Student Number:

Cheltenham Girls High School

2025 **HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 2

Assessment Task 4 – Trial HSC

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen only
- Calculators approved by NESA may be used
- A NESA reference sheet is provided at the back of this paper
- For all questions in Section II, show relevant mathematical reasoning and/or calculations to obtain maximum marks.

Total marks: 100

Section I – 10 marks (pages 2-6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section
- Use the multiple choice answer sheet attached to answer Questions 1–10

Section II – 90 marks (pages 7-13)

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section
- Use separate writing booklets to answer each of the Questions 11 to 16.

Multiple Choice Q1-10	Question 11	Question 12	Question 13	Question 14	Question 15	Question 16	Total
/10	/15	/15	/15	/15	/15	/15	/100

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Section I

10 marks

Attempt Questions 1-10

Allow about 15 minutes for this section

(USE THE MULTIPLE CHOICE ANSWER SHEET PROVIDED for **Questions 1 – 10**)

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1. Which complex number is a square root of $7 + 24i$?

(A) $-3 - 4i$

(B) $4 - 3i$

(C) $-4 + 3i$

(D) $-4 - 3i$

2. Which of the following is the negation of "$y > 6$ or $y \leq -2$"?

(A) $-2 < y < 6$

(B) $-2 \leq y < 6$.

(C) $-2 < y \leq 6$

(D) $-2 \leq y \leq 6$.

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5. A cubic polynomial $P(z)$ has all real coefficients. Given that $z = 3$ and $z = 2 + i$ are two roots such that $P(z) = 0$, which equation best describes $P(z)$?

(A) $P(z) = z^3 - 7z^2 - 17z + 15$

(B) $P(z) = z^3 + 7z^2 - 35z + 15$

(C) $P(z) = z^3 - 7z^2 + 17z - 15$

(D) $P(z) = z^3 + 7z^2 + 35z - 15$

6. Find the expression that best represents:

$$\int \frac{1}{\sqrt{x}} \operatorname{cosec}(\sqrt{x}) \cot(\sqrt{x})\, dx$$

(A) $-2\operatorname{cosec}(\sqrt{x}) + C$

(B) $-\frac{1}{2}\operatorname{cosec}(\sqrt{x}) + C$

(C) $\frac{1}{2}\operatorname{cosec}(\sqrt{x}) + C$

(D) $2\operatorname{cosec}(\sqrt{x}) + C$

7. A particle is moving along a straight line. Initially its displacement is $x = 1$, its velocity is $v = 2$ and its acceleration is $a = 4$. Which equation could describe the motion of the particle?

(A) $v = 2 + 4\ln x$

(B) $v = 2\sin(x - 1) + 2$

(C) $v^2 = 4(x^2 - 2)$

(D) $v^2 = x^2 + 2x + 4$

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5. A cubic polynomial $P(z)$ has all real coefficients. Given that $z = 3$ and $z = 2 + i$ are two roots such that $P(z) = 0$, which equation best describes $P(z)$?

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$$\int \frac{1}{\sqrt{x}} \operatorname{cosec}(\sqrt{x}) \cot(\sqrt{x})\, dx$$

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(C) $v^2 = 4(x^2 - 2)$

(D) $v^2 = x^2 + 2x + 4$

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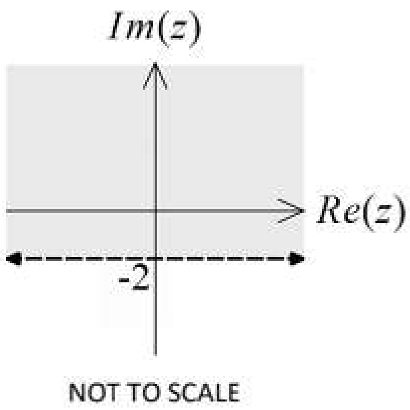
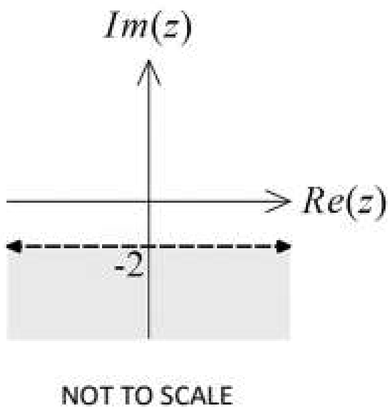
8. Which of the following is the region in the Argand diagram specified by $i(z - \bar{z}) > 4$?

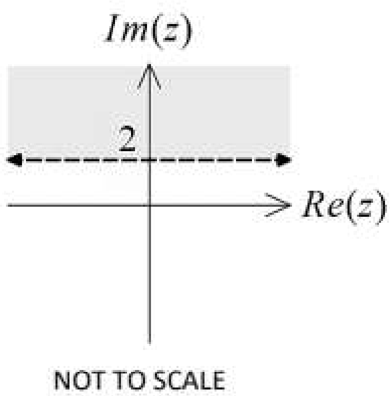
(A)



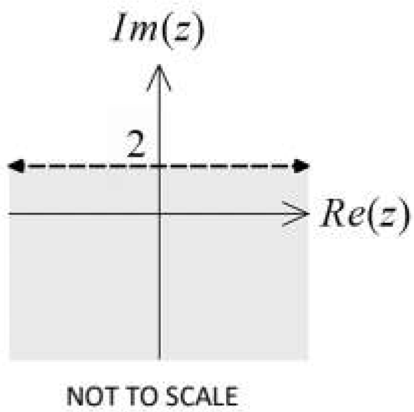
(B)



(C)



(D)



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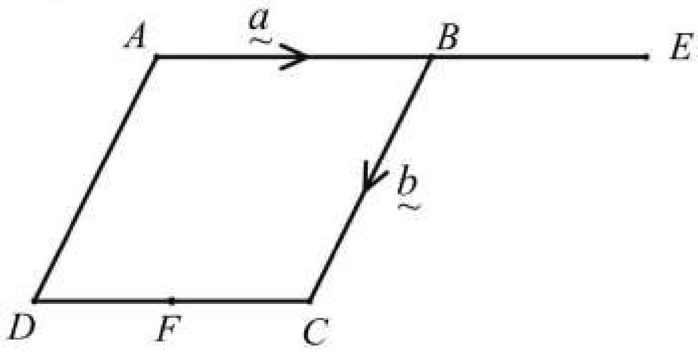
9. Consider the diagram below, where $ABCD$ is a rhombus, F is the midpoint of CD and BE is an extension of AB.



NOT TO SCALE

Given that $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{b}$ and point B divides $\overrightarrow{AE}$ internally in the ratio $4:3$, which of the following best represents $\overrightarrow{FE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$?

(A) $\frac{5}{4}\underset{\sim}{a} - \underset{\sim}{b}$

(B) $\frac{5}{4}\underset{\sim}{a} + \underset{\sim}{b}$

(C) $\frac{11}{6}\underset{\sim}{a} - \underset{\sim}{b}$

(D) $\frac{11}{6}\underset{\sim}{a} + \underset{\sim}{b}$

10. A sphere of unit radius is tangent to the xy-plane, the xz-plane, and the yz-plane.
i.e. The surface of the sphere meets each plane at exactly one point.

If a point A on the sphere, is to be joined to the origin, what is the shortest possible distance?

(A) 1

(B) $\sqrt{2}$

(C) $\sqrt{2} - 1$

(D) $\sqrt{3} - 1$

End of Section I

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Section II

90 marks

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a new writing booklet to answer this question.

(a) Consider the complex numbers $z = 1 - i$ and $w = \sqrt{3} + i$.

(i) Evaluate $|z| + |w|$, leaving answers in exact form. **2**

(ii) Evaluate $\arg\left(\frac{z^2}{w^3}\right)$ in terms of π, showing all lines of working out. **3**

(b) Prove by contradiction that $\sqrt{5}$ is irrational. **3**

(c) Evaluate the following definite integral: **3**

$$\int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$$

(d) Let $z = \cos\theta + i\sin\theta$ be a complex number.

(i) Solve $z^5 - 1 = 0$ over the complex field, giving your answer in modulus-argument form. **2**

(ii) Hence, or otherwise, express $z^5 - 1$ as a product of linear and quadratic factors. **2**

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Question 12 (15 marks) Use a new writing booklet to answer this question.

(a) Evaluate the following indefinite integrals

(i) $\int \frac{4x-10}{x^2+2x+2}\,dx$ **2**

(ii) $\int 2x^3\sqrt{4-x^2}\,dx$ **2**

(iii) $\int e^{2x}\cos 2x\;dx$ **2**

(b) Let $z = \cos\theta + i\sin\theta$ and $z^n + z^{-n} = 2\cos n\theta$.

(i) By expanding $(z+z^{-1})^4$, prove that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. **2**

(ii) Hence, solve $8x^4 - 8x^2 + 1 = 0$ and show that the solutions are given by $x = \cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}$. **1**

(iii) Hence, explain why $\cos\frac{\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{7\pi}{8} = 0$. **1**

(c) Solve the equation $x^3 - 5x^2 + 8x - 6 = 0$ over the set of complex numbers. **3**

(d) Prove that the statement "If n^2 is even, then n is even", is logically true, by considering its contrapositive. **2**

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Question 13 (15 marks) Use a new writing booklet to answer this question.

(a) The equation of a sphere is given by $x^2 - 8x + y^2 + 4y + z^2 + 2z = 4$. **3**
Find the vector equation of the sphere and state its centre and radius.

(b) Prove by mathematical induction that $(1+\sqrt{2})^n = p_n + q_n\sqrt{2}$, where p_n and q_n are unique positive integers, for n is an integer, $n \geq 2$. **3**

(c) Consider the vectors $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, where A, B are points and O is the origin.

(i) Find the vector projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$. **2**

(ii) Find the vector equation of a line parallel to $\overrightarrow{OB}$ and intersecting point A. **1**

(d) Given $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$, where n is a positive integer.

(i) Show that $I_n = \frac{1}{n-1}[(\sqrt{2})^{n-2} + (n-2)I_{n-2}]$ for $n \geq 2$ **4**

(ii) Hence, evaluate I_4, leaving your answer in exact form **2**

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Question 14 (15 marks) Use a new writing booklet to answer this question.

(a) Let $z = x + iy$, where $x, y \in \mathbb{R}$, be some complex number such that $\arg\left(\frac{z-w_1}{z-w_2}\right) = \frac{\pi}{6}$.

(i) Given that $w_1 = 2(3+i)$ and $w_2 = 2(1+i)$, realise the denominator in the expression $\frac{z-w_1}{z-w_2}$ and prove that $\left|z - \left(4 + 2\left(1+\sqrt{3}\right)i\right)\right| = 4$ by considering $\arg\left(\frac{z-w_1}{z-w_2}\right) = \frac{\pi}{6}$. **4**

(ii) Hence, or otherwise, sketch the locus of z on the argand diagram specified by $\arg\left(\frac{z-w_1}{z-w_2}\right) = \frac{\pi}{6}$, showing all important features including the point in which the locus meets the y-axis. **3**

(b) Given that the inequality $\frac{9}{17} < \frac{n}{n+k} < \frac{8}{15}$ holds true for a unique integer k and n is a positive integer,

(i) Prove that the largest positive value of n is given by $n = 144$. **2**

(ii) Hence, show that the inequality indeed holds true when $n = 144$, by finding the unique integer k. **2**

(c) It is known that $a + b + c \geq 1$ where a, b and c are positive numbers. **4**

Prove that:

$$\left(\frac{a(a+1)+b+c}{a}\right)\left(\frac{b(b+1)+c+a}{b}\right)\left(\frac{c(c+1)+a+b}{c}\right) \geq 8$$

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Question 15 (15 marks) Use a new writing booklet to answer this question.

(a) A particle is performing Simple Harmonic Motion in a straight line such that at time t seconds, it has displacement x metres, velocity v ms^{-1} and acceleration $\ddot{x}$ ms^{-2} given by $\ddot{x} = -2(x-1)$.

Initially the particle is at $x = 2$ metres and has a velocity of $-\sqrt{6}$ ms^{-1}.

(i) Show by integration that $v^2 = -2x^2 + 4x + 6$. **2**

(ii) Hence, by letting $x = B + A\cos(nt + \theta)$ where $A > 0, n > 0$ and $0 < \theta < \pi$, find the values of B, A, n and θ. **4**

(b) The line L_1 passes through the points A and B and the line L_2 passes through C and D, where $A = (5, 4, 0)$, $B = (10, 8, 3)$, $C = (-1, -6, -10)$ and $D = (9, 2, -4)$. **3**

Show that L_1 and L_2 do not intersect and find the shortest distance between them.

(c) Prove that if $\frac{d}{dx}(f(x)) = \frac{d}{dx}(g(x))$, then $f(x) = g(x)$, is not always true by counterexample. **1**

(d) A line $\underset{\sim}{r} = \binom{-2}{0} + \lambda\binom{1}{3}$ intersect a circle with equation $(x+1)^2 + (y-2)^2 = 4$ at two points A and B, to form a chord AB. **5**

Given that M is the midpoint of chord AB and C is the centre of the circle, prove that $\overrightarrow{CM}$ is perpendicular to $\overrightarrow{AB}$ and find the perpendicular distance from the centre of the circle to chord AB.

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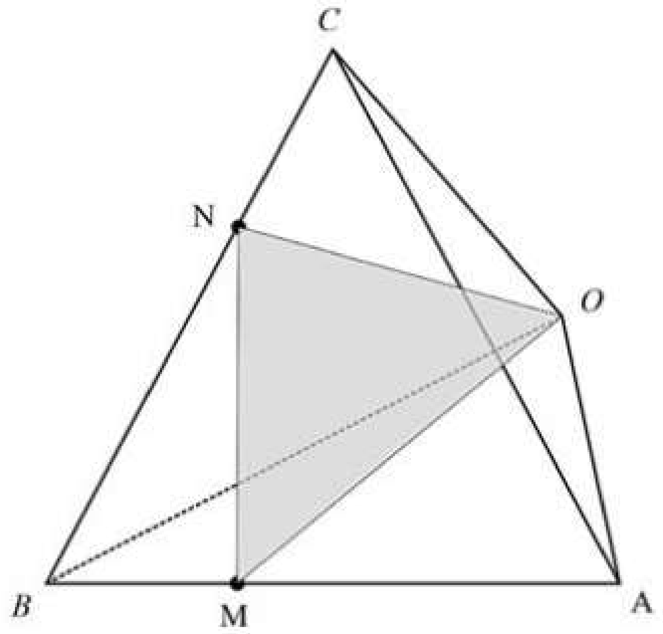
Question 16 (15 marks) Use a new writing booklet to answer this question.

(a) A regular tetrahedron $OABC$ consists of four congruent equilateral triangle faces and has sides of length 1, where O is the origin.



NOT TO SCALE

Let $\overrightarrow{OA} = \underset{\sim}{\boldsymbol{a}}$, $\overrightarrow{OB} = \underset{\sim}{\boldsymbol{b}}$, $\overrightarrow{OC} = \underset{\sim}{\boldsymbol{c}}$, and M and N be the points that divides sides AB and BC in the ratio $x : 1 - x$, where $0 < x < 1$.

(i) Show that $\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} = \underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{c}} = \underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{c}} = \frac{1}{2}$. **1**

(ii) Show that $\left|\overrightarrow{OM}\right| = \sqrt{1 - x + x^2}$. **2**

(iii) Let $f(x) = \frac{1+x-x^2}{1-x+x^2}$, where $0 < x < 1$. **3**

Without using calculus, prove that $1 < f(x) \leq \frac{5}{3}$.

(iv) Hence, show that $\cos^{-1}\frac{5}{6} \leq \angle MON < \frac{\pi}{3}$. **3**

Question 16 continues on page 13

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Question 16 (Continued)

(b) Describe and sketch the general shape of parametric curve given by: **2**

$$\underset{\sim}{r} = \begin{pmatrix} 3\sin\theta \\ 3\cos\theta \\ \theta \end{pmatrix} \text{ for } 0 \le \theta \le 4\pi.$$

(c) Consider the following definite integral.

$$I = \int_2^3 \tan^{-1}\left(\frac{1}{x^2 - 5x + 7}\right) dx$$

(i) Show that $\tan^{-1}\left(\frac{1}{x^2-5x+7}\right) = \tan^{-1}(x-2) - \tan^{-1}(x-3)$. **2**

(ii) Hence, evaluate I, leaving your answer in exact form. **2**

End of Section II

End of Assessment Task

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Cheltenham Girls High School

2025 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Assessment Task 4 – Trial HSC

Section I - Multiple Choice

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

Start Here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

2025 Year 12 Extension 2 Mathematics Assessment Task 4
Marking Guidelines

Section I (1 mark each)

Multiple-choice Answer Key

Question	Answer
1	D
2	C
3	A
4	B
5	C
6	A
7	B
8	B
9	A
10	D

Question 1

Let $(x + iy)^2 = 7 + 24i$

$x^2 + 2xyi - y^2 = 7 + 24i$

$x^2 - y^2 = 7$ and $2xy = 24$

$$xy = 12$$

Hence, $x = 4$ and $y = 3$

$\therefore \pm(4 + 3i)$ are the square roots

$\therefore -4 - 3i$ is a solution.

Answer: D

Question 2

The negation is $y \leq 6$ and $y > -2$.

The inequality that best represents this is $-2 < y \leq 6$.

Answer: C

Question 3

Let $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$dx = \frac{2}{\tan^2\frac{x}{2}+1}dt$

$dx = \frac{2}{1+t^2}dt$

Also $\cos x = \frac{1-t^2}{1+t^2}$

$\therefore \int \frac{2}{2+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\ dt$

$= \int \frac{4}{2(1+t^2)+1-t^2}dt$

$= \int \frac{4}{3+t^2}dt$

Answer: A

Question 4

$$\cos\angle POR = \frac{\overrightarrow{OP}\cdot \underset{\sim}{i}}{\left|\overrightarrow{OP}\right|\left|\underset{\sim}{i}\right|}$$

$$= \frac{1}{\sqrt{14}}$$

$$\angle POR = \cos^{-1}\frac{1}{\sqrt{14}}$$

$= 74^o$ to the nearest degree.

Answer: B

Question 5

Since $P(z)$ has all real coefficients, complex roots must come in conjugate pairs.

$\therefore z = 3, z = 2 + i$ and $z = 2 - i$ are roots.

$P(z) = (z-3)(z-(2+i))(z-(2-i))$

$= (z-3)(z^2 - z(2-i) - z(2+i) + 5)$

$= (z-3)(z^2 - 4z + 5)$

$= z^3 - 4z^2 + 5z - 3z^2 + 12z - 15$

$= z^3 - 7z^2 + 17z - 15$

Answer: C

Question 6

Note that $\frac{d}{dx}(\operatorname{cosec}\sqrt{x}) = \frac{d}{dx}\left((\sin\sqrt{x})^{-1}\right)$

$$= -\frac{1}{2\sqrt{x}}\cos\sqrt{x}\,(\sin\sqrt{x})^{-2}$$

$$= -\frac{1}{2\sqrt{x}}\operatorname{cosec}\sqrt{x}\cot\sqrt{x}$$

$$\therefore \int \frac{1}{\sqrt{x}}\operatorname{cosec}\sqrt{x}\cot\sqrt{x} = -2\int -\frac{1}{2\sqrt{x}}\operatorname{cosec}\sqrt{x}\cot\sqrt{x}$$

$$= -2\operatorname{cosec}\sqrt{x} + C$$

Answer: A

Question 7

Check each equation and see if $x = 1, v = 2$ and $a = 4$. Through checking we can see that:

For $v = 2\sin(x-1) + 2$

Let $x = 1$

$v = 2\sin 0 + 2$

$v = 2$

$a = 2\cos(x-1)\,(2\sin(x-1) + 2)$

$a = 4$

Answer: B

Question 8

Let $z = x + iy$

$\therefore i(z - \bar{z}) = i(x + iy - (x - iy))$

$= i(2iy)$

$= -2y$

$\therefore -2y > 4$

$y < -2$

$Im(z) < -2$

Answer: B

Question 9

Given point B divides $\overrightarrow{AE}$ internally in the ratio $4:3$, then $\overrightarrow{BE} = \frac{3}{4}\underset{\sim}{a}$.

Also $\overrightarrow{FC} = \frac{1}{2}\underset{\sim}{a}$ and $\overrightarrow{CB} = -\underset{\sim}{b}$.

$\therefore \overrightarrow{FE} = \overrightarrow{FC} + \overrightarrow{CB} + \overrightarrow{BE}$

$= \frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b} + \frac{3}{4}\underset{\sim}{a}$

$= \frac{5}{4}\underset{\sim}{a} - \underset{\sim}{b}$

Answer: A

Question 10

Point A must lie on the line joining the centre of sphere to the origin to have the shortest possible distance.

One possible centre of sphere is $(1,1,1)$ (does not matter which octant it is in as they will all yield same results).

Distance to origin from centre is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Distance of Point A to origin = $\sqrt{3} - radius\ of\ sphere$

$= \sqrt{3} - 1$

Answer: D

Section II

Question 11 (a)(i)

Criteria	Marks
• Correctly evaluates $\lvert z\rvert + \lvert w\rvert$ in exact form.	2
• Evaluates $\lvert z\rvert$ or $\lvert w\rvert$ correctly only but did not evaluate $\lvert z\rvert + \lvert w\rvert$.	1

Sample answer:

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$|w| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$$
$$\therefore |z| + |w| = \sqrt{2} + 2$$

Question 11 (a)(ii)

Criteria	Marks
• Correctly evaluates the argument of $\left(\frac{z^2}{w^3}\right)$ in terms of π, showing all steps of working out.	3
• Applies De Moivre's Theorem to find the argument of z^2 and w^3.	2
• Finds the argument for z and w only.	1

Sample answer:

$$\arg(z) = -\tan^{-1}\frac{1}{1} = -\frac{\pi}{4}$$

$$\arg(w) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$\arg(z^2) = 2(-\tfrac{\pi}{4}) = -\frac{\pi}{2}$$

$$\arg(w^3) = 3\left(\frac{\pi}{6}\right) = \frac{\pi}{2}$$

$$\therefore \arg\left(\frac{z^2}{w^3}\right) = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi = \pi \text{ (principal argument)}$$

Question 11 (b)

Criteria	Marks
• Correctly proves that $\sqrt{5}$ is irrational using proof by contradiction with all working out shown.	3
• Makes significant progress in completing the proof by contradiction including providing a conclusion, however, makes errors such as not clearly showing all lines leading up to contradiction or incorrect algebraic evidence to suggest a contradiction.	2
• Makes progress such as setting up the correct initial assumption line.	1

Sample answer:

Assume that $\sqrt{5}$ is rational, such that there exists $p, q \in \mathbb{N}$, $q \neq 0$, such that $\sqrt{5} = \frac{p}{q}$, where the highest common factor between p and q is 1.

$\sqrt{5} = \dfrac{p}{q}$

$5 = \dfrac{p^2}{q^2}$

$5q^2 = p^2$

Hence, p^2 is divisible by 5, which implies p is divisible by 5.

Let $p = 5k$

$5q^2 = (5k)^2$

$5q^2 = 25k^2$

$q^2 = 5k^2$

Hence, q^2 is divisible by 5, which implies q is divisible by 5.

Therefore, both p and q are divisible by 5.

This is a contradiction, since p and q do not have any common factor except 1 in our initial assumption.

Therefore, our initial assumption of $\sqrt{5}$ being rational is false.

$\therefore \sqrt{5}$ must be irrational.

Question 11 (c)

Criteria	Marks
• Correctly calculates the definite integral with full working out shown.	3
• Makes significant progress and correctly integrates but makes an error during substitution of boundaries.	2
• Makes progress such as correctly using partial fraction decomposition to yield two simpler integrals.	1

Sample answer:

Let $\frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$

$1 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 1)$

$1 = (A + C)x^3 + (B + D)x^2 + (4A + C)x + 4B + D$

$A + C = 0$ **(1)**

$B + D = 0$ **(2)**

$4A + C = 0$ **(3)**

$4B + D = 1$ **(4)**

$\therefore$ From (1) and (3) we get $A = 0$, and $C = 0$

$\therefore$ From (2) and (4) we get $B = \frac{1}{3}$ and $D = -\frac{1}{3}$

Hence,

$\int_0^\infty \frac{1}{(x^2+1)(x^2+4)} dx = \int_0^\infty \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)} dx$

$= \frac{1}{3}\int_0^\infty \frac{1}{x^2+1} - \frac{1}{x^2+4} dx$

$= \frac{1}{3}\left[\tan^{-1} x - \frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^\infty$

$= \frac{1}{3}\left[\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - (0 - 0)\right]$

$= \frac{\pi}{12}$

Question 11 (d)(i)

Criteria	Marks
• Correctly finds all 5 solutions to the equation with full working out shown.	2
• Makes progress such as finding one pair of complex conjugate solutions.	1

Sample answer:

$z^5 - 1 = 0$ are the 5th roots of unity.

Let z_n represent the solutions to the equation.

$z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$z_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$z_3 = \cos\frac{-4\pi}{5} + i\sin\frac{-4\pi}{5}$

$z_4 = \cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5}$

$z_5 = 1$

Question 11 (d)(ii)

Criteria	Marks
• Correctly expresses $z^5 - 1$ as a product of linear and quadratic factors.	2
• Makes progress such as expressing $z^5 - 1$ as a product of linear factors only.	1

Sample answer:

$z^5 - 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)$

Note that: $z_4 = \overline{z_1}$ and $z_3 = \overline{z_2}$

$= (z - z_1)(z - \overline{z_1})(z - z_2)(z - \overline{z_2})(z - 1)$

$= (z^2 - z(z_1 + \overline{z_1}) + z_1\overline{z_1})\ (z^2 - z(z_2 + \overline{z_2}) + z_2\overline{z_2})(z - 1)$

$= (z - 1)(z^2 - z(2Re(z_1)) + |z_1|^2)(z^2 - z(2Re(z_2)) + |z_2|^2)$

$= (z - 1)(z^2 - 2\cos\frac{2\pi}{5}z + 1)(\ z^2 - 2\cos\frac{4\pi}{5}z + 1)$

Question 12 (a)(i)

Criteria	Marks
• Correct answer with full working out shown.	2
• Makes progress such as separating the integral and correctly evaluating the log component OR the inverse tan component of the answer.	1

Sample answer:

$$\int \frac{4x-10}{x^2+2x+2}dx = \int \frac{4x+4}{x^2+2x+2} - \frac{14}{x^2+2x+2}dx$$

$$= 2\ln|x^2+2x+2| - 14\int \frac{1}{(x+1)^2+1}dx$$

$$= 2\ln|x^2+2x+2| - 14\tan^{-1}(x+1) + C$$

Question 12 (a)(ii)

Criteria	Marks
• Correct answer with full working out shown. Full marks awarded if students could achieve up to line marked with *** or any subsequent lines.	2
• Makes progress such as finding the correct u-substitution or equivalent and finds an equivalent integral with respect to u.	1

Sample answer:

$$\int 2x^3\sqrt{4-x^2}\,dx$$

Let $u = 4 - x^2$

Hence, $x^2 = 4 - u$

$$\frac{du}{dx} = -2x$$

$du = -2x\,dx$

$$-\int (4-u)\sqrt{u}\,du = -\int 4u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$$

$$= -\left(\frac{8u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5}\right)$$

$$= \frac{2}{5}\sqrt{(4-x^2)^5} - \frac{8}{3}\sqrt{(4-x^2)^3} + C \quad ***$$

$$= \sqrt{4-x^2}\left(\frac{2}{5}(4-x^2)^2 - \frac{8}{3}(4-x^2)\right) + C$$

$$= \frac{2}{15}\sqrt{4-x^2}(3x^4 - 4x^2 - 32) + C$$

Note: Students who used trigonometric substitution $x = 2\sin\theta$ followed by $u = \cos\theta$, will yield the solution, which can be modified to equal above solution.

$$I = -64\left(\frac{\cos^3(\sin^{-1}\left(\frac{x}{2}\right))}{3} - \frac{\cos^5(\sin^{-1}(\frac{x}{2}))}{5}\right) + C$$

Question 12 (a)(iii)

Criteria	Marks
• Correct answer with full working out shown.	2
• Constructs a correct resultant integral using one application of integration by parts.	1

Sample answer:

$$\int e^{2x}\cos 2x\ dx$$

u	v'
e^{2x}	$\cos 2x$
$2e^{2x}$	$\frac{1}{2}\sin 2x$
$4e^{2x}$	$-\frac{1}{4}\cos 2x$

$$\int e^{2x}\cos 2x\ dx = \tfrac{1}{2}e^{2x}\sin 2x + \tfrac{1}{2}e^{2x}\cos 2x - \int e^{2x}\cos 2x\ dx + C$$

$$2\int e^{2x}\cos 2x\ dx = \tfrac{1}{2}e^{2x}(\sin 2x + \cos 2x) + C$$

$$\int e^{2x}\cos 2x = \tfrac{1}{4}e^{2x}(\sin 2x + \cos 2x) + C$$

Question 12 (b)(i)

Criteria	Marks
• Correctly applies the trigonometric identities and appropriate substitutions to yield the final solution with full working out shown.	2
• Makes progress such as applying the binomial expansion correctly.	1

Sample answer:

$$(z + z^{-1})^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$$

$$(z + z^{-1})^4 = z^4 + z^{-4} + 4(z^2 + z^{-2}) + 6$$

$$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$16\cos^4\theta = 2\cos 4\theta + 8(2\cos^2\theta - 1) + 6$$

$$2\cos 4\theta = 16\cos^4\theta - 8(2\cos^2\theta - 1) - 6$$

$$2\cos 4\theta = 16\cos^4\theta - 16\cos^2\theta + 8 - 6$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

Question 12 (b)(ii)

Criteria	Marks
• Correctly shows the solutions are as given by solving the given equation first, with full working out shown.	1

Sample answer:

$8x^4 - 8x^2 + 1 = 0$

Let $x = \cos\theta$

$8\cos^4\theta - 8\cos^2\theta + 1 = 0$

$\cos 4\theta = 0$

$4\theta = \cdots, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \ldots$

$\theta = \cdots, \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{7\pi}{8}, \ldots$

Hence, $x = \cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}$

Question 12 (b)(iii)

Criteria	Marks
• Provides correct explanation such as the sum of roots of a polynomial.	1

Sample answer:

Roots of $8x^4 - 8x^2 + 1 = 0$ are $x = \cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}$

Hence, the sum of roots is equal to 0 in this case, since $\frac{-b}{a} = \frac{0}{8} = 0$.

$\therefore \cos\frac{\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{7\pi}{8} = 0.$

Question 12 (c)

Criteria	Marks
• Correctly calculates all three solutions of the equation, with full working out shown.	3
• Correctly calculates one real solution and makes significant progress in finding the complex conjugate pair solution, but makes one error in calculation.	2
• Correctly calculates one real solution only.	1

Sample answer:

Let $P(x) = x^3 - 5x^2 + 8x - 6$

$\therefore x = 3$ is a zero of $P(x)$

Hence, $P(x) = (x-3)(x^2 - 2x + 2)$
Let $P(x) = 0$
$x^2 - 2x + 2 = 0$
$$x = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2}$$
$$x = \frac{2 \pm \sqrt{-4}}{2}$$
$$x = \frac{2 \pm 2i}{2}$$
$x = 1 \pm i$

$\therefore x = 3, 1 + i, 1 - i$

Question 12 (d)

Criteria	Marks
• Correctly demonstrates algebraically that the contrapositive is true, hence, by equivalence, the original statement must be true.	2
• Makes progress such as writing the correct contrapositive statement.	1

Sample answer:

Consider "If n^2 is even, then n is even",

Contrapositive: "If n is not even, then n^2 is not even".

Let $n = 2k + 1$ where k is an integer.

$n^2 = (2k+1)^2$

$\quad = 4k^2 + 4k + 1$

$\quad = 2(2k^2 + 2k) + 1$

$\quad = 2P + 1 \quad$ where $P = 2k^2 + 2k$ is an integer.

$\therefore n^2$ is not even.

Hence, if the contrapositive is true, then by logical equivalence, the original statement must also be true.

Question 13 (a)

Criteria	Marks
• Correct vector equation, centre and radius of sphere stated, with full working out shown.	3
• Makes significant progress in constructing the vector equation of the sphere and stating its centre and radius, but makes one error in calculation.	2
• Makes progress such as correctly completing the square of the given cartesian equation.	1

Sample answer:

$x^2 - 8x + y^2 + 4y + z^2 + 2z = 4.$

$x^2 - 8x + 16 + y^2 + 4y + 4 + z^2 + 2z + 1 = 25$

$(x-4)^2 + (y+2)^2 + (z+1)^2 = 25$

$\therefore \left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \right| = 5$ is the vector equation of a sphere with centre $(4, -2, -1)$ and $r = 5$

Question 13 (b)

Criteria	Marks
• Correctly proves the statement by induction, with full working out shown and justifies correctly, with mathematical calculations, why p_{k+1} and q_{k+1} are unique positive integers for the $n = k + 1$ case.	3
• Correctly substitutes the assumption case $n = k$ in the process of proving true for the $n = k + 1$ case, however, does not justify whether p_{k+1} and q_{k+1} are unique positive integers with appropriate mathematical calculations.	2
• Correctly shows true for $n = 2$ case with full working out shown.	1

Sample answer:

Show true for $n = 2$

$(1+\sqrt{2})^2 = 3 + 2\sqrt{2}$

$\therefore p_2 = 3$ and $q_2 = 2$ hence unique positive integers.

True for $n = 2$.

Assume true for $n = k$, where k is an integer.

i.e. $(1+\sqrt{2})^k = p_k + q_k\sqrt{2}$ where p_k and q_k are unique positive integers.

Prove true for $n = k + 1$

RTP: $(1+\sqrt{2})^{k+1} = p_{k+1} + q_{k+1}\sqrt{2}$ where p_{k+1} and q_{k+1} are unique positive integers.

$LHS = (1+\sqrt{2})^{k+1}$

$= (1+\sqrt{2})(1+\sqrt{2})^k$

$= (1+\sqrt{2})(\, p_k + q_k\sqrt{2})$ from our assumption case.

$= p_k + q_k\sqrt{2} + p_k\sqrt{2} + 2q_k$

$= p_k + 2q_k + (p_k + q_k)\sqrt{2}$

$= p_{k+1} + q_{k+1}\sqrt{2}$ where $p_{k+1} = p_k + 2q_k$ which is a positive integer and $q_{k+1} = p_k + q_k$ which is also a positive integer.

Also, if $p_{k+1} = q_{k+1}$, then $p_k + 2q_k = p_k + q_k$ which implies that $p_k = p_k - q_k$ which cannot be true since $q_k \neq 0$. Hence, $p_{k+1} \neq q_{k+1}$ and thus are unique.

Hence, true for $n = k + 1$

$\therefore$ Proven true for $n = k + 1$ by showing true for $n = 2$ and assuming true for $n = k$. Therefore, by mathematical induction, must be true for all $n \geq 2$ where n is an integer.

Question 13 (c)(i)

Criteria	Marks
• Correctly calculates the vector projection with full working out shown.	2
• Makes progress such as substituting into the correct vector projection formula, but makes one error in calculation.	1

Sample answer:

$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

$$Proj_{\overrightarrow{OB}}\overrightarrow{OA} = \frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OB}|^2}\overrightarrow{OB}$$

$$= \frac{(3\times -1)+(-2\times 3)+(5\times 2)}{(-1)^2+(3)^2+(2)^2}\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

Question 13 (c)(ii)

Criteria	Marks
• Correct answer provided.	1

Sample answer:

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

Question 13 (d)(i)

Criteria	Marks
• Correctly proves the given statement, with full working out shown.	4
• Makes significant progress by evaluating and rearranging terms and yielding results involving I_n and I_{n-2}, but makes algebraic errors leading up to final result.	3
• Makes progress by correctly applying integration by parts to $\int_0^{\frac{\pi}{4}} \sec^{n-2} x \sec^2 x \ dx$.	2
• Makes some initial progress such as applying the trigonometric identity $\sec^n x = \sec^{n-2} x \sec^2 x$	1

Sample answer:

$$I_n = \int_0^{\frac{\pi}{4}} \sec^n x \ dx$$

$$I_n = \int_0^{\frac{\pi}{4}} \sec^{n-2} x \sec^2 x \ dx$$

Let $u = \sec^{n-2} x$, $v' = \sec^2 x$

$u' = (n-2)(\tan x \sec x)(\sec^{n-3} x)$, $v = \tan x$

$$I_n = [\tan x \sec^{n-2} x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (n-2)(\tan^2 x \sec x)(\sec^{n-3} x)\, dx$$

$$I_n = \left[\left(\sqrt{2}\right)^{n-2} - 0\right] - (n-2)\int_0^{\frac{\pi}{4}} \tan^2 x \sec^{n-2} x \ dx$$

$$I_n = (\sqrt{2})^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\sec^{n-2} x \ dx$$

$$I_n = (\sqrt{2})^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^n x - \sec^{n-2} x \ dx$$

$$I_n = (\sqrt{2})^{n-2} - (n-2)(I_n - I_{n-2})$$

$$I_n = (\sqrt{2})^{n-2} - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n + (n-2)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$$

$$(n-1)I_n = \left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}$$

$I_n = \frac{1}{n-1}\left[\left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}\right]$ for $n \geq 2$

Question 13 (d)(ii)

Criteria	Marks
• Correctly evaluates I_4, with full working out shown.	2
• Makes progress such as finding an expression for I_4 in terms of I_2.	1

Sample answer:

$$I_4 = \frac{1}{3}\left[\left(\sqrt{2}\right)^2 + 2I_2\right] = \frac{1}{3}(2 + 2I_2)$$

$$I_2 = \left(\sqrt{2}\right)^0 = 1$$

$$\therefore I_4 = \frac{1}{3}(2 + 2)$$

$$= \frac{4}{3}$$

Question 14 (a)(i)

Criteria	Marks
• Correctly calculates with full working out shown, the cartesian equation of the circle represented by the equation connecting $Re\left(\frac{z-w_1}{z-w_2}\right)$ and $Im\left(\frac{z-w_1}{z-w_2}\right)$, identifies its centre and radius and expresses in complex modulus form to yield the final result.	4
• Makes significant progress rearranging the equation connecting $Re\left(\frac{z-w_1}{z-w_2}\right)$ and $Im\left(\frac{z-w_1}{z-w_2}\right)$ to yield the cartesian equation of a circle, but makes one errors in calculation.	3
• Uses the property that $\arg\left(\frac{z-w_1}{z-w_2}\right) = \frac{\pi}{6}$ to find an equation connecting $Re\left(\frac{z-w_1}{z-w_2}\right)$ and $Im\left(\frac{z-w_1}{z-w_2}\right)$.	2
• Correctly realises the denominator of $\frac{z-w_1}{z-w_2}$.	1

Sample answer:

$$\frac{z - w_1}{z - w_2} = \frac{(x + iy) - 2(3 + i)}{(x + iy) - 2(1 + i)}$$

$$= \frac{(x-6) + i(y-2)}{(x-2) + i(y-2)} \times \frac{(x-2) - i(y-2)}{(x-2) - i(y-2)}$$

$$= \frac{((x-6) + i(y-2))((x-2) - i(y-2))}{(x-2)^2 + (y-2)^2}$$

$$= \frac{(x-6)(x-2) - i(y-2)(x-6) + i(y-2)(x-2) + (y-2)^2}{(x-2)^2 + (y-2)^2}$$

$$= \frac{(x-6)(x-2) + 4(y-2)i + (y-2)^2}{(x-2)^2 + (y-2)^2}$$

$$\therefore Re\left(\frac{z - w_1}{z - w_2}\right) = \frac{(x-6)(x-2) + (y-2)^2}{(x-2)^2 + (y-2)^2}$$

$$Im\left(\frac{z-w_1}{z-w_2}\right)=\frac{4(y-2)}{(x-2)^2+(y-2)^2}$$

Since $\arg\left(\frac{z-w_1}{z-w_2}\right)=\frac{\pi}{6}$

$$\tan\frac{\pi}{6}=\frac{Im\left(\frac{z-w_1}{z-w_2}\right)}{Re\left(\frac{z-w_1}{z-w_2}\right)}$$

$$\frac{1}{\sqrt{3}}=\frac{Im\left(\frac{z-w_1}{z-w_2}\right)}{Re\left(\frac{z-w_1}{z-w_2}\right)}$$

$$Re\left(\frac{z-w_1}{z-w_2}\right)=\sqrt{3}\,Im\left(\frac{z-w_1}{z-w_2}\right)$$

$$\frac{(x-6)(x-2)+(y-2)^2}{(x-2)^2+(y-2)^2}=\frac{4\sqrt{3}(y-2)}{(x-2)^2+(y-2)^2}$$

$$(x-6)(x-2)+(y-2)^2=4\sqrt{3}(y-2)$$

$$x^2-8x+12+y^2-4y+4=4\sqrt{3}y-8\sqrt{3}$$

$$x^2-8x+y^2-4y-4\sqrt{3}y=-16-8\sqrt{3}$$

$$x^2-8x+y^2-y(4+4\sqrt{3})=-16-8\sqrt{3}$$

$$x^2-8x+16+y^2-y\left(4+4\sqrt{3}\right)+\left(2+2\sqrt{3}\right)^2=\left(2+2\sqrt{3}\right)^2-8\sqrt{3}$$

$$(x-4)^2+\left(y-\left(2+2\sqrt{3}\right)\right)^2=4+8\sqrt{3}+12-8\sqrt{3}$$

$$(x-4)^2+\left(y-2\left(1+\sqrt{3}\right)\right)^2=16$$

Circle with centre $\left(4,2\left(1+\sqrt{3}\right)\right)$ and radius 4 on the complex plane.

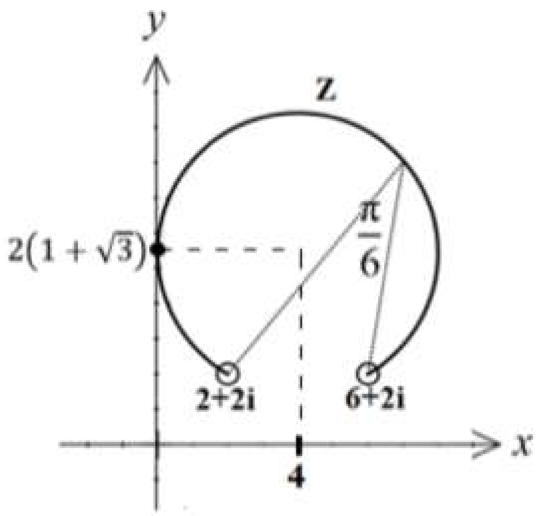
$$\therefore\left|z-\left(4+2\left(1+\sqrt{3}\right)i\right)\right|=4$$

Question 14 (a)(ii)

Criteria	Marks
• Correctly sketches the locus of z, with all important features shown including the y-intercept.	3
• Makes significant progress such as drawing the locus as a major arc of a circle with centre $\left(4, 2(1+\sqrt{3})\right)$ and radius 4 units, where $z = 2 + 2i$ and $z = 6 + 2i$ undefined, but does not label the y-intercept.	2
• Makes progress such as drawing the locus as a major arc of some circle where $z = 2 + 2i$ and $z = 6 + 2i$ are undefined.	1

Sample answer:

From before, we know z must be a point that lies on part of the circumference of a circle with centre $(4, 2(1+\sqrt{3})$ and radius 4 units. Since $\arg\left(\frac{z-w_1}{z-w_2}\right) = \frac{\pi}{6}$, z must lie on the major arc and $z = 2 + 2i$ and $z = 6 + 2i$ must be undefined, with y-intercept at $2(1+\sqrt{3})$.



Question 14 (b)(i)

Criteria	Marks
• Correctly proves that $n = 144$ by considering the conditions required to achieve a unique integer k, with full working out shown.	2
• Makes progress such as achieving the line $\frac{7}{8} < \frac{k}{n} < \frac{8}{9}$ with full working out shown.	1

Sample answer:

$$\frac{9}{17} < \frac{n}{n+k} < \frac{8}{15}$$

Rearrange $\frac{n}{n+k}$ into $\frac{1}{1+\frac{k}{n}}$ to get only ONE term involving k and n.

$$\frac{9}{17} < \frac{1}{1+\frac{k}{n}} < \frac{8}{15}$$

Note that $1 + \frac{k}{n}$ must be positive due to original inequality holding true, so inequality preserved when taking reciprocals.

$$\frac{15}{8} < 1 + \frac{k}{n} < \frac{17}{9}$$

$$\frac{7}{8} < \frac{k}{n} < \frac{8}{9}$$

Multiply both sides by n, inequality is still preserved since n is a positive integer.

$$\frac{7n}{8} < k < \frac{8n}{9}$$

Since there is only one unique integer k that should satisfy the inequality then the maximum difference between the upper and lower boundary can only be 2 units at greatest to ensure that we can still yield a scenario where a single integer lies between them. The case where we have the one unique integer, is if both the upper and lower boundaries are integers as well.

$\therefore \frac{8n}{9} - \frac{7n}{8} \leq 2$ and n must be a number divisible by both 9 and 8.

$\frac{n}{72} \leq 2$ and n must be a number divisible by both 9 and 8.

$n \leq 144$ and n must be a number divisible by both 9 and 8.

Hence, $n = 144$ is divisible by both 9 and 8, so it is the largest possible value to yield a unique integer k that satisfies the inequality.

Question 14 (b)(ii)

Criteria	Marks
• Correctly evaluates the unique integer k with full working out shown.	2
• Makes progress towards solving k by substituting $n = 144$ into the inequality.	1

Sample answer:

Let $n = 144$

$$\therefore \frac{9}{17} < \frac{144}{144 + k} < \frac{8}{15}$$

Consider right side of inequality.

$$144(15) < 8(144) + 8k$$

$$k > 126$$

Consider left side of inequality.

$$9(144) + 9k < 144(17)$$

$$k < 128$$

Hence, $126 < k < 128$

$\therefore$ Only one unique integer $k = 127$ when $n = 144$.

Question 14 (c)

Criteria	Marks
• Correctly proves the statement, by simplifying the expression, using $a+b+c \geq 1$, identifying appropriate AM-GM inequalities with two terms and full working out shown.	4
• Makes significant progress such as correctly identifying and using the AM-GM inequality explicitly to recognise that the terms $a+\frac{1}{a} \geq 2$, $b+\frac{1}{b} \geq 2$ and $c+\frac{1}{c} \geq 2$.	3
• Makes further progress by simplifying the result using $a+b+c \geq 1$ to yield $\left(\frac{a^2+a+b+c}{a}\right)\left(\frac{b^2+a+b+c}{b}\right)\left(\frac{c^2+a+b+c}{c}\right) \geq \left(\frac{a^2+1}{a}\right)\left(\frac{b^2+1}{b}\right)\left(\frac{c^2+1}{c}\right)$ or $\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)$.	2
• Makes some progress such as expanding to yield $\left(\frac{a^2+a+b+c}{a}\right)\left(\frac{b^2+a+b+c}{b}\right)\left(\frac{c^2+a+b+c}{c}\right)$	1

Sample answer:

Consider $\left(\frac{a(a+1)+b+c}{a}\right)\left(\frac{b(b+1)+c+a}{b}\right)\left(\frac{c(c+1)+a+b}{c}\right) = \left(\frac{a^2+a+b+c}{a}\right)\left(\frac{b^2+a+b+c}{b}\right)\left(\frac{c^2+a+b+c}{c}\right)$

Since $a+b+c \geq 1$

$\geq \left(\frac{a^2+1}{a}\right)\left(\frac{b^2+1}{b}\right)\left(\frac{c^2+1}{c}\right)$

$= \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)$

Using AM-GM Inequality we get:

$$\frac{a+\frac{1}{a}}{2} \geq \sqrt{(a)\left(\frac{1}{a}\right)}$$

$$a+\frac{1}{a} \geq 2\sqrt{1}$$

$$a+\frac{1}{a} \geq 2$$

And thus, similarly, we can conclude that
$b+\frac{1}{b} \geq 2$ and $c+\frac{1}{c} \geq 2$

Hence, $\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \geq (2)(2)(2)$

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \geq 8$$

$$\therefore \left(\frac{a(a+1)+b+c}{a}\right)\left(\frac{b(b+1)+c+a}{b}\right)\left(\frac{c(c+1)+a+b}{c}\right) \geq 8$$

Question 15 (a)(i)

Criteria	Marks
• Correctly proves the given result including showing all working out involving integration and substituting initial conditions to find C.	2
• Makes progress such as letting $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and attempts to integrate but makes errors.	1

Sample answer:

$a = -2(x-1)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2(x-1)$

$d\left(\frac{1}{2}v^2\right) = -2(x-1)\,dx$

$\int d\left(\frac{1}{2}v^2\right) = \int -2(x-1)\,dx$

$\frac{1}{2}v^2 = -2\left(\frac{x^2}{2} - x\right) + C$

$v^2 = -2x^2 + 4x + C$

Let $x = 2$ and $v = -\sqrt{6}$ when $t = 0$.

$6 = -2(2)^2 + 4(2) + C$

$6 = -8 + 8 + C$

$C = 6$

$\therefore v^2 = -2x^2 + 4x + 6$

Question 15 (a)(ii)

Criteria	Marks
• Correctly finds all four values, with full working out shown.	4
• Correctly finds three of the 4 required values, with full working out shown.	3
• Correctly finds two of the 4 required values, with full working out shown.	2
• Correctly finds one of the 4 required values, with full working out shown.	1

Sample answer:

$x = B + A\cos(nt + \theta)$

Since
$\ddot{x} = -2(x - 1)$

Then centre of motion at $x = 1$ and $n^2 = 2$.

$\therefore B = 1$ and $n = \sqrt{2}$

Also, when $v = 0$,

$0 = -2x^2 + 4x + 6$

$x = -1$ or 3

Hence, maximum is $x = 3$ and minimum is $x = -1$, thus amplitude of 2 units.

$\therefore A = 2$

$x = 1 + 2\cos(\sqrt{2}t + \theta)$

When $t = 0, x = 2$

$\therefore 2 = 1 + 2\cos\theta$

$\cos\theta = \frac{1}{2}$ for $0 < \theta < \pi$

$\therefore \theta = \dfrac{\pi}{3}$

Hence, $B = 1$, $A = 2$, $n = \sqrt{2}$ and $\theta = \frac{\pi}{3}$.

Question 15 (b)

Criteria	Marks
• Correctly proves that L_1 does not intersect L_2 and finds the shortest distance between them, with full working out shown.	3
• Correctly proves that L_1 does not intersect L_1 by constructing the correct vector equations and demonstrating that they are parallel, by showing that their direction vectors are scalar multiples numerically and stating why they are not concurrent.	2
• Makes some progress such as finding the correct vector equations of L_1 and L_2.	1

Sample answer:

Find the vector equations of L_1 and L_2

$$\underset{\sim}{L_1} = \begin{pmatrix}5\\4\\0\end{pmatrix} + \lambda\begin{pmatrix}5\\4\\3\end{pmatrix} \text{ and } \underset{\sim}{L_2} = \begin{pmatrix}-1\\-6\\-10\end{pmatrix} + \mu\begin{pmatrix}10\\8\\6\end{pmatrix}$$

The direction vectors $\begin{pmatrix}5\\4\\3\end{pmatrix}$ and $\begin{pmatrix}10\\8\\6\end{pmatrix}$ are scalar multiples of each other, where $2\begin{pmatrix}5\\4\\3\end{pmatrix} = \begin{pmatrix}10\\8\\6\end{pmatrix}$

$\therefore L_1 \parallel L_2$

And since there does not exist a λ and μ such that $L_1 = L_2$, then the lines are not concurrent.
Hence, we can conclude that L_1 does not intersect L_2.

Finding shortest distance between L_1 and L_2.

Method 1:

Consider the projection of $\overrightarrow{CA}$ onto $\overrightarrow{CD}$, the distance of point A to the end point of the projection is the shortest distance (perpendicular).

$$\overrightarrow{CA} = \begin{pmatrix}-1\\-6\\-10\end{pmatrix} \text{ and } \overrightarrow{CD} = \begin{pmatrix}10\\8\\6\end{pmatrix}$$

$$Proj_{\overrightarrow{CD}}\overrightarrow{CA} = \frac{(6)(10) + (10)(8) + (10)(6)}{10^2 + 8^2 + 6^2}\begin{pmatrix}10\\8\\6\end{pmatrix} = \begin{pmatrix}10\\8\\6\end{pmatrix}$$

Hence $\overrightarrow{AD} \perp \overrightarrow{CD}$

$|\overrightarrow{AD}|$ will yield the shortest distance

$$|\overrightarrow{AD}| = \sqrt{4^2 + (-2)^2 + (-4)^2}$$
$$= 6 \text{ units}$$

Method 2:

Let P be some point on L_2 such that $|\overrightarrow{AP}|$ is the distance between L_1 and L_2.

$$\overrightarrow{AP} = \begin{pmatrix}-1\\-6\\-10\end{pmatrix} + \mu\begin{pmatrix}10\\8\\6\end{pmatrix} - \begin{pmatrix}5\\4\\0\end{pmatrix}$$

$$\overrightarrow{AP} = \begin{pmatrix}-6\\-10\\-10\end{pmatrix} + \mu\begin{pmatrix}10\\8\\6\end{pmatrix}$$

$$\begin{aligned}|\overrightarrow{AP}| &= \sqrt{(10\mu - 6)^2 + (8\mu - 10)^2 + (6\mu - 10)^2} \\ &= \sqrt{100\mu^2 - 120\mu + 36 + 64\mu^2 - 160\mu + 100 + 36\mu^2 - 120\mu + 100} \\ &= \sqrt{200\mu^2 - 400\mu + 236}\end{aligned}$$

$200\mu^2 - 400\mu + 236$ is a quadratic (concave up parabola) that is minimised when $\mu = \frac{400}{400} = 1$

$$\therefore \text{Minimum value of } |\overrightarrow{AP}| = \sqrt{200(1)^2 - 400(1) + 236}$$
$$= \sqrt{36}$$
$$= 6 \text{ units}$$

Question 15 (c)

Criteria	Marks
• Provides a correct counterexample to disprove the statement being always true.	1

Sample answer:

Let $\frac{d}{dx}\big(f(x)\big) = \frac{d}{dx}\big(g(x)\big) = 1$

Clearly $f(x) = x + 1$ yields $\frac{d}{dx}\big(f(x)\big) = 1$

And $g(x) = x + 2$ yields $\frac{d}{dx}\big(g(x)\big) = 1$

But $f(x) \neq g(x)$.

Hence, by counterexample, statement is not always true.

Question 15 (d)

Criteria	Marks
• Correctly calculates the perpendicular distance from the centre of the circle to chord AB, including proving that $\overrightarrow{CM}$ is perpendicular to $\overrightarrow{AB}$, and calculates all required subsequent steps with full working out shown.	5
• Correctly shows that the dot product between $\overrightarrow{CM}$ and a direction vector of $\overrightarrow{AB}$ equals to 0, hence, proven perpendicular, with full working out shown.	4
• Correctly finds the λ that corresponds to midpoint M and finds the components of vector $\overrightarrow{CM}$, with full working out shown.	3
• Correctly finds the λ that corresponds to points A and B, with full working out shown.	2
• Attempts to substitute the parametric equations of the line into the cartesian equation, with full working out shown.	1

Sample answer:

Let $x = -2 + \lambda$ and $y = 3\lambda$ into Cartesian equation of circle to find λ that will yield point A and B.

$(-2 + \lambda + 1)^2 + (3\lambda - 2)^2 = 4$

$(\lambda - 1)^2 + (3\lambda - 2)^2 = 4$

$$\lambda^2 - 2\lambda + 1 + 9\lambda^2 - 12\lambda + 4 = 4$$

$$10\lambda^2 - 14\lambda + 1 = 0$$

$$\lambda = \frac{14 \pm \sqrt{196 - 4 \times 10 \times 1}}{20}$$

$$\lambda = \frac{7 \pm \sqrt{39}}{10}$$

Hence, midpoint M found by letting $\lambda = \frac{7}{10}$

$$M = \left(-2 + \frac{7}{10}, 3 \times \frac{7}{10}\right)$$

$$= \left(-\frac{13}{10}, \frac{21}{10}\right)$$

Also, $C = (-1, 2)$

$$\overrightarrow{CM} = \begin{pmatrix} -\frac{3}{10} \\ \frac{1}{10} \end{pmatrix}$$

Use direction vector for $\overrightarrow{AB}$ which is $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, check their scalar products.

$$\overrightarrow{CM} \cdot \underset{\sim}{b} = \left(-\frac{3}{10} \times 1\right) + \left(\frac{1}{10} \times 3\right) = -\frac{3}{10} + \frac{3}{10} = 0$$

Hence, $\overrightarrow{CM}$ is perpendicular to $\overrightarrow{AB}$.

$\therefore$ Perpendicular distance from centre of circle to chord AB is the length of vector $\overrightarrow{CM}$.

$$|\overrightarrow{CM}| = \sqrt{\left(-\frac{3}{10}\right)^2 + \left(\frac{1}{10}\right)^2}$$

$= \frac{1}{\sqrt{10}}$ units

$= \frac{\sqrt{10}}{10}$ units

.

Question 16 (a)(i)

Criteria	Marks
• Correctly shows with algebraic calculations that the three scalar products all yield $\frac{1}{2}$.	1

Sample answer:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right| \left|\underset{\sim}{b}\right| \cos \angle AOB = 1 \times 1 \times \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\underset{\sim}{b} \cdot \underset{\sim}{c} = \left|\underset{\sim}{b}\right| \left|\underset{\sim}{c}\right| \cos \angle BOC = 1 \times 1 \times \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{c} = \left|\underset{\sim}{a}\right| \left|\underset{\sim}{c}\right| \cos \angle AOC = 1 \times 1 \times \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = \frac{1}{2}.$$

Question 16 (a)(ii)

Criteria	Marks
• Correctly proves the result using $\left\|\overrightarrow{OM}\right\|^2 = \overrightarrow{OM} \cdot \overrightarrow{OM}$ and showing all necessary algebraic steps including using identities such as $\left\|\overrightarrow{OA}\right\| = \left\|\overrightarrow{OB}\right\| = 1$ and $\overrightarrow{OA} \cdot \overrightarrow{OB} = \frac{1}{2}$, with full working out shown.	2
• Finds a correct expression for $\overrightarrow{OM}$ with respect to $\overrightarrow{OA}$, $\overrightarrow{OB}$ and x.	1

Sample answer:

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$

$$\overrightarrow{OM} = \overrightarrow{OA} + x\overrightarrow{AB}$$

$$\overrightarrow{OM} = \overrightarrow{OA} + x(\overrightarrow{OB} - \overrightarrow{OA})$$

$$\overrightarrow{OM} = (1-x)\overrightarrow{OA} + x\overrightarrow{OB}$$

$$\left|\overrightarrow{OM}\right|^2 = ((1-x)\overrightarrow{OA} + x\overrightarrow{OB}) \cdot ((1-x)\overrightarrow{OA} + x\overrightarrow{OB})$$

$$\left|\overrightarrow{OM}\right|^2 = |(1-x)\overrightarrow{OA}|^2 + 2\left((1-x)\overrightarrow{OA} \cdot x\overrightarrow{OB}\right) + |x\overrightarrow{OB}|^2$$

$$\left|\overrightarrow{OM}\right|^2 = (1-x)^2|\overrightarrow{OA}|^2 + 2x(1-x)\left(\overrightarrow{OA} \cdot \overrightarrow{OB}\right) + x^2|\overrightarrow{OB}|^2$$

Since $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| = 1$ and $\overrightarrow{OA} \cdot \overrightarrow{OB} = \frac{1}{2}$

$$\left|\overrightarrow{OM}\right|^2 = (1-x)^2 + x(1-x) + x^2$$

$$\left|\overrightarrow{OM}\right|^2 = 1 - 2x + x^2 + x - x^2 + x^2$$

$$\left|\overrightarrow{OM}\right|^2 = 1 - x + x^2$$

$$\therefore \left|\overrightarrow{OM}\right| = \sqrt{1 - x + x^2}$$

Question 16 (a)(iii)

Criteria	Marks
• Correctly proves the inequality statement without calculus, with full working out shown.	3
• Makes significant progress such as identifying correct expressions for $f(x)-1$ and $f(x)-\frac{5}{3}$ and claiming that $f(x)-1>0$ and $f(x)-\frac{5}{3}\le 0$, but does not provide justification as to why. No calculus allowed.	2
• Makes some progress such as recognising to use expressions $f(x)-1$ and $f(x)-\frac{5}{3}$ to begin the proof. No calculus allowed.	1

Sample answer:

$f(x)>1$ if and only if $f(x)-1>0$

$\therefore f(x)-1=\frac{1+x-x^2}{1-x+x^2}-1$

$=\frac{(1+x-x^2)-(1-x+x^2)}{1-x+x^2}$

$=\frac{2x-2x^2}{1-x+x^2}$

$=\frac{2x(1-x)}{x^2-x+1}$

Since $2x(1-x)>0$ when $0<x<1$ and $x^2-x+1>0$ for all x, due to positive definite parabola, then $\frac{2x(1-x)}{x^2-x+1}>0$.

Hence, $f(x)-1>0, \therefore f(x)>1$

$f(x)\le\frac{5}{3}$ if and only if $f(x)-\frac{5}{3}\le 0$

$f(x)-\frac{5}{3}=\frac{1+x-x^2}{1-x+x^2}-\frac{5}{3}$

$=\frac{(3+3x-3x^2)-(5-5x+5x^2)}{3(1-x+x^2)}$

$=\frac{-2+8x-8x^2}{3(1-x+x^2)}$

$=\frac{-2(1-4x+4x^2)}{3(1-x+x^2)}$

Sine $1-4x+4x^2\ge 0$, concave up with one root at $x=\frac{1}{2}$, then $-2(1-4x+4x^2)\le 0$ and $3(x^2-x+1)>0$ for all x, due to positive definite parabola, then $\frac{-2(1-4x+4x^2)}{3(1-x+x^2)}\le 0$.

Hence, $f(x)-\frac{5}{3}\le 0, \therefore f(x)\le\frac{5}{3}$

Therefore, proven that $1<f(x)\le\frac{5}{3}$.

Question 16 (a)(iv)

Criteria	Marks
• Correctly proves the result by using known identity from part)iii) to construct an inequality involving $\cos\angle MON$, with full working out shown.	3
• Makes significant progress such as showing that $\cos\angle MON = \frac{1+x-x^2}{2(1-x+x^2)}$, with full working out shown.	2
• Makes progress such as substituting known results into the dot product involving $\angle MON$ to yield: $\cos\angle MON = \frac{((1-x)\overrightarrow{OA}+x\overrightarrow{OB})\cdot((1-x)\overrightarrow{OB}+x\overrightarrow{OC})}{1-x+x^2}$, with full working out shown.	1

Sample answer:

$\cos\angle MON = \frac{\overrightarrow{OM}\cdot\overrightarrow{ON}}{|\overrightarrow{OM}||\overrightarrow{ON}|}$

Note that $\overrightarrow{OM} = (1-x)\overrightarrow{OA} + x\overrightarrow{OB}$ and thus using the same procedure we can say that $\overrightarrow{ON} = (1-x)\overrightarrow{OB} + x\overrightarrow{OC}$ and also $|\overrightarrow{OM}| = |\overrightarrow{ON}|$ due to congruent faces on the regular tetrahedron.

$$\cos\angle MON = \frac{((1-x)\overrightarrow{OA}+x\overrightarrow{OB})\cdot((1-x)\overrightarrow{OB}+x\overrightarrow{OC})}{1-x+x^2}$$

$$= \frac{(1-x)^2(\overrightarrow{OA}\cdot\overrightarrow{OB}) + x(1-x)(\overrightarrow{OA}\cdot\overrightarrow{OC}) + x(1-x)|\overrightarrow{OB}|^2 + x^2(\overrightarrow{OB}\cdot\overrightarrow{OC})}{1-x+x^2}$$

$$= \frac{\frac{1}{2}(1-x)^2+\frac{1}{2}x(1-x)+x(1-x)+\frac{1}{2}x^2}{1-x+x^2}$$

$$= \frac{1-2x+x^2+x-x^2+2x-2x^2+x^2}{2(1-x+x^2)}$$

$$= \frac{1+x-x^2}{2(1-x+x^2)}$$

From part iii) we know that $1 < \frac{1+x-x^2}{1-x+x^2} \leq \frac{5}{3}$

Hence $\frac{1}{2} < \frac{1+x-x^2}{2(1-x+x^2)} \leq \frac{5}{6}$

$\therefore \frac{1}{2} < \cos\angle MON \leq \frac{5}{6}$

$\cos^{-1}\frac{5}{6} \leq \angle MON < \frac{\pi}{3}$

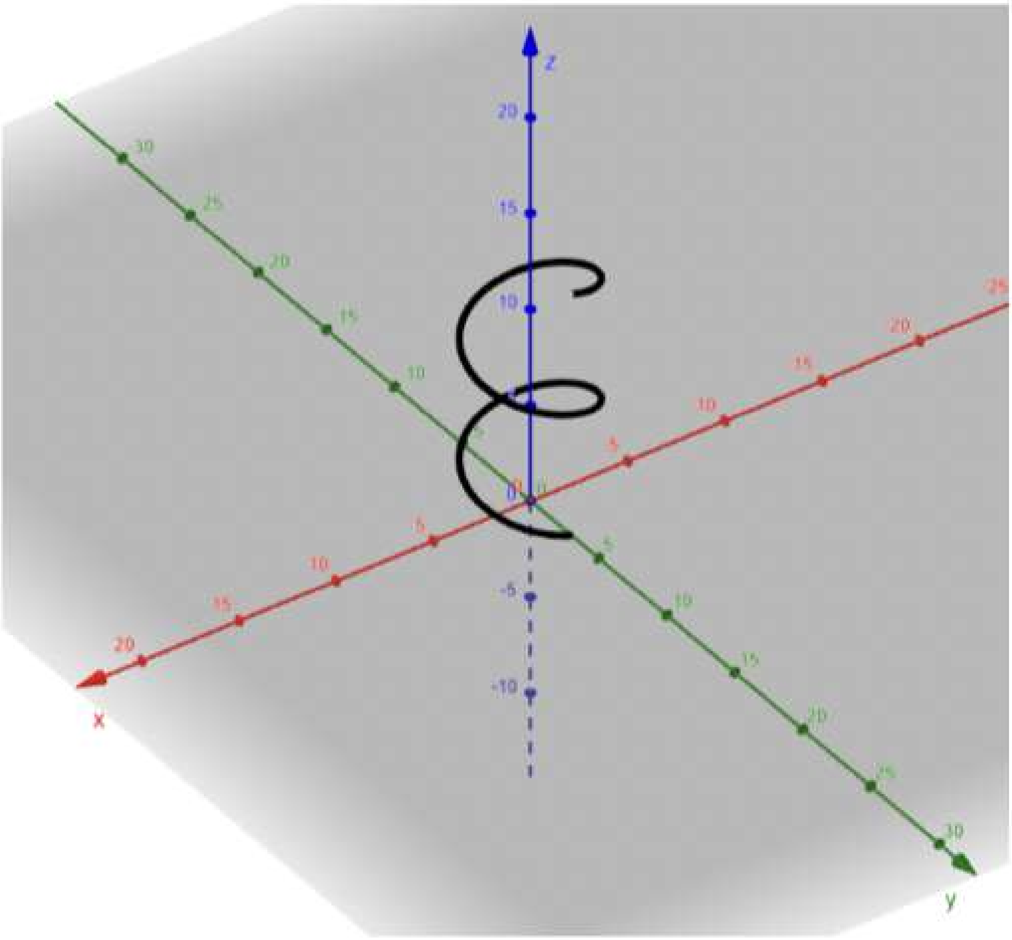
Proven as required.

Question 16 (b)

Criteria	Marks
• Correctly describes the shape of the parametric curve AND sketches the general shape of the graph correctly.	2
• Correctly describes the shape of the parametric curve OR sketches the general shape of the graph only.	1

Sample answer:

A uniform spiral starting from (0,3,0) and completing two cycles with height 4π



Question 16 (c)(i)

Criteria	Marks
• Correctly proves the given result by showing that $LHS = RHS$ using appropriate algebraic manipulations and trigonometric identities with full working out shown.	2
• Makes progress such as completing the square on the denominator and using known trigonometric identities to yield $LHS = \tan^{-1}\left(\frac{\tan A - \tan B}{1+\tan A \tan B}\right)$, where $\tan A = x-2$ and $\tan B = x-3$.	1

Sample answer:

$$LHS = \tan^{-1}\left(\frac{1}{x^2-5x+7}\right)$$

$$= \tan^{-1}\left(\frac{1}{1+(x-2)(x-3)}\right)$$

$$= \tan^{-1}\left(\frac{(x-2)-(x-3)}{1+(x-2)(x-3)}\right)$$

Let $\tan A = x-2$ and $\tan B = x-3$

$$= \tan^{-1}\left(\frac{\tan A - \tan B}{1+\tan A \tan B}\right)$$

Using $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$

$$= \tan^{-1}(\tan(A-B))$$

$$= A - B$$

$$= \tan^{-1}(x-2) - \tan^{-1}(x-3)$$

$$= RHS$$

Question 16 (c)(ii)

Criteria	Marks
• Correctly evaluates I in exact form, with full working out shown.	2
• Makes progress such using integration by parts to correctly evaluate the general integral form of $\tan^{-1} x$, but makes errors during substitution and does not achieve a final solution.	1

Sample answer:

$$I = \int_2^3 \tan^{-1}\left(\frac{1}{x^2-5x+7}\right) dx$$

$$I = \int_2^3 \tan^{-1}(x-2)\ dx - \int_2^3 \tan^{-1}(x-3)\ dx$$

Let $\alpha = x - 2$

$x: 2{\sim}3,\ \alpha: 0{\sim}1$

$\frac{d\alpha}{dx} = 1$

$d\alpha = dx$

Also let $\beta = x - 3$

$x: 2{\sim}3,\ \beta = -1{\sim}0$

$\frac{d\beta}{dx} = 1$

$d\beta = dx$

$I = \int_0^1 \tan^{-1} \alpha \ d\alpha - \int_{-1}^0 \tan^{-1} \beta \ d\beta$

Consider the solution generally to $\int \tan^{-1} x \ dx$ using integration by parts.

Let $u = \tan^{-1} x$ and $v' = 1$

$u' = \frac{1}{1+x^2}$ and $v = x$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \int \frac{x}{1+x^2} \ dx$$

$$= x \tan^{-1} x - \tfrac{1}{2} \ln |1 + x^2|$$

Hence,

$$I = \left[x \tan^{-1} x - \frac{1}{2} \ln|1 + x^2| \right]_0^1 - \left[x \tan^{-1} x - \frac{1}{2} \ln|1 + x^2| \right]_{-1}^0$$

$$I = [(\tan^{-1}(1) - \frac{1}{2} \ln |2|) - 0] - [0 - (-\tan^{-1}(-1) - \frac{1}{2} \ln |2|)]$$

$$I = \left(\frac{\pi}{4} - \frac{1}{2} \ln 2\right) - \left(-\frac{\pi}{4} + \frac{1}{2} \ln 2\right)$$

$$I = \frac{\pi}{2} - \ln 2$$