Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer question sheet for Questions 1-10.

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1 What are the centre and radius of the sphere with equation :
$x^2 + y^2 + z^2 - 2x + 6y + 4z - 2 = 0$?

A. Centre (1, –3, –2), Radius = 4

B. Centre (–1, 3, 2), Radius = 4

C. Centre (1, –3, –2), Radius = 16

D. Centre (–1, 3, –2), Radius = 16

2 Consider the following statement:

"$\forall\, n \in \mathbb{N}$, if n^2 is odd then n is odd."

Which of the following is the negation of the statement?

A. $\forall\, n \in \mathbb{N}$, if n^2 is not odd then n is also not odd.

B. $\exists\, n \in \mathbb{N}$, if n is not odd then n^2 is also not odd.

C. $\forall\, n \in \mathbb{N}$, if n is not odd then n^2 is not odd.

D. $\exists\, n \in \mathbb{N}$, if n^2 is not odd then n is also not odd.

3 If $\omega = \frac{1+2i}{3}$, then what is the modulus of ω?

A. $\frac{\sqrt{5}}{\sqrt{3}}$

B. $\frac{\sqrt{3}}{5}$

C. $\frac{\sqrt{5}}{3}$

D. $\frac{3}{\sqrt{5}}$

4 Evaluate : $\int_1^e \frac{(\ln x)^3}{x}\, dx$.

A. $\frac{1}{5}$

B. $\frac{1}{4}$

C. $\frac{1}{3}$

D. $\frac{1}{2}$

5 Consider the vector $\underset{\sim}{a} = 2\underset{\sim}{i} - b\underset{\sim}{j} + 6\underset{\sim}{k}$.

If $\underset{\sim}{a}$ has a magnitude of $4\sqrt{3}$ units, what is the exact value of b ?

A. $2\sqrt{2}$

B. $4\sqrt{2}$

C. 4

D. 8

6 A particle is moving in simple harmonic motion. The displacement x of the particle is given by $x = 1 + 4\sin\left(2\text{t} - \frac{\pi}{5}\right)$.

Which of the following is the first time the velocity of the particle is at a minimum?

A. $\frac{3\pi}{5}$

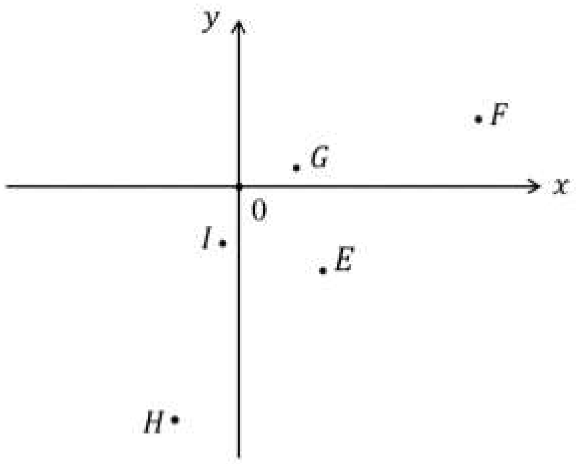
B. $\frac{\pi}{10}$

C. $\frac{7\pi}{10}$

D. $\frac{2\pi}{5}$

7 In the Argand diagram the point E represents a complex number.

When this number is multiplied by $\left(\sqrt{3}+i\right)^4\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i\right)^3$ it gave a new complex number.



Which of the following points represents the new complex number?

A. F

B. G

C. H

D. I

8 The equations of two lines ℓ_1 and ℓ_2 are

$$\vec{r_1}=\begin{pmatrix}-1+t\left(2^{\lambda}\right)\\ -5+t\left(6-2^{\lambda}\right)\\ -2+t\left(7-2^{\lambda}\right)\end{pmatrix} \text{ and } \vec{r_2}=\begin{pmatrix}1+t\left(8+2^{\lambda}\right)\\ 2+6t\\ -3+t\left(5+2^{\lambda}\right)\end{pmatrix},$$

where t is a real number and λ is a constant.

For what value of λ are the two lines ℓ_1 and ℓ_2 parallel?

A. 4

B. 3

C. 2

D. 1

9 A particle initially at rest at the origin starts to move along the x axis. Its velocity at any time t in seconds is $v\ m\ s^{-1}$.

The acceleration of the particle is given by $a = (1 + v^2)^{\frac{3}{2}}$.

Which of the following is the correct expression of the velocity v ?

A. $v = \tan\left(\cos^{-1} t\right)$

B. $v = \tan\left(\sin^{-1} t\right)$

C. $v = \sin^{-1}(\tan t)$

D. $v = \cos\left(\sin^{-1} t\right)$

10 The complex number z satisfies $|z + a| = a$, where a is a positive real number. The greatest distance that z can be from the point P representing the complex number $ka + a(k+1)i$, where k is a positive real number, is $(3\sqrt{2} + 1)a$.

What is the value of k ?

A. 2

B. 3

C. 1

D. 4

Section II

90 marks
Attempt Questions 11 - 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available

For each question in Section II, your responses should include relevant mathematical reasoning and/or calculations

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Question 11 (16 marks) Use a separate Writing Booklet

(a) Consider the complex numbers $\alpha = 2i$ and $\beta = \sqrt{3} + i$. **3**

Evaluate $(\alpha - \beta)^2$, leaving your answer in exponential form.

(b) Consider the line ℓ with equation $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$. **2**

Show that the point $A\ (2, -5, 3)$ lies on the line ℓ.

(c) Solve $z^2 - z + (4 - 2i) = 0$. Give your answers in the form $x + yi$, where x and y are real. **4**

(d) PQRS is a parallelogram with vertices $P\ (2, 4, 3)$, $Q\ (4, 3, 1)$, $R\ (3, 2, 5)$ and $S\ (x, y, z)$. **3**

Find the coordinates of S and the length of the diagonal $|\overrightarrow{QS}|$.

(e) A particle is moving along the x axis in simple harmonic motion centred at $x = 2$ and with amplitude 3. **4**
The velocity of the particle is given by $v^2 = p + qx - 6x^2$, where p and q are constants

Find the value of p and q, and the period of the motion.

End of Question 11

Question 12 (15 marks) Use a separate Writing Booklet

(a) Prove by counterexample, that the assertion $|2x+1| \le 5 \Rightarrow |x| \le 2$, is false. **2**

(b) Consider the statement P in the set of integers. **3**

"If $5n^4 + 8n$ is a multiple of 16, then n is even."

Prove that the converse of P is true.

(c) Find all the 4th roots of $1 + \sqrt{3}\,i$. Give your answer in exponential form. **3**

(d) The polynomial $Q(z) = z^4 - 8z^3 + pz^2 + qz - 80$ has root $3 + i$, where p and q are real numbers.

(i) Find all the roots of $Q(z)$. **3**

(ii) Write $Q(z)$ as a product of two real quadratic factors. **1**

(e) A particle starts from $x = ln\,(e^4 + 1)$ and moves with initial velocity e and acceleration given by $v^2\,(e^4 + \ln v)$, where v is the velocity of the particle. **3**

Find an expression for x, the displacement of the particle, in terms of v.

End of Question 12

Question 13 (15 marks) Use a separate Writing Booklet

(a) On the Argand diagram provided in the ***Question 13 Writing Booklet***, sketch the region represented by the intersection of the inequalities: **3**

$$1 \leq |z+1| < 2 \text{ and } \frac{\pi}{4} < \arg(z+1) < \frac{2\pi}{3}.$$

(b) Find $\int \frac{dx}{\sqrt{6-5x-x^2}}$. **3**

(c) Use contrapositive proof to prove that " If p is divisible by 5 then p can be expressed as a sum of five consecutive integers". **2**

(d) Find $\int \frac{(2\tan\theta + 3)\ \sec^2\theta}{\sec^2\theta + \tan\theta}\ d\theta$. **3**

(e) (i) Show $\frac{2k}{k+2} < \frac{2k+2}{k+3}$ for k > 0. **1**

(ii) Use mathematical induction to prove that **3**

$$\frac{1}{3!} + \frac{2}{4!} + \frac{3}{5!} + \cdots + \frac{n}{(n+2)!} < \frac{2n}{n+2} - \frac{1}{(n+2)!} \text{ for } n \geq 1$$

End of Question 13

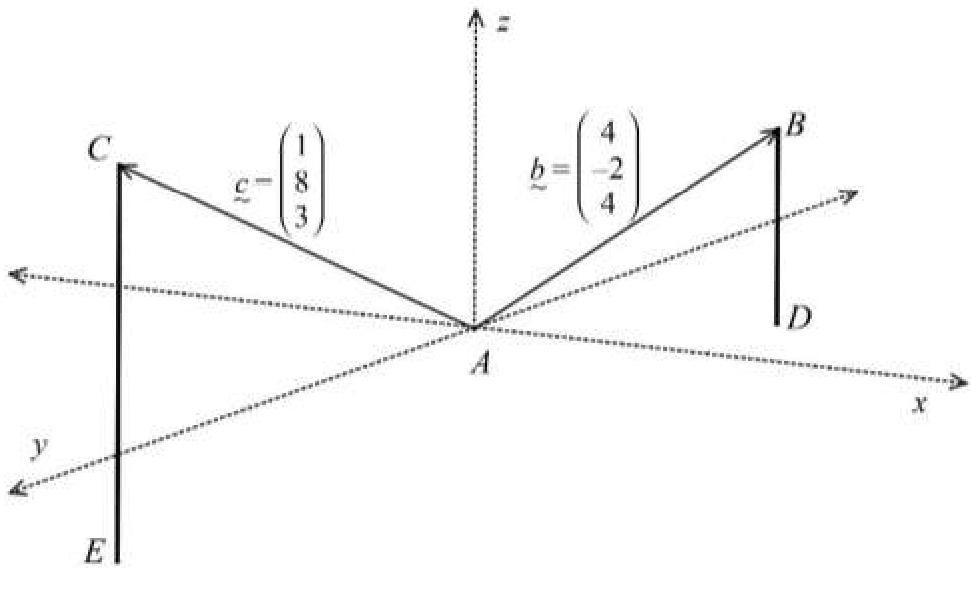
Question 14 (16 marks) Use a separate Writing Booklet

(a) Given that $z^n + \frac{1}{z^n} = 2\cos n\theta$, where $z = \cos\theta + i\sin\theta$. **3**

Express $\cos^6\theta$ in terms of $\cos n\theta$.

(b) Find the intersections of the line $\underset{\sim}{r} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + t\left(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$ with the sphere $(x-3)^2 + (y-2)^2 + (z+1)^2 = 36$. **3**

(c) Two vertical posts CE and BD stand on a horizontal plane ADE. **2**
Using A as the origin, the vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ represent the locations of the top of each post, B and C.



Show that $\Delta\, ABC$ is right angled at A.

Question 14 continues on page 9

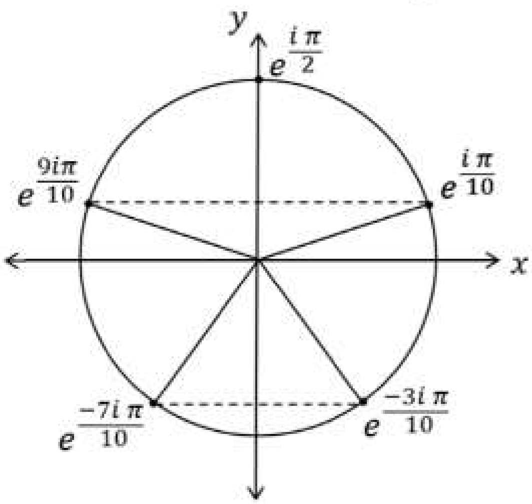
Question 14 (continued)

(d) (i) Using the AM-GM inequality, or otherwise, prove that, if $x > 1$, then $\dfrac{x}{\sqrt{x-1}} \geq 2$. **2**

(ii) Hence, or otherwise, prove that, for $a > 1, b > 1$ the following inequality holds **2**

$$\frac{a^2}{b-1} + \frac{b^2}{a-1} \geq 8$$

(e) The Argand diagram below shows the five roots of the equation $z^5 = i$ or $z^5 - i = 0$.



(i) Show that $e^{\frac{9i\pi}{10}} + e^{\frac{i\pi}{10}} + e^{\frac{-7i\pi}{10}} + e^{\frac{-3i\pi}{10}} = 2i\sin\frac{\pi}{10} - 2i\sin\frac{3\pi}{10}$ **1**

(ii) Hence, show that $\sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = \frac{1}{2}$. **1**

(iii) Hence, by using the identity $\sin(A+B) - \sin(A-B) = 2\cos A \sin B$, show that $\cos\frac{\pi}{5}\sin\frac{\pi}{10} = \frac{1}{4}$. **2**

End of Question 14

Question 15 (15 marks) Use a separate Writing Booklet

(a) Two lines, L_1 and L_2, are represented by position vectors r_1 and r_2 respectively. **4**

The vector equations of these two lines are

$$\underset{\sim}{r_1} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{r_2} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

where λ and μ are real numbers.

Show that L_1 and L_2 are neither parallel nor intersecting lines.

(b) (i) Show that if $\cos x \equiv A(3\cos x + 4\sin x) + B(4\cos x - 3\sin x)$, **1**

then $A = \frac{3}{25}$ and $B = \frac{4}{25}$.

(ii) Hence, determine $\int \frac{e^x \cos x}{\sqrt[3]{3\cos x + 4\sin x}}\, dx$ **4**

(c) The numbers a_n, for integers $n \geq 1$, are defined as $a_1 = 2$, $a_2 = 56$ and $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 3$. **3**

Use mathematical induction to prove that, for all integers $n \geq 1$,

$$a_n = 5(-2)^n + 4 \times 3^n.$$

(d) Use the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\sin x + 1}\, dx$ **3**

End of Question 15

Question 16 (13 marks) Use a separate Writing Booklet

(a) Let $I_n = \int_0^1 x^{2n}\sqrt{1-x^2}\,dx$ where $n = 0, 1, 2, \ldots$.

(i) State the value of I_0. **1**

(ii) Use integration by parts to show that for $n = 1, 2, 3, \ldots$ **3**

$$I_n = \frac{2n-1}{2n+2} I_{n-1}$$

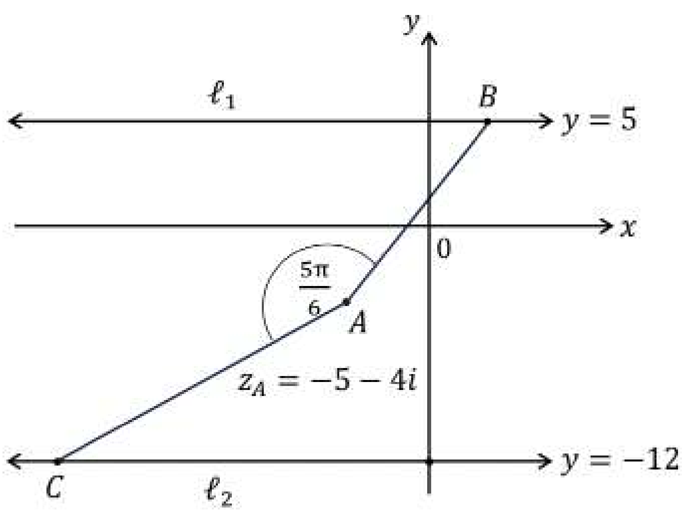
(iii) Hence, calculate $\int_0^1 x^6\sqrt{1-x^2}\,dx$. **2**

(b) Prove by contradiction that for $a > 6$; $\sqrt{2a+2} > \sqrt{a+8}$. **3**

(c) The point A, which represents the complex number $z_A = -5 - 4i$, lies between two parallel lines ℓ_1 and ℓ_2.

The equations of ℓ_1 and ℓ_2 are $y = 5$ and $y = -12$ respectively.

The point B, which represents the complex number z_B, lies on ℓ_1. The point C, which represents the complex number z_C, lies on ℓ_2 such that $AC = 2AB$ and $\angle BAC = \frac{5\pi}{6}$.



Find the exact value of the complex number z_C. **4**

End of paper

2024 Year 12 Extension 2 Mathematics Assessment Task 4
Marking Guidelines

Section I (1 mark each)

Multiple-choice Answer Key

Question	Answer
1	A
~~2~~	~~D~~
3	C
4	B
5	A
6	A
7	C
8	C
9	B
10	A

Question 1

$x^2 + y^2 + z^2 - 2x + 6y + 4z - 2 = 0$

$x^2 - 2x + y^2 + 6y + z^2 + 4z = 2$

$x^2 - 2x + 1 + y^2 + 6y + 9 + z^2 + 4z + 4 = 2 + 1 + 9 + 4$

$(x - 1)^2 + (y + 3)^2 + (z + 2)^2 = 16$

Centre $(1, -3, -2)$

Radius = 4

Answer: A

Question 2

For a statement "$\forall\, n \in \mathbb{N}$, if P then Q" its negation statement is " $\exists\, n \in \mathbb{N}$ if not P then not Q".

So, as the statement is "$\forall\, n \in \mathbb{N}$, if n^2 is odd then n is odd ", the negation statement would be "$\exists\, n \in \mathbb{N}$, if n^2 is not odd then n is also not odd."

Answer: D

Question 3

$\omega = \frac{1+2i}{3} = \frac{1}{3} + \frac{2i}{3}$

$$|\omega| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$$

$$= \sqrt{\frac{5}{9}}$$

$$= \frac{\sqrt{5}}{3}$$

Answer: C

Question 4

Let $u = \log_e x$

$\frac{du}{dx} = \frac{1}{x}, \therefore du = \frac{1}{x}\, dx$

when $x = e, u = \log_e e = 1$

when $x = 1, u = \log_e 1 = 0$

$$\int_1^e \frac{(\log_e x)^3}{x}\, dx = \int_0^1 u^3\, du$$

$$= \left[\frac{1}{4}u^4\right]_0^1$$

$$= \frac{1}{4}[1^4 - 0^4]$$

$$= \frac{1}{4}$$

Answer: B

Question 5

$$|\underset{\sim}{a}| = \sqrt{2^2 + (-b)^2 + 6^2} = 4\sqrt{3}$$

$$4 + b^2 + 36 = 48$$

$$b^2 + 40 = 48$$

$$b^2 = 8$$

$$b = \sqrt{8} = 2\sqrt{2}$$

Answer: A

Question 6

As $x = 1 + 4\sin\left(2\text{t} - \frac{\pi}{5}\right)$ then

$v = 8\,cos\left(2\text{t} - \frac{\pi}{5}\right) = 8\cos 2\left(\text{t} - \frac{\pi}{10}\right)$

A cos curve without any horizontal translation when $\text{t} = 0$ it has maximum and its first minimum will be at $\text{t} = \frac{1}{2}T$, where T is the period. As this curve is translated by $\frac{\pi}{10}$ to the right then the first maximum is at $t = \frac{\pi}{10}$ and as its period is $T = \frac{2\pi}{2} = \pi$, then the first minimum is at

$t = \frac{\pi}{10} + \frac{1}{2} \times \pi = \frac{3\pi}{5}$.

Answer: A

Question 7

$\sqrt{3}+i \ = 2\, cis\, \frac{\pi}{6}$ this means

$\left(\sqrt{3}+i\right)^4 = (2\,)^4 cis\, \dfrac{4\,\pi}{6} = 16\, cis\, \dfrac{2\,\pi}{3}$

$\frac{1}{4} - \frac{\sqrt{3}}{4}\, i = \sqrt{\left(\frac{1}{4}\right)^2 + \left(-\frac{\sqrt{3}}{4}\right)^2}\ cis\left(-\frac{\pi}{3}\right)$

$= \frac{1}{2} cis\left(-\frac{\pi}{3}\right)$ this means

$\left(\dfrac{1}{4} - \dfrac{\sqrt{3}}{4}\, i\ \right)^3 = \left(\dfrac{1}{2}\right)^3 cis\left(3 \times -\dfrac{\pi}{3}\right) = \dfrac{1}{8}\ cis(-\pi)$

Hence, $\left(\sqrt{3}+i\ \right)^4 \left(\frac{1}{4} - \frac{\sqrt{3}}{4}\, i\ \right)^3$

$16\, cis\, \frac{2\,\pi}{3} \times \frac{1}{8}\ cis(-\pi)\ \ = 2\, cis\, \left(-\frac{\pi}{3}\right)$

So, the new complex number will be twice as far from the origin as E. Also, the new complex number will have an argument which is rotated $-\frac{\pi}{3}\ radians$

from E. The point H is in this position.

Answer: C

Question 8

ℓ_1 and ℓ_2 are parallel if the components of their directional vectors are in the same ratio.

This means $\begin{pmatrix} 2^{\lambda} \\ 6-2^{\lambda} \\ 7-2^{\lambda} \end{pmatrix} = k \begin{pmatrix} 8+2^{\lambda} \\ 6 \\ 5+2^{\lambda} \end{pmatrix}$

So, $2^{\lambda} = k\left(8+2^{\lambda}\right)$ that is $k = \dfrac{2^{\lambda}}{8+2^{\lambda}}$ (1)

Also, $6-2^{\lambda} = 6k$ that is $k = \dfrac{6-2^{\lambda}}{6}$ (2)

From (1) and (2), we get

$$\frac{2^{\lambda}}{8+2^{\lambda}} = \frac{6-2^{\lambda}}{6}$$

$6 \times 2^{\lambda} = 48 - 8 \times 2^{\lambda} + 6 \times 2^{\lambda} - 2^{2\lambda}$
$2^{2\lambda} + 8 \times 2^{\lambda} - 48 = 0$
$(2^{\lambda} + 12)(2^{\lambda} - 4) = 0$
$2^{\lambda} = -12$ or $2^{\lambda} = 4$
Note: $2^{\lambda} = -12$ is invalid this indicates that $2^{\lambda} = 4$,
that is $\lambda = 2$ is the only solution.
Note: To check solutions, we substitute $\lambda = 2$ into
the directional vectors of ℓ_1 and ℓ_2, we get

$\begin{pmatrix} 2^{\lambda} \\ 6-2^{\lambda} \\ 7-2^{\lambda} \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$ and

$\begin{pmatrix} 8+2^{\lambda} \\ 6 \\ 5+2^{\lambda} \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$.

This indicates that ℓ_1 and ℓ_2 are parallel.

Answer: C

Question 9

$$\frac{dv}{dt} = (1 + v^2)^{\frac{3}{2}}$$

$$\frac{dt}{dv} = \frac{1}{(1 + v^2)^{\frac{3}{2}}}$$

$$\int dt = \int \frac{1}{(1 + v^2)^{\frac{3}{2}}} dv$$

$$t = \int \frac{1}{(1 + v^2)^{\frac{3}{2}}} dv$$

Let $v = tan\,\theta$, so $dv = sec^2\theta\; d\theta$
and $1 + v^2 = 1 + tan^2\theta = sec^2\theta$

$$t = \int \frac{1}{(sec^2\theta)^{\frac{3}{2}}} sec^2\theta\, d\theta$$

$$t = \int \frac{1}{sec^3\theta} sec^2\theta\, d\theta$$

$$t = \int \frac{1}{sec\,\theta} d\theta$$

$$t = \int cos\,\theta\; d\theta$$

$$t = sin\,\theta + c$$

So, $t = \sin(\tan^{-1} v) + c$

When $t = 0, v = 0$

so $0 = \sin(\tan^{-1} 0) + c$, that is

$c = 0$. This indicates that

$$t = \sin(\tan^{-1} v)$$

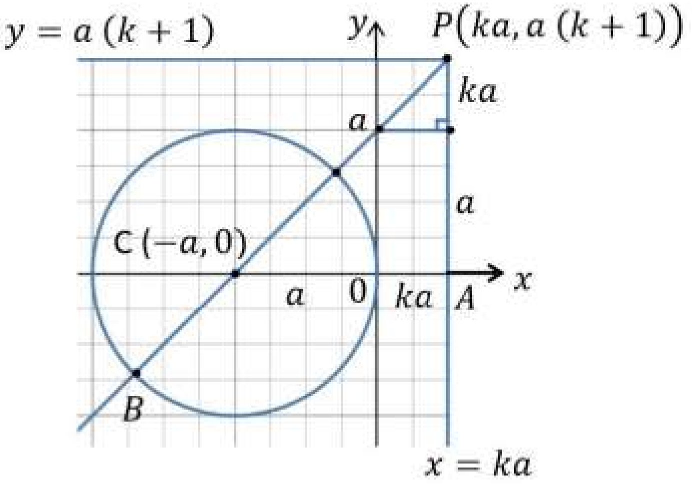
$$\sin^{-1} t = \tan^{-1} v$$

$$v = \tan(\sin^{-1} t)$$

Answer: B

Question 10

As the complex number z satisfies $|\, z + a \,| = a$,
then it lies on a circle with centre $C(-a, 0)$ and
radius a. The diagram shows the circle with centre C and the point P.



From the diagram above we can see that PB is the greatest distance that z can be from the point P.
Using Pythagoras' theorem in triangle APC, we get

$PC = \sqrt{a^2\,(k+1)^2 + a^2\,(k+1)^2}$

$PC = \sqrt{2\,a^2\,(k+1)^2}$, that is $PC = a\,(k+1)\sqrt{2}$.

So, $PB = PC + CB = a\,(k+1)\sqrt{2} + a$

But $PB = \left(3\sqrt{2} + 1\right) a$, this means

$a\,(k+1)\sqrt{2} + a = \left(3\sqrt{2} + 1\right) a$

Dividing by a, which is a positive real number, we get

$(k+1)\sqrt{2} + 1 = 3\sqrt{2} + 1$

$(k+1)\sqrt{2} = 3\sqrt{2}$ that is $k + 1 = 3$

Therefore, $k = 2$.

Answer: A

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution in exponential form.	3
• Finds correct argument or modulus of $(\alpha-\beta)^2$	2
• Finds correct expression for $(\alpha-\beta)^2$ in $x+iy$ form.	1

Sample answer:

$$(\alpha-\beta)^2 = 2-2i\sqrt{3}$$
$$r=\sqrt{4+12}=4$$
$$\theta=-\frac{\pi}{3}$$
$$(\alpha-\beta)^2=4e^{-\frac{\pi}{3}i}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Justify that the point $A\,(2,-5,3)$ lies on the line ℓ.	1

Sample answer:

To show that $A\,(2,-5,3)$ lies on the line ℓ we substitute its coordinates into the equation of ℓ.
We get

$$\begin{pmatrix}2\\-5\\3\end{pmatrix}=\begin{pmatrix}3\\-2\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\6\\4\end{pmatrix}$$

Now, we need to verify that there is only one value for λ in the solutions of the following three equations.

$$\begin{array}{lll} 2=3+2\lambda \text{ and } & -5=-2+6\lambda \text{ and } & 3=5+4\lambda \\ -1=2\lambda & -3=6\lambda & -2=4\lambda \\ \lambda=-\frac{1}{2} & \lambda=-\frac{1}{2} & \lambda=-\frac{1}{2} \end{array}$$

As $\lambda=-\frac{1}{2}$ is the common solution for the above equations then A lies on the line ℓ.

Question 11 (c)

Criteria	Marks
• Provides correct solutions	4
• Substitutes $\pm(1+4i)$ back into the equation, however, makes minor errors in solution.	3
• Indicate that $\sqrt{-15+8i} = \pm(1+4i)$.	2
• Indicate that $z = \frac{1 \pm \sqrt{-15+8i}}{2}$.	1

Sample answer:

We can solve $z^2 - z + (4-2i) = 0$

Using the quadratic formula.

$$z = \frac{1 \pm \sqrt{1-4(4-2i)}}{2}$$

$$z = \frac{1 \pm \sqrt{1-16+8i}}{2}$$

$$z = \frac{1 \pm \sqrt{-15+8i}}{2} \quad (1)$$

Now, let $(a+ib)^2 = -15+8i$

$a^2 - b^2 + 2ab\,i = -15+8i$

Equating reals, we get

$a^2 - b^2 = -15 \quad (2)$

Equating imaginaries, we get

$2ab = 8 \quad (3)$

Equating moduli, we get

$a^2 + b^2 = 17 \quad (4)$

(2) + (4) gives

$2a^2 = 2$, that is $a^2 = 1$.

So, $a = \pm 1$.

Now, when $a = 1, b = 4$ and

when $a = -1, b = -4$,

So, $\sqrt{-15+8i} = \pm(1+4i) \quad (5)$

By substituting (5) in (1), we get

$$z = \frac{1 \pm (1+4i)}{2}$$

$$z = \frac{1+(1+4i)}{2} \text{ or } z = \frac{1-(1+4i)}{2}$$

$$z = \frac{2+4i}{2} \text{ or } z = \frac{-4i}{2}$$

$z = 1+2i$ or $z = -2i$.

Question 11 (d)

Criteria	Marks
• Provides correct solution	3
• Indicate that $S = (1, 3, 7)$	2
• Indicate that $\begin{pmatrix} x-3 \\ y-2 \\ z-5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ or equivalent expression.	1

Sample answer:

As PQRS is a parallelogram then $\overrightarrow{QP} = \overrightarrow{RS}$.

$$\overrightarrow{QP} = \begin{pmatrix} 2-4 \\ 4-3 \\ 3-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$\overrightarrow{RS} = \begin{pmatrix} x-3 \\ y-2 \\ z-5 \end{pmatrix}$$

Now, we equate the components of the two vectors, we get

$x - 3 = -2$ and $y - 2 = 1$ and $z - 5 = 2$
$x = 1$ $\quad y = 3$ $\quad z = 7$
So, $S\ (1, 3, 7)$.

Also, $\overrightarrow{QS} = \begin{pmatrix} 1-4 \\ 3-3 \\ 7-1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix}$

$|\overrightarrow{QS}| = \sqrt{(-3)^2 + 0^2 + 6^2} = 3\sqrt{5}$ units.

Question 11 (e)

Criteria	Marks
• Provides correct solution	4
• Indicate $\frac{1}{2}v^2 = 15 + 12x - 3x^2$.	3
• Indicate $0 = p + 5q - 150$ and $0 = p - q - 6$.	2
• Indicate that $v = 0$ when $x = 5$ and when $x = -1$.	1

Sample answer:

As the motion is simple harmonic motion with centre at $x = 2$ and amplitude 3 this indicates that $v = 0$ when $x = 5$ and when $x = -1$.

By substituting into $v^2 = p + qx - 6x^2$, we get
$0 = p + 5q - 150$ (1) and $0 = p - q - 6$ (2)
By subtracting (2) from (1), we get
$0 = 6q - 144$, that is $q = 24$.
By substituting into (2), we get
$0 = p - 24 - 6$, that is $p = 30$.
So, $v^2 = 30 + 24x - 6x^2$

$\frac{1}{2}v^2 = 15 + 12x - 3x^2$

$a = \frac{\mathrm{d}\left(\frac{1}{2}v^2\right)}{\mathrm{d}x} = 12 - 6x$

$a = -6(x - 2)$ this means

$n^2 = 6$, that is $n = \sqrt{6}$ as $n > 0$.

Hence, the period is $T = \frac{2\pi}{\sqrt{6}} = \frac{\pi\sqrt{6}}{3}$.

Question 12 (a)

Criteria	Marks
• Provides correct solution	2
• Makes some progress	1

Sample answer:

Consider $x = -3$:

then $|2(-3)+1| = 5$ but $|x| > 2$

Hence the assertion is false.

Question 12 (b)

Criteria	Marks
• Prove that $5n^4 + 8n = 16\,(5m^4 + m)$ which is s divisible by 16.	3
• Let $n = 2m$, show that $5n^4 + 8n = 5(2m)^4 + 8 \times 2m$ or similar progress.	2
• Indicate that the converse of *P* would be " If n is even, then $5n^4 + 8n$ will be a multiple of 16."	1

Sample answer:

The converse of *P* would be " If n is even, then
$5n^4 + 8n$ will be a multiple of 16."

Proof: As n is even, let $n = 2m$, where m is a positive integer.

So $5n^4 + 8n = 5(2m)^4 + 8 \times 2m$

$= 5 \times 16\, m^4 + 16m$

$= 16\,(5m^4 + m)$

Let $5m^4 + m = k$, where k is a positive integer.

So, $5n^4 + 8n = 16k$

Hence, the converse of *P* is true.

Question 12 (c)

Criteria	Marks
• State that the roots are $2^{\frac{1}{4}} e^{\frac{i\pi}{12}},\ 2^{\frac{1}{4}} e^{\frac{i7\pi}{12}},\ 2^{\frac{1}{4}} e^{\frac{i13\pi}{12}} = 2^{\frac{1}{4}} e^{\frac{-i11\pi}{12}}$ $or\ 2^{\frac{1}{4}} e^{\frac{i19\pi}{12}} = 2^{\frac{1}{4}} e^{\frac{-i5\pi}{12}}$.	3
• Indicate equating $r^4 e^{4i\theta} = 2e^{\frac{i\pi}{3}}$.	2
• Indicate that $1 + \sqrt{3}i = 2e^{\frac{i\pi}{3}}$.	1

Sample answer:

Let $z = re^{i\theta}$ be a fourth root of $1 + \sqrt{3}i$.

This means $z^4 = 1 + \sqrt{3}i$.

Now, as $1 + \sqrt{3}i = 2e^{\frac{i\pi}{3}}$ then

$z^4 = 1 + \sqrt{3}\,i$ can be written as $r^4 e^{4i\theta} = 2e^{\frac{i\pi}{3}}$.

This indicates that $r^4 = 2$, that is $r = 2^{\frac{1}{4}}\ as\ r > 0$,

and $cos\ 4\theta = cos\ \frac{\pi}{3}$.

So, $4\theta = \frac{\pi}{3} + 2n\pi$, that is $\theta = \frac{\pi}{12} + \frac{2n\pi}{4}$.

Therefore, $\theta = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}\ or\ \frac{19\pi}{12}$.

Hence, the four roots of $z^4 = 1 + \sqrt{3}\,i$ are

$2^{\frac{1}{4}} e^{\frac{i\pi}{12}},\ 2^{\frac{1}{4}} e^{\frac{i7\pi}{12}},\ 2^{\frac{1}{4}} e^{\frac{i13\pi}{12}} = 2^{\frac{1}{4}} e^{\frac{-i11\pi}{12}}$

$or\ 2^{\frac{1}{4}} e^{\frac{i19\pi}{12}} = 2^{\frac{1}{4}} e^{\frac{-i5\pi}{12}}$.

Question 12 (d)(i)

Criteria	Marks
• Provides correct solutions	3
• Indicate $\alpha\beta = -8$.	2
• State the 4 roots are $3+i, 3-i, \alpha, \beta$. and indicate that $\alpha + \beta = 2$, or similar progress.	1

Sample answer:

$Q(z) = z^4 - 8z^3 + pz^2 + qz - 80$

As the coefficients of this polynomial are real, the conjugate of the root $3+i$, which is $3-i$, is also a root.

Now, let the roots be $3+i, 3-i, \alpha, \beta$.

The sum of the roots is

$3+i+3-i + \alpha + \beta = 8$ that is $\alpha + \beta = 2$.

The product of the roots is

$\alpha\,\beta\,(3+i)(3-i) = -80$

$10\,\alpha\,\beta = -80$, this means $\alpha\,\beta = -8$.

The quadratic equation with α and β is

$z^2 - 2z - 8 = 0$

$(z-4)(z+2) = 0$. So, $z = 4 \; or -2$.

Hence, the four roots are $3+i,\ 3-i,\ 4, -2$.

Question 12 (d)(ii)

Criteria	Marks
• Provides correct solution.	1

Sample answer:

Using part (i), we can write

$$\begin{aligned}Q(z) &= (z-3-i)(z-3+i)(z-4)(z+2)\\ &= ((z-3)^2 - i^2)(z^2-2z-8)\\ &= (z^2-6z+9+1)(z^2-2z-8)\\ &= (z^2-6z+10)(z^2-2z-8).\end{aligned}$$

Question 12 (e)

Criteria	Marks
• Provides correct solution	3
• Indicate that $u = e^4 + ln\, v$, so $x = \int \frac{1}{u}\, du.$	2
• Indicate that $\frac{dx}{dv} = \frac{1}{v\,(e^4 + \ln v)}$.	1

Sample answer:

Note: the initial velocity is e which is positive.
So, the particle will start moving in the positive direction and as its acceleration a at the that time is $e^2(e^4 + \ln e) = e^2(e^4 + 1)$,which is also positive, then the velocity will increase and a will always be positive.

$a = v\frac{dv}{dx} = v^2\,(e^4 + \ln v)$, that is

$\frac{dv}{dx} = v\,(e^4 + \ln v)$, so $\frac{dx}{dv} = \frac{1}{v\,(e^4 + \ln v)}$.

Integrating both sides, we get

$$\int dx = \int \frac{1}{v\ (e^4\ +\ ln\, v)}\ dv$$

$$x = \int \frac{1}{v\ (e^4\ +\ ln\, v)}\ dv$$

Let $u = e^4 + ln\, v$ this indicates that $du = \frac{1}{v}dv$

$x = \int \frac{1}{u}\ du = ln\ u\ + c$ this means

$x = ln\,(e^4\ +\ ln\, v)\ + c.$

When $t = 0,\ x = ln\,(\,e^4\ +\ 1)$ and $v = e.$

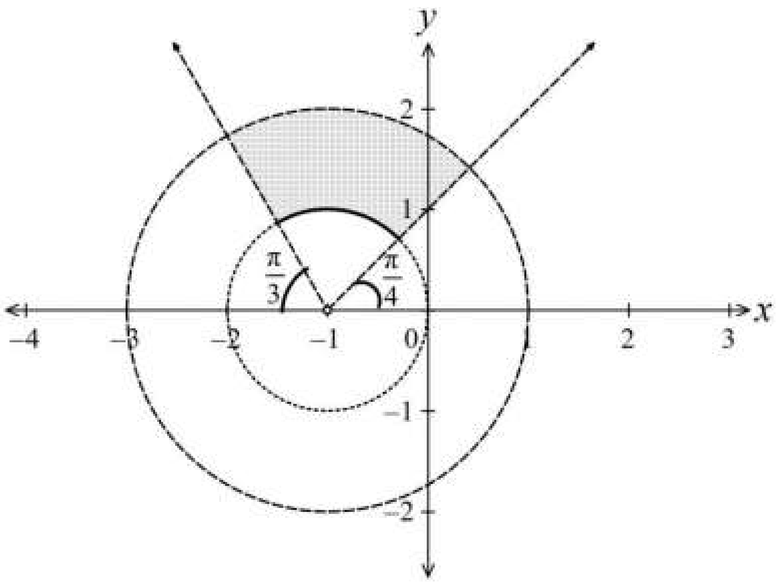
This indicates $ln\,(\,e^4\ +\ 1) = ln\,(\,e^4\ +\ ln\, e\,) + c$ So, $c = 0.$

Hence, $x = ln\,(e^4 + ln\, v).$

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Graph with minor error such as incorrect line marking or shading	2
• One correct graph	1

Sample answer:



Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Completing the square and forming the integral or equivalent merit	2
• Completing the square and attempt to form integral or equivalent merit	1

Sample answer:

$$6-5x-x^2=6-(x^2+5x)$$

$$=6-\left(x^2+5x+\tfrac{25}{4}\right)+\tfrac{25}{4}$$

$$=\tfrac{49}{4}-\left(x+\tfrac{5}{2}\right)^2$$

$$\int\frac{dx}{\sqrt{6-5x-x^2}}=\int\frac{dx}{\sqrt{\frac{49}{4}-\left(x+\frac{5}{2}\right)^2}}$$

$$=\sin^{-1}\frac{\left(x+\frac{5}{2}\right)}{\frac{7}{2}}+c$$

$$=\sin^{-1}\left[\frac{2\left(x+\frac{5}{2}\right)}{7}\right]+c$$

$$=\sin^{-1}\left(\frac{2x+5}{7}\right)+c$$

Question 13 (c)

Criteria	Marks
• Use contrapositive proof to prove that the statement is true.	2
• Indicate that there are no integer q such that $p=q+(q+1)+(q+2)+(q+3)+(q+4)$.	1

Sample answer:

To prove the given statement by contrapositive we need to prove "If p can not be expressed as a sum of five consecutive integers then p is not divisible by 5".

Let us assume that p can not be expressed as a sum of five consecutive integers, this indicates there are no integer q such that $p=q+(q+1)+(q+2)+(q+3)+(q+4)$.
This means that $p\neq 5q+10$, that is $p\neq 5(q+2)$. So, $p\neq 5r$, where $r=q+2$ is an integer and this shows that p is not divisible by 5.
Hence, by contrapositive the statement is true.

Question 13 (d)

Criteria	Marks
• Provides correct solution	3
• Indicate that $I = \ln(1 + u + u^2) + \int \frac{2}{\left(u + \frac{1}{2}\right)^2 + \frac{3}{4}}\, du$ or similar progress.	2
• Indicate the use of the substitution $u = \tan\theta$ so $du = \sec^2\theta\, d\theta$. Hence, $I = \int \frac{2u + 3}{1 + u + u^2} du$.	1

Sample answer:

$$I = \int \frac{(2\tan\theta + 3)\sec^2\theta}{\sec^2\theta + \tan\theta}\, d\theta$$

$$I = \int \frac{(2\tan\theta + 3)\sec^2\theta}{1 + \tan^2\theta + \tan\theta}\, d\theta$$

Let $u = \tan\theta$ so $du = \sec^2\theta\, d\theta$

So, $I = \int \frac{2u + 3}{1 + u + u^2} du = \int \frac{2u + 1 + 2}{1 + u + u^2} du$

$$= \int \frac{2u + 1}{1 + u + u^2} du + \int \frac{2}{1 + u + u^2} du$$

$$= \ln|1 + u + u^2| + \int \frac{2}{1 + u + u^2} du$$

Note: $1 + u + u^2 = \left(u + \frac{1}{2}\right)^2 + \frac{3}{4}$ which is positive.

$$I = \ln(1 + u + u^2) + \int \frac{2}{\left(u + \frac{1}{2}\right)^2 + \frac{3}{4}} du$$

$$I = \ln(1 + u + u^2) + \frac{4}{\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}}\left(u + \frac{1}{2}\right) + k$$

$$I = \ln(1 + \tan\theta + \tan^2\theta) + \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan\theta + 1}{\sqrt{3}}\right) + k$$

Question 13 (e)(i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Consider $\dfrac{2k}{k+2} - \dfrac{2k+2}{k+3}$

$= \dfrac{2k}{k+2} \times \dfrac{k+3}{k+3} - \dfrac{2k+2}{k+3} \times \dfrac{k+2}{k+2}$

$= \dfrac{2k^2+6k-(2k^2+6k+4)}{(k+2)(k+3)}$

$= \dfrac{-4}{(k+2)(k+3)} < 0$ as $k > 0$.

So, $\dfrac{2k}{k+2} - \dfrac{2k+2}{k+3} < 0$

Hence, $\dfrac{2k}{k+2} < \dfrac{2k+2}{k+3}$ for $k > 0$.

Question 13 (e)(ii)

Criteria	Marks
• Satisfactorily use Mathematical Induction to prove the statement	3
• Indicate $\frac{1}{3!} + \cdots + \frac{k}{(k+2)!} + \frac{k+1}{(k+3)!} < \frac{2k}{k+2} - \frac{1}{(k+2)!} + \frac{k+1}{(k+3)!}$ or similar progress.	2
• Indicate an attempt to prove $\frac{1}{3!} + \frac{2}{4!} + \cdots + \frac{k}{(k+2)!} + \frac{k+1}{(k+3)!} < \frac{2k+2}{k+3} - \frac{1}{(k+3)!}$	1

Sample answer:

$\dfrac{1}{3!} + \dfrac{2}{4!} + \cdots + \dfrac{n}{(n+2)!} < \dfrac{2n}{n+2} - \dfrac{1}{(n+2)!}$ for $n \geq 1$

For $n = 1$

LHS = $\dfrac{1}{3!} = \dfrac{1}{6}$ RHS = $\dfrac{2}{3} - \dfrac{1}{3!} = \dfrac{1}{2}$

As $\dfrac{1}{6} < \dfrac{1}{2}$, the statement is true for n = 1.

Assume the statement is true for $n = k$, that is

$\dfrac{1}{3!} + \dfrac{2}{4!} + \cdots + \dfrac{k}{(k+2)!} < \dfrac{2k}{k+2} - \dfrac{1}{(k+2)!}$ (1)

Our aim is to prove the statement is true for $n = k+1$, that is

$\dfrac{1}{3!} + \dfrac{2}{4!} + \cdots + \dfrac{k}{(k+2)!} + \dfrac{k+1}{(k+3)!} < \dfrac{2k+2}{k+3} - \dfrac{1}{(k+3)!}$

Starting from the assumption (1) and adding $\dfrac{k+1}{(k+3)!}$ on both sides, we get:

$$\frac{1}{3!}+\frac{2}{4!}+\cdots+\frac{k}{(k+2)!}+\frac{k+1}{(k+3)!}<\frac{2k}{k+2}-\frac{1}{(k+2)!}+\frac{k+1}{(k+3)!}$$

$$<\frac{2k}{k+2}-\frac{1(k+3)}{(k+3)!}+\frac{k+1}{(k+3)!}$$

$$<\frac{2k}{k+2}-\frac{(k+3)-(k+1)}{(k+3)!}$$

$$<\frac{2k}{k+2}-\frac{2}{(k+3)!}$$

$$<\frac{2k}{k+2}-\frac{1}{(k+3)!}$$

Note: The above inequality is valid as we are subtracting a smaller term.

Now, from part i) $\frac{2k}{k+2}<\frac{2k+2}{k+3}$, by subtracting $\frac{1}{(k+3)!}$ on both sides, we get:

$$\frac{2k}{k+2}-\frac{1}{(k+3)!}<\frac{2k+2}{k+3}-\frac{1}{(k+3)!}$$

Now, from the above, we can state that

$$\frac{1}{3!}+\frac{2}{4!}+\cdots+\frac{k}{(k+2)!}+\frac{k+1}{(k+3)!}<\frac{2k+2}{k+3}-\frac{1}{(k+3)!}$$

Hence, if the statement is true for $n=k$, it is also true for $n=k+1$.

The statement is true for $n=1$, and by mathematical induction it is also true for `
$n=2,3,4$ and so on.
Hence, the statement is true for all integers $n\geq 1$.

Question 14 (a)

Criteria	Marks
• Correct expression for $\cos^6\theta$ in terms of $\cos n\theta$.	3
• Correctly applies $z^n + \frac{1}{z^n} = 2\cos n\theta$ to the expanded form.	2
• Expands $(z + \frac{1}{z})^6$ correctly using Binomial expansion.	1

Sample answer:

$$(z + \frac{1}{z})^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}.$$

$$64\cos^6\theta = z^6 + \frac{1}{z^6} + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$$64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$$

$$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

Question 14 (b)

Criteria	Marks
• Provides correct solutions	3
• Solution with a minor error or equivalent merit	2
• Obtaining parametric equations or equivalent merit	1

Sample answer:

Parametric Equations are:

$x = 3 + 2t$

$y = 2 - t$

$z = -1 - 2t$

$(3 + 2t - 3)^2 + (2 - t - 2)^2 + (-1 - 2t + 1)^2 = 36$

$(2t)^2 + (-t)^2 + (-2t)^2 = 36$

$4t^2 + t^2 + 4t^2 = 36$

$9t^2 = 36$

$t^2 = 4$

$t = \pm 2$

$t = -2; x = 3 + 2(-2) = -1$

$t = 2;\ x = 3 + 2(2) = 7$

$t = -2;\ y = 2 - (-2) = 4$

$t = 2;\ y = 2 - (2) = 0$

$t = -2;\ z = -1 - 2(-2) = 3$

$t = 2;\ z = -1 - 2(2) = -5$

$\therefore$ Points of intersection are $(-1, 4, 3)$ and $(7, 0, -5)$

Question 14 (c)

Criteria	Marks
• Correctly using dot product to show sides at right angles	2
• finding the dot product or equivalent merit	1

Sample answer:

$\angle ABC$ can be found using the dot product of $\underset{\sim}{b}$ and $\underset{\sim}{c}$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = |\underset{\sim}{b}|.|\underset{\sim}{c}| \cos\theta$

If $\cos 90° = 0$, then $\underset{\sim}{b} \cdot \underset{\sim}{c} = 0$ and $\angle ABC = 90°$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = 1 \times 4 + 8 \times (-2) + 3 \times 4$

$= 4 - 16 + 12$

$= 0$

$\therefore \angle ABC = 90°$ and $\Delta\, ABC$ is a right triangle.

Question 14 (d)(i)

Criteria	Marks
• Provides correct solution	2
• Makes some progress	1

Sample answer:

Method 1: using the AM-GM inequality,

$$\begin{aligned}\frac{x}{\sqrt{x-1}} &= \frac{x-1+1}{\sqrt{x-1}} \\ &= \sqrt{x-1} + \frac{1}{\sqrt{x-1}} \\ &\geq 2\sqrt{\sqrt{x-1} \times \frac{1}{\sqrt{x-1}}} \\ &= 2\end{aligned}$$

Method 2:

$$\left(\sqrt{x-1}-1\right)^2 \geq 0$$

$$x-1-2\sqrt{x-1}+1 \geq 0$$

$$x \geq 2\sqrt{x-1}$$

$$\frac{x}{\sqrt{x-1}} \geq 2, \text{ since } \sqrt{x-1} > 0$$

Question 14 (d)(ii)

Criteria	Marks
• Provides correct solution	2
• Makes some progress	1

Sample answer:

Using the AM-GM inequality,

$$\begin{aligned}
\frac{a^2}{b-1}+\frac{b^2}{a-1} &\geq 2\sqrt{\frac{a^2}{b-1}\times\frac{b^2}{a-1}} \\
&= 2\sqrt{\frac{a^2}{a-1}}\sqrt{\frac{b^2}{b-1}} \\
&= 2\frac{a}{\sqrt{a-1}}\frac{b}{\sqrt{b-1}}, \quad \text{since } a,b>0 \\
&\geq 2\times 2\times 2, \quad \text{using part (i)} \\
&=8
\end{aligned}$$

Question 14 (e)(i)

Criteria	Marks
• Correctly evaluates the expression with full working out shown to get the result.	1

Sample answer:

$$e^{\frac{9i\pi}{10}}+e^{\frac{i\pi}{10}}+e^{\frac{-7i\pi}{10}}+e^{\frac{-3i\pi}{10}}=(\cos\frac{9\pi}{10}+i\sin\frac{9\pi}{10}+\cos\frac{\pi}{10}+i\sin\frac{\pi}{10})+(\cos\frac{-7\pi}{10}+i\sin\frac{-7\pi}{10}+\cos\frac{-3\pi}{10}+i\sin\frac{-3\pi}{10})$$

Note: $\cos\frac{9\pi}{10}+\cos\frac{\pi}{10}=0$ and $\cos\frac{-7\pi}{10}+\cos\frac{-3\pi}{10}=0$

$$=i(\sin\tfrac{9\pi}{10}+\sin\tfrac{\pi}{10})+i(\sin\tfrac{-7\pi}{10}+\sin\tfrac{-3\pi}{10})$$

Note: $\sin\frac{9\pi}{10}=\sin\frac{\pi}{10}$ and $\sin\frac{-7\pi}{10}=\sin\frac{-3\pi}{10}=-\sin\frac{3\pi}{10}$

$$=2i\sin\frac{\pi}{10}-2i\sin\frac{3\pi}{10}$$

Question 14 (e)(ii)

Criteria	Marks
• Correctly evaluates the expression with full working out shown to get the result.	1

Sample answer:

Recall that the sum of the roots $z^5-i=0$

$$e^{\frac{9i\pi}{10}}+e^{\frac{i\pi}{10}}+e^{\frac{-7i\pi}{10}}+e^{\frac{-3i\pi}{10}}+i=0$$

$$2i\sin\frac{\pi}{10}-2i\sin\frac{3\pi}{10}+i=0$$

$$2\sin\frac{\pi}{10}-2\sin\frac{3\pi}{10}+1=0$$

$$\sin\frac{\pi}{10}-\sin\frac{3\pi}{10}+\frac{1}{2}=0$$

$$\sin\frac{3\pi}{10}-\sin\frac{\pi}{10}=\frac{1}{2}$$

Question 14 (e)(iii)

Criteria	Marks
• Correctly proves the given result	2
• Makes progress such as evaluating A and B correctly when equating result from part e)ii) with the given trigonometric identity or equivalent.	1

Sample answer:

Using the given identity

$sin(A + B) - sin(A - B) = 2\, cosA\, sinB$

Let $A + B = \frac{3\pi}{10}$ (1) and $A - B = \frac{\pi}{10}$ (2)

By adding (1) and (2), we get

$2A = \frac{4\pi}{10}$ that is $A = \frac{\pi}{5}$.

By substituting in (1), we get $B = \frac{\pi}{10}$.

From the above, we can state that

$\sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = 2\cos\frac{\pi}{5}\sin\frac{\pi}{10} = \frac{1}{2}$

Hence, $\cos\frac{\pi}{5}\sin\frac{\pi}{10} = \frac{1}{4}$.

Alternative method

Using $\sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = \frac{1}{2}$, we get

$$2\cos\left(\frac{\frac{3\pi}{10}+\frac{\pi}{10}}{2}\right)\sin\left(\frac{\frac{3\pi}{10}-\frac{\pi}{10}}{2}\right) = \frac{1}{2}$$

$2\cos\frac{\pi}{5}\sin\frac{\pi}{10} = \frac{1}{2}$

Hence, $\cos\frac{\pi}{5}\sin\frac{\pi}{10} = \frac{1}{4}$.

Question 15 (a)

Criteria	Marks
• Justify that the lines are not parallel and do not meet indicating they are skew lines.	4
• Indicate that the solution of $-1-\lambda=-1+\mu$ and $-1+2\lambda=1-\mu$ does not satisfy $1+\lambda=-1+\mu$	3
• Justify that the lines are not parallel.	2
• Indicate testing $\begin{pmatrix}-1\\2\\1\end{pmatrix}=k\begin{pmatrix}1\\-1\\1\end{pmatrix}$.	1

Sample answer:

L_1 and L_2 are skew lines if they are not parallel and not intersecting.

First, let us check if they are parallel by considering the components of their directional vectors and suppose they are in the same ratio.

This means $\begin{pmatrix}-1\\2\\1\end{pmatrix}=k\begin{pmatrix}1\\-1\\1\end{pmatrix}$

So, $-1=k$ (1)

Also, $2=-k$, that is $k=-2$. (2)

From (1) and (2), we can see that k is not the same

So, their directional vectors are not in the same ratio and this indicates that they are not parallel.

Now, we check if they are intersecting by equating their parametric equations.

The parametric equation of the line L_1 or the coordinates of any point on L_1 are

$A(-1-\lambda,-1+2\lambda,1+\lambda)$ (A)

The parametric equation of the line L_2 or the coordinates of any point on L_2 are

$B(-1+\mu,1-\mu,-1+\mu)$ (B)

Now, we equate the coordinates of the above two points, we get

$-1-\lambda=-1+\mu$, that is $\mu=-\lambda$. (3)

Also, $-1+2\lambda=1-\mu$, that is $\mu=2-2\lambda$. (4)

By equating (3) in (4), we get

$-\lambda=2-2\lambda$. So, $\lambda=2$. Also, as $\mu=-\lambda=-2$.

Now, by substituting $\lambda=2$ into (A) the parametric equations of L_1, that is $(-1-\lambda,-1+2\lambda,1+\lambda)$.

We get $(-3,3,3)$ and then we substitute $\mu=-2$

into (B) the parametric equations of L_2 that is

$(-1+\mu,1-\mu,-1+\mu)$, we get $(-3,3,-3)$.

As the points obtained are not the same, then there

is no common point between the lines L_1 and L_2.

Hence, the lines L_1 and L_2 do not intersect. From the above, we can conclude that L_1 and L_2 are skew lines.

Question 15 (b)(i)

Criteria	Marks
• Provides correct proof to solution	1

Sample answer:

$$\cos x \equiv A(3\cos x + 4\sin x) + B(4\cos x - 3\sin x)$$
$$\cos x \equiv (3A + 4B)\cos x + (4A - 3B)\sin x$$
$$x = 0 \Rightarrow 1 = 3A + 4B \quad (1)$$
$$x = \frac{\pi}{2} \Rightarrow 0 = 4A - 3B \quad (2)$$
$$3\times(1) + 4\times(2) \Rightarrow A = \frac{3}{25} \text{ and } 4\times(1) - 3\times(2) \Rightarrow B = \frac{4}{25}$$

Question 15(b)(ii)

Criteria	Marks
• Provides the correct solution	4
• Recognises the integral of $uv' + u'v$ is $uv + c$	3
• Rearranges the terms to yield $uv' + u'v$	2
• Makes some progress such as replacing $\cos x$ with the identity from part b)i)	1

Sample answer:

$$\int \frac{e^{x/2}\cos x}{\sqrt[3]{3\cos x + 4\sin x}}\,dx = \int \frac{e^{x/2}\left[\frac{3}{25}(3\cos x + 4\sin x) + \frac{4}{25}(4\cos x - 3\sin x)\right]}{\sqrt[3]{3\cos x + 4\sin x}}\,dx$$

$$= \frac{3}{25}\int\left(\frac{(3\cos x + 4\sin x)e^{x/2}}{\sqrt[3]{3\cos x + 4\sin x}} + \frac{\frac{4}{3}(4\cos x - 3\sin x)e^{x/2}}{\sqrt[3]{3\cos x + 4\sin x}}\right)dx$$

$$= \frac{6}{25}\int\left(\underbrace{(3\cos x + 4\sin x)^{2/3}}_{u}\underbrace{\left(\frac{1}{2}e^{x/2}\right)}_{\frac{dv}{dx}} + \underbrace{\frac{2}{3}(3\cos x + 4\sin x)^{-1/3}(-3\sin x + 4\cos x)}_{\frac{du}{dx}}\underbrace{e^{x/2}}_{v}\right)dx$$

$$= \frac{6}{25}\int(uv' + u'v)\,dx$$

$$= \frac{6}{25}\int(uv)'\,dx$$

$$= \frac{6}{25}uv + c$$

$$= \frac{6}{25}(3\cos x + 4\sin x)^{2/3}e^{x/2} + c$$

Question 15 (c)

Criteria	Marks
• Provides correct solution	3
• Attempts to prove true for $n = k$	2
• Proves true for $n = 1$ and $n = 2$	1

Sample answer:

Let P_n be the proposition that $a_n = 5(-2)^n + 4\times 3^n$ for $n \in \mathbb{Z}^+$.

Since $5(-2)^1 + 4\times 3^1 = -10+12 = 2$ and $5(-2)^2 + 4\times 3^2 = 20+36 = 56$, both P_1 and P_2 are true.

Assume both P_{k-2} and P_{k-1} are true, where $k-2 \geq 1$, $i.e. k \geq 3$, that is

$$a_{k-2} = 5(-2)^{k-2} + 4\times 3^{k-2} \text{ and } a_{k-1} = 5(-2)^{k-1} + 4\times 3^{k-1}$$

Then:

$$\begin{aligned}
a_k &= a_{k-1} + 6a_{k-2} \\
&= 5(-2)^{k-1} + 4\times 3^{k-1} + 6\left(5(-2)^{k-2} + 4\times 3^{k-2}\right) \\
&= 5(-2)^{k-1} + 30(-2)^{k-2} + 4\times 3^{k-1} + 24\times 3^{k-2} \\
&= (-2)^{k-2}[-10+30] + 3^{k-2}[12+24] \\
&= 20\times(-2)^{k-2} + 36\times 3^{k-2} \\
&= 5(-2)^k + 4\times 3^k
\end{aligned}$$

$\therefore P_k$ is true whenever both P_{k-2} and P_{k-1} are true.

Since P_1 and P_2 are true, then P_k is true by mathematical induction.

Question 15 (d)

Criteria	Marks
• Provides correct solution	3
• Correctly uses the t −substitution and the limits for integration	2
• Attempts to use t −substitution	1

Sample answer:

$$t = \tan\frac{x}{2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2 x = \frac{1+\tan^2 x}{2}$$

$$\frac{dt}{dx} = \frac{1+t^2}{2}$$

When $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$

When $x = \frac{\pi}{3}$, $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$$\sin x = \frac{2t}{1+t^2}$$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\sin x + 1}\,dx$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{\frac{2t}{1+t^2}+1} \times \frac{2}{1+t^2}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{\frac{1+2t+t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2(1+t^2)}{(1+t)^2} \times \frac{2}{(1+t^2)}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{4}{(1+t)^2}\,dt$$

$$= -4\left[\frac{1}{(1+t)}\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= -4\left[\frac{1}{(1+1)} - \frac{1}{\left(1+\frac{1}{\sqrt{3}}\right)}\right]$$

$$= -4\left[\frac{1}{2} - \frac{1}{\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)}\right]$$

$$= -4\left[\frac{1}{2} - \frac{\sqrt{3}}{1+\sqrt{3}}\right]$$

$$= -4\left[\frac{1}{2} - \frac{\sqrt{3}-3}{-2}\right]$$

$$= -2(\sqrt{3}-2)$$

$$= 4 - 2\sqrt{3}$$

Question 16 (a)(i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$I_0 = \int_0^1 x^{2\times 0}\sqrt{1-x^2}\,dx = \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$ as it represents the area inside the unit circle in the first quadrant.

Question 16 (a)(ii)

Criteria	Marks
• Provides correct solution	3
• Correctly integrates by parts or equivalent merit	2
• Makes some progress	1

Sample answer:

$$\begin{aligned}
I_n &= \int_0^1 x^{2n}\sqrt{1-x^2}\,dx \\
&= \int_0^1 x^{2n-1}\times x\sqrt{1-x^2}\,dx \\
&= \int_0^1 \underbrace{x^{2n-1}}_{u}\times \underbrace{\frac{d}{dx}\left(-\frac{1}{3}\left(1-x^2\right)^{3/2}\right)}_{v'}dx \\
&= \left[\underbrace{x^{2n-1}\times -\frac{1}{3}\left(1-x^2\right)^{3/2}}_{uv}\right]_0^1 - \int_0^1 \underbrace{(2n-1)x^{2n-2}}_{u'}\times \underbrace{-\frac{1}{3}\left(1-x^2\right)^{3/2}}_{v}dx \\
&= 0 + \frac{(2n-1)}{3}\int_0^1 x^{2n-2}\left(1-x^2\right)^{3/2}dx \\
&= \frac{(2n-1)}{3}\int_0^1 x^{2n-2}\left(1-x^2\right)\sqrt{1-x^2}\,dx \\
&= \frac{(2n-1)}{3}\left(I_{n-1}-I_n\right)
\end{aligned}$$

$$I_n = \frac{(2n-1)}{3}\left(I_{n-1}-I_n\right)$$

$$3I_n = (2n-1)I_{n-1} - (2n-1)I_n$$

$$(2n+2)I_n = (2n-1)I_{n-1} \Rightarrow I_n = \frac{2n-1}{2n+2}I_{n-1}$$

Question 16 (a)(iii)

Criteria	Marks
• Provides correct solution	2
• Makes some progress	1

Sample answer:

$$I_3 = \int_0^1 x^6 \sqrt{1-x^2}\,dx$$

$$I_0 = \frac{\pi}{4} \text{ and } I_n = \frac{2n-1}{2n+2} I_{n-1}$$

$$I_1 = \frac{1}{4} I_0 \quad \Rightarrow \quad I_1 = \frac{\pi}{16}$$

$$I_2 = \frac{3}{6} I_1 \quad \Rightarrow \quad I_2 = \frac{\pi}{32}$$

$$I_3 = \frac{5}{8} I_2 \quad \Rightarrow \quad I_3 = \frac{5\pi}{256}$$

$$\int_0^1 x^6 \sqrt{1-x^2}\,dx = \frac{5\pi}{256}$$

Question 16 (b)

Criteria	Marks
• Correctly identifies the contradiction within the results and writes a concluding statement.	3
• Correctly squares both sides of the inequality and obtains correct expression for a.	2
• Establishes a correct proof by contradiction statement or equivalent merit.	1

Sample answer

Assume that the converse is true, that is $\sqrt{2a+2} \leq \sqrt{a+8}$ for $a > 6$.

Since LHS and RHS are both positive numbers, squaring both sides yields:

$2a + 2 \leq a + 8$

$a \leq 6$

However, this is a contradiction, since statement is true for $a > 6$ only.

Hence, our original assumption must be false. Therefore, the original statement $\sqrt{2a+2} > \sqrt{a+8}$ for $a > 6$, must be true.

Question 16 (c)

Criteria	Marks
• State that $z_C = 8\sqrt{3} - 41 - 12i$.	4
• Show that by equating real parts we get $c + 5 = -\sqrt{3}\left(9\sqrt{3} - 13\right) - 5\sqrt{3} - 9.$	3
• Show that by equating imaginary parts we get $-8 = -9\sqrt{3} + \mathrm{b} + 5$.	2
• Indicate that $\overrightarrow{AC} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\overrightarrow{AB}$.	1

Sample answer:

We are given $A\,(-5, -4)$ and let B be $(b\,, 5\,)$, and $C\,(c, -12)$.

This indicates that

$\overrightarrow{AB} = (b + 5\,) + (5 + 4)i = (b + 5\,) + 9i$ and $\overrightarrow{AC} = (c + 5) + (-12 + 4)i = (c + 5) - 8i$

By rotating vector $\overrightarrow{AB}$ by $\frac{5\pi}{6}$ in an anticlockwise direction and multiply its magnitude by 2, we get vector $\overrightarrow{AC}$.

This means $\overrightarrow{AC} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\overrightarrow{AB}$

By substituting $\overrightarrow{AC}$ and $\overrightarrow{AB}$, we get

$$(c + 5) - 8i = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\,i\right)\left((b + 5\,) + 9i\right)$$

$$(c + 5) - 8i = \left(-\sqrt{3} + i\right)\left((b + 5\,) + 9i\right)$$

$$(c + 5) - 8i = -\sqrt{3}\,(b + 5\,) - 9\sqrt{3}\,i + (b + 5)i - 9$$

$$(c + 5) - 8i = -\sqrt{3}\,b - 5\sqrt{3} - 9\sqrt{3}\,i + bi + 5i - 9$$

$$(c + 5) - 8i = -\sqrt{3}\,b - 5\sqrt{3} - 9 + \left(-9\sqrt{3} + b + 5\,\right)i \quad (1)$$

Equating imaginary parts in (1), *we get*

$-8 = -9\sqrt{3} + b + 5$, that is

$-13 = -9\sqrt{3} + b$

Therefore, $b = 9\sqrt{3} - 13$.

Now, equating real parts in (1), *we get*

$c + 5 = -\sqrt{3}\,b - 5\sqrt{3} - 9$

By substituting, $b = 9\sqrt{3} - 13$, we get

$c + 5 = -\sqrt{3}\left(9\sqrt{3} - 13\right) - 5\sqrt{3} - 9$

$c + 5 = -27 + 13\sqrt{3} - 5\sqrt{3} - 9$

Therefore, $c = 8\sqrt{3} - 41$.

Hence, $z_C = 8\sqrt{3} - 41 - 12i$.