CRANBROOK SCHOOL

Student Number: ____________________

2023 **YEAR 12 TASK 4**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 180 Minutes
- Write using blue or black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 20 minutes for this section

Section II – 90 marks

- Attempt Questions 11 – 14
- Allow about 160 minutes for this section

Section I

10 marks
Attempt Questions 1–10

Use the multiple-choice answer sheet for Questions 1–10.

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1 In an Argand diagram the points $A(-3,2)$ and $B(5,-4)$ lie at opposite ends of a diameter of a circle. What is the equation of the circle?

A. $|z-1+i|=5$

B. $|z+1-i|=5$

C. $|z-1+i|=10$

D. $|z+1-i|=10$

2 What is the size of the acute angle θ between the vectors $\underset{\sim}{a}=2\underset{\sim}{i}-\underset{\sim}{j}-\underset{\sim}{k}$ and $\underset{\sim}{b}=2\underset{\sim}{i}-2\underset{\sim}{k}$?

A. $\theta=\dfrac{\pi}{6}$

B. $\theta=\dfrac{\pi}{5}$

C. $\theta=\dfrac{\pi}{4}$

D. $\theta=\dfrac{\pi}{3}$

3 Which of the following is an expression for $\displaystyle\int \frac{1}{x^2-\sqrt{3}x+1}\,dx$?

A. $\tan^{-1}\left(x-\sqrt{3}\right)+c$

B. $2\tan^{-1}\left(x-\sqrt{3}\right)+c$

C. $\tan^{-1}\left(2x-\sqrt{3}\right)+c$

D. $2\tan^{-1}\left(2x-\sqrt{3}\right)+c$

4 The amount of apples, bananas and oranges sold by a fruit seller over a year is shown in the table below

Fruit	Amount Sold (tonnes)	Profit ($/tonne)
Apples	25	530
Bananas	55	380
Oranges	10	410

Let $\underset{\sim}{a}=\begin{bmatrix}25\\55\\10\end{bmatrix}$, $\underset{\sim}{b}=\begin{bmatrix}530\\380\\410\end{bmatrix}$ and $\underset{\sim}{c}=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Which of the following expressions calculates the average profit in dollars per tonne of fruit sold over the year?

A. $\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{c}}{\underset{\sim}{b}\cdot\underset{\sim}{c}}$

B. $\dfrac{\underset{\sim}{b}\cdot\underset{\sim}{c}}{\underset{\sim}{a}\cdot\underset{\sim}{c}}$

C. $\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{c}}$

D. $\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{a}\cdot\underset{\sim}{c}}$

5 Which of the following expressions is equivalent to $\int \ln\left(x^2+1\right) dx$?

A. $x\ln\left(x^2+1\right)-2x+2\tan^{-1}x+c$

B. $x\ln\left(x^2+1\right)-2\ln\left(x^2+1\right)+c$

C. $\ln\left(x^2+1\right)-2x+2\tan^{-1}x+c$

D. $\ln\left(x^2+1\right)-x\ln\left(x^2+1\right)+c$

6 Consider the complex, non-real cube roots of unity ω and ω^2.

What is the value of $\left(1-\omega+\omega^2\right)\left(1+\omega-\omega^2\right)$?

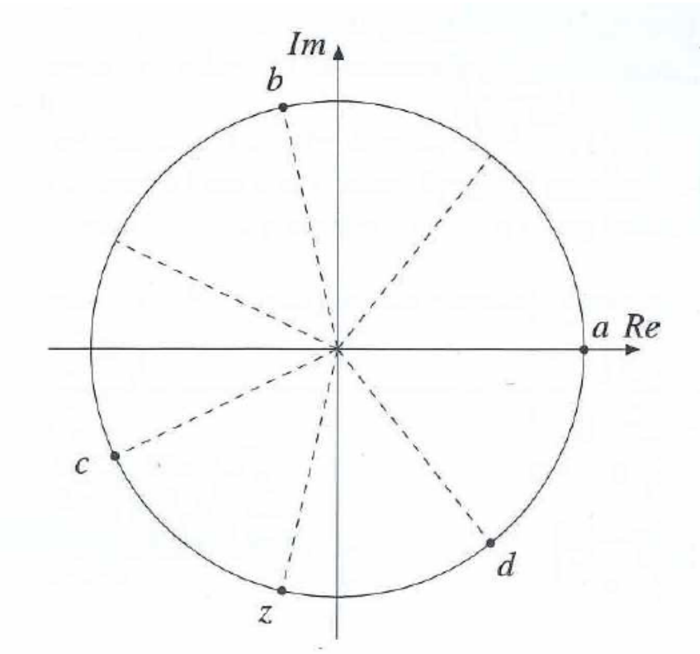
A. 0

B. 1

C. 2

D. 4

7 The complex numbers a, b, c, d and z are solutions to $z^7 = 1$ as shown in the Argand diagram below.



Which of the following is a cube root of z?

A. a

B. b

C. c

D. d

8 A sequence of complex numbers $z_1, z_2, z_3, z_4, \ldots$ is given by the rule $z_1 = Z$ and $z_{n+1} = c\bar{z}_n + c - 1$ for $n \in \mathbb{Z}^+$ where c is a complex number with modulus 1. What is the value of z_3?

A. $z_3 = Z$

B. $z_3 = -2 + Z$

C. $z_3 = 2c + Z$

D. $z_3 = -2 + 2c + Z$

9 Consider a line that passes through the point $(5, 2, 1)$ and is parallel to the $x-y$ plane and the $x-z$ plane.

Which of the following is the vector equation of the line?

A. $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

10 The function $F(x)$ is the primitive of $f(x)$, that is $F'(x) = f(x)$. Which of the following is true?

A. $\displaystyle \int \left(\frac{d}{dx} \int_a^b f(x)\,dx \right) dx = f(b) - f(a)$

B. $\displaystyle \int_a^b \left(\frac{d}{dx} \int f(x)\,dx \right) dx = f(b) - f(a)$

C. $\displaystyle \frac{d}{dx} \int \left(\int_a^b f(x)\,dx \right) dx = F(b) - F(a)$

D. $\displaystyle \frac{d}{dx} \int_a^b \left(\int f(x)\,dx \right) dx = F(b) - F(a)$

Section II

90 marks
Attempt Questions 11 – 16

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Please begin a new Writing Booklet.

(a) Express $2\sqrt{2}e^{-\frac{3\pi}{4}i}$ in the form $x+iy$. **2**

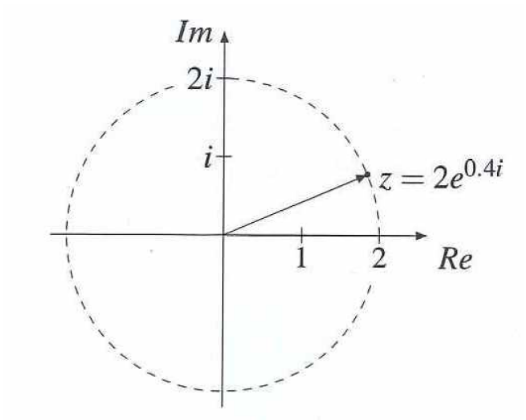
(b) Consider the two points $A(2,2,2)$ and $B(2,-2,2)$.

(i) Find $\overrightarrow{AB}$ **1**

(ii) Find $|\overrightarrow{AB}|$ **1**

(iii) Find $\angle AOB$. Give your answer to the nearest degree. **2**

(c) Consider the complex number $z = 2e^{0.4i}$, as sketched below. **3**



Copy and clearly label this Argan diagram, and sketch the four points represented by z, $\bar{z}$, $-\bar{z}$ and $z-\bar{z}$.

(d) Use the substitution $t = \tan\dfrac{x}{2}$ to find $\displaystyle\int \frac{1}{1+\sin x}\,dx$. **3**

(e) Show for any complex numbers z and w, that $\overline{zw} = \overline{z} \times \overline{w}$. **3**

End of Question 11

Question 12 (15 marks) Please begin a new Writing Booklet.

(a) Consider the two lines

$$L_1 : \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} \text{ and } L_2 : \begin{bmatrix} -1 \\ 2 \\ -6 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}.$$

(i) Find the values of k given $B(9, k, 24)$ lies on L_2. **2**

(ii) Find the point of intersection of L_1 and L_2. **3**

(b) Find $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1-x^2}}\,dx$ **4**

(c) Solve $z^2 + (7-i)z + 16 + 4i = 0$ **3**

(d) Find $\displaystyle\int \frac{x^3 - 2}{x^3 - x}\,dx.$ **3**

End of Question 12

Question 13 (15 marks) Please begin a new Writing Booklet.

(a) On an Argand diagram shade the region containing all points representing the complex numbers z that satisfy both $|z-2i| \leq 2$ and $\dfrac{\pi}{6} \leq \arg z \leq \dfrac{\pi}{4}$. **2**

(b) The graph of a polynomial function $f(x) = (x+3)(x-2)(x^2+bx+c)$ has a y intercept of -6 and passes through $(1,-4)$.

(i) Find the two complex roots of the equation $f(x) = 0$. **2**

(ii) Express these two complex roots in the form $r(\cos\theta + i\sin\theta)$. **2**

(iii) Plot all four solutions to $f(x) = 0$ on an Argand diagram. **2**

(iv) Write down the name of the quadrilateral formed by these four points. **1**

(c) Consider the sphere with vector equation $\left| \underset{\sim}{r} - \begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix} \right| = 3$.

(i) Show that the point $(5,-10,3)$ lies on the sphere. **1**

(ii) Find the point on the sphere farthest from the origin. **2**

(d) Prove by Mathematical induction that $3n^2 - 3n \leq 2^n - 1$ for $n \geq 7$. **3**

End of Question 13

Question 14 (15 marks) Please begin a new Writing Booklet.

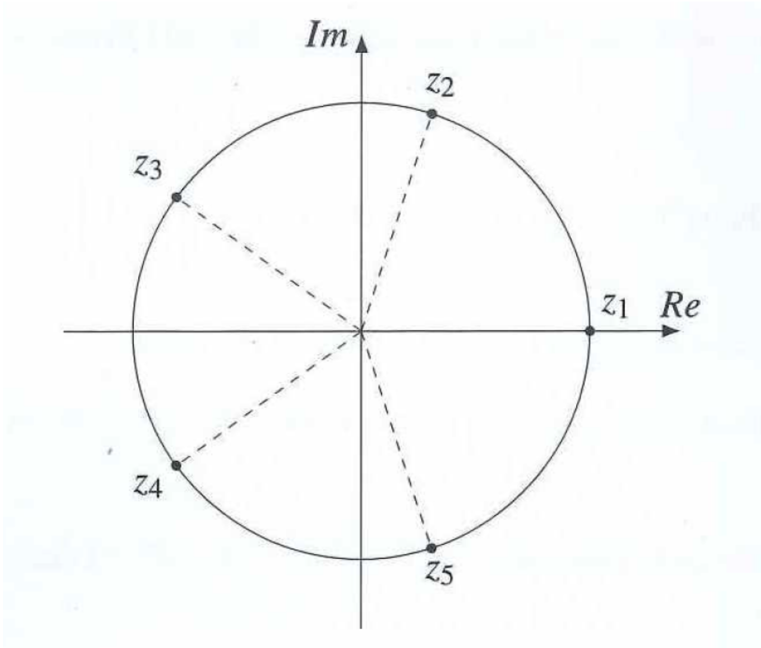
(a) The number z is a fifth root of unity where $z \neq 1$, that is $z^5 = 1$

(i) Show that $\left(z+z^{-1}\right)^2 + \left(z+z^{-1}\right) - 1 = 0$. **2**

(ii) If $z = e^{i\theta}$, show $\cos\theta = \dfrac{z+z^{-1}}{2}$. **1**

(iii) Hence show $\cos\dfrac{2\pi}{5} = \dfrac{\sqrt{5}-1}{4}$ **2**

(iv) Consider the five fifth roots of unity z_1, z_2, z_3, z_4 and z_5 as shown in the diagram below. **3**



Show that $\left|\dfrac{z_3 - z_1}{z_2 - z_1}\right| = \dfrac{1+\sqrt{5}}{2}$.

Question 14 continues on page 13

Question 14 (continued)

(b) The position of an object afer t seconds is given by the vector equation

$$\underset{\sim}{r} = \cos\frac{\pi t}{4}\underset{\sim}{i} + \left(\cos\frac{\pi t}{4} + \sin\frac{\pi t}{4}\right)\underset{\sim}{j} + \sin\frac{\pi t}{4}\underset{\sim}{k}.$$

(i) What is the position of the object after 3 seconds? **1**

(ii) Find the vector equation of the tangent to the path taken by the object after 3 seconds. **3**

(c) Find $\displaystyle\int_0^9 \frac{1}{\sqrt{1+\sqrt{x}}}\,dx$ **3**

End of Question 14

Question 15 (15 marks) Please begin a new Writing Booklet.

(a) Consider the function $f(x) = e^{-x}\cos x$, with domain $x \in \left[0, \dfrac{3\pi}{2}\right]$. Show that the ratio of the area above the x-axis to the area below the x-axis is **4**

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}$$

(b) A series is defined by $T_n = 6T_{n-1} - 9T_{n-2}$ for $n \geq 2$ where $T_0 = 1$ and $T_1 = 6$. **4**

Use Mathematical induction to prove $T_n = 3^n + n \times 3^n$.

(c) Let $I_n = \displaystyle\int \frac{\cos nx}{\sin x}\,dx$.

(i) Show that $\cos((n-2)x) - \cos nx = 2\sin((n-1)x)\sin x$. **2**

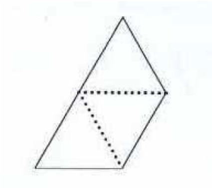
(ii) Show that $I_n - I_{n-2} = \dfrac{2\cos((n-1)x)}{n-1} + c$ **2**

(iii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\cos 2x - \cos 6x}{\sin x}\,dx$ **3**

End of Question 15

Question 16 (15 marks) Please begin a new Writing Booklet.

(a) Consider the tile below consisting of three equilateral triangles of side length 1 unit. **3**



Prove the following result for positive integers n, using Mathematical induction:

An equilateral triangle of side length 2^n units may be covered by the tiles shown above (in any orientation) such that a single equilateral triangle of side length 1 unit is left over at one of the vertices. The tiles may not overlap.

(b) The limiting sum of series is be given by $S = \dfrac{a}{1-r}$ where a is the first term of the series and r is the common ratio between consecutive terms. It only exists when $|r| \leq 1$.

(i) Show that the limiting sum S of the series $1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \frac{1}{8}e^{3i\theta} + ...$ is given by $S = \dfrac{2}{2 - e^{i\theta}}$. **2**

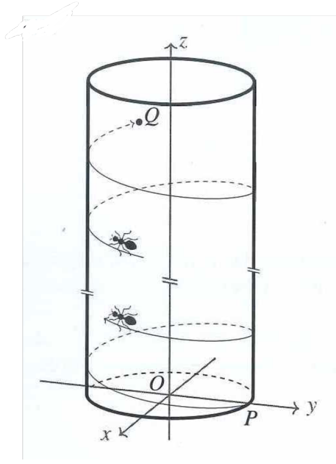
(ii) Hence show that $\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + ... = \dfrac{2\sin\theta}{5 - 4\cos\theta}$. **2**

(iii) Show that there exists no real values of θ such that S is purely imaginary. **1**

Question 16 continues on page 16

Question 16 (continued)

(c) An ant follows a spiral path up a cylindrical column from $P(0,7,0)$ to the point Q as shown in the diagram below.



The ant's position $\underset{\sim}{r}$ in centimetres after t seconds is given by the vector equation below.

$$\underset{\sim}{r}(t) = 7\sin\frac{\pi t}{32}\underset{\sim}{i} + 7\cos\frac{\pi t}{32}\underset{\sim}{j} + \frac{t}{15}\underset{\sim}{k}$$

(i) Find the coordinates of the point Q if $\left|\overrightarrow{OQ}\right| = 25$. **3**

(ii) How many times has the ant crossed the line $x = 7$ on its journey to Q? **2**

(iii) The ant crawls back from Q to P along the shortest path possible. How far did it crawl on this leg of its journey? Give you answer correct to one decimal place. **2**

End of Exam

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2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

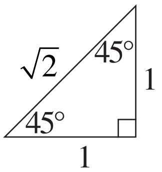
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

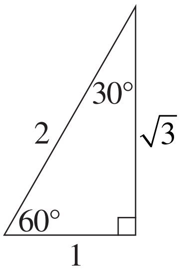
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

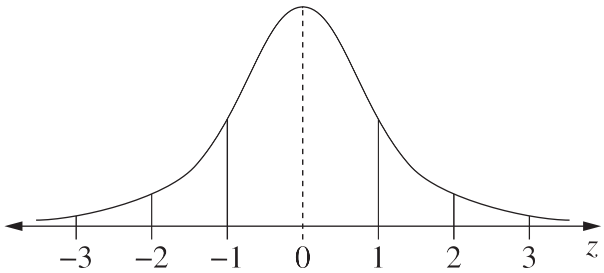
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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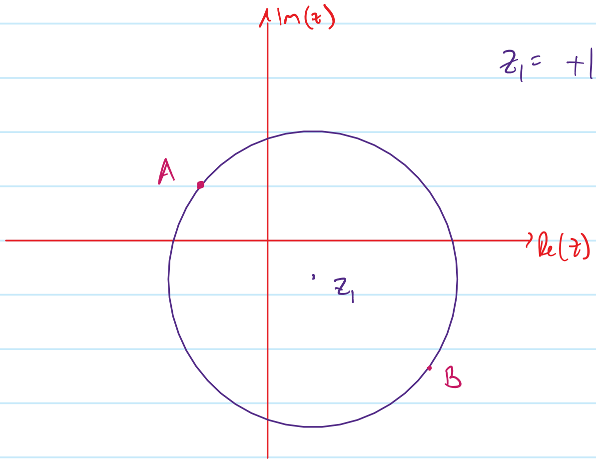
12 Ext 2 2023 Task 4 - RDS Solutions MC

Monday, 31 July 2023 12:23 PM

Q1



$z_1 = +1 - i$

$\therefore$ Centre @ $z = 1 - i$

Radius $= \frac{1}{2}\left(\sqrt{8^2 + 6^2}\right)$

$= 5$

$\therefore \quad |z - (1 - i)| = 5 \quad \therefore$ A

Q2

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$|\underset{\sim}{a}| = \sqrt{6} \qquad |\underset{\sim}{b}| = 2\sqrt{2}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 4 + 0 + 2$

$= 6$

$\therefore \quad \cos\theta = \dfrac{6}{2\sqrt{2} \cdot \sqrt{6}}$

$= \sqrt{3}/2$

$\theta = \pi/6 \quad \therefore$ A

Q3

$$\int \frac{1}{x^2 - \sqrt{3}x + 1} \, dx$$

$$= \int \frac{1}{\frac{1}{4}\left((2x - \sqrt{3})^2 + 1\right)} \, dx$$

$$= 4 \tan^{-1}\left(\frac{2x - \sqrt{3}}{1}\right) \div 2 + C$$

$x^2 - \sqrt{3}x + 1$

$= x^2 - \sqrt{3}x + \left(\frac{\sqrt{3}}{2}\right)^2 + 1 - \left(\frac{\sqrt{3}}{2}\right)^2$

$= \left(x - \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}$

$= \frac{1}{4}\left(2x - \sqrt{3}\right)^2 + \frac{1}{4}$

$= \frac{1}{4}\left[(2x - \sqrt{3})^2 + 1\right]$

$= 2\tan^{-1}(2x-\sqrt{3}) + c \quad \therefore$ D

Q4 $\bar{x} = \dfrac{\text{Amount} \times \text{Profit}}{\text{total fruit sold}}$

$\text{Amount} \times \text{Profit} = \underset{\sim}{a} \cdot \underset{\sim}{b}$

$\text{total fruit} = \underset{\sim}{a} \cdot \underset{\sim}{c}$

$\therefore \bar{x} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{a} \cdot \underset{\sim}{c}} \quad \therefore$ D

Q5 $\int \ln(x^2+1)\, dx$

$u = \ln(x^2+1) \qquad v = x$

$\dfrac{du}{dx} = \dfrac{2x}{x^2+1} \qquad \dfrac{dv}{dx} = 1$

$\therefore I = x\ln(x^2+1) - \int \dfrac{2x^2}{x^2+1}\, dx$

$= x\ln(x^2+1) - 2\int \dfrac{x^2+1}{x^2+1} - \dfrac{1}{x^2+1}\, dx$

$= x\ln(x^2+1) - 2(x - \tan^{-1}x) + c$

$\therefore$ A

Q6 $(1-\omega+\omega^2)(1+\omega-\omega^2)$

$= (-2\omega)(-2\omega^2)$

$= 4\omega^3$

$= 4 \quad \therefore$ D

$1+\omega+\omega^2 = 0$

$\therefore 1+\omega^2 = -\omega$

$1+\omega = -\omega^2$

$\omega^3 = 1$

Q7 $\text{Arg}(a) = 0 \quad \times 3 = 0$

$\text{Arg}(b) = \frac{4\pi}{7} \times 3 = \frac{12\pi}{7}$

$\text{Arg}(c) = \frac{8\pi}{7} \times 3 = \frac{24\pi}{7} = \frac{10\pi}{7}$

$\text{Arg}(d) = \frac{12\pi}{7} \times 3 = \frac{36\pi}{7} = \frac{12\pi}{7}$

$\text{Arg}(z) = \frac{10\pi}{7}$

$\therefore$ C

Q8 $z_{n+1} = c\bar{z}_n + c - 1$

$z_2 = c \times \bar{z} + c - 1$

$z_3 = c \times \left[\overline{c\bar{z} + c - 1}\right] + c - 1$

$= c \times [\bar{c}z + \bar{c} - 1] + c - 1$

$= c \times \bar{c}\, z + \cancel{c\bar{c}} - \cancel{c} + \cancel{c} - \cancel{1}$ $\qquad c \times \bar{c} = 1$

$= z \quad \therefore$ A

Q9 To be parallell to x-y plane & the x-z plane the line must be parallel to the x axis.

$\therefore$ line : $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\therefore$ C

Q10 $\frac{d}{dx}\int_a^b f(x)\, dx = 0$ $\quad$ (As $\int_a^b f(x)\, dx$ gives a constant

$\therefore$ Not A or D

$\frac{d}{dx}\int f(x)\, dx = \frac{d}{dx}\left(F(x) + c\right)$

$= f(x)$

$$= f(x)$$

$$\therefore \int_a^b f(x)\, dx = F(b) - F(a) \quad \therefore \text{Not } B$$

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

$$\int F(b) - F(a)\, dx = [F(b) - F(a)]x + C$$

$$\frac{d}{dx}[F(b) - F(a)]x = F(b) - F(a) \quad \therefore C$$

Answers.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
A	A	D	D	A	D	C	A	C	C

12 Ext 2 2023 Task 4 - RDS Solutions Q11

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(a)
$$2\sqrt{2}e^{-\frac{3\pi}{4}i} = 2\sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin-\frac{3\pi}{4}\right)$$
$$= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \quad (1)$$
$$= -2-2i \quad (1)$$

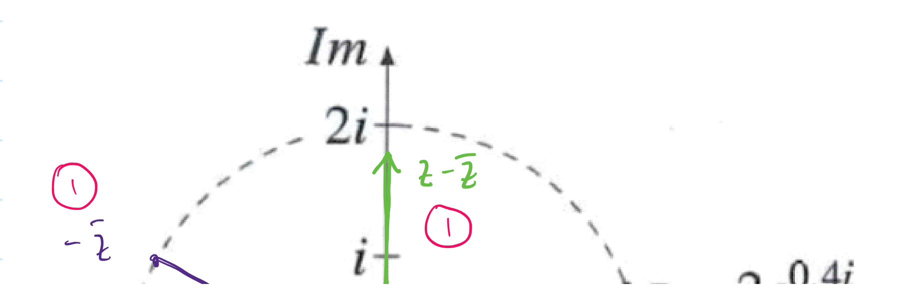
(b) $A(2,2,2)$, $B(2,-2,2)$

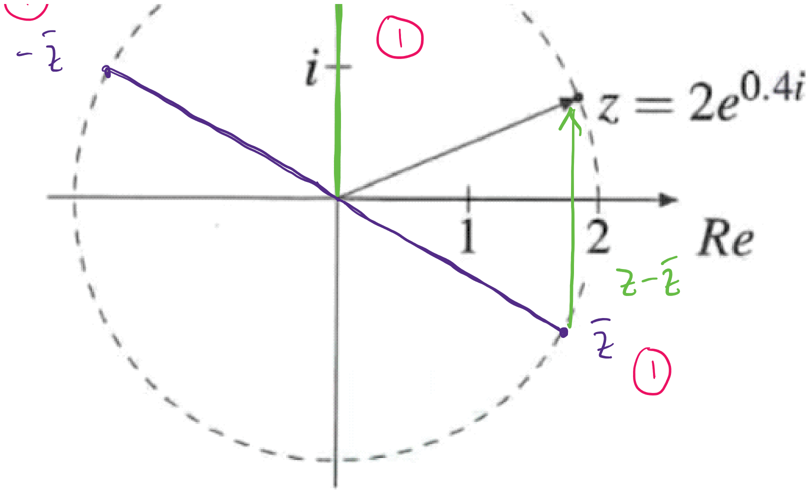
(i)
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= -4\underset{\sim}{j}$$

(ii)
$$|\overrightarrow{AB}| = 4$$

(iii)
$$\cos\theta = \frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|} \quad (1)$$
$$= \frac{4}{2\sqrt{3}\times2\sqrt{3}}$$
$$= \frac{1}{3}$$
$$\therefore \theta = \cos^{-1}\left(\frac{1}{3}\right)$$
$$\doteqdot 71^\circ \text{ (Nearest degree)} \quad (1)$$

(c)





(d) $t = \tan\frac{x}{2}$

$\therefore x = 2\tan^{-1} t$

$\frac{dx}{dt} = 2\left(\frac{1}{1+t^2}\right)$

$\therefore dx = \frac{2dt}{1+t^2}$ 1

$$\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$$ 1

$$= \int \frac{2}{t^2+2t+1}\,dt$$

$$= \int \frac{2}{(t+1)^2}\,dt$$

$$= -2(t+1)^{-1} + c$$

$$= -2\left[\tan\left(\frac{x}{2}\right) + 1\right]^{-1} + c$$ 1

(e) $\overline{zw} = \bar{z} \times \bar{w}$

let $z = a+ib$, $w = x+iy$ 1

let $z = a + ib$, $\omega = x + iy$ 1

$$\overline{z\omega} = \overline{(a+ib)(x+iy)}$$

$$= \overline{(ax + ibx + iay - by)}$$

$$= \overline{(ax - by) + i(bx + ay)}$$

$$= (ax - by) - i(bx + ay)$$

1

$$\bar{z} \times \bar{\omega} = (a - ib)(x - iy)$$

$$= ax - iay - ibx - by$$

$$= (ax - by) - i(ay + bx)$$

$$= \overline{z\omega}$$

1

As required.

12 Ext 2 2023 Task 4 - RDS Solutions Q12

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(a) (i) $L_2 : \begin{bmatrix} -1 \\ 2 \\ -6 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$ $B \begin{bmatrix} 9 \\ k \\ 24 \end{bmatrix}$

Comparing x : $-1 + 2\mu = 9$

$\therefore \mu = 5$ (1)

Comparing y : $2 - 3\mu = k$

$2 - 15 = k$

$k = -13$ (1)

(ii) $L_1 : x = 7 + \lambda$ $L_2 : x = -1 + 2\mu$

$y = -1 + 3\lambda$ $y = 2 - 3\mu$

$z = 2 - 5\lambda$ $z = -6 + 6\mu$

Solve x/y

$7 + \lambda = -1 + 2\mu$ $-1 + 3\lambda = 2 - 3\mu$

$\lambda - 2\mu = -8$ ... (1) $3\lambda + 3\mu = 3$

$\lambda + \mu = 1$ ... (2)

(1) − (2)

$-3\mu = -9$

$\mu = 3$

$\therefore \lambda = -2$

Sub into z equations

When $\lambda = -2$ $\mu = 3$

$L_1 : z = 12$ $z = 12$ $\therefore$ Intersect.

$\therefore$ Point of intersection (sub $\lambda = -2$ into L_1)

$P \begin{pmatrix} 5 \\ -7 \\ 12 \end{pmatrix}$

(b) $I = \int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1 - x^2}} \, dx$ let $x = \sin\theta$ $\frac{dx}{d\theta} = \cos\theta$ $x = \frac{1}{\sqrt{2}}, \ \theta = \frac{\pi}{4}$ $x = 0, \ \theta = 0$ (1)

$\therefore I = \int_0^{\frac{\pi}{4}} \frac{\sin^3\theta}{\sqrt{1 - \sin^2\theta}} \, d\theta \cdot \cos\theta$ (1)

$= \int_0^{\frac{\pi}{4}} \frac{\sin^3\theta}{\cancel{\sin\theta}} \cancel{\sin\theta} \, d\theta$

$= \int_0^{\frac{\pi}{4}} \sin\theta \left(1 - \cos^2\theta\right) d\theta$

$$= \int_0^{\frac{\pi}{4}} \sin\theta\,(1-\cos^2\theta)\,d\theta$$

$$= \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{4}} \quad (1)$$

$$= \left(-\frac{1}{\sqrt{2}} + \frac{1}{3}\times\frac{1}{2\sqrt{2}}\right) - \left(-1+\frac{1}{3}\right)$$

$$= \frac{8-5\sqrt{2}}{12} \quad (1)$$

(c) $z^2 + (7-i)z + 16+4i = 0$

$$\Delta = (7-i)^2 - 4(16+4i)$$

$$= 49 - 14i + 1 - 64 - 16i$$

$$= -16 - 30i \quad (1)$$

let $(x+iy)^2 = -16-30i \qquad x, y \in \mathbb{R}$

$\therefore x^2 - y^2 = -16 \quad \ldots (2)$ & $\quad 2xy = -30$

$y = \frac{-15}{x} \quad \ldots (1)$

sub (1) into (2)

$$x^2 - \frac{225}{x^2} = -16$$

$$x^4 + 16x^2 - 225 = 0$$

$$(x^2+25)(x^2-9) = 0 \qquad x \in \mathbb{R}$$

$\therefore x = \pm 3, \quad y = \mp 5$

$\therefore \sqrt{\Delta} = \pm(3-5i) \quad (1)$

$$\therefore z = \frac{-(7-i) \pm (3-5i)}{2}$$

$$z = \frac{-10+6i}{2}, \quad \frac{-4-4i}{2}$$

$$= -5+3i, \quad -2-2i \quad (1)$$

(d) $\displaystyle\int \frac{x^3-2}{x^3-x}\,dx = \int \frac{x^3-2}{x(x^2-1)}\,dx$

$$= \int \frac{x^3-2}{x(x-1)(x+1)}\,dx$$

let $\displaystyle\frac{x^3-2}{x(x-1)(x+1)} = A + \frac{B}{x} + \frac{C}{x+1} + \frac{D}{x-1}$ (1) Attempting partial fractions

$$x^3-2 = Ax(x-1)(x+1) + B(x+1)(x-1) + Cx(x-1) + Dx(x+1)$$

When $x=0$ $\quad -2 = 0 - B + 0 + 0$

$B = 2$

$x=1$ $\quad -1 = 0 + 0 + 0 + 2D$

$D = -\frac{1}{2}$

$x=-1$ $\quad -3 = 0 + 0 + 2C + 0$

$C = -\frac{3}{2}$

$x=2$ $\quad 6 = 6A + 3B + 2C + 6D$

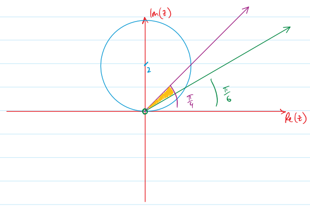
$6A = \cancel{6} - \cancel{6} + 3 + 3$

$= 1$

$$\therefore \frac{x^3-2}{x^3-x} = 1 + \frac{2}{x} - \frac{3}{2(x+1)} - \frac{1}{2(x-1)} \quad 1$$

$$\therefore \int 1 + \frac{2}{x} - \frac{3}{2(x+1)} - \frac{1}{2(x-1)} \, dx = x + 2\ln|x| - \frac{3}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C \quad 1$$

(a)



① region
① All graphs correct

(b) $f(x) = (x+3)(x-2)(x^2+bx+c)$

(i) $f(0) = -6$, $f(1) = -4$

$\therefore -6 = -6c$

$\therefore c = 1$

$-4 = -4(2+b)$

$2+b = 1$

$b = -1$ ①

$\therefore f(x) = (x+3)(x-2)(x^2-x+1)$

Solving $x^2-x+1 = 0$

$x = \dfrac{1 \pm \sqrt{1-4}}{2}$

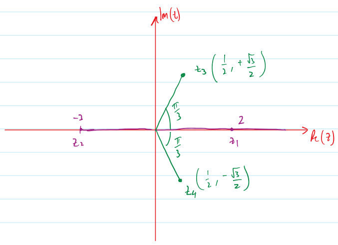
$= \dfrac{1 \pm \sqrt{3}i}{2}$ ①

(ii) $\left|\dfrac{1 \pm \sqrt{3}i}{2}\right| = 1$

$\text{Arg}\left(\dfrac{1+\sqrt{3}i}{2}\right) = \dfrac{\pi}{3}$ $\text{Arg}\left(\dfrac{1-\sqrt{3}i}{2}\right) = \dfrac{-\pi}{3}$ Any correct working ①

$\therefore$ roots are $1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ & $1\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)$ ①

(iii)



$z_1 = -3$

$z_2 = +2$

$z_3 = \text{cis}\,\frac{\pi}{3}$

$z_4 = \text{cis}\,\frac{-\pi}{3}$ Where $\text{cis}\theta = \cos\theta + i\sin\theta$

① All points correct

① All information added.

(iv) Kite → Diagonal are perpendicular

→ Adjacent sides equal ①

(c) (i) $\begin{bmatrix} 5-3 \\ -10+12 \\ 3-4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

$\left|\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}\right| = \sqrt{9} = 3$

$\therefore$ Satisfies the equation

$\therefore$ lies on the sphere. ①

(ii) let the point on the sphere with maximum distance from the origin be $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$

As this goes through the centre of the sphere

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = k\begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix} \cdots ①$ As the vector will be parallel to the position vector of the centre.

We also know that our vector lies on the sphere

$\therefore \left|\begin{matrix} a-3 \\ b+12 \\ c-4 \end{matrix}\right| = 3 \cdots ②$ ①

→ OR $\underset{\sim}{r_0} = \text{centre} = \begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix}$

$\hat{\underset{\sim}{r_0}} = \dfrac{1}{\sqrt{3^2+(12)^2+(4)^2}}$

$= \dfrac{1}{13}\begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix}$

$\therefore$ Furthest point is scalar multiple of centre position vector, (+3) to magnitude

$\therefore \dfrac{1}{13}\begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix} \times (3+13)$

$\begin{vmatrix} b+12 \\ c-4 \end{vmatrix}$

Substituting ① into ②

$$\begin{vmatrix} 3k-3 \\ -12k+12 \\ 4k-4 \end{vmatrix} = 3$$

$\therefore \quad 9(k-1)^2 + 144(k-1)^2 + 16(k-1)^2 = 9$

$169(k-1)^2 = 9$

$(k-1)^2 = \frac{9}{169}$

$k-1 = \frac{3}{13}$ ($k>0$ As we want maximum)

$k = \frac{16}{13}$

$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{16}{13}\begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix}$

vector, (+3) to magnitude

$\therefore \frac{1}{13}\begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix} \times (3+13)$

$= \frac{16}{13}\begin{bmatrix} 3 \\ -12 \\ 4 \end{bmatrix}$

(d) Prove by Mathematical induction that $3n^2 - 3n \leq 2^n - 1$ for $n \geq 7$. **3**

STEP ①: Prove true for $n=7$.

$n=7$

LHS $= 3(7)^2 - 3(7) = 147 - 21 = 126$

RHS $= 2^7 - 1 = 128 - 1 = 127$

RHS $\geq$ LHS

$\therefore$ True for $n=7$. ①

STEP ②: Assume true for $n=k$

ie: Assume $3k^2 - 3k \leq 2^k - 1$

$3k^2 \leq 2^k + 3k - 1$

$k^2 \leq \frac{2^k}{3} + k - \frac{1}{3}$

OR $2^k \geq 3k^2 - 3k + 1$

STEP ③: Prove true for $n=k+1$

RTP: $3(k+1)^2 - 3(k+1) \leq 2^{k+1} - 1$

LHS $= 3(k+1)^2 - 3(k+1)$

$= 3(k^2+2k+1) - 3k - 3$

$= 3k^2 + 6k + 3 - 3k - 3$

$= 3k^2 + 3k$

$= 3k^2 - 3k + 6k$

$\leq 2^k - 1 + 6k$, from assumption.

$\leq 2^{k+1} - 1$

↳ since: $2^{k+1} - 1 \geq 2^k - 1 + 6k$. $2^{k+1} \geq 2^k + 6k$, true for $k \geq 7$

$\therefore$ True for $n=k+1$

① use of assumption correctly.

① Proof. correct.

CONCLUSION:

$\therefore$ Proven true by MI.

③ RTP: $3(k+1)^2 - 3(k+1) \leq 2^{k+1} - 1$

$3(k^2+2k+1) - 3k - 3 \leq 2^{k+1} - 1$

$3k^2 + 6k + 3 - 3k - 3 \leq 2^{k+1} - 1$

$3k^2 + 3k + 1 \leq 2^{k+1}$

RHS $= 2^{k+1}$

$= 2(2^k)$

$\geq 2(3k^2 - 3k + 1)$

$\geq 3k^2 + 3k + 1$

↳ since $2(3k^2 - 3k + 1) \geq 3k^2 + 3k + 1$

$3k^2 - 9k + 1 \geq 0$, true for $k \geq 7$.

$\therefore$ True for $n=k+1$

(a) $z^5 = 1$

$\therefore z^5 - 1 = 0 \quad , \quad z \neq 1$

(i) RTP: $(z + z^{-1})^2 + (z + z^{-1}) - 1 = 0$

$$\begin{aligned}
LHS &= z^2 + 2 + z^{-2} + z + z^{-1} - 1 \\
&= \underbrace{z^{-1}}_{\times z^5} + \underbrace{z^{-2}}_{\times z^5} + z^2 + z + 1 \\
&= z^4 + z^3 + z^2 + z + 1
\end{aligned}$$

1

$$= \frac{(z-1)(z^4 + z^3 + z^2 + z + 1)}{z-1} \quad (\text{As } z \neq 1)$$

$$= \frac{z^5 - 1}{z - 1} \qquad \text{But } z^5 = 1$$

$= 0 = RHS$ As required.

1 or equivalent.

(ii) $z = e^{i\theta} = \cos\theta + i\sin\theta$

$$\begin{aligned}
z^{-1} &= (e^{i\theta})^{-1} \\
&= e^{-i\theta} \\
&= \cos(-\theta) + i\sin(-\theta) \\
&= \cos\theta - i\sin\theta
\end{aligned}$$

cosine = even function
sine = odd function

$$\begin{aligned}
\therefore z + z^{-1} &= \cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta} \\
&= 2\cos\theta
\end{aligned}$$

$\therefore \cos\theta = \dfrac{z + z^{-1}}{2}$ As required.

(iii) from (i) $(z + z^{-1})^2 + (z + z^{-1}) - 1 = 0$

let $\alpha = z + z^{-1}$

$$\begin{aligned}
\alpha^2 + \alpha - 1 &= 0 \\
\alpha &= \frac{-1 \pm \sqrt{1 + 4}}{2} \\
&= \frac{-1 \pm \sqrt{5}}{2}
\end{aligned}$$

1

z can be any complex root of $z^5 = 1$ $\qquad cis\,\theta = \cos\theta + i\sin\theta$

let $z = cis\,\frac{2\pi}{5}$

$$\begin{aligned}
\left(z^5 \right. &= \left(cis\,\tfrac{2\pi}{5}\right)^5 \\
&= cis\,2\pi \quad (\text{By DMT}) \\
&= 1 \quad \therefore z \text{ can be } cis\,\tfrac{2\pi}{5}
\end{aligned}$$

$\therefore 2\cos\frac{2\pi}{5} = \dfrac{-1 + \sqrt{5}}{2}$ or $\dfrac{-1 - \sqrt{5}}{2}$

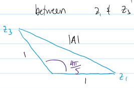
$\frac{2\pi}{5}$ is in the first quadrant $\therefore$ +ve

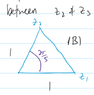
$$\begin{aligned}
\therefore 2\cos\frac{2\pi}{5} &= \frac{\sqrt{5} - 1}{2} \\
\cos\frac{2\pi}{5} &= \frac{\sqrt{5} - 1}{4}
\end{aligned}$$

1

(iv) let $A = z_3 - z_1 \qquad B = z_2 - z_1 \qquad$ NB $z_3 = cis\,\frac{4\pi}{5}$

$|A|$ is the distance of between z_1 & z_3 $\qquad$ $|B|$ is the distance between z_2 & z_3





$$\begin{aligned}
\therefore |A|^2 &= 1 + 1 - 2 \times 1 \times 1 \cos\frac{4\pi}{5} \\
&= 2 - 2\cos\frac{4\pi}{5}
\end{aligned} \qquad |B|^2 = 2 - 2\cos\frac{2\pi}{5}$$

1

$\cos 2\theta = 2\cos^2\theta - 1$

$$\begin{aligned}
\frac{|A|^2}{|B|^2} &= \frac{2 - 2\cos\frac{4\pi}{5}}{2 - 2\cos\frac{2\pi}{5}} \\
&= \frac{\cancel{2}\left[1 - \left(2\cos^2\frac{2\pi}{5} - 1\right)\right]}{\cancel{2}\left(1 - \cos\frac{2\pi}{5}\right)} \\
&= 2\left[\frac{1 - \cos^2\frac{2\pi}{5}}{1 - \cos\left(\frac{2\pi}{5}\right)}\right] \\
&= 2\left[\frac{\left(1 + \cos\frac{2\pi}{5}\right)\cancel{\left(1 - \cos\frac{2\pi}{5}\right)}}{\cancel{1 - \cos\frac{2\pi}{5}}}\right] \\
&= 2\left[1 + \frac{\sqrt{5} - 1}{4}\right] \quad \text{from (iii)} \\
&= \cancel{2}\left[\frac{4 + \sqrt{5} - 1}{\cancel{4}\,2}\right]
\end{aligned}$$

$$\left(\frac{|A|}{|B|}\right)^2 = \frac{3 + \sqrt{5}}{2}$$

1

$$\begin{aligned}
\left(\frac{1 + \sqrt{5}}{2}\right)^2 &= \frac{1 + 5 + 2\sqrt{5}}{4} \\
&= \frac{6 + 2\sqrt{5}}{4} \\
&= \frac{3 + \sqrt{5}}{2}
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{|A|}{|B|} &= \sqrt{\frac{3 + \sqrt{5}}{2}} \\
&= \frac{1 + \sqrt{5}}{2} \quad \text{As required.}
\end{aligned}$$

1

(b) $\underset{\sim}{r}(t) = \cos\frac{\pi}{4}t\,\underset{\sim}{i} + \left(\cos\frac{\pi}{4}t + \sin\frac{\pi}{4}t\right)\underset{\sim}{j} + \sin\frac{\pi}{4}t\,\underset{\sim}{k}$

(i) $\underset{\sim}{r}(3) = -\frac{1}{\sqrt{2}}\underset{\sim}{i} + 0\,\underset{\sim}{j} + \frac{1}{\sqrt{2}}\underset{\sim}{k}$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad ✓$$

(ii) Direction of tangent $\Rightarrow \frac{d}{dt}\left[\underset{\sim}{r}(t)\right]$ @ $t=3$.

$$\frac{d}{dt}\underset{\sim}{r}(t) = -\frac{\pi}{4}\sin\left(\frac{\pi}{4}t\right)\underset{\sim}{i} + \left(\frac{\pi}{4}\cos\frac{\pi}{4}t - \frac{\pi}{4}\sin\frac{\pi}{4}t\right)\underset{\sim}{j} + \frac{\pi}{4}\cos\frac{\pi}{4}t\,\underset{\sim}{k}$$

$$\therefore \frac{d}{dt}\underset{\sim}{r}(3) = \frac{\pi}{4}\left[-\frac{1}{\sqrt{2}}\underset{\sim}{i} - \frac{2}{\sqrt{2}}\underset{\sim}{j} - \frac{1}{\sqrt{2}}\underset{\sim}{k}\right]$$

$$= \frac{\pi}{4\sqrt{2}}\begin{bmatrix}-1\\-2\\-1\end{bmatrix}$$

$\therefore$ Vector equation of tangent $L = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} + \frac{\pi}{4\sqrt{2}}\begin{bmatrix}-1\\-2\\-1\end{bmatrix}$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} + \lambda\begin{bmatrix}1\\2\\1\end{bmatrix} \text{ when } \lambda \in \mathbb{R}.$$

(c) $\int_0^9 \frac{1}{\sqrt{1+\sqrt{x}}}$ let $u = 1+\sqrt{x}$, $u-1 = \sqrt{x}$, $x = (u-1)^2$; $x=9$ $u=4$, $x=0$ $u=1$ 1

$\therefore I = \int_1^4 \frac{1}{\sqrt{u}} \cdot 2(u-1)\,du$ $\frac{dx}{du} = 2(u-1)$

$= 2\int_1^4 \sqrt{u} - \frac{1}{\sqrt{u}}\,du$

$= 2\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4$ 1

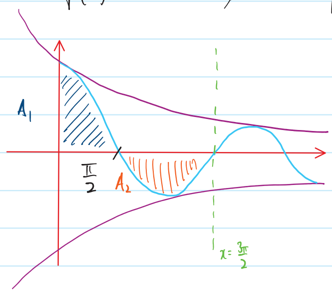
$= 2\left[\left(\frac{16}{3}-4\right) - \left(\frac{2}{3}-2\right)\right]$

$= \frac{16}{3}$ units^2 1

12 Ext 2 2023 Task 4 - RDS Solutions Q15

Monday, 31 July 2023 12:23 PM

(a) $f(x) = e^{-x}\cos x \quad , \quad x \in \left[0, \frac{3\pi}{2}\right]$



RTP $\left|\dfrac{A_1}{A_2}\right| = \dfrac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}$

$$\int \underbrace{e^{-x}}\,\underbrace{\cos x}\, dx \qquad u = e^{-x} \quad v = \sin x$$
$$u' = -e^{-x} \quad v' = \cos x$$

$$\therefore I = \left(e^{-x}\sin x\right) + \int e^{-x}\sin x \, dx \qquad u = e^{-x} \quad v = -\cos x$$
$$u' = -e^{-x} \quad v' = \sin x$$

$$= e^{-x}\sin x + \left[-e^{-x}\cos x - \underbrace{\int e^{-x}\cos x \, dx}_{\Downarrow \; I}\right]$$

$$\therefore 2I = e^{-x}\left[\sin x - \cos x\right]$$

$$I = \frac{e^{-x}}{2}\left[\sin x - \cos x\right] \quad 1$$

Splitting @ $x = \frac{\pi}{2}$ 1

switched as A_2 should be positive.

$$\therefore A_1 = \left[\frac{e^{-x}}{2}\left(\sin x - \cos x\right)\right]_0^{\frac{\pi}{2}} \qquad A_2 = \left[\frac{e^{-x}}{2}\left(\cos x - \sin x\right)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left[\frac{e^{-\frac{\pi}{2}}}{2}\left(1 - 0\right) - \frac{1}{2}\left(0 - 1\right)\right] \qquad = \left(\frac{e^{-\frac{3\pi}{2}}}{2}\left(0+1\right) - \frac{e^{-\frac{\pi}{2}}}{2}\left(0-1\right)\right)$$

$$= \frac{1}{2}\left[e^{-\frac{\pi}{2}} + 1\right] \qquad = \frac{1}{2}\left(e^{-\frac{3\pi}{2}} + e^{-\frac{\pi}{2}}\right)$$

1

$$\frac{A_1}{A_2} = \frac{\cancel{\frac{1}{2}}\left(e^{-\frac{\pi}{2}}+1\right)}{\cancel{\frac{1}{2}}\left(e^{-\frac{3\pi}{2}}+e^{-\frac{\pi}{2}}\right)}$$

$$= \frac{e^{-\frac{\pi}{2}}\left(1+e^{\frac{\pi}{2}}\right)}{e^{-\frac{3\pi}{2}}\left(1+e^{\pi}\right)}$$

$$= e^{-\frac{\pi}{2}+\frac{3\pi}{2}}\left(\frac{1+e^{\frac{\pi}{2}}}{1+e^{\pi}}\right) \quad 1$$

A_2 $\quad \cancel{\frac{1}{2}}\left(e^{-\frac{3\pi}{2}} + e^{-\frac{\pi}{2}}\right)$

$$= \frac{e^{-\frac{\pi}{2}}\left(1 + e^{\frac{\pi}{2}}\right)}{e^{-\frac{3\pi}{2}}\left(1 + e^{\pi}\right)}$$

$$= e^{-\frac{\pi}{2} + \frac{3\pi}{2}}\left(\frac{1 + e^{\frac{\pi}{2}}}{1 + e^{\pi}}\right)$$

$$= \frac{e^{\pi}\left(1 + e^{\frac{\pi}{2}}\right)}{1 + e^{\pi}} \quad \text{As required.}$$

1

(b) $T_n = 6T_{n-1} - 9T_{n-2}$ for $n \geq 2$ $\quad T_0 = 1$, $T_1 = 6$

RTP $\quad T_n = 3^n + n \times 3^n$

1. Show true for $n = 0$ & $n = 1$

$T_0 = 3^0 + 0 \times 3^0$ $\qquad T_1 = 3^1 + 1 \times 3^1$

$= 1$ $\quad \therefore$ true for $n = 0$ $\qquad = 6$ $\quad \therefore$ true for $n = 1$

1

2. Assume true for $n = k$ & $n = k-1$ (k integer, $k \geq 1$)

$T_k = 3^k + k \times 3^k$ $\qquad T_{k-1} = 3^{k-1} + (k-1) \times 3^{k-1}$

1

3. Prove true for $n = k+1$

RTP $\quad T_{k+1} = 3^{k+1} + (k+1)3^{k+1}$

$T_{k+1} = 6T_k - 9T_{k-1}$

$= 6\left(3^k + k \times 3^k\right) - 9\left(3^{k-1} + (k-1)3^{k-1}\right)$ $\quad$ from Assumption. $\quad$ 1

$= 6 \times 3^k + 6k \times 3^k - \underset{3\times 3}{9} \times 3^{k-1} - \underset{3\times 3}{9}(k-1)3^{k-1}$

$= 6 \times 3^k + 6k \times 3^k - 3 \times 3^k - 3(k-1)3^k$

$= 3 \times 3^k + (6k - 3k + 3)3^k$

$= 3^{k+1} + (3k+3)3^k$

$$= 3^{k+1} + (3k+3)3^k$$

$$= 3^{k+1} + 3(k+1)3^k$$

$$= 3^{k+1} + (k+1)3^{k+1} \quad \text{As required}$$

1

$\therefore$ true for $n = k+1$

$\therefore$ Proven by Mathematical Induction.

(c) let $I_n = \int \frac{\cos nx}{\sin x}\,dx$

(i) RTP $\cos((n-2)x) - \cos nx = 2\sin((n-1)x)\sin x$

from reference sheet.

$$2\sin A \sin B = [\cos(A-B) - \cos(A+B)] \quad 1$$

let $A = (n-1)x \quad B = x$

$$2\sin[(n-1)x]\sin x = \cos[(n-1)x - x] - \cos[(n-1)x + x]$$

$$= \cos(n-2)x - \cos(nx) \quad \text{As required.}$$

1

(ii) $I_n = \int \frac{\cos nx}{\sin x}\,dx$

$$I_{n-2} = \int \frac{\cos(n-2)x}{\sin x}\,dx$$

$$I_n - I_{n-2} = \int \frac{\cos nx - \cos(n-2)x}{\sin x}$$

$$= \int \left[- \frac{(2\sin(n-1)x)\cancel{\sin x}}{\cancel{\sin x}} \right] dx \quad \text{from (i)}$$

$$= -\int 2\sin(n-1)x \quad dx$$

$$= \cos(n-1)x \div (n-1) + c$$

$$= \frac{2\cos(n-1)x}{n-1} + c \quad \text{As required.}$$

(iii) $\int^{\frac{\pi}{2}}$ [cut off]

$$\overline{\quad n-1 \quad}$$

(iii) $$\int_0^{\frac{\pi}{3}} \frac{\cos 2x - \cos 6x}{\sin x}\, dx$$

$$= I_2 - I_6 + I_4 - I_4$$ (1)

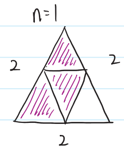
$$= -[I_6 - I_4] - [I_4 - I_2]$$

$$= -\left[\left[\frac{2\cos(5)x}{5} \right] + \left[\frac{2\cos(3)x}{3} \right] \right]_0^{\frac{\pi}{3}}$$ (1)

$$= -\left[\left(\frac{2}{5} \times \frac{1}{2} + \frac{2}{3} \times -1 \right) - \left(\frac{2}{5} + \frac{2}{3} \right) \right]$$

$$= {}^{23}/_{15} \text{ units}^2$$ (1)

When $n=1$

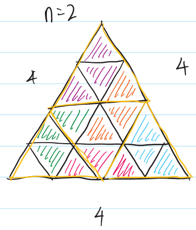


- 1 piece needed
- 1 empty space

Triangle Area $= 4$

$\therefore$ true for $n=1$

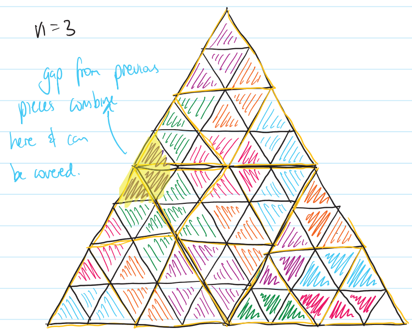
$n=2$



- 5 pieces needed
- 1 empty space.

Triangle area $= 4\times4$
$= 16$

$n=3$



- 21 pieces needed
- 1 empty space

Triangle area $= 64$

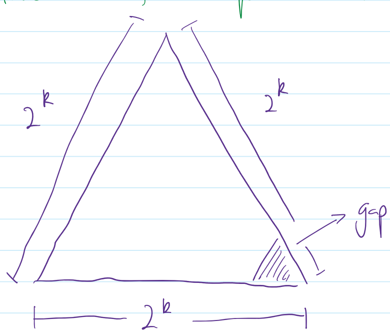
$n=4$, pieces needed $= 4\times21+1$
$= 85$

Triangle Area $= 256$

Each increase in n requires 4 of the previous triangle. 3 can rotate to create a new piece

$\therefore$ If area is true the gap always in a corner.

Assume true for $n=k$

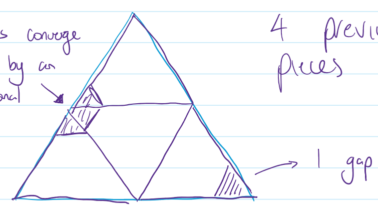


(1) True for $n=1$

(2) Displaying the kth piece

(3) Forming the $(k+1)$th piece from 4 previous pieces.

Prove true for $n=k+1$



$\therefore$ True for $n=k+1$

Thus proven by Mathematical induction.

(b) $S = \dfrac{a}{1-r}$

(i) $a=1$ $\quad r=\frac{1}{2}e^{i\theta}$ (1)

$$\therefore S = \frac{1}{1-\frac{1}{2}e^{i\theta}}$$

$$= \frac{1}{\frac{2-e^{i\theta}}{2}}$$

$$= \frac{2}{2-e^{i\theta}} \quad \text{As required.}$$ (1)

(ii) As $e^{i\theta} = \cos\theta + i\sin\theta$

$\dfrac{2}{2-e^{i\theta}}$

(ii) As $e^{i\theta} = \cos\theta + i\sin\theta$

$Im(S) = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta \ldots$ 1

$\therefore \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots = Im\left(\dfrac{2}{2-e^{i\theta}}\right)$

$$RHS = Im\left(\frac{2}{2-\cos\theta - i\sin\theta} \times \frac{2-\cos\theta + i\sin\theta}{2-\cos\theta + i\sin\theta}\right)$$

$$= Im\left[\frac{2(2-\cos\theta) + 2i\sin\theta}{(2-\cos\theta)^2 + \sin^2\theta}\right]$$

$$= \frac{2\sin\theta}{4 - 4\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}}$$

$$= \frac{2\sin\theta}{5-4\cos\theta}$$

1

(iii) For the sum to be purely imaginary then

$Re(S) = 0$

$Re(S) = \dfrac{2(2-\cos\theta)}{5-4\cos\theta}$ 1

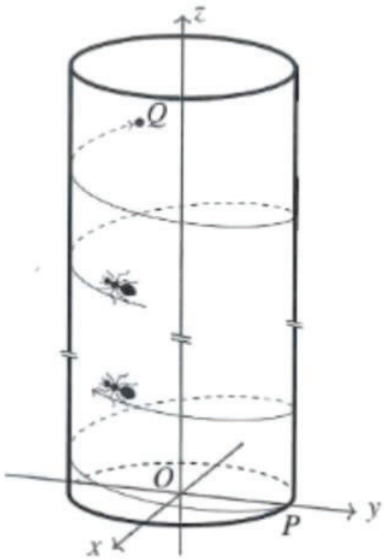
$Re(S) = 0$ when $2-\cos\theta = 0$

$-1 \leq \cos\theta \leq 1 \quad \therefore \quad 2-\cos\theta \neq 0$

$\therefore$ Sum can never be purely imaginary.

1

(c)



$$\underset{\sim}{r}(t) = \left(7\sin\frac{\pi}{32}t\right)\underset{\sim}{i} + \left(7\cos\frac{\pi}{32}t\right)\underset{\sim}{j} + \frac{t}{15}\underset{\sim}{k}$$

(i) $|OQ| = |\underset{\sim}{r}(t)|$

$$|\underset{\sim}{r}(t)|^2 = 49\left[\sin^2\frac{\pi}{32}t + \cos^2\frac{\pi}{32}t\right] + \frac{t^2}{225}$$

$$\therefore 49 + \frac{t^2}{225} = 25^2$$

$$t^2 = 576 \times 225$$

$t = 360$ (1) $(t > 0$ as time is positive$)$ (1)

(1)

Position of $Q = \underset{\sim}{r}(360)$

$$= 7\sin\frac{360\pi}{32}\underset{\sim}{i} + 7\cos\frac{360\pi}{32}\underset{\sim}{j} + \frac{360}{15}\underset{\sim}{k}$$

$$= \left(\frac{-7}{\sqrt{2}}, \frac{-7}{\sqrt{2}}, 24\right)$$

(ii) let $\hat{i}$ component of $\underset{\sim}{r}(t) = 7$

$7 = 7\sin\frac{\pi t}{32}$ (1)

$$\sin\frac{\pi t}{32} = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2} \ldots$$

$\therefore t = 16, 80, 144, 208, 272, 336, \boxed{400} \ldots$

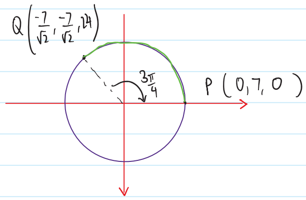
1, 2, 3, 4, 5, 6

$> 360 \therefore$ Ant has stopped moving

$\therefore$ Crosses $x = 7$ 6 times (1)

(iii) From $Q \to P$ the ant walks along the curved edge of the cylinder

From above.

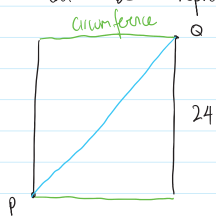


That is the ant's trip can be considered in 2 parts.

- Along the circumference of the circle in the x/y plane
- 24 units down in the z plane.

① Any working towards the components of the trip

This can be represented as a rectangle



$$\text{Circumference} = r\theta = 7 \times \frac{3\pi}{4} = \frac{21\pi}{4}$$

$$\therefore \text{Shortest path} = \sqrt{24^2 + \left(\frac{21\pi}{4}\right)^2}$$

$$\doteqdot 29.1 \text{ cm.}$$ ①