion. **2**

(ii) Verify that $\dot{y} = \dfrac{1}{k}((ku\sin\alpha + g)e^{-kt} - g)$ satisfies the appropriate equation of motion. **2**

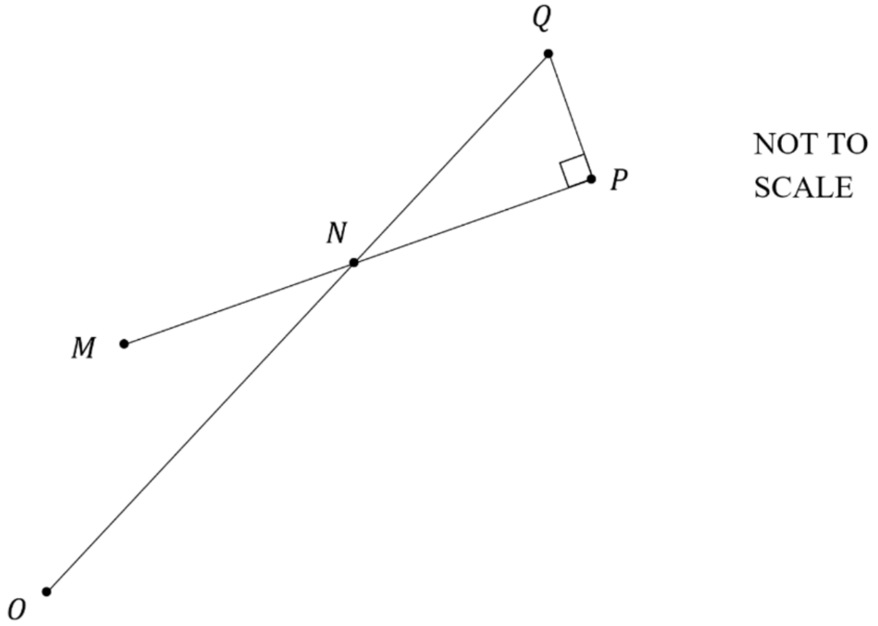
(iii) Find the value of t when the particle reaches its maximum height. **1**

(iv) What is the limiting value of the horizontal displacement of the particle? **2**

Question 15 continues on page 16

(d) M and P are two points with position vectors $\overrightarrow{OM} = \underset{\sim}{m}$ and $\overrightarrow{OP} = \underset{\sim}{p}$ respectively. N is the midpoint of MP and has position vector $\overrightarrow{ON} = \underset{\sim}{n}$. **4**

PQ is perpendicular to MN and $\overrightarrow{NQ} = \lambda\overrightarrow{ON}$.



Show that

$$\lambda = \frac{|\underset{\sim}{m}|^2 + |\underset{\sim}{n}|^2 - 2\underset{\sim}{m} \cdot \underset{\sim}{n}}{|\underset{\sim}{n}|^2 - \underset{\sim}{m} \cdot \underset{\sim}{n}}$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of unit mass is allowed to fall under gravity from rest in a medium which exerts a resistance proportional to the speed, v, of the particle.

(i) Show that the particle reaches a terminal velocity T, given by $T = \frac{g}{k}$ where k is a positive constant. **1**

(ii) Show that the distance fallen to reach half of its terminal velocity, $\frac{T}{2}$ is given by **4**

$$x = \frac{T^2}{g}\ln(2) - \frac{T^2}{2g}$$

(b) Let $P(x)$ be a monic polynomial of degree n with rational coefficients and zeroes $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$. **2**

Show that

$$P'(x) = P(x)\left(\frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \frac{1}{x-\alpha_3} + \ldots + \frac{1}{x-\alpha_n}\right)$$

Question 16 continues on page 18

(c) **(i)** Given that for $k > 0$, $2k + 3 > 2\sqrt{(k+1)(k+2)}$, **4**

use mathematical induction to show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > 2\left(\sqrt{n+1} - 1\right)$$

for all positive integers n.

(ii) Use the graph of $y = \dfrac{1}{\sqrt{x}}$ to show that **2**

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} < 1 + \int_{1}^{n} \frac{1}{\sqrt{x}}\, dx$$

(iii) Hence show that **2**

$$198 < \sum_{r=1}^{10\,000} \frac{1}{\sqrt{r}} < 199$$

End of Examination

2025 Extension 2 Trial Solutions

Section 1

Q1) $\dfrac{1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}$

$1 = A(x+1) + Bx$

let $x = 0$, $A = 1$ $\qquad$ $x = -1$, $B = -1$

B.

Q1) B
Q2) C
Q3) B
Q4) D
Q5) A
Q6) C
Q7) D
Q8) A
Q9) B
Q10) B

Q2) $z = \dfrac{3+4i}{1+2i} \times \dfrac{1-2i}{1-2i}$

$= \dfrac{3-6i+4i+8}{1+4}$

$= \dfrac{11-2i}{5}$

$\therefore \operatorname{Im}(z) = \dfrac{-2}{5}$

C.

Q3) $\dfrac{4z\bar{z}}{(z+\bar{z})^2} = \dfrac{4(a^2+b^2)}{4a^2}$

$= \dfrac{a^2+b^2}{a^2}$

$= 1 + \left(\dfrac{b}{a}\right)^2$

$1 + \left(\dfrac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)^2 = 1 + \left(\dfrac{b}{a}\right)^2$

B.

Q4) To negate a statement of the form "If P then Q," the negation is "P and not Q."

A. Common error
B. Contrapositive
C. Converse
D. Negation

D.

Q5) $i + i^2 + i^3 + \dots + i^{2024} = 0$

$i + i^2 + i^3 + \dots + i^{2024} + i^{2025}$

$= 0 + i^{2025}$
$= 0 + i$
$= i$

A.

Q6) $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix} + \lambda \overrightarrow{BA}$

$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -9 \\ 8 \end{pmatrix}$

C.

Q7) $v = e^x \sin x$

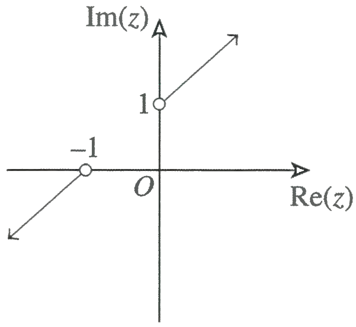
$\frac{dv}{dx} = e^x \sin x + e^x \cos x$

$v\frac{dv}{dx} = e^x \sin x \times e^x (\sin x + \cos x)$

$a = e^{2x}(\sin^2 x + \frac{1}{2}\sin 2x)$

D.

Q8)



A.

Q9) let $\underset{\sim}{n} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{n}| = x^2 + y^2 + z^2 = 1$

$\underset{\sim}{a} \cdot \underset{\sim}{n} = 0 \qquad \underset{\sim}{b} \cdot \underset{\sim}{n} = 0$

$x + y = 0 \qquad x - y = 0$

$x = y = 0$

$\therefore \quad \underset{\sim}{n} = z\underset{\sim}{k}$

$\underset{\sim}{c} \cdot \underset{\sim}{n} = 3z$

$|\underset{\sim}{c} \cdot \underset{\sim}{n}| = 3$

B.

Q10) $\displaystyle\int_0^a f(a-x) + f(a+x) \, dx$

$\displaystyle = \int_0^a f(a-x)\,dx + \int_0^a f(a+x)\,dx$

$\displaystyle = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

$\displaystyle = \int_0^{2a} f(x)\,dx$

B.

Section 2

Q11)a)

$$z + \frac{1}{z} = \cos\theta + i\sin\theta + (\cos\theta + i\sin\theta)^{-1}$$

$$= \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta) \qquad \text{(De Moivre)}$$

$$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

b) $$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{2+2-2}{\sqrt{6} \times \sqrt{9}}$$

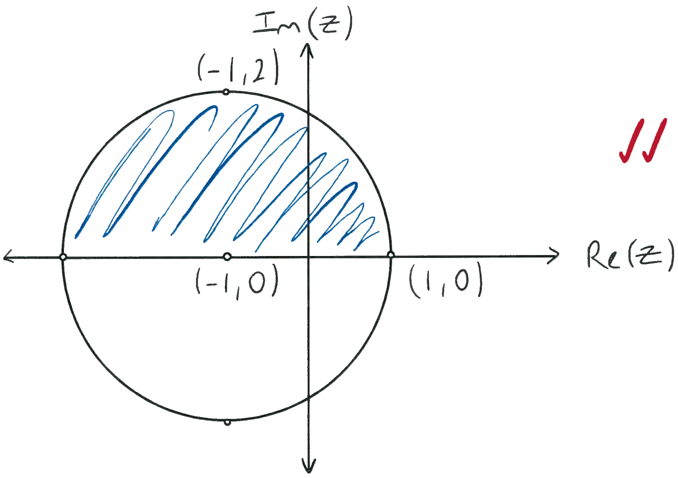
$$= \frac{2}{\sqrt{54}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{54}}\right)$$

$$= 74^\circ$$

c) $|z - (-1)| \leq 2$

circle with centre $(-1,0)$ and radius 2.



d) $\displaystyle\int \frac{1}{x^2+6x+13}\,dx$

$$= \int \frac{1}{x^2+6x+9+13-9}\,dx$$

$$= \int \frac{1}{(x+3)^2+4}\,dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + c$$

e) i) $(x+iy)^2 = 3+4i$

$x^2-y^2 = 3$ ① $\qquad$ $x^2+y^2=5$ ②

①+②, $\qquad$ ①−②,

$2x^2 = 8$ $\qquad$ $2y^2 = 2$

$x = \pm 2$ $\qquad$ $y = \pm 1$

Since $\text{Im}(3+4i) > 0$

$2+i$ and $-2-i$ are the square roots.

ii) $z^2 + iz - (1+i) = 0$

$$z = \frac{-i \pm \sqrt{i^2 - 4\times 1\times -(1+i)}}{2}$$

$$z = \frac{-i \pm \sqrt{3+4i}}{2}$$

$$z = \frac{-i}{2} \pm \frac{2+i}{2} \quad \text{or} \quad \cancel{\frac{-i}{2} \pm \frac{(-2-i)}{2}}$$

duplicate solutions

$$= \frac{-i}{2} + 1 + \frac{i}{2}, \quad \frac{-i}{2} - 1 - \frac{i}{2}$$

$$= 1, \quad -1-i$$

f) $V = \pi \int_1^e (\ln x)^2 \, dx$

$u = (\ln x)^2 \qquad v' = 1$
$u' = 2\ln x \times \frac{1}{x} \qquad v = x$

$V = \pi \left[x(\ln x)^2 \right]_1^e - 2\pi \int_1^e \ln x \, dx$ ✓

$u = \ln x \qquad v' = 1$
$u' = \frac{1}{x} \qquad v = x$

$V = \pi(e - 0) - 2\pi \left[x \ln x \right]_1^e - \int_1^e 1 \, dx$ ✓

$V = \pi e - 2\pi(e - 0) + 2\pi(e - 1)$ ✓
$= \pi e - 2\pi e + 2\pi e - 2\pi$
$= \pi e - 2\pi$ ✓

Q12) a) $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx$

let $u = \cos x$

$-du = \sin x\,dx$

$$= -\int \frac{1-u^2}{u^2}\,du$$ ✓

$$= -\int u^{-2} - 1\,du$$

$$= u^{-1} + u + c$$

$$= \sec x + \cos x + c$$ ✓

Alternatively

$$\int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \sin x \tan^2 x\,dx$$

$$= \int \sin x(\sec^2 x - 1)$$

$$= \int \frac{\sin x}{\cos^2 x} - \sin x\,dx$$

$$= \int \sin x(\cos x)^{-2} - \sin x\,dx$$

$$= \sec x + \cos x + c$$

b) $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$$

$$\int \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \int -4x\,dx$$

$$\frac{1}{2}v^2 = -2x^2 + c$$ ✓

When $v = -2\,m/s$, $x = 0$.

$\therefore\ c = 2$

$$v^2 = -4x^2 + 4$$ ✓

$$v = -\sqrt{4-4x^2}$$ since $v=-2$ when $x=0$.

$$= -2\sqrt{1-x^2}$$ ✓

c) Assume that $\sqrt{11} - \sqrt{5} \geq \sqrt{2}$

$$(\sqrt{11} - \sqrt{5})^2 \geq 2$$

$$11 - 2\sqrt{55} + 5 \geq 2$$

$$2\sqrt{55} \leq 14$$

$$\sqrt{55} \leq 7$$

$$55 \leq 49 \quad \text{(which is incorrect)} \checkmark$$

$\therefore$ By contradiction, $\sqrt{11} - \sqrt{5} < \sqrt{2}$ ✓

d) Area of a quarter of a circle.

$$= \frac{1}{4} \times \pi \times 6^2 \quad \checkmark$$

$$= 9\pi \quad \checkmark$$

Alternatively

let $x = 6\sin\theta$ $\quad$ $x = 0,$ $\quad$ $x = 6,$

$dx = 6\cos\theta \, d\theta$ $\quad$ $\theta = 0$ $\quad$ $\theta = \frac{\pi}{2}$

$$= 6\int_0^{\frac{\pi}{2}} \cos\theta \times 6\cos\theta \, d\theta$$

$$= 36\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2} \, d\theta$$

$$= 18\int_0^{\frac{\pi}{2}} \cos 2\theta + 1 \, d\theta$$

$$= 18\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}} \quad \checkmark$$

$$= 18\left(0 + \frac{\pi}{2} - 0\right)$$

$$= 9\pi \quad \checkmark$$

e) let $z = x + iy$

$$Re\left(\frac{x + iy - ib}{x + iy - a}\right) = 0$$

$$Re\left(\frac{x + i(y-b)}{(x-a) + iy} \times \frac{(x-a) - iy}{(x-a) - iy}\right) = 0$$

$$Re\left(\frac{x(x-a) - ixy + i(y-b)(x-a) + y(y-b)}{(x-a)^2 + y^2}\right) = 0$$

$$\frac{x(x-a) + y(y-b)}{(x-a)^2 + y^2} = 0 \quad ✓$$

$$x(x-a) + y(y-b) = 0$$

$$x^2 - ax + y^2 - by = 0$$

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} \quad ✓$$

centre $\left(\frac{a}{2}, \frac{b}{2}\right)$ radius $\sqrt{\frac{a^2+b^2}{4}}$ ✓

)i) $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$

$= x\underset{\sim}{i} + \sqrt{a^2 - (x-a)^2}\,\underset{\sim}{j}$ ✓

$\overrightarrow{QP} = (x-2a)\underset{\sim}{i} + y\underset{\sim}{j}$

$= (x-2a)\underset{\sim}{i} + \sqrt{a^2 - (x-a)^2}\,\underset{\sim}{j}$

ii) $\overrightarrow{OP} \cdot \overrightarrow{QP} = x(x-2a) + a^2 - (x-a)^2$ ✓

$= x^2 - 2ax + a^2 - x^2 - 2ax + a^2$

$= 0$ ✓

Q13) a) i) $x = 2 + \sin 2t + \sqrt{3}\cos 2t$

$\dot{x} = 2\cos 2t - 2\sqrt{3}\sin 2t$

$\ddot{x} = -4\sin 2t - 4\sqrt{3}\cos 2t$ ✓

$\ddot{x} = -4(\sin 2t + \sqrt{3}\cos 2t)$

$\ddot{x} = -4(x-2)$

which is in the form of $\ddot{x} = -n^2(x-a)$. ✓

ii) $a = 2$ ✓

iii) Max speed occurs at the centre of motion.

$\therefore$ at $x = 2$

$2 = 2 + \sin 2t + \sqrt{3}\cos 2t$

$\sin 2t + \sqrt{3}\cos 2t = 0$

$\tan\alpha = \sqrt{3}$ $\qquad r = \sqrt{1+3}$

$\qquad\qquad\qquad = 2$

$\alpha = \frac{\pi}{3}$

$\therefore \quad 2\sin\left(2t + \frac{\pi}{3}\right) = 0$ ✓

$\sin\left(2t + \frac{\pi}{3}\right) = 0$

$2t + \frac{\pi}{3} = 0, \pi, \dots$

$t = \cancel{-\frac{\pi}{6}}, \frac{\pi}{3}$

$\therefore t = \frac{\pi}{3}$ ✓

b) $(a^2-1) \geq 0$

$a^4 - 2a^2 + 1 \geq 0$

$a^4 + a^2 + 1 \geq 3a^2$

$\therefore \quad \frac{a^4+a^2+1}{a^2} \geq 3$ ✓

c) i) $\left(z-\frac{1}{z}\right)^5 = z^5 - 5\frac{z^4}{z} + 10\frac{z^3}{z^2} - 10\frac{z^2}{z^3} + 5\frac{z}{z^4} - \frac{1}{z^5}$ ✓

$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

$= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$ ✓

ii) Let $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$
$z^{-n} = \cos n\theta - i\sin n\theta$

$z^n - z^{-n} = 2i\sin n\theta$

From (i), $(z - z^{-1})^5 = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$

$(2i\sin\theta)^5 = 2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta$ ✓

$32i^5\sin^5\theta = i(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta)$

$32i\sin^5\theta = i(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta)$

$32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$ ✓

iii) $\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} \sin 5\theta - 5\sin 3\theta + 10\sin\theta \, d\theta$

$= \frac{1}{16}\left[-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\frac{\pi}{2}}$ ✓

$= \frac{1}{16}\left[\left(-\frac{1}{5}\cos\frac{5\pi}{2} + \frac{5}{3}\cos\frac{3\pi}{2} - 10\cos\frac{\pi}{2}\right) - \left(-\frac{1}{5}\times 1 + \frac{5}{3}\times 1 - 10\times 1\right)\right]$

$= \frac{1}{16}\left[0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right]$

$= \frac{8}{15}$ ✓

d) let the consecutive numbers be $k-1$, k and $k+1$ for $k \in \mathbb{Z}$

$$\therefore \quad P = (k-1) \times k \times (k+1)$$
$$= k^3 - k$$

Case 1: k is even

$k = 2m$, $m \in \mathbb{Z}$

$$\therefore \quad P = (2m)^3 - 2m$$
$$= 8m^3 - 2m$$
$$= 2(4m^3 - m)$$
$$= 2P, \quad P \in \mathbb{Z}$$

$\therefore$ true if k is even

Case 2: k is odd

let $k = 2m+1$

$$\therefore \quad P = (2m+1)^3 - (2m+1)$$
$$= 8m^3 + 12m^2 + 6m + 1 - 2m - 1$$
$$= 2(4m^3 + 6m^2 + 2m)$$
$$= 2Q, \quad Q \in \mathbb{Z}$$

$\therefore$ true if k is odd

$\therefore$ the product of three consecutive numbers is even.

Q14) a) i) $P(-1, 2, 3)$
$Q(-2, 4, 2)$

$$\overrightarrow{PQ} = (-2--1)\underset{\sim}{i} + (4-2)\underset{\sim}{j} + (2-3)\underset{\sim}{k}$$
$$= -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$$

$$\underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$ ✓

ii) $$\cos\left(\cos^{-1}\frac{1}{6}\right) = \frac{-1 \times -1 + 2 \times b + -1 \times 2}{\sqrt{b^2+5} \times \sqrt{6}}$$

$$\frac{2b-1}{\sqrt{b^2+5} \times \sqrt{6}} = \frac{1}{6}$$ ✓

$$2b - 1 = \frac{1}{\sqrt{6}}\sqrt{b^2+5}$$

$$(2b-1)^2 = \frac{1}{6}(b^2+5)$$

$$24b^2 - 24b + 6 = b^2 + 5$$
$$23b^2 - 24b + 1 = 0$$
$$(b-1)(23b-1) = 0$$

$b = 1$, since $b = \frac{1}{23}$ does not produce an acute angle when substituting ✓

b) i) If $y > x$, then $y^3 + yx^2 > x^3 + xy^2$ ✓

ii) $$y > x$$
$$y(x^2+y^2) > x(x^2+y^2)$$
$$y^3 + yx^2 > x^3 + xy^2$$ ✓

$\therefore$ the contrapositive is true.

Hence, if $y \leq x$, then $y^3 + yx^2 \leq x^3 + xy^2$ ✓

c) i) $\overrightarrow{OA} = \begin{pmatrix} -p \\ 5+4p \\ 4+3p \end{pmatrix}$ $\overrightarrow{OB} = \begin{pmatrix} -2-q \\ 4+2q \\ 1+2q \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{pmatrix} -2-q+p \\ -1+2q-4p \\ -3+2q-3p \end{pmatrix}$ ✓

ii) $|\overrightarrow{AB}|$ is a minimum when $\overrightarrow{AB}$ is perpendicular to both lines.

$\overrightarrow{AB} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0$

$2+q-p-4+8q-16p-9+6q-9p = 0$
$-11+15q-26p = 0$ (1)

$\overrightarrow{AB} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$

$2+q-p-2+4q-8p-6+4q-6p = 0$
$-6+9q-15p = 0$
$3q = 5p+2$ (2)

sub (2) into (1), $-11+5(5p+2)-26p = 0$
$-11+25p+10-26p = 0$
$p = -1$ sub into (2)

$3q = -5+2$
$3q = -3$
$q = -1$ ✓

$\therefore \overrightarrow{AB} = \begin{pmatrix} -2-(-1)-1 \\ -1+2(-1)-4(-1) \\ -3+2(-1)-3(-1) \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$ ✓

$|\overrightarrow{AB}| = \sqrt{(-2)^2+1^2+(-2)^2}$
$= 3$ units ✓

d) For S_1, $x^2 - 2x + y^2 + 4y + z^2 - 4z = 0$

$$x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 - 4z + 4 = 9$$

$$(x-1)^2 + (y+2)^2 + (z-2)^2 = 9$$

centre $= (-1, -2, -2)$ radius $= 3$

For S_2, $x^2 + 10x + y^2 + z^2 + 2z = -10$

$$x^2 + 10x + 100 + y^2 + z^2 + 2z + 1 = 16$$

$$(x+5)^2 + y^2 + (z+1)^2 = 16$$

centre $= (-5, 0, -1)$ radius $= 4$ ✓

The distance the centres $= \sqrt{(-5-1)^2 + (-2)^2 + (-1-2)^2}$
$= 7$

The distance between the centres is equal to the sum of the radii. Therefore, the spheres intersect at a point ✓

e) let $u = e^x$
$du = e^x \, dx$

$$= \int \frac{1}{u^2 + 3u + 2} \, du$$

$$= \int \frac{1}{(u+2)(u+1)} \, du$$ ✓

$$\frac{1}{(u+2)(u+1)} = \frac{a}{u+2} + \frac{b}{u+1}$$

$$1 = a(u+1) + b(u+2)$$

let $u = -1$, $b = 1$; $u = -2$, $a = -1$

$$= \int \frac{-1}{u+2} + \frac{1}{u+1} \, du$$ ✓

$$= \ln|u+1| - \ln|u+2| + c$$

$$= \ln\left|\frac{e^x + 1}{e^x + 2}\right| + c$$ ✓

Q15) a) $\int (1+2x^2)e^{x^2}\,dx = \int e^{x^2}\,dx + \int 2x^2 e^{x^2}\,dx$

$= \int e^{x^2}\,dx + \int x \times 2xe^{x^2}\,dx$ ✓

$u = x \qquad v' = 2xe^{x^2}$
$u' = 1 \qquad v = e^{x^2}$

$= \int e^{x^2}\,dx + xe^{x^2} - \int e^{x^2}\,dx$

$= xe^{x^2} + c$ ✓

b) i) $|\sin(\alpha+\beta)| = |\sin\alpha\cos\beta + \cos\alpha\sin\beta|$
$\leq |\sin\alpha\cos\beta| + |\cos\alpha\sin\beta|$
$= |\sin\alpha||\cos\beta| + |\cos\alpha||\sin\beta|$
$\leq |\sin\alpha| + |\sin\beta| \qquad$ since $|\cos\theta| \leq 1$ ✓

ii) Equality holds if one of α or β is a multiple of π. ✓

c) i) $\ddot{x} = -k\dot{x}$

$\frac{d\dot{x}}{dt} = -k\dot{x}$

$\frac{dt}{d\dot{x}} = \frac{-1}{k\dot{x}}$

$t = \int \frac{-1}{k\dot{x}}\,dt$

$t = -\frac{1}{k}\ln(\dot{x}) + c$

$\dot{x} = u\cos\alpha$ when $t = 0$

$\therefore c = \frac{1}{k}\ln(u\cos\alpha)$ ✓

$\therefore t = -\frac{1}{k}\ln(\dot{x}) + \frac{1}{k}\ln(u\cos\alpha)$

$kt = \ln\left(\frac{u\cos\alpha}{\dot{x}}\right)$

$\frac{u\cos\alpha}{\dot{x}} = e^{kt}$

$\therefore \dot{x} = u\cos\alpha e^{-kt}$ ✓

ii) $\dot{y} = \frac{1}{k}\left[(ku\sin\alpha + g)e^{-kt} - g\right]$

$\ddot{y} = -k \times \frac{1}{k}(ku\sin\alpha + g)e^{-kt}$

$= -(ku\sin\alpha + g)e^{-kt}$ ✓

$-k\dot{y} - g = -k \times \frac{1}{k}\left[(ku\sin\alpha + g)e^{-kt} - g\right] - g$

$= -\left[(ku\sin\alpha + g)e^{-kt} - g\right] - g$

$= -(ku\sin\alpha + g)e^{-kt}$

$\therefore \ddot{y} = -k\dot{y} - g$ ✓

Hence $\dot{y} = \frac{1}{k}\left[(ku\sin\alpha + g)e^{-kt} - g\right]$ satisfies $\ddot{y} = -k\dot{y} - g$

iii) At max height, $\dot{y} = 0$

$\therefore \frac{1}{k}\left[(ku\sin\alpha + g)e^{-kt} - g\right] = 0$

$(ku\sin\alpha + g)e^{-kt} = g$

$e^{kt} = \frac{ku\sin\alpha + g}{g}$

$\therefore t = \frac{1}{k}\ln\left(\frac{ku\sin\alpha + g}{g}\right)$ ✓

iv) From (i), $\dot{x} = u\cos\alpha e^{-kt}$

$$x = \int u\cos\alpha e^{-kt}\,dt$$

$$= -\frac{u}{k}\cos\alpha e^{-kt} + C_2$$

$x = 0$ when $t = 0$,

$$\therefore C_2 = \frac{u}{k}\cos\alpha$$

$$\therefore x = \frac{u\cos\alpha}{k}\left(1 - e^{-kt}\right) ✓$$

As $t \to \infty$, $e^{-kt} \to 0$.

The limiting value of the horizontal displacement is $x = \dfrac{u\cos\alpha}{k}$ ✓

c) $\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP}$

$$\underset{\sim}{p} = ON + MN$$
$$= \underset{\sim}{n} + \underset{\sim}{n} - \underset{\sim}{m}$$
$$= 2\underset{\sim}{n} - \underset{\sim}{m}$$

$$\overrightarrow{OQ} = \overrightarrow{ON} + \overrightarrow{NQ}$$
$$\underset{\sim}{q} = ON + \lambda\overrightarrow{ON}$$
$$= (1+\lambda)\overrightarrow{ON}$$
$$= (1+\lambda)\underset{\sim}{n} ✓$$

$$\overrightarrow{QP} = \underset{\sim}{p} - \underset{\sim}{q}$$
$$= 2\underset{\sim}{n} - \underset{\sim}{m} - (1+\lambda)\underset{\sim}{n}$$
$$= (1-\lambda)\underset{\sim}{n} - \underset{\sim}{m} ✓$$

$$\overrightarrow{PQ} \cdot \overrightarrow{MN} = 0$$
$$(\underset{\sim}{m} - (1-\lambda)\underset{\sim}{n}) \cdot (\underset{\sim}{n} - \underset{\sim}{m}) = 0 ✓$$
$$\underset{\sim}{m}\cdot\underset{\sim}{n} - \underset{\sim}{m}\cdot\underset{\sim}{m} - (1-\lambda)\underset{\sim}{n}\cdot\underset{\sim}{n} + (1-\lambda)\underset{\sim}{n}\cdot\underset{\sim}{m} = 0$$
$$(\lambda-1)|\underset{\sim}{n}|^2 + (2-\lambda)\underset{\sim}{m}\cdot\underset{\sim}{n} - |\underset{\sim}{m}|^2 = 0$$
$$\lambda(|\underset{\sim}{n}|^2 - \underset{\sim}{m}\cdot\underset{\sim}{n}) + 2\underset{\sim}{m}\cdot\underset{\sim}{n} - |\underset{\sim}{m}|^2 - |\underset{\sim}{n}|^2 = 0$$

$$\lambda(|\underset{\sim}{n}|^2 - \underset{\sim}{m}\cdot\underset{\sim}{n}) = |\underset{\sim}{m}|^2 + |\underset{\sim}{n}|^2 - 2\underset{\sim}{m}\cdot\underset{\sim}{n}$$

$$\lambda = \frac{|\underset{\sim}{m}|^2 + |\underset{\sim}{n}|^2 - 2\underset{\sim}{m}\cdot\underset{\sim}{n}}{|\underset{\sim}{n}|^2 - \underset{\sim}{m}\cdot\underset{\sim}{n}} ✓$$

Q16) a) i) $\ddot{x} = g - kv$, $k > 0$

At terminal velocity, $\ddot{x} = 0$

$$\therefore \quad g - kv = 0$$
$$g = kv$$
$$v = \frac{g}{k}$$

$\therefore$ terminal velocity, $T = \frac{g}{k}$ ✓

ii) $\ddot{x} = g - kv$

$$v\frac{dv}{dx} = g - kv$$

$$\frac{dv}{dx} = \frac{g-kv}{v}$$

$$\frac{dx}{dv} = \frac{v}{g-kv}$$

$$= \frac{1}{k}\left(\frac{v}{\frac{g}{k} - v}\right)$$

$$= \frac{1}{k}\left(\frac{v}{T-v}\right) \qquad \text{since } T = \frac{g}{k}$$ ✓

$$x = \frac{-1}{k}\int \frac{T-v}{T-v} - \frac{T}{T-v} \, dv$$

$$x = \frac{-1}{k}\int 1 - \frac{T}{T-v} \, dv$$

$$= \frac{-1}{k}\left[v + T\ln(T-v)\right] + c$$

When $x = 0$, $v = 0$

$$\therefore \quad c = \frac{T\ln T}{k}$$

$$\therefore\ x = \frac{-1}{k}\left[v + T\ln(T-v) - T\ln T \right]$$

$$= \frac{-T}{k}\ln\left(\frac{T-v}{T}\right) - \frac{v}{k}$$

$$= \frac{T}{k}\ln\left(\frac{T}{T-v}\right) - \frac{v}{k}$$

When $v = \frac{T}{2}$, $x = \frac{T}{k}\ln 2 - \frac{T}{2k}$

$$k = \frac{g}{T}$$

$$\therefore\ x = \frac{T^2}{g}\ln 2 - \frac{T^2}{2g}$$

<u>Alternatively</u>

$$v\frac{dv}{dx} = g - kv$$

$$\frac{dx}{dv} = \frac{v}{g-kv}$$

$$\int_0^x dx = \int_0^{\frac{T}{2}} \frac{v}{g-kv}\, dv$$

$$x = \frac{-1}{k}\int_0^{\frac{T}{2}} \frac{g-kv-g}{g-kv}\, dv$$

$$x = \frac{-1}{k}\int_0^{\frac{T}{2}} dv + \frac{1}{k}\int_0^{\frac{T}{2}} \frac{g}{g-kv}\, dv$$

$$x = \frac{-1}{k}\int_0^{\frac{T}{2}} dv - \frac{g}{k^2}\int_0^{\frac{T}{2}} \frac{-k}{g-kv}\, dv$$

$$x = \frac{-1}{k}\times\frac{T}{2} - \frac{g}{k^2}\Big[\ln|g-kv|\Big]_0^{\frac{T}{2}}$$

$$x = \frac{-T}{2k} - \frac{g}{k^2}\ln\left|g - \frac{Tk}{2}\right| + \frac{g}{k^2}\ln g$$

$$x = \frac{-T}{2k} + \frac{g}{k^2} \ln \left| \frac{2g}{2g - Tk} \right|$$

Let $k = \frac{g}{T}$

$$\therefore \quad x = \frac{-T^2}{2g} + \frac{T^2}{g} \ln \left| \frac{2g}{2g - g} \right|$$

$$x = \frac{-T^2}{2g} + \frac{T^2}{g} \ln 2$$

b) $P(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)$

$\ln P(x) = \ln[(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)]$

$\ln P(x) = \ln(x - \alpha_1) + \ln(x - \alpha_2) + \ln(x - \alpha_3) + \ldots + \ln(x - \alpha_n)$

differentiating both sides with respect to x,

$$\frac{P'(x)}{P(x)} = \frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + \frac{1}{x - \alpha_3} + \cdots + \frac{1}{x - \alpha_n}$$

$$P'(x) = P(x) \left[\frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + \frac{1}{x - \alpha_3} + \cdots + \frac{1}{x - \alpha_n} \right]$$

c) i) test $n=1$

$$LHS = \sum_{r=1}^{1} \frac{1}{\sqrt{r}} \qquad\qquad RHS = 2(\sqrt{1+1} - 1)$$
$$= 1 \qquad\qquad = 2(\sqrt{2} - 1)$$
$$\doteqdot 0.828$$

$$1 > 0.828$$

$\therefore$ true for $n=1$.

Assume true for $n=k$,

$$\sum_{r=1}^{k} \frac{1}{\sqrt{r}} > 2(\sqrt{k+1} - 1)$$ where k is an in and > 0 ✓

Prove true for $n=k+1$,

i.e. prove $$\sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} > 2(\sqrt{k+2} - 1)$$

$$LHS = \sum_{r=1}^{k} \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{k+1}}$$

$$> 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}}$$ (from assumption) ✓

$$= 2\sqrt{k+1} - 2 + \frac{1}{\sqrt{k+1}}$$

$$= \frac{2\sqrt{k+1} \times \sqrt{k+1}}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} - 2$$

$$= \frac{2(k+1) + 1}{\sqrt{k+1}} - 2$$

$$= \frac{2k+3}{\sqrt{k+1}} - 2$$ ✓

$$> \frac{2\sqrt{(k+1)(k+2)}}{\sqrt{k+1}} - 2$$ (given) ✓

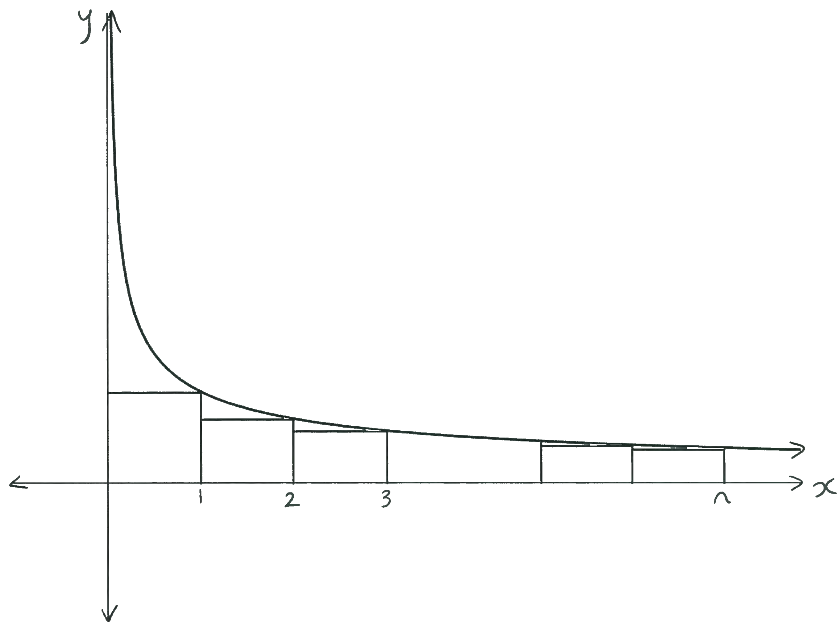
$$= \frac{2\sqrt{k+1} \times \sqrt{k+2}}{\sqrt{k+1}} - 2$$

$= 2\sqrt{k+2} - 2$

$\therefore$ true for $n=k+1$ ✓

$\therefore$ by induction, the result is true for all positive integers of n.

ii)



Exact area under the curve from 1 to n is $\int_1^n \frac{1}{\sqrt{x}}\,dx$

Area of rectangles under the curve in the same interval =

$$1 \times \frac{1}{\sqrt{2}} + 1 \times \frac{1}{\sqrt{3}} + 1 \times \frac{1}{\sqrt{4}} + \cdots + 1 \times \frac{1}{\sqrt{n}}$$

$$\therefore \quad \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}} < \int_1^n \frac{1}{\sqrt{x}}\,dx$$ ✓

$$\therefore \quad 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}} < \int_1^n \frac{1}{\sqrt{x}}\,dx + 1$$

$$\therefore \quad \sum_{r=1}^{n} \frac{1}{\sqrt{r}} < \int_1^n \frac{1}{\sqrt{x}}\,dx + 1$$ ✓

iii) From (i) and (ii),

$$2\left(\sqrt{n+1}-1\right) < \sum_{r=1}^{n} \frac{1}{\sqrt{r}} < \int_{1}^{n} \frac{1}{\sqrt{x}}\,dx + 1$$ ✓

let $n = 10000$,

$$2\left(\sqrt{10001}-1\right) < \sum_{r=1}^{10000} \frac{1}{\sqrt{r}} < \left[2\sqrt{x}\right]_{1}^{10000} + 1$$

$$198 < \sum_{r=1}^{10000} \frac{1}{\sqrt{r}} < 199$$ ✓