NESA Number:

Fort Street High School

Name:

Teacher	Mr Choi	Ms Hussein	Ms Kaur

2025 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Marks may be deducted for careless or badly arranged work.
- In Questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1 – 10 (pages 2–5)
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11 – 16 (pages 6–13)
- Allow about 2 hours and 45 minutes for this section
- Write your student number on each answer booklet.

Question	MC	Q11	Q12	Q13	Q14	Q15	Q16	Total
Mark	/10	/14	/16	/15	/16	/16	/13	/100

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 Which of the following vectors is parallel to the vector $3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$?

A. $27\underset{\sim}{i} - 12\underset{\sim}{j} + 6\underset{\sim}{k}$

B. $21\underset{\sim}{i} - 28\underset{\sim}{j} + 14\underset{\sim}{k}$

C. $15\underset{\sim}{i} - 20\underset{\sim}{j} - 10\underset{\sim}{k}$

D. $24\underset{\sim}{i} - 28\underset{\sim}{j} + 16\underset{\sim}{k}$

2 Consider the following statement.

‘If it rains, then the ground will be wet.’

Which of the following is logically equivalent to the statement above?

A. If the ground is wet, then it has rained.

B. If the ground is not wet, then it will rain.

C. If the ground is not wet, then it has not rained.

D. If it does not rain then the ground will not be wet.

3 It is given that $|z - 1 - i| = 1$.

What is the maximum value of Arg (z)?

A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. π

4 Consider the following proposition.

'If a positive integer is not a perfect square, then it is not prime.'

If a number was found which had the following properties, which would be a counterexample to the proposition?

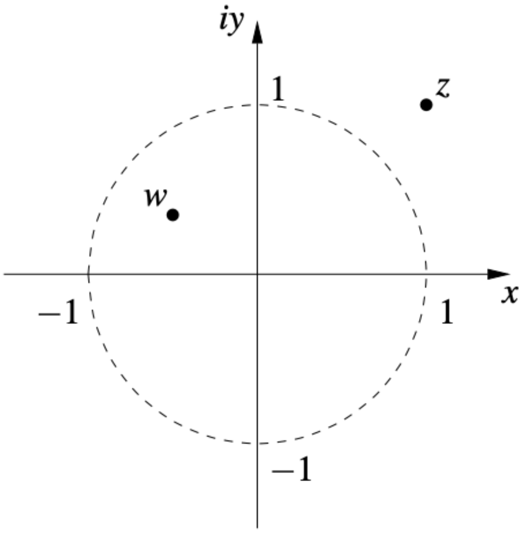
A. A positive integer that is a perfect square and is prime.

B. A positive integer that is not a perfect square and is prime.

C. A positive integer that is a perfect square and is not prime.

D. A positive integer that is not a perfect square and is not prime.

5 Consider the complex numbers z and w shown on the complex plane below.



Which of the following could be true?

A. $w = z \div \left(2e^{-i\frac{\pi}{2}}\right)$

B. $w = z \times \left(2e^{-i\frac{\pi}{2}}\right)$

C. $w = z \div \left(2e^{i\frac{\pi}{2}}\right)$

D. $w = z \times \left(2e^{i\frac{\pi}{2}}\right)$

6 Which of these algebraic fractions cannot be written as partial fractions in the form

$$\frac{a}{x+b}+\frac{c}{x+d}$$

where $a,\ b,\ c$ and d are real numbers?

A. $\dfrac{2x+6}{x^2+6x+5}$

B. $\dfrac{2x-8}{x^2-8x+16}$

C. $\dfrac{2x+2}{x^2+2x-3}$

D. $\dfrac{2x+4}{x^2-4x+8}$

7 A point lies on the sphere such that its position vector, $\mathbf{r}$, satisfies $|\mathbf{r}|^2 = \mathbf{r}\cdot\mathbf{c}$, where $\mathbf{c}$ is a constant vector.

What is the centre of the sphere?

A. $\mathbf{c}$

B. $\frac{1}{2}\mathbf{c}$

C. $-\mathbf{c}$

D. $-\frac{1}{2}\mathbf{c}$

8 A vector makes equal angles of $\dfrac{\pi}{4}$ with the positive directions of the $x-$axis and the $y-$ axis.

What angle does the vector make with the positive direction of the $z-$ axis?

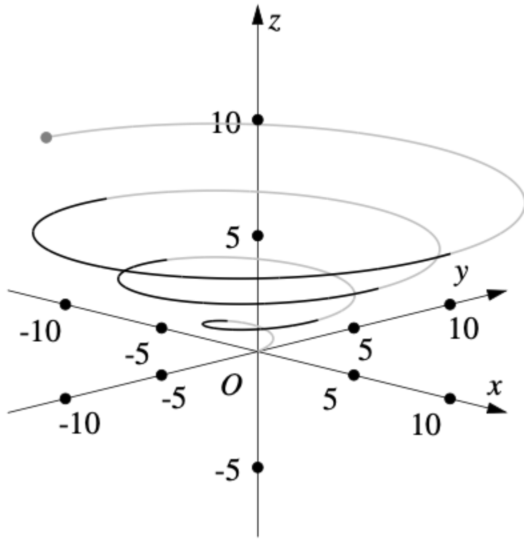
A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. $\frac{3\pi}{4}$

9 Consider the diagram of the curve below for $0 \leq t \leq 7\pi$.



Which of the following best describes the position vector of the points on the curve?

A. $\underset{\sim}{r}(t) = \frac{1}{2}\sin(t)\underset{\sim}{i} + \frac{1}{2}\cos(t)\underset{\sim}{j} + \left(\frac{t}{\pi}\right)\underset{\sim}{k}$

B. $\underset{\sim}{r}(t) = \frac{1}{2}\cos(t)\underset{\sim}{i} + \frac{1}{2}\sin(t)\underset{\sim}{j} + \left(\frac{t}{\pi}\right)\underset{\sim}{k}$

C. $\underset{\sim}{r}(t) = \frac{t}{2}\sin(t)\underset{\sim}{i} + \frac{t}{2}\cos(t)\underset{\sim}{j} + \left(\frac{t}{\pi}\right)\underset{\sim}{k}$

D. $\underset{\sim}{r}(t) = \frac{t}{2}\cos(t)\underset{\sim}{i} + \frac{t}{2}\sin(t)\underset{\sim}{j} + \left(\frac{t}{\pi}\right)\underset{\sim}{k}$

10 Which of the following is a necessary and sufficient condition for $f(x)$ to satisfy

$$\int_0^a f(x-a) + f(a-x)\,dx = 0$$

for all positive value of a?

A. $f(x)$ is odd

B. $f(x)$ is even

C. $f(x)$ is monotonically increasing

D. $f(x)$ is monotonically decreasing

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a new writing booklet.

(a) Let $z = 3 - i$ and $w = 2 + 5i$.

(i) Find $|z + w|$. **2**

(ii) Express $\frac{w}{z}$ in the form $a + ib$, where $a, b \in \mathbb{R}$. **2**

(b) Find the angle between the vectors $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$ correct to the nearest minute. **2**

(c) Sketch the region defined by $\text{Im}(z) \geq 2$ and $\frac{\pi}{2} \leq \arg(z) \leq \frac{3\pi}{4}$. **2**

(d) (i) Find A, B, and C such that **2**

$$\frac{1-2x}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}.$$

(ii) Hence find $\displaystyle\int \frac{1-2x}{(x+2)(x^2+1)}\,dx$. **2**

(e) Find $\displaystyle\int \tan^4 x\,dx$. **2**

End of Question 11

Question 12 (16 marks) Use a new writing booklet.

(a) Find the square roots of $3 + 2i\sqrt{10}$. **3**

(b) Given $z \neq 0$, show that $\phi = \dfrac{2z}{z^2 + |z|^2}$ is real. **3**

(c) Find **3**

$$\int e^{\sqrt{x}}\, dx.$$

(d) The vector equations of lines ℓ_1 and ℓ_2 are given below.

$$\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

$$\ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

(i) Show that ℓ_1 and ℓ_2 do not intersect. **3**

(ii) Find a vector of the form $\begin{pmatrix} 0 \\ a \\ b \end{pmatrix}$ that is perpendicular to both ℓ_1 and ℓ_2, **1**

(iii) Hence find the minimum distance between ℓ_1 and ℓ_2. **3**

End of Question 12

Question 13 (15 marks) Use a new writing booklet.

(a) Evaluate **3**

$$\int_0^{\frac{\pi}{3}} \frac{5}{3 + 2\cos x}\, dx.$$

(b) Explain why there is no integer p such that $p^{11} - (p + 2)^7 = 2025$. **2**

(c) Find **2**

$$\int \frac{e^{2x} - e^x}{e^x + 1} dx\,.$$

(d) Let z be a complex number and λ a real constant. **2**

Find the values of $|z|$ that satisfy the equation,

$$\lambda(\overline{z} + z^n) = i(\overline{z} - z^n)$$

for $n \geq 2$.

(e) The line L with equation $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ intersects with the sphere with **3**

equation $(x + 1)^2 + (\,y - 2\,)^2 + z^2 = 45$ at the two points A and B.

Find the midpoint of AB.

(f) Given $a_{n+1} = \sqrt{2a_n}$ and $a_1 = 1$. **3**

Prove by mathematical induction that for all integers $n \geq 1$

$$a_n = 2^{\left(1-2^{(1-n)}\right)}.$$

End of Question 13

Question 14 (16 marks) Use a new writing booklet.

(a) Consider the expansion of z^5 for $z = \cos\theta + i\sin\theta$.

(i) Using De Moivre's theorem to show that **3**

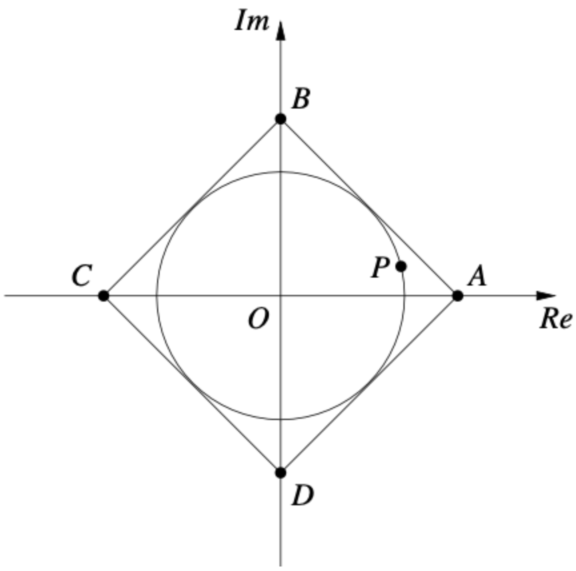
$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(ii) Hence show that $x = \pm\cos\frac{\pi}{10}$ and $x = \pm\cos\frac{3\pi}{10}$ are the solutions to the equation $16x^4 - 20x^2 + 5 = 0$. **3**

(iii) Deduce the exact value of $\cos\frac{\pi}{10}$. **3**

(b) Consider a square $ABCD$, of side length a, in the complex plane. **4**

The point P lies on a circle centred at the origin, of radius $\frac{a}{2}$, inscribed in square $ABCD$ as shown.



Find, in terms of a, the value of $PA^2 + PB^2 + PC^2 + PD^2$.

(c) Consider the vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ in 3 dimensions with magnitudes $\sqrt{n},\ 2\sqrt{n},\ 3\sqrt{n}$ respectively, where n is positive real number. **3**

The magnitude of the resultant vector is $\left|\underset{\sim}{p}+\underset{\sim}{q}+\underset{\sim}{r}\right| = n$.

What is the maximum value of $\left|\underset{\sim}{p}-\underset{\sim}{q}\right|^2 + \left|\underset{\sim}{q}-\underset{\sim}{r}\right|^2 + \left|\underset{\sim}{r}-\underset{\sim}{p}\right|^2$?

End of Question 14

Question 15 (16 marks) Use a new writing booklet.

(a) Consider the integral $I_n = \int_0^1 (x^2+1)^{\frac{n}{2}}\, dx$.

(i) Show that $I_n = \frac{1}{n+1}\left(2^{\frac{n}{2}} + nI_{n-2}\right)$. **3**

(ii) Show that **3**

$$\int_0^{\frac{\pi}{4}} \sec^3\theta \; d\theta = \frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right).$$

(iii) Hence evaluate I_3 leaving your answer in the form $a\sqrt{2} + b\ln\left(1+\sqrt{2}\right)$. **2**

(b) The complex number w lies on the unit circle with $\operatorname{Arg}(w) = \theta$.

(i) Show that **2**

$$\frac{1+w}{1-w} = i\cot\frac{\theta}{2}.$$

(ii) Hence find all the solutions of **3**

$$(z-1)^5 + (z+1)^5 = 0$$

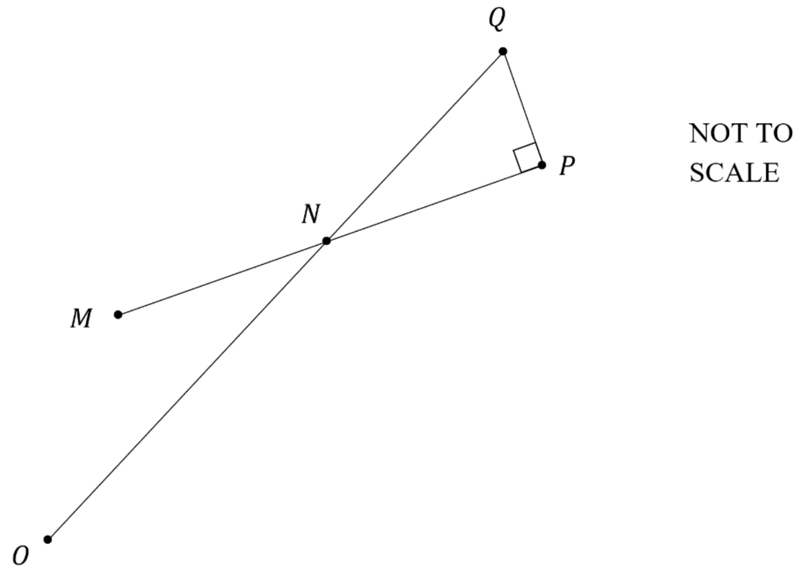
where z is a complex number.

Question 15 continues on page 11

Question 15 (continued)

(c) M and P are two points with position vectors $\overrightarrow{OM} = \underset{\sim}{m}$ and $\overrightarrow{OP} = \underset{\sim}{p}$ respectively. **3**

N is the midpoint of MP and has position vector $\overrightarrow{ON} = \underset{\sim}{n}$.
Q lies on the line ON such that PQ is perpendicular to MP.
$\overrightarrow{NQ} = \lambda \overrightarrow{ON}$ where λ is a real number.



Show that

$$\lambda = \frac{|\underset{\sim}{m}|^2 + |\underset{\sim}{n}|^2 - 2\underset{\sim}{m} \cdot \underset{\sim}{n}}{|\underset{\sim}{n}|^2 - \underset{\sim}{m} \cdot \underset{\sim}{n}}.$$

End of Question 15

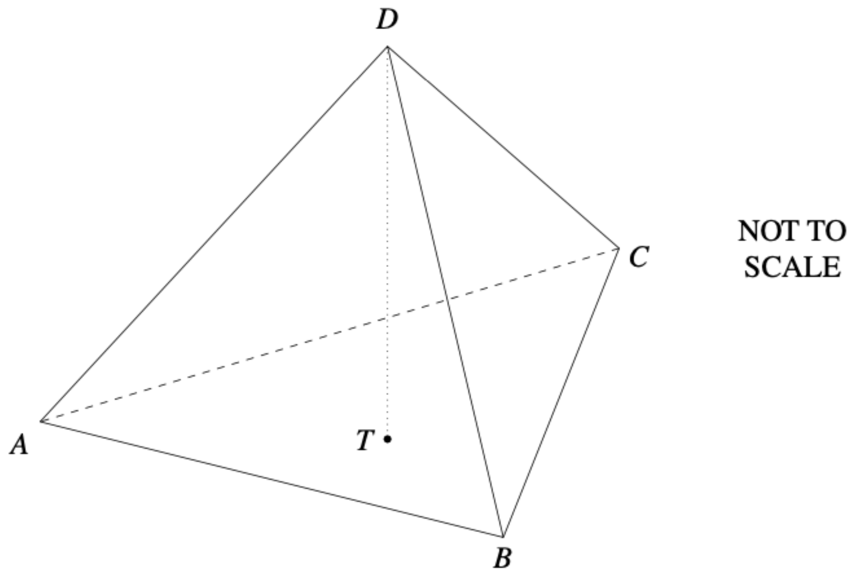
Question 16 (13 marks) Use a new writing booklet.

(a) (i) By considering vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$, show that **2**

$$(p+q+r)^2 \le 3(p^2+q^2+r^2)$$

where p, q and r are positive real numbers.

(ii) Consider the triangular pyramid $ABCD$, where T is a point in the plane of triangle ABC, as shown. **3**



Let O be the origin. It is known that $\overrightarrow{OT} = \frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right)$

Using the result in part (i), show that $\left|\overrightarrow{DT}\right|^2 \le \frac{1}{3}\left(\left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{BD}\right|^2 + \left|\overrightarrow{CD}\right|^2\right)$

(b) The three positive real numbers r, s, and t are such that **3**

$$(r+n)(s+n)(t+n) = n^6$$

show that $rst \le (n^2 - n)^3$.

You may use the result $(a+b+c)^3 \ge 27abc$ where a, b and c are positive.
[DO NOT PROVE THIS]

Question 16 continues on page 13

(c) Let z_1, z_2, z_3 be complex numbers such that **5**

$$|z_1| = |z_2| = |z_3| = 1$$

$$z_1 + z_2 + z_3 \neq 0$$

$$z_1^2 + z_2^2 + z_3^2 = 0$$

Show that, for all integers $n \geq 1$,

$$|z_1^n + z_2^n + z_3^n|$$

can only take specific integer values, and state those values.

End of paper

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Question	MC	Q11	Q12	Q13	Q14	Q15	Q16	Total
Mark	/10	/14	/14	/15	/16	/16	/15	/100

Section I
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following vectors is parallel to the vector $3\underset{\sim}{i}-4\underset{\sim}{j}+2\underset{\sim}{k}$?

A. $27\underset{\sim}{i}-12\underset{\sim}{j}+6\underset{\sim}{k}$

B. $21\underset{\sim}{i}-28\underset{\sim}{j}+14\underset{\sim}{k}$

C. $15\underset{\sim}{i}-20\underset{\sim}{j}-10\underset{\sim}{k}$

D. $24\underset{\sim}{i}-28\underset{\sim}{j}+16\underset{\sim}{k}$

$7\times(3\underset{\sim}{i}-4\underset{\sim}{j}+2\underset{\sim}{k}) = 21\underset{\sim}{i}-28\underset{\sim}{j}+14\underset{\sim}{k}$

Therefore, option B.

2 Consider the following statement.

‘If it rains, then the ground will be wet.’

Which of the following is logically equivalent to the statement above?

A. If the ground is wet, then it has rained.

B. If the ground is not wet, then it will rain.

C. If the ground is not wet, then it has not rained.

D. If it does not rain then the ground will not be wet.

A possible logical equivalent is the contrapositive of a statement.
The contrapositive of ‘If it rains, then the ground will be wet’ is ‘If the ground is not wet, then it has not rained.’
Therefore, option C.

3 It is given that $|z-1-i|=1$.

What is the maximum value of $Arg\ (z)$?

A. 0

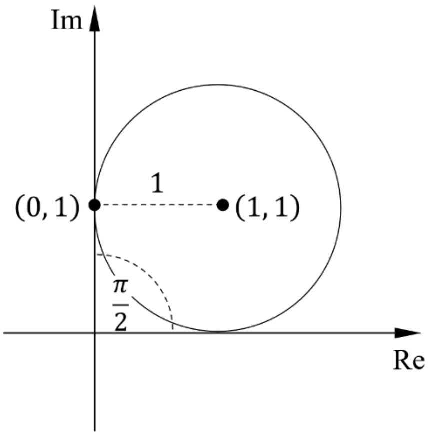
B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. π

$z-1-i=1$ is a circle of radius one unit centred at $(1,1)$.

The maximum value of $Arg(z)$ occurs when z is directly above zero at $(0,1)$, where it takes a value of $\frac{\pi}{2}$ as shown.



Therefore answer is C

4 Consider the following proposition.

'If a positive integer is not a perfect square, then it is not prime.'

If a number was found which had the following properties, which would be a counterexample to the proposition?

A. A positive integer that is a perfect square and is prime.

B. A positive integer that is not a perfect square and is prime.

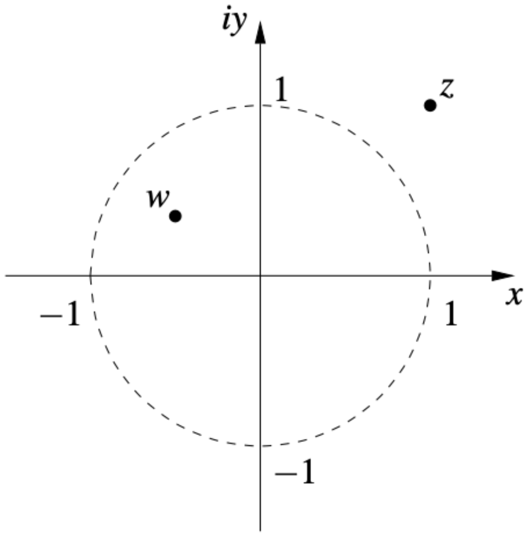
C. A positive integer that is a perfect square and is not prime.

D. A positive integer that is not a perfect square and is not prime.

The negation of $P \Rightarrow Q$ is P and $\neg Q$, so in this case 'not a perfect square and is prime'.

Therefore answer is B

5 Consider the complex numbers z and w shown on the complex plane below.



Which of the following could be true?

A. $w = z \div \left(2e^{-i\frac{\pi}{2}}\right)$

B. $w = z \times \left(2e^{-i\frac{\pi}{2}}\right)$

C. $w = z \div \left(2e^{i\frac{\pi}{2}}\right)$

D. $w = z \times \left(2e^{i\frac{\pi}{2}}\right)$

w is a rotation of z in an anti-clockwise direction, and has a smaller modulus (approximately half).

The situation where z is multiplied by a complex number $re^{i\theta}$ with $|\,r\,| < 1$ and $\theta > 0$ is $w = z \div 2e^{-i\frac{\pi}{2}}$.

Therefore, option A.

6 Which of these algebraic fractions cannot be written as partial fractions in the form

$$\frac{a}{x+b}+\frac{c}{x+d}$$

where $a,\ b,\ c$ and d are real numbers?

A. $$\frac{2x+6}{x^2+6x+5}$$

B. $$\frac{2x-8}{x^2-8x+16}$$

C. $$\frac{2x+2}{x^2+2x-3}$$

D. $$\frac{2x+4}{x^2-4x+8}$$

$$A.\quad \frac{2x+6}{x^2+6x+5}=\frac{1}{x+5}+\frac{1}{x+1}$$

$$B.\quad \frac{2x-8}{x^2-8x+16}=\frac{1}{x-4}+\frac{1}{x-4}$$

$$C.\quad \frac{2x+2}{x^2+2x-3}=\frac{1}{x-1}+\frac{1}{x+3}$$

$$D.\quad \frac{2x+4}{x^2-4x+8}=\frac{2x+4}{(x-2)^2+2^2}$$

The denominator cannot be written as the product of two real linear factors as it is not the difference of two squares.

Alternative solution I:
If the denominator is the product of two real linear factors, then it can be written in this form. We can do this if the discriminant is non-negative.

$$A.\quad \Delta=6^2-4(1)(5)=16\geq 0\ ✓$$

$$B.\quad \Delta=(-8)^2-4(1)(16)=0\geq 0\ ✓$$

$$C.\quad \Delta=2^2-4(1)(-3)=16\geq 0\ ✓$$

$$D.\quad \Delta=(-4)^2-4(1)(8)=-16<0\ ✗$$

Therefore answer is D

7 A point lies on the sphere such that its position vector, r, satisfies $|r|^2 = r \cdot c$, where c is a constant vector.

What is the centre of the sphere?

A. c

B. $\frac{1}{2}c$

C. $-c$

D. $-\frac{1}{2}c$

$$|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{c}$$
$$|\mathbf{r}|^2 - \mathbf{r} \cdot \mathbf{c} = 0$$
$$|\mathbf{r}|^2 - \mathbf{r} \cdot \mathbf{c} + \frac{|\mathbf{c}|^2}{4} = \frac{|\mathbf{c}|^2}{4}$$
$$\left|\mathbf{r} - \frac{\mathbf{c}}{2}\right|^2 = \frac{|\mathbf{c}|^2}{4}$$

This is a sphere with centre $\frac{1}{2}\mathbf{c}$ and radius $\frac{1}{2}|\mathbf{c}|$.

Therefore answer is B

8 A vector makes equal angles of $\frac{\pi}{4}$ with the positive directions of the $x-$axis and the $y-$ axis. What angle does the vector make with the positive direction of the $z-$ axis?

A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. $\frac{3\pi}{4}$

$\underline{v} \cdot \underline{i} = |\underline{v}| \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}|v|$

$\underline{v} \cdot \underline{j} = |\underline{v}| \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}|v|$

$\therefore \underline{v} = \begin{pmatrix} \frac{1}{\sqrt{2}}|v| \\ \frac{1}{\sqrt{2}}|v| \\ a \end{pmatrix}$

$\therefore |\underline{v}| = \frac{1}{2}|\underline{v}|^2 + \frac{1}{2}|\underline{v}|^2 + a^2$

$\therefore 0 = a^2$

$a = 0$

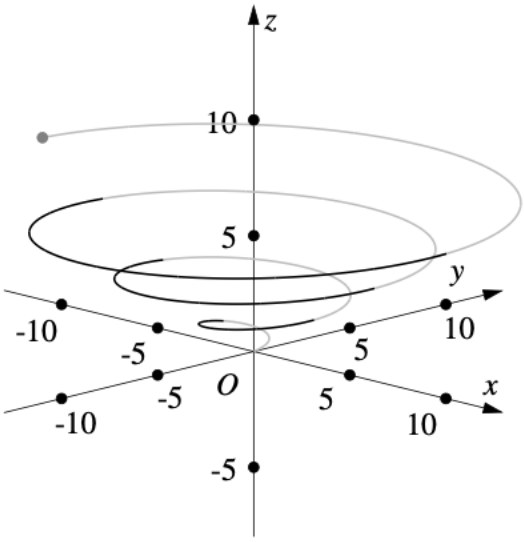
$\therefore V = \begin{pmatrix} \frac{1}{\sqrt{2}}|v| \\ \frac{1}{\sqrt{2}}|v| \\ 0 \end{pmatrix}$

$\therefore \underline{v} \cdot \underline{k} = 0$

$\therefore \underline{v} \perp \underline{k}$

Therefore answer is C

9 Consider the diagram of the curve below for $0 \leq t \leq 7\pi$.



Which of the following best describes the position vector of the points on the curve?

A. $\underset{\sim}{r}(t) = \frac{1}{2}\sin(t)\underset{\sim}{i} + \frac{1}{2}\cos(t)\underset{\sim}{j} + \left(\frac{t}{\pi}\right)\underset{\sim}{k}$

B. $\underset{\sim}{r}(t) = \frac{1}{2}\cos(t)\underset{\sim}{i} + \frac{1}{2}\sin(t)\underset{\sim}{j} + \left(\frac{t}{\pi}\right)\underset{\sim}{k}$

C. $\underset{\sim}{r}(t) = \frac{t}{2}\sin(t)\underset{\sim}{i} + \frac{t}{2}\cos(t)\underset{\sim}{j} + \left(\frac{t}{\pi}\right)\underset{\sim}{k}$

D. $\underset{\sim}{r}(t) = \frac{t}{2}\cos(t)\underset{\sim}{i} + \frac{t}{2}\sin(t)\underset{\sim}{j} + \left(\frac{t}{\pi}\right)\underset{\sim}{k}$

From the top view, the point on the curve is "moving" in an anti-clockwise direction and it is also increasing in radius, hence the $\underset{\sim}{i}$ component is proportional to $t\cos(t)$ and the the $\underset{\sim}{j}$ component is proportional to $t\sin(t)$.

Therefore answer is D

10 Which of the following is a necessary and sufficient condition for $f(x)$ to satisfy

$$\int_0^a f(x-a)+f(a-x)\,dx=0$$

for all positive value of a?

A. $f(x)$ is odd

B. $f(x)$ is even

C. $f(x)$ is monotonically increasing

D. $f(x)$ is monotonically decreasing

$$\int_0^a f(x-a)+f(a-x)\,dx$$

$$=\int_0^a f(x-a)\,dx+\int_0^a f(a-x)\,dx$$

$$\int_0^a f(x-a)\,dx=\int_{-a}^0 f(u)\,du \text{ if } u=x-a$$

$$\int_0^a f(a-x)\,dx=-\int_a^0 f(u)\,du \text{ if } u=a-x$$

$$=\int_0^a f(u)\,du$$

$$\therefore \int_{-a}^0 f(u)\,du+\int_0^a f(u)\,du=\int_{-a}^a f(u)\,du$$

By changing variables in the two parts, the given integral is equivalent to $\int_{-a}^{a} f(u)\,du=0$.

Therefore the definite integral will always be zero if f(x) is odd.

Therefore answer is A

Section II

Question 11 (14 marks) Use a new writing booklet.

(a) Let $z = 3 - i$ and $w = 2 + 5i$.

(i) Find $|z + w|$. **2**

$$z + w = 3 - i + 2 + 5i$$
$$= 5 + 4i$$
$$|z+w| = \sqrt{5^2 + 4^2}$$
$$= \sqrt{25+16}$$
$$= \sqrt{41}$$

Marking criteria	**Marks**
Correct solution	**2**
Correctly evaluated z + w or correct modulus of a complex number	**1**

(ii) Express $\frac{w}{z}$ in the form $a + ib$, where $a, b \in R$. **2**

$$\begin{aligned}\frac{w}{z} &= \frac{2+5i}{3-i} \\ &= \frac{(2+5i)(3+i)}{(3-i)(3+i)} \\ &= \frac{6+2i+15i-5}{9+1} \\ &= \frac{1}{10} + \frac{17}{10}i\end{aligned}$$

Marking criteria	**Marks**
Correct solution	**2**
Correct numerator or correct denominator	**1**

(b) Find the angle between the vectors $\underset{\sim}{a} = (2\ \ -1\ 3\ )$ and $\underset{\sim}{b} = (-1\ 4\ 5\ )$ correct to the nearest minute. **2**

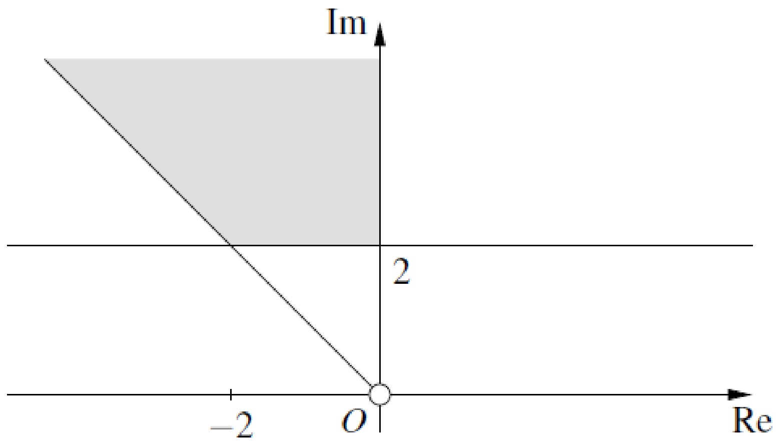
Marking Criteria	**Marks**
• Provides correct solution	2
• Finds the dot product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, or equivalent merit	1

Let θ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$$\begin{aligned}
\cos\theta &= \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\\
\cos\theta &= \frac{-2-4+15}{\sqrt{14}\times\sqrt{42}}\\
\theta &= \cos^{-1}\left(\frac{9}{14\sqrt{3}}\right)\\
\therefore\ \theta &= 68^\circ 13' \quad \text{(nearest minute)}
\end{aligned}$$

(c) Sketch the region defined by $Im(z) \geq 2$ and $\frac{\pi}{2} \leq arg\ arg\ (z)\ \leq \frac{3\pi}{4}$. **2**

Marking Criteria	**Marks**
• Provides correct sketch	2
• Provides correct sketch of either $\mathrm{Im}(z) \geqslant 2$ or $\frac{\pi}{2} \leqslant \arg(z) \leqslant \frac{3\pi}{4}$	1



(d) (i) Find A, B, and C such that **2**

$$\frac{1-2x}{(x+2)(x^2+1)} = \frac{A}{(x+2)} + \frac{Bx+C}{(x^2+1)}.$$

$1-2x = A(x^2+1)+(Bx+C)(x+2)$

$x=-2$

$5 = 5A+0$

$A=1$

comparing coefficient of x^2 yields

$A+B=0$

$\therefore B=-1$

comparing the constant term $1=A+2C$

$A=1 \Rightarrow C=0$

$$\therefore \frac{1-2x}{(x+2)(x^2+1)} = \frac{1}{x+2} - \frac{x}{x^2+1}$$

Marking criteria	**Marks**
Correct solution	**2**
Correct value of A or set up 3 correct simultaneous equations, A+B=0, A+2C=1, 2B+C=-2	**1**

(ii) Hence find $\int \frac{1-2x}{(x+2)(x^2+1)}dx$. **2**

$$\int \frac{1}{x+2} - \frac{x}{x^2+1}dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2+1| + C$$

Marking criteria	**Marks**
Correct solution	**2**
Correctly integrated A/(x+2) or or Bx/(x^2+1)	**1**

(e) Find $\int \tan^4 x\,dx$. **2**

$$\int \tan^2 x(\sec^2 x - 1)\,dx$$
$$= \int \tan^2 x \sec^2 x - \tan^2 x\,dx$$
$$= \int \tan^2 x \sec^2 x - \sec^2 x + 1\,dx$$
$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Marking criteria	**Marks**
Correct solution	**2**
Correctly integrated $\tan^2 x \sec^2 x$	**1**

End of Question 11

Question 12 (14 marks) Use a new writing booklet.

(a) Find the square roots of $3 + 2i\sqrt{10}$. **3**

Marking Criteria	Marks
• Provides correct solution	3
• Finds one possible value of z	2
• Lets $z^2 = x^2 - y^2 + 2ixy$ or uses De Moivre's Theorem	1

Let $z = x + iy$

$$x^2 - y^2 + 2ixy = 3 + 2\sqrt{10}i$$

equating real and imaginary parts

$$x^2 - y^2 = 3$$

$$2xy = 2\sqrt{10}$$

$$\text{so } x^2 - \left(\frac{\sqrt{10}}{x}\right)^2 = 3$$

$$(x^2-5)(x^2+2) = 0$$

$$x \pm \sqrt{5}$$

$$\text{when } x = \pm\sqrt{5} y = \pm\sqrt{2}$$

Hence, $z = \pm\left(\sqrt{5} + i\sqrt{2}\right)$.

(b) Given $z \neq 0$, show that $\phi = \frac{2z}{z^2 + |z|^2}$ is real. **3**

$\phi = \frac{2z}{z^2 + z\bar{z}}$ $= \frac{2\cancel{z}}{\cancel{z}(z+\bar{z})}$ $= \frac{2}{z+\bar{z}}$ $= \frac{2}{2\text{Re}z}$ $= \frac{1}{\text{Re}z}$ whic is real	Alternative solution a) As $z = re^{i\theta}$ and $\mid z\mid^2 = r^2$ then $\phi = \frac{2z}{z^2 + \|z\|^2}$ can be expressed as $\phi = \frac{2re^{i\theta}}{r^2 e^{i2\theta} + r^2} = \frac{2re^{i\theta}}{r^2 (e^{i2\theta}+1)}$ this means $\phi = \frac{2e^{i\theta}}{r (e^{i2\theta} +1)}$ as $r \neq 0$. Now, $e^{i\theta} = cos\ cos\ \theta + i\ sin\ sin\ \theta$ this indicates that $\phi = \frac{2\ cos\ cos\ \theta + 2i\ sin\ sin\ \theta}{r\ (\ cos\ cos\ 2\theta + i\ sin\ sin\ 2\theta\ \ + 1)}$ $\phi = \frac{2\ cos\ cos\ \theta + 2i\ sin\ sin\ \theta}{r\ (\ cos^2\theta - 1 + i\theta\ cos\ cos\ \theta\ \ + 1)}$ $\phi = \frac{2(cos\ cos\ \theta + i\ sin\ sin\ \theta\ )}{2r(\ \theta + i\ sin\ sin\ \theta\ cos\ cos\ \theta\ \ )}$ $\phi = \frac{cos\ cos\ \theta + i\ sin\ sin\ \theta}{r\ cos\ cos\ \theta\ (\ cos\ cos\ \theta + i\ sin\ sin\ \theta\ \ )}$ Hence, $\phi = \frac{1}{rcoscos\,\theta}$ which is purely real.

(c) Find **3**

$$\int e^{\sqrt{x}} dx.$$

Let $u = \sqrt{x}$

$x = u^2$

$dx = 2u\,du$

$\int e^{\sqrt{x}} dx = \int 2ue^{u} du$

$= 2ue^{u} - \int e^{u} \times 2\, du$

$= 2ue^{u} - 2e^{u} + C$

$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

(d) The vector equations of lines l_1 and l_2 are given below.

$$l_1: \underset{\sim}{r} = (2\ 1\ 0\ ) + \lambda(3\ \ -2\ 4\ )$$

$$l_2: \underset{\sim}{r} = (-2\ 1\ 2\ ) + \mu(2\ 1\ \ -2\ )$$

(i) Show that l_1 and l_2 do not intersect. **3**

Marking Criteria	**Marks**
• Provides correct solution	3
• Makes progress towards solution. e.g solves a system of appropriate equations	2
• Finds the parametric equations of the lines, or equivalent merit	1

Let $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, such that:

$$\begin{aligned} 2+3\lambda &= -2+2\mu && (1) \\ 1-2\lambda &= 1+\mu && (2) \\ 4\lambda &= 2-\mu && (3) \end{aligned}$$

Using (2) and (3), $\lambda = 1$ and $\mu = -2$.

Verifying this result in (1): $2+3(1) \neq -2+2(-2)$.

Therefore ℓ_1 and ℓ_2 do not intersect.

(ii) Find a vector of the form $\begin{pmatrix} 0 \\ a \\ b \end{pmatrix}$ that is perpendicular to both l_1 and l_2, **1**

by inspection

$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is perpendicular to both l_1 & l_2

Alternative

$\begin{pmatrix} 0 \\ a \\ b \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = 0 \Rightarrow -2a+4b=0$

$\begin{pmatrix} 0 \\ a \\ b \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0 \Rightarrow a-2b=0$

Let $b=t$

$a=2t$

$\therefore \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 2t \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$\therefore$ any vectors parallel to $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ will be perpendicular to both l_1 & l_2.

(iii) Hence find the minimum distance between l_1 and l_2. **3**

The vector connecting the two lines is

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$$

Then project $\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$ onto the vector found in C(ii)

$$\therefore \operatorname{Proj}_{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}} \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} = \frac{0+0-2}{0+4+1} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$= \frac{-2}{5} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

The minimum distance is the perpendicular distance between the two lines

$$\therefore \text{minimum distance} = \left| \frac{-2}{5} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right| = \frac{2}{5} \times \sqrt{2^2+1^2} = \frac{2\sqrt{5}}{5}$$

End of Question 12

Question 13 (15 marks) Use a new writing booklet.

(a) Evaluate **3**

$$\int_0^{\frac{\pi}{3}} \frac{5}{3 + 2\,cos\,cos\,x}\,dx.$$

Marking Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards solution. e.g. finds the integral in terms of t	2
• Uses an appropriate substitution, or equivalent merit	1

Let $t = \tan\frac{x}{2} \Rightarrow dx = \frac{2\,dt}{1+t^2}$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{3}$, $t = \frac{1}{\sqrt{3}}$

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} \frac{5}{3+2\cos x}\,dx &= \int_0^{\frac{1}{\sqrt{3}}} \frac{5}{3+2\cdot\frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{10\,dt}{3+3t^2+2-2t^2} \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{10}{5+t^2}\,dt \\
&= \left[\frac{10}{\sqrt{5}}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right)\right]_0^{\frac{1}{\sqrt{3}}} \\
&= \frac{10}{\sqrt{5}}\tan^{-1}\frac{1}{\sqrt{15}}
\end{aligned}$$

(b) Explain why there is no integer p such that $p^{11} - (p+2)^7 = 2025$. **2**

Criteria	Marks
• Provides correct solution	2
• Shows difference of even or odd numbers are even.	1

Sample answer:

Suppose p be even

$\therefore$ both p^{11} and $(p+2)^7$ are even

$\therefore p^{11} = 2M$ and $(p+2)^7 = 2L$ for some integer M and L

$\therefore p^{11} - (p+2)^7 = 2M - 2L$

$= 2(M-L)$ which is even

but RHS is odd

$\therefore$ contradiction

Suppose p is odd

$\therefore$ both p^{11} and $(p+2)^7$ are odd

$\therefore p^{11} = 2Q+1$ and $(p+2)^7 = 2R+1$

for some integer Q and R

but $p^{11} - (p+2)^7 = 2Q - 2R$

$= 2(Q-R)$ which is even

but RHS is odd

$\therefore$ contradiction

Hence there is no integer p which satisfies the equation

(c) Find **2**

$$\int \frac{e^{2x} - e^x}{e^x + 1} dx.$$

Criteria	Marks
• Provides correct solution	2
• Rearranges the numerator in terms of $e^x + 1$, or correctly rewrites the integral in terms of their u, or equivalent merit	1

Sample answer:

$$\int \frac{e^{2x} - e^x}{e^x + 1} dx$$

$$= \int \frac{(e^x(e^x + 1) - 2e^x)}{e^x + 1} dx \qquad A = \int \left(e^x - \frac{2e^x}{e^x + 1}\right) dx = e^x - 2\ ln\ ln\ |e^x + 1|\ + c \qquad B$$

The absolute value signs are not needed as $e^x + 1 > 0$ for all x, so $e^x - 2\ ln\ ln\ (e^x + 1)\ + c$ is acceptable for full marks.

Alternative solution I:

$$\int \frac{e^{2x} - e^x}{e^x + 1} dx \qquad u = e^x\ \frac{du}{dx} = e^x\ dx = \frac{du}{u} = \int \frac{u^2 - u}{u + 1} \times \frac{du}{u} \qquad A = \int \frac{u - 1}{u + 1} du$$

$$= \int \frac{u + 1 - 2}{u + 1} du = \int \left(1 - \frac{2}{1 + u}\right) du = u - 2\ ln\ ln\ |1 + u|\ + c$$

$$= e^x - 2\ ln\ ln\ |1 + e^x|\ + c \qquad B$$

(d) Let z be a complex number and λ a real constant. **2**

Find the values of $|z|$ that satisfy the equation,

$$\lambda(\underline{z} + z^n) = i(\underline{z} - z^n)$$

for $n \geq 2$.

Marking Criteria	**Marks**
• Provides correct solution	2
• Finds $\lvert z^n \rvert = \lvert \overline{z} \rvert$, or equivalent merit	1

$$\lambda(\overline{z} + z^n) = i(\overline{z} - z^n)$$

rearranging and factorising:

$$z^n(\lambda + i) = \overline{z}(i - \lambda)$$

since $|\lambda + i| = |i - \lambda|$

$$|z^n| = |\overline{z}|$$

$$\therefore |z| = 0 \text{ or } |z| = 1$$

(e) The line L with equation $\underset{\sim}{r} = (-1 \;\; 2\; 6\,) + \lambda\,(\,1\;2\;1\;\,)$ intersects with the sphere with **3**

equation $(x+1)^2 + (\,y-2\,)^2 + z^2 = 45$ at the two points A and B.

Find the midpoint of AB.

Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards finding a quadratic equation for λ, or equivalent merit	2
• Sets up expressions for x and y in terms of λ, or equivalent merit	1

$L = \begin{pmatrix} -1+\lambda \\ 2+2\lambda \\ 6+\lambda \end{pmatrix}$ substitute that into the equation of sphere

$\lambda^2 + (2\lambda)^2 + (6+\lambda)^2 = 45$

$6\lambda^2 + 12\lambda + 36 - 45 = 0$

$6\lambda^2 + 12\lambda - 9 = 0$

$2\lambda^2 + 4\lambda - 3 = 0$

suppose roots of the equation are α and β

$\therefore$ A and B lies on the line with $\lambda = \alpha$ and $\lambda = \beta$ respectively

$\therefore M_{AB}$ will have the parameter $\lambda = \frac{\alpha+\beta}{2} = \frac{1}{2}\left(-\frac{4}{2}\right) = -1$

$\therefore M_{AB} = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$

(f) Given $a_{n+1} = \sqrt{2a_n}$ and $a_1 = 1$, prove by mathematical induction that for all integers $n \geq 1$ **3**

$$a_n = 2^{\left(1-2^{(1-n)}\right)}.$$

Criteria	**Marks**
• Provides correct solution	3
• Proves the base case and uses $P(k)$ correctly in the body of the proof, or equivalent merit	2
• Proves the base case, or uses $P(k)$ correctly in the body of the proof without proving the base case, or equivalent merit	1

Sample answer:

Let $P(n)$ represent the proposition.

$P(1)$ is true since $a_1 = 2^{1-2^{1-1}} = 2^{1-1} = 1$ $\quad A$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = 2^{\left(1-2^{(1-k)}\right)}$

RTP $P(k+1)$ $\quad a_{k+1} = 2^{\left(1-2^{-k}\right)}$

$$LHS = a_{k+1} = \sqrt{2a_k} = \sqrt{2 \times 2^{\left(1-2^{(1-k)}\right)}} \ from\ P(k) \qquad B = \sqrt{2^{1+1-2^{1-k}}} = \sqrt{2^{2-2^{1-k}}}$$

$$= \sqrt{2^{2\left(1-2^{-k}\right)}} = 2^{1-2^{-k}} \qquad C = RHS$$

$$\therefore P(k) \Rightarrow P(k+1)$$

Hence $P(n)\ is\ true\ for\ n \geq 1\ by\ induction$

End of Question 13

Question 14 (16 marks) Use a new writing booklet.

(a) Consider the expansion of z^5 for $z = cos\ cos\ \theta\ + i\ sin\ sin\ \theta$.

(i) Using De Moivre's theorem show that $cos\ cos\ 5\theta = 16\theta\ - 20\theta\ + 5\ cos\ cos\ \theta$. **3**

Marking Criteria	Marks
• Provides correct solution	3
• Finds the correct binomial expansion of z^5	2
• Applies De Moivre's theorem	1

Using De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

Using the binomial theorem:

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2 2\theta - 10\cos^2\theta\, i\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

Equating the real parts of both expansions gives:

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)^2\\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\left(1 - 2\cos^2\theta + \cos^4\theta\right)\end{aligned}$$

$$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

(ii) Hence show that $x = \pm\, cos\, cos\, \frac{\pi}{10}$ and $x = \pm\, cos\, cos\, \frac{3\pi}{10}$ are the solutions to the equation $16x^4 - 20x^2 + 5 = 0$. **3**

Marking Criteria	Marks
• Provides correct solution	3
• Attempts to relate roots and coefficients	2
• Solves $\cos 5\theta = 0$, or equivalent merit	1

$$\begin{aligned} \cos 5\theta &= 0 \\ 5\theta &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2} \\ \therefore \theta &= \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10} \end{aligned}$$

Hence, $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$ when $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$.

Since $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right)$,

$16\cos^4\theta - 20\cos^2\theta + 5 = 0$ when $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

Let $x = \cos\theta$ such that $16x^4 - 20x^2 + 6 = 0$ when:

$$\begin{aligned} x &= \cos\frac{\pi}{10} \\ x &= \cos\frac{3\pi}{10} \\ x &= \cos\frac{7\pi}{10} = \cos\left(\pi - \frac{3\pi}{10}\right) = -\cos\frac{3\pi}{10} \\ x &= \cos\frac{9\pi}{10} = \cos\left(\pi - \frac{\pi}{10}\right) = -\cos\frac{\pi}{10} \end{aligned}$$

(iii) Deduce the exact value of $\cos \cos \frac{\pi}{10}$. **3**

Marking Criteria	**Marks**
• Provides correct solution	3
• Finds all solutions to the quartic polynomial equation, or equivalent merit	2
• Finds some solutions to the quartic polynomial equation, or equivalent merit	1

Let $u = x^2$,

$$16u^2 - 20u + 5 = 0$$
$$u = \frac{20 \pm \sqrt{400 - 4 \times 16 \times 5}}{2 \times 16}$$
$$u = \frac{5 \pm \sqrt{5}}{8}$$
$$\text{hence } x^2 = \frac{5 \pm \sqrt{5}}{8}$$
$$\therefore x = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}}$$

the solutions:

$$x = -\sqrt{\frac{5+\sqrt{5}}{8}},\ x = -\sqrt{\frac{5-\sqrt{5}}{8}},\ x = \sqrt{\frac{5-\sqrt{5}}{8}} \text{ and } x = \sqrt{\frac{5+\sqrt{5}}{8}}$$

correspond to the solutions:

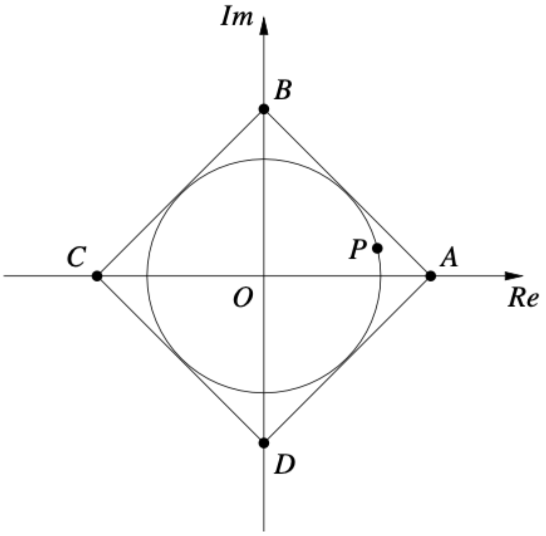
$$x = -\cos\frac{3\pi}{10},\ x = -\cos\frac{\pi}{10},\ x = \cos\frac{\pi}{10} \text{ and } x = \cos\frac{3\pi}{10}$$

since $\cos\frac{\pi}{10} > \cos\frac{3\pi}{10}$

$\therefore \cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ as $\sqrt{\frac{5+\sqrt{5}}{8}} > \sqrt{\frac{5-\sqrt{5}}{8}}$

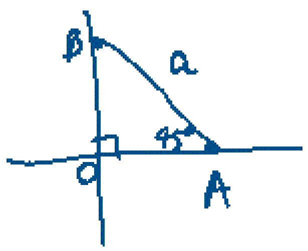
(b) Consider a square $ABCD$, of side length a, in the complex plane. **4**

The point P lies on a circle centred at the origin, of radius $\frac{a}{2}$, inscribed in square $ABCD$ as shown.



Find, in terms of a, the value of $PA^2 + PB^2 + PC^2 + PD^2$.

Marking Criteria	Marks
• Provides correct solution	4
• Finds an expression for PA^2 in terms of a and θ, or equivalent merit	3
• Finds the complex number represented by A, B, C, or D	2
• Expresses P in modulus-argument form, or equivalent merit	1



$OB = a \sin 45°$

$= \frac{a}{\sqrt{2}}$

by symmetry

$A = \left(\frac{a}{\sqrt{2}}, 0\right)$ $B = \left(0, \frac{a}{\sqrt{2}}\right)$ $C = \left(-\frac{a}{\sqrt{2}}, 0\right)$ $D = \left(0, -\frac{a}{\sqrt{2}}\right)$

Let $P = (x, y)$ Since it lies on the circle with radius $\frac{a}{2}$

$\therefore x^2 + y^2 = \left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$

$PA^2 + PB^2 + PC^2 + PD^2 = \left(x - \frac{a}{\sqrt{2}}\right)^2 + y^2 + x^2 + \left(y - \frac{a}{\sqrt{2}}\right)^2 + \left(x + \frac{a}{\sqrt{2}}\right)^2 + y^2 + x^2 + \left(y + \frac{a}{\sqrt{2}}\right)^2$

$= 4x^2 + 4y^2 + \frac{a^2}{2} \times 4$

$= 4(x^2 + y^2) + 2a^2 = 4\left(\frac{a^2}{4}\right) + 2a^2 = 3a^2$

Alternative Solution

Point P represents the complex number $\frac{a}{2}(\cos\theta + i\sin\theta)$.

$$\begin{aligned}
\cos\frac{\pi}{4} &= \frac{\frac{a}{2}}{OA} \\
OA &= \frac{\sqrt{2}}{2}a \\
\Rightarrow \overrightarrow{OA} &= \frac{\sqrt{2}}{2}a + 0i
\end{aligned}$$

$$\begin{aligned}
\text{Hence, } \overrightarrow{OB} &= 0 + \frac{\sqrt{2}}{2}ai \\
\overrightarrow{OC} &= -\frac{\sqrt{2}}{2}a + 0i \text{ and} \\
\overrightarrow{OD} &= 0 - \frac{\sqrt{2}}{2}ai
\end{aligned}$$

$$\begin{aligned}
PA^2 &= \left|\frac{a}{2}(\cos\theta + i\sin\theta) - \frac{\sqrt{2}}{2}a\right|^2 \\
&= \frac{a^2}{4}\left|\cos\theta + i\sin\theta - \sqrt{2}\right|^2 \\
&= \frac{a^2}{4}\left[\left(\cos\theta - \sqrt{2}\right)^2 + (\sin\theta)^2\right] \\
&= \frac{a^2}{4}\left(\cos^2\theta - 2\sqrt{2}\cos\theta + 2 + \sin^2\theta\right) \\
\therefore PA^2 &= \frac{a^2}{4}\left(3 - 2\sqrt{2}\cos\theta\right)
\end{aligned}$$

Similarly,

$BP^2 = \dfrac{a^2}{4}(3 - 2\sqrt{2}\sin\theta)$, $CP^2 = \dfrac{a^2}{4}(3 + 2\sqrt{2}\cos\theta)$ and $DP^2 = \dfrac{a^2}{4}(3 + 2\sqrt{2}\sin\theta)$.

Hence,

$$
\begin{aligned}
&PA^2 + PB^2 + PC^2 + PD^2 \\
&= \frac{a^2}{4}\left(3 - 2\sqrt{2}\cos\theta\right) + \frac{a^2}{4}\left(3 - 2\sqrt{2}\sin\theta\right) + \frac{a^2}{4}\left(3 + 2\sqrt{2}\cos\theta\right) + \frac{a^2}{4}\left(3 + 2\sqrt{2}\sin\theta\right) \\
&= \frac{a^2}{4}\left(3 - 2\sqrt{2}\cos\theta + 3 - 2\sqrt{2}\sin\theta + 3 + 2\sqrt{2}\cos\theta + 3 + 2\sqrt{2}\sin\theta\right) \\
&= 3a^2
\end{aligned}
$$

(c) Consider the vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ in 3 dimensions with magnitudes $\sqrt{n},\ 2\sqrt{n},\ 3\sqrt{n}$ respectively, where n is positive real number. **3**

The magnitude of the resultant vector is $\left|\underset{\sim}{p}+\underset{\sim}{q}+\underset{\sim}{r}\right|=n$.

What is the maximum value of $\left|\underset{\sim}{p}-\underset{\sim}{q}\right|^2+\left|\underset{\sim}{q}-\underset{\sim}{r}\right|^2+\left|\underset{\sim}{r}-\underset{\sim}{p}\right|^2$?

Criteria	Marks
• Provides correct solution	3
• Find quadratic relationship or equivalent merit	2
• Attempts to find magnitudes	1

Sample answer:

$$y=|\underset{\sim}{p}-\underset{\sim}{q}|^2+|\underset{\sim}{q}-\underset{\sim}{r}|^2+|\underset{\sim}{p}-\underset{\sim}{r}|^2$$

$$y=|\underset{\sim}{p}|^2-2\underset{\sim}{p}\underset{\sim}{q}+|\underset{\sim}{q}|^2+|\underset{\sim}{q}|^2-2\underset{\sim}{q}\underset{\sim}{r}+|\underset{\sim}{r}|^2$$
$$+|\underset{\sim}{p}|^2-2\underset{\sim}{p}\underset{\sim}{r}+|\underset{\sim}{r}|^2$$

$$y=2|\underset{\sim}{p}|^2+2|\underset{\sim}{q}|^2+2|\underset{\sim}{r}|^2-2\underset{\sim}{p}\underset{\sim}{q}-2\underset{\sim}{q}\underset{\sim}{r}-2\underset{\sim}{p}\underset{\sim}{r}$$

$$y=2n+8n+18n-2\underset{\sim}{p}\underset{\sim}{q}-2\underset{\sim}{q}\underset{\sim}{r}-2\underset{\sim}{p}\underset{\sim}{r}$$

$$y=28n+|\underset{\sim}{p}|^2+|\underset{\sim}{q}|^2+|\underset{\sim}{r}|^2-(\underset{\sim}{p}+\underset{\sim}{q}+\underset{\sim}{r})^2$$

$$y=28n+n+4n+9n-n^2$$

So, $y=42n-n^2$which is a parabola concaving down
With a maximum occurring when $n=21$. The Maximum value is 21^2 = 441.

End of Question 14

Question 15 (16 marks) Use a new writing booklet.

(a) Consider the integral $I_n = \int_0^1 (x^2+1)^{\frac{n}{2}}\, dx$.

(i) Show that $I_n = \frac{1}{n+1}\left(2^{\frac{n}{2}} + nI_{n-2}\right)$. **3**

Marking Criteria	Marks
• Provides correct solution	3
• Correctly uses integration by parts to obtain a single integral term, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Using integration by parts :

$$\begin{aligned}
I_n &= \int_0^1 (x^2+1)^{\frac{n}{2}}\, dx \\
&= \left[x\left(x^2+1\right)^{\frac{n}{2}}\right]_0^1 - n\int_0^1 x^2\left(x^2+1\right)^{\frac{n}{2}-1} dx \\
&= 2^{\frac{n}{2}} - n\left[\int_0^1 \left(x^2+1-1\right)\left(x^2+1\right)^{\frac{n}{2}-1} dx\right] \\
&= 2^{\frac{n}{2}} - n\left[\int_0^1 \left(x^2+1\right)^{\frac{n}{2}} dx - \int_0^1 \left(x^2+1\right)^{\frac{n}{2}-1} dx\right] \\
&= 2^{\frac{n}{2}} - n\left[I_n - I_{n-2}\right] \\
(n+1)\, I_n &= 2^{\frac{n}{2}} + nI_{n-2} \\
\therefore I_n &= \frac{1}{n+1}\left(2^{\frac{n}{2}} + nI_{n-2}\right)
\end{aligned}$$

(ii) Show that **3**

$$\int_0^{\frac{\pi}{4}} \theta \; d\theta = \frac{1}{2}\left(\sqrt{2} + ln\, ln\,(1+\sqrt{2})\right)$$

$$\int_0^{\frac{\pi}{4}} \sec^2\theta \sec\theta \, d\theta = \Big[\tan\theta \sec\theta\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan\theta(\sec\theta\tan\theta)\, d\theta$$

$$= 1\times\sqrt{2} - 0 - \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\sec\theta \, d\theta$$

$$= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta + \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

$$2\int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \sqrt{2} + \int_0^{\frac{\pi}{4}} \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} \, d\theta$$

$$= \sqrt{2} + \Big[\ln(\sec\theta + \tan\theta)\Big]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2} + \left[\ln(\sqrt{2}+1) - \ln(1)\right]$$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)$$

(iii) Hence evaluate I_3 leaving your answer in the form $a\sqrt{2}+b\ln\left(1+\sqrt{2}\right)$. **2**

Marking Criteria	**Marks**
• Provides correct solution	2
• Identifies the integral to be evaluated after applying the recurrence relation to be I_1	1

$$I_3 = \frac{1}{4}\left(2^{\frac{3}{2}} + 3I_1\right)$$

$$I_1 = \int_0^1 (x^2+1)^{\frac{1}{2}}\,dx$$

using the substitution $x = \tan\theta$:

$$dx = \sec^2\theta\,d\theta \quad \text{and} \quad \left(x^2+1\right)^{\frac{1}{2}} \to \left(\tan^2\theta+1\right)^{\frac{1}{2}} = \sec\theta$$

$$I_1 = \int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta$$

$$= \frac{1}{2}\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)$$

$$\therefore I_3 = \frac{1}{4}\left[2^{\frac{3}{2}} + \frac{3}{2}\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)\right]$$

$$= \frac{1}{8}\left(7\sqrt{2}+3\ln\left(1+\sqrt{2}\right)\right)$$

(b) The complex number w lies on the unit circle with $Arg(w) = \theta$.

(i) Show that **2**

$$\frac{1+w}{1-w} = i\cot\cot\frac{\theta}{2}.$$

Criteria	**Marks**
• Provides correct solution	2
• Substitutes $w = e^{i\theta}$ or $w = \cos\cos\theta + i\sin\sin\theta$, or draws a diagram showing vectors $1-w$ and $1+w$ where w is on the unit circle, or equivalent merit	1

Sample answer:

$$\frac{1+w}{1-w} = \frac{1 + \cos\cos\theta + i\sin\sin\theta}{1 - \cos\cos\theta - i\sin\sin\theta}$$

$$= \frac{1+ti}{t(t-i)}$$

$$= i\cot\cot\frac{\theta}{2}$$

$$A = \frac{1 + \frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}}{1 - \frac{1-t^2}{1+t^2} - i\frac{2t}{1+t^2}} = \frac{1+t^2+1-t^2+2ti}{1+t^2-1+t^2-2ti} = \frac{2+2ti}{2t^2-2ti}$$

$$B = \frac{1+ti}{t(t-i)} \times \frac{t+i}{t+i} = \frac{t+i+t^2i-t}{t(t^2+1)} = \frac{i(1+t^2)}{t(1+t^2)} = \frac{i}{t}$$

C

Alternative solutions next two pages

Alternative solution I:

$$\frac{1+w}{1-w}=\frac{1+e^{i\theta}}{1-e^{i\theta}} \qquad A=\frac{e^{-\frac{i\theta}{2}}+e^{\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}}-e^{\frac{i\theta}{2}}}=\frac{2Re\left(e^{-\frac{i\theta}{2}}\right)}{2i\,Im\left(e^{-\frac{i\theta}{2}}\right)}=\frac{cos\,cos\,\left(-\frac{\theta}{2}\right)}{\left(-\frac{\theta}{2}\right)} \qquad B=\frac{cos\,cos\,\left(-\frac{\theta}{2}\right)}{\left(-\frac{\theta}{2}\right)}\times\frac{i}{i}$$

$$=\frac{i\,cos\,cos\,\frac{\theta}{2}}{sin\,sin\,\frac{\theta}{2}} \qquad since\ cosine\ is\ even,\ sine\ is\ odd\ and\ i^2=-1$$

$$=i\,cot\,cot\,\frac{\theta}{2} \qquad C$$

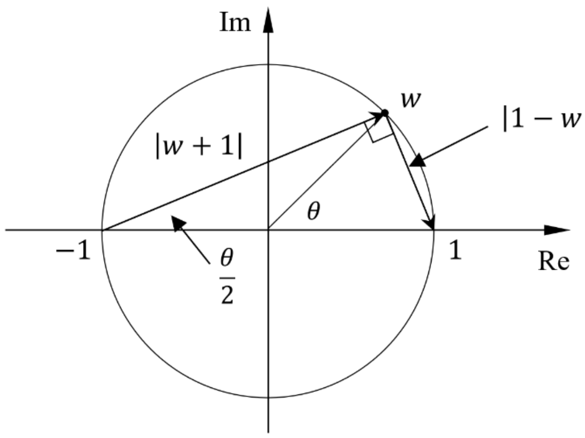
Alternative solution II:

$$Arg(1-w)=Arg(w+1)-\frac{\pi}{2} \qquad AArg(w+1)-Arg(1-w)$$

$$=\frac{\pi}{2}Arg\left(\frac{w+1}{1-w}\right)=\frac{\pi}{2} \qquad B$$

$$\therefore\frac{1+w}{1-w}\ is\ purely\ imaginaryNow\ \left|\frac{1+w}{1-w}\right|=\frac{|1+w|}{|1-w|}=cot\,cot\,\frac{\theta}{2}\ \ from\ diagram\ \therefore\ \frac{1+w}{1-w}$$

$$=i\,cot\,cot\,\frac{\theta}{2} \qquad C$$



- Angle in a semicircle is a right-angle
- Angle at the circumference is half the angle at the centre

Alternative solution III:

$$Arg\left(\frac{1+w}{1-w}\right) = Arg(1+w) - Arg(1+w) = Arg(1 + \cos\cos\theta\ + i\sin\sin\theta\ ) - Arg(1 - \cos\cos\theta\ - i\sin\sin\theta\ ) \qquad A$$

$$= \left(\frac{\sin\sin\theta}{1+\cos\cos\theta}\right) - \left(\frac{-\sin\sin\theta}{1-\cos\cos\theta}\right) = \left[\tan\tan\left\{\left(\frac{\sin\sin\theta}{1+\cos\cos\theta}\right) - \left(\frac{-\sin\sin\theta}{1-\cos\cos\theta}\right)\right\}\right]$$

$$= \left[\frac{\tan\tan\left\{\left(\frac{\sin\sin\theta}{1+\cos\cos\theta}\right)\right\} - \tan\tan\left\{\left(\frac{-\sin\sin\theta}{1-\cos\cos\theta}\right)\right\}}{1+\tan\tan\left\{\left(\frac{\sin\sin\theta}{1+\cos\cos\theta}\right)\right\} \times \tan\tan\left\{\left(\frac{-\sin\sin\theta}{1-\cos\cos\theta}\right)\right\}}\right] = \left[\frac{\frac{\sin\sin\theta}{1+\cos\cos\theta} - \frac{-\sin\sin\theta}{1-\cos\cos\theta}}{1+\frac{\sin\sin\theta}{1+\cos\cos\theta} \times \frac{-\sin\sin\theta}{1-\cos\cos\theta}}\right]$$

$$= \left[\frac{\frac{\sin\sin\theta\ - \sin\sin\theta\ \cos\cos\theta\ + \sin\sin\theta\ + \sin\sin\theta\ \cos\cos\theta}{1-\theta}}{1-\frac{\theta}{1-\theta}}\right] = \left[\frac{\frac{2\sin\sin\theta}{1-\theta}}{\frac{1-\theta\ -\theta}{1-\theta}}\right] = \left[\frac{2\sin\sin\theta}{0}\right]$$

$$= \pm\frac{\pi}{2} \qquad (1) \qquad B$$

$$\left|\frac{1+w}{1-w}\right| = \left|\frac{1+\cos\cos\theta\ + i\sin\sin\theta}{1-\cos\cos\theta\ - i\sin\sin\theta}\right| = \frac{\sqrt{(1+\cos\cos\theta\ )^2+\theta}}{\sqrt{(1-\cos\cos\theta\ )^2+\theta}} = \frac{\sqrt{1+2\cos\cos\theta\ +\theta\ +\theta}}{\sqrt{1-2\cos\cos\theta\ +\theta\ +\theta}} = \frac{\sqrt{2+2\cos\cos\theta}}{\sqrt{2-2\cos\cos\theta}}$$

$$= \sqrt{\frac{1+\cos\cos\theta}{1-\cos\cos\theta}} = \sqrt{\frac{1+\frac{1-t^2}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}} = \sqrt{\frac{1+t^2+1-t^2}{1+t^2-1+t^2}} = \sqrt{\frac{1}{t^2}} = \left|\frac{1}{\tan\tan\frac{\theta}{2}}\right| = \left|\cot\cot\frac{\theta}{2}\right| \qquad (2)$$

From (1) and (2):

$$w = i\cot\cot\frac{\theta}{2} \qquad C$$

(ii) Hence find all the solutions of **3**

$$(z-1)^5 + (z+1)^5 = 0$$

where z is a complex number.

Criteria	Marks
• Provides correct solution	3
• Shows $z = \frac{1+w}{1-w}$ where $\lvert w\rvert = 1$ and $Arg(w) = \theta$, or shows the solutions satisfy $\lvert z-1\rvert = \lvert z+1\rvert$ and $arg\left[\frac{z-1}{z+1}\right] = \frac{(2k+1)\pi}{5}$, or equivalent merit	2
• Shows $\frac{z-1}{z+1}$ is equal to one the fifth roots of -1, or rearranges the equation to $\left(\frac{1+z}{1-z}\right)^5 = -1$, or shows the solutions satisfy $\lvert z-1\rvert = \lvert z+1\rvert$ or $arg\left[\frac{z-1}{z+1}\right] = \frac{(2k+1)\pi}{5}$, or equivalent merit	1

Sample answer:

$$(z-1)^5 + (z+1)^5 = 0$$

$$\left(\frac{z-1}{z+1}\right)^5 + 1 = 0$$

$$\left(\frac{z-1}{z+1}\right)^5 = -1$$

Let $u = \frac{z-1}{z+1}$

$\therefore u^5 = -1$

let $u = r \operatorname{cis} \theta$

$\therefore r^5 = 1 \quad \operatorname{cis} 5\theta = \operatorname{cis}(\pi + 2k\pi) \quad k = 0, \pm 1, \pm 2$

$5\theta = (1+2k)\pi$

$\theta = \frac{(1+2k)\pi}{5}$

$\therefore \theta = \frac{\pi}{5}, -\frac{\pi}{5}, \frac{3\pi}{5}, -\frac{3\pi}{5}, \pi$

$\therefore u = \operatorname{cis}\left(\pm\frac{\pi}{5}\right), \operatorname{cis}\left(\pm\frac{3\pi}{5}\right), \operatorname{cis}\pi$

$\frac{z-1}{z+1} = u$

$z - 1 = uz + u$

$$z(1-u) = u+1$$

$$z = \frac{1+u}{1-u}$$

since $|u| = 1$

$$\therefore z = i\cot\frac{\theta}{2}$$

since $\theta = \pi, \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}$

$$\therefore z = i\cot\frac{\pi}{2},\ i\cot\left(\pm\frac{\pi}{10}\right),\ i\cot\left(\pm\frac{3\pi}{10}\right)$$

$$= 0,\ \pm i\cot\frac{\pi}{10},\ \pm i\cot\frac{3\pi}{10}$$

Alternative

$$\left(\frac{z-1}{z+1}\right)^5 = -1 \quad = cis\ (2k+1)\pi \therefore \frac{z-1}{z+1} = cis\frac{(2k+1)\pi}{5} \quad for\ k = 0, \pm1, \pm2 \qquad A \therefore z-1$$

$$= z\ cis\frac{(2k+1)\pi}{5} + cis\frac{(2k+1)\pi}{5} z\left(1 - cis\frac{(2k+1)\pi}{5}\right) = \left(1 + cis\frac{(2k+1)\pi}{5}\right) zz$$

$$= \frac{1 + cis\frac{(2k+1)\pi}{5}}{1 - cis\frac{(2k+1)\pi}{5}} \qquad B = i\ cot\ cot\ \frac{(2k+1)\pi}{10} \quad with\ w$$

$$= cis\frac{(2k+1)\pi}{5}\ in\ part\ (i) = i\ cot\ cot\ \left(\pm\frac{\pi}{10}\right),\ i\ cot\ cot\ \left(\pm\frac{3\pi}{10}\right), i\ cot\ cot\ \left(\frac{\pi}{2}\right) \qquad C$$

$$= \pm\ i\ cot\ cot\ \left(\frac{\pi}{10}\right), \pm i\ cot\ cot\ \left(\frac{3\pi}{10}\right), 0$$

Alternatively:

$$z = i\ cot\ cot\ \left(\frac{\pi}{16}\right),\ i\ cot\ cot\ \left(\frac{3\pi}{16}\right), \ldots, i\ cot\ cot\ \left(\frac{15\pi}{16}\right)$$

Alternative solutions next two pages

Alternative solution I:

$$\left(\frac{z-1}{z+1}\right)^8 = -1\left(\frac{z-1}{z+1}\right)^8 = e^{(2k+1)\pi i} \quad for\ k=-4,-3,\dots,3\frac{z-1}{z+1}$$

$$= e^{\frac{(2k+1)\pi}{5}i} \qquad Az-1$$

$$= e^{\frac{(2k+1)\pi}{5}i}z + e^{\frac{(2k+1)\pi}{5}i}z\left(1-e^{\frac{(2k+1)\pi}{5}i}\right) = 1+e^{\frac{(2k+1)\pi}{5}i}z$$

$$= \frac{1+e^{\frac{(2k+1)\pi}{5}i}}{1-e^{\frac{(2k+1)\pi}{5}i}} \qquad B = i\ cot\ cot\ \left(\frac{(2k+1)\pi}{10}\right)\ \ with\ w$$

$$= e^{\frac{(2k+1)\pi}{5}i}\ in\ part\ (i) = i\ cot\ cot\ \left(\pm\frac{\pi}{10}\right),\ i\ cot\ cot\ \left(\pm\frac{3\pi}{10}\right), i\ cot\ cot\ \left(\frac{\pi}{2}\right) \qquad C$$

$$= \pm\ i\ cot\ cot\ \left(\frac{\pi}{10}\right), \pm i\ cot\ cot\ \left(\frac{3\pi}{10}\right), 0$$

Alternative solution II:

$$\left|\left(\frac{z-1}{z+1}\right)^5\right| = |-1|\left|\frac{z-1}{z+1}\right|^5 = 1\left|\frac{z-1}{z+1}\right| = 1|z-1| = |z+1| \qquad A$$

The solutions lie on the perpendicular bisector of the interval joining $(1,0)$ and $(-1,0)$, which is the imaginary axis. The solutions are all purely real.

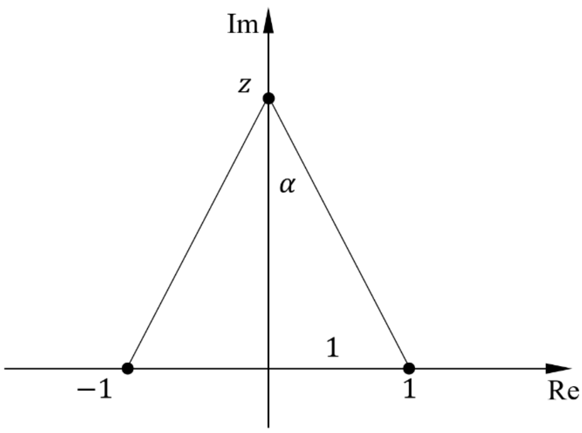
$$arg\left[\left(\frac{z-1}{z+1}\right)^5\right] = arg(-1)5\ arg\left[\frac{z-1}{z+1}\right] = (2k+1)\pi arg\left[\frac{z-1}{z+1}\right] = \frac{(2k+1)\pi}{5} \qquad B$$

In the diagram $\alpha = \frac{(2k+1)\pi}{10}$

$$tan\ tan\ \alpha\ = \frac{1}{|z|}|z| = cot\ cot\ \alpha\ = cot\ cot\ \left(\frac{(2k+1)\pi}{10}\right)$$

Since z lies on the imaginary axis above or below zero,

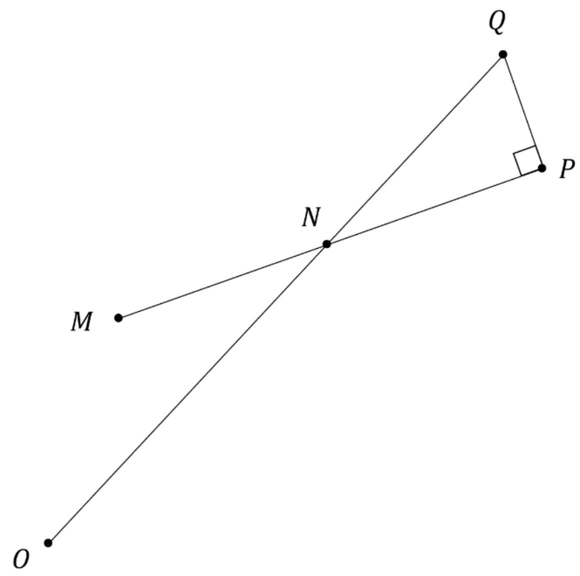
$$z = i\ cot\ cot\ \left(\pm\frac{\pi}{10}\right),\ i\ cot\ cot\ \left(\pm\frac{3\pi}{10}\right), i\ cot\ cot\ \left(\frac{\pi}{2}\right) \qquad C == \pm\ i\ cot\ cot\ \left(\frac{\pi}{10}\right), \pm i\ cot\ cot\ \left(\frac{3\pi}{10}\right), 0$$



Question 15 (continued)

(c) M and P are two points with position vectors $\overrightarrow{OM} = \underset{\sim}{m}$ and $\overrightarrow{OP} = \underset{\sim}{p}$ respectively. **3**

N is the midpoint of MP and has position vector $\overrightarrow{ON} = \underset{\sim}{n}$. Q lies on the line ON.
PQ is perpendicular to MN and $\overrightarrow{NQ} = \lambda\overrightarrow{ON}$ where λ is a real number.



NOT TO SCALE

Show that

$$\lambda = \frac{|\underset{\sim}{m}|^2 + |\underset{\sim}{n}|^2 - 2\underset{\sim}{m} \cdot \underset{\sim}{n}}{|\underset{\sim}{n}|^2 - \underset{\sim}{m} \cdot \underset{\sim}{n}}.$$

Question 16 (c)

Criteria	Marks
• Provides correct solution	3
• Finds an equation for the dot product of $\overrightarrow{QP}$ and $\overrightarrow{MN}$ in terms of $\underset{\sim}{m}$ and $\underset{\sim}{n}$, or equivalent merit	2
• Finds expressions for $\underset{\sim}{p}$ and $\underset{\sim}{q}$ in terms of $\underset{\sim}{m}$ and $\underset{\sim}{n}$, or equivalent merit	1

Sample answer:

$\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP} p_{\sim} = \overrightarrow{ON} + \overrightarrow{MN} = n_{\sim} + n_{\sim} - m_{\sim} = 2n_{\sim} - m_{\sim} \overrightarrow{OQ} = \overrightarrow{ON} + \overrightarrow{NQ} q_{\sim} = \overrightarrow{ON} + \lambda \overrightarrow{ON} = (1+\lambda)\overrightarrow{ON}$

$= (1+\lambda) n_{\sim}$ $A \overrightarrow{QP} = p_{\sim} - q_{\sim}$

$= 2n_{\sim} - m_{\sim} - (1+\lambda) n_{\sim} = (1-\lambda) n_{\sim} - m_{\sim}$

$\overrightarrow{PQ} \cdot \overrightarrow{MN} = 0 (m_{\sim} - (1-\lambda) n_{\sim}) \cdot (n_{\sim} - m_{\sim})$

$= 0$ $B m_{\sim} \cdot n_{\sim} - m_{\sim} \cdot m_{\sim} - (1-\lambda) n_{\sim} \cdot n_{\sim} + (1-\lambda) n_{\sim} \cdot m_{\sim}$

$= 0 (\lambda - 1)|n_{\sim}|^2 + (2-\lambda) m_{\sim} \cdot n_{\sim} - |m_{\sim}|^2 = 0 \lambda (|n_{\sim}|^2 - m_{\sim} \cdot n_{\sim}) + 2m_{\sim} \cdot n_{\sim} - |m_{\sim}|^2 - |n_{\sim}|^2$

$= 0 \lambda (|n_{\sim}|^2 - m_{\sim} \cdot n_{\sim}) = |m_{\sim}|^2 + |n_{\sim}|^2 - 2m_{\sim} \cdot n_{\sim} \lambda = \dfrac{|m_{\sim}|^2 + |n_{\sim}|^2 - 2m_{\sim} \cdot n_{\sim}}{|n_{\sim}|^2 - m_{\sim} \cdot n_{\sim}}$ C

Question 16 (15 marks) Use a new writing booklet.

(a) (i) By considering vectors $(1\ 1\ 1)$ and $(p\ q\ r)$, show that **2**

$$(p+q+r)^2 \leq 3(p^2+q^2+r^2)$$

where p, q and r are positive real numbers.

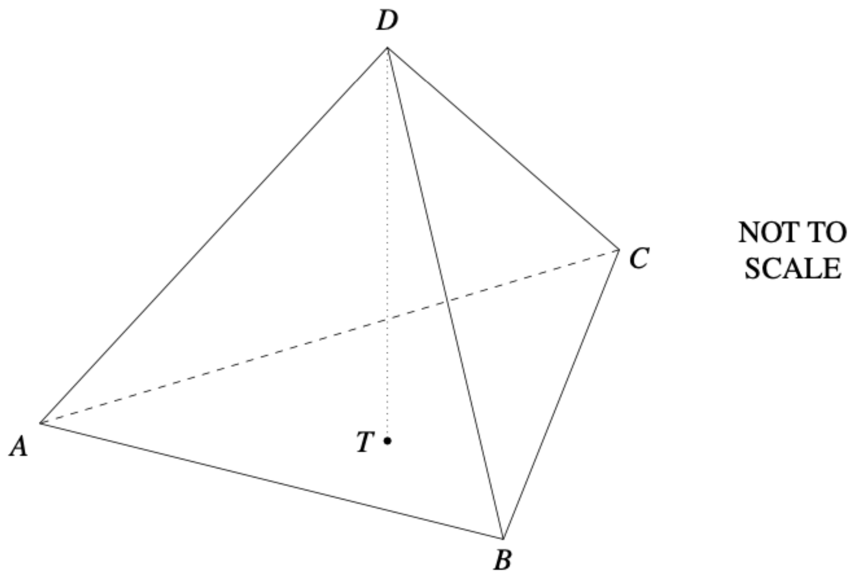
$$\text{since } |\mathbf{u}\cdot\mathbf{v}|^2 \leqslant |\mathbf{u}|^2|\mathbf{v}|^2$$

$$\text{then } \left(\sqrt{p+q+r}\right)^2 \leqslant \left(\sqrt{p^2+q^2+r^2}\right)^2 \times \left(\sqrt{1^2+1^2+1^2}\right)^2$$

$$\therefore (p+q+r)^2 \leqslant 3\left(p^2+q^2+r^2\right)$$

Marking criteria	**Marks**
Correct solution	**2**
Correctly found p + q + r using dot product or correctly proved the inequality using non vector method.	**1**

(ii) Consider the triangular pyramid $ABCD$, where T is a point in the plane of triangle ABC, as shown. **3**



It is known that $\overrightarrow{OT} = \frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right)$

Using the result in part (i), show that $\left|\overrightarrow{DT}\right|^2 \leq \frac{1}{3}\left(\left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{BD}\right|^2 + \left|\overrightarrow{CD}\right|^2\right)$

Marking Criteria	Marks
• Provides correct solution, including the geometrical interpretation of the equality case	3
• Makes significant progress towards solution	2
• Expresses $\lvert \overrightarrow{DT} \rvert$ in terms of other vectors	1

$$
\begin{aligned}
\lvert \overrightarrow{DT} \rvert &= \left\lvert \overrightarrow{OD} - \frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right) \right\rvert \\
&= \left\lvert \frac{1}{3}\overrightarrow{OD} - \frac{1}{3}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OD} - \frac{1}{3}\overrightarrow{OB} + \frac{1}{3}\overrightarrow{OD} - \frac{1}{3}\overrightarrow{OC} \right\rvert \\
\text{so } \lvert \overrightarrow{DT} \rvert^2 &= \left\lvert \frac{1}{3}\overrightarrow{OD} - \frac{1}{3}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OD} - \frac{1}{3}\overrightarrow{OB} + \frac{1}{3}\overrightarrow{OD} - \frac{1}{3}\overrightarrow{OC} \right\rvert^2 \\
&= \frac{1}{9} \lvert \overrightarrow{AD} + \overrightarrow{BD} + \overrightarrow{CD} \rvert^2 \\
&\leqslant \frac{1}{9} \times 3\left(\lvert \overrightarrow{AD} \rvert^2 + \lvert \overrightarrow{BD} \rvert^2 + \lvert \overrightarrow{CD} \rvert^2\right) \quad \text{using part (i)} \\
&= \frac{1}{3}\left(\lvert \overrightarrow{AD} \rvert^2 + \lvert \overrightarrow{BD} \rvert^2 + \lvert \overrightarrow{CD} \rvert^2\right) \quad \text{as required}
\end{aligned}
$$

In the case of $\lvert \overrightarrow{DT} \rvert^2 = \frac{1}{3}\left(\lvert \overrightarrow{AD} \rvert^2 + \lvert \overrightarrow{BD} \rvert^2 + \lvert \overrightarrow{CD} \rvert^2\right)$, D is equidistant from A, B and C.

(b) The three positive real numbers r, s, and t are such that **3**

$$(r+n)(s+n)(t+n) = n^6$$

show that $rst \leq (n^2 - n)^3$.

You may use the result $(a + b + c)^3 \geq 27abc$ where a, b and c are positive.
[DO NOT PROVE THIS]

Marking criteria	**Marks**
Correct solution	**3**
Indicate that $r+s+t \geq 3\sqrt[3]{rst}$ or $rs+rt+st \geq 3\sqrt[3]{r^2s^2t^2}$	**2**
Indicate that $(r+n)(s+n)(t+n) = rst + n(rs+rt+st) + n^2(r+s+t) + n^3$	**1**

$(r+n)(s+n)(t+n)$

$$= (rs + rn + sn + n^2)(t+n)$$
$$= rst + rsn + rnt + r\,n^2 + snt + sn^2 + tn^2 + n^3$$

$= rst + n(rs+rt+st) + n^2(r+s+t)$

$$+n^3 = n^6 \ (1)$$

Given that $(a+b+c)^3 \geq 27abc$

Let $a = r$, $b = s$ and $c = t$ we get

$(r+s+t)^3 \geq 27rst$ this means $r+s+t \geq 3\sqrt[3]{rst}$ (2)

Also, let $a = rs,\ b = rt\ and\ c = st$. we get

$(rs+rt+st)^3 \geq 27r^2s^2t^2$ this means $rs+rt+st \geq 3\sqrt[3]{r^2s^2t^2}$ (3)

By substituting (2) and (3) in (1), we get

$$rst + 3n\sqrt[3]{r^2s^2t^2} + 3n^2\sqrt[3]{rst} + n^3 \leq n^6$$
$$((rst)^{\frac{1}{3}})^3 + 3\times n((rst)^{\frac{1}{3}})^2 + 3\times n^2(rst)^{\frac{1}{3}} + n^3 \leq n^6$$

$((rst)^{\frac{1}{3}} + n)^3 \leq n^6$

By applying the cube root on both sides, we get

$(rst)^{\frac{1}{3}} + n \leq n^2$ that is $(rst)^{\frac{1}{3}} \leq n^2 - n$.

By cubing on both sides, we get

$$rst \leq (n^2-n)^3.$$

Question 16 (continued)

(c) Let z_1, z_2, z_3 be complex numbers such that **5**

$$|z_1| = |z_2| = |z_3| = 1$$

$$z_1 + z_2 + z_3 \neq 0$$

$$z_1^2 + z_2^2 + z_3^2 = 0$$

Show that, for all integers $n \geq 1$,

$$|z_1^n + z_2^n + z_3^n|$$

can only take specific integer values, and state those values.

Marking Criteria	Marks
• Provides correct solution	5
• Finds a value for $\lvert z_1^n + z_2^n + z_3^n \rvert$, or equivlent merit	4
• Solves a cubic polynomial formed from the relationship, or equivalent merit	3
• Finds a relationship between the expressions given in the question	2
• Considers the complex numbers as roots of a polynomial, or equivalent merit	1

Let z_1, z_2, z_3 be the roots of the polynomial

$z^3 - az^2 + bz - c = 0$

$\therefore z_1 + z_2 + z_3 = a$

$z_1z_2 + z_2z_3 + z_3z_1 = b$

$z_1z_2z_3 = c$

$a^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1)$

$= 0 + 2b$

$= 2b$

$$\bar{a} = \overline{z_1+z_2+z_3}$$

$$= \bar{z}_1+\bar{z}_2+\bar{z}_3 \qquad \text{since } |z_1|=|z_2|=|z_3|=1$$

$$= \frac{\bar{z}_1 z_1}{z_1}+\frac{\bar{z}_2 z_2}{z_2}+\frac{\bar{z}_3 z_3}{z_3}$$

$$= \frac{|z_1|^2}{z_1}+\frac{|z_2|^2}{z_2}+\frac{|z_3|^2}{z_3}$$

t

$$= \frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}$$

$$= \frac{z_1z_2+z_2z_3+z_3z_1}{z_1z_2z_3}$$

$$= \frac{b}{c}$$

$\therefore b = c\bar{a}$

but $a^2 = 2b$

$\therefore a^2 = 2c\bar{a}$

$\therefore |a|^2 = 2|c||\bar{a}|$

but $|c| = |z_1z_2z_3| = 1$ and $|\bar{a}| = |a|$

$\therefore |a|^2 = 2|a| \Rightarrow |a| = 2$

$\therefore$ let $a = 2w$ where $w \in \mathbb{C}$ $|w|=1$

since $b = \frac{1}{2}a^2$ $\qquad c = \frac{b}{\bar{a}}$

$$= \frac{1}{2}(2w)^2 \qquad = \frac{2w^2}{2(\frac{1}{w})}$$

$$= 2w^2 \qquad = w^3$$

$\therefore$ the polynomial is

$$z^3 - 2wz^2 + 2w^2z - w^3 = 0$$

$$(z-w)(z^2 - zw + w^2)$$

$z = w, \quad z^2 - zw + w^2 = 0$

$$z = \frac{w \pm \sqrt{w^2 - 4w^2}}{2}$$

$$= \frac{w \pm \sqrt{3}iw}{2}$$

$$= \left(\frac{1 \pm \sqrt{3}i}{2}\right)w$$

$$= we^{\pm i\frac{\pi}{3}}$$

$\therefore$ The three roots are $w, \; we^{i\frac{\pi}{3}}, \; we^{-\frac{\pi}{3}}$

$$|z_1^n + z_2^n + z_3^n| = |w^n + e^{n\frac{\pi}{3}}w^n + w^n e^{-\frac{\pi}{3}n}|$$

$$= |w^n| \, |1 + e^{n\frac{\pi}{3}} + e^{-\frac{\pi}{3}n}|$$

$$= |w|^n |1 + 2\cos\tfrac{n\pi}{3}|$$

$$= |1 + 2\cos\tfrac{n\pi}{3}| \qquad \text{since } |w| = 1$$

$$\cos\frac{n\pi}{3} = \begin{cases} \frac{1}{2} & n = 1 \\ -\frac{1}{2} & n = 2 \\ -1 & n = 3 \\ -\frac{1}{2} & n = 4 \\ \frac{1}{2} & n = 5 \\ 1 & n = 6 \end{cases}$$

which repeat every multiple of 6.

$$\therefore |z_1^n + z_2^n + z_3^n| = \begin{cases} |1+1| = 2 & n = 1 \\ |1-1| = 0 & n = 2 \\ |1-2| = 1 & n = 3 \\ |1-1| = 0 & n = 4 \\ |1+1| = 2 & n = 5 \\ |1+2| = 3 & n = 6 \end{cases}$$

$\therefore |z_1^n + z_2^n + z_3^n|$ only take 4 integer values

0, 1, 2, 3, 4.

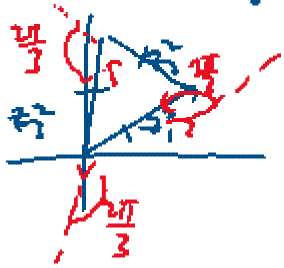
Alternative 2

Since $z_1^2+z_2^2+z_3^2=0$

$\therefore z_1^2, z_2^2, z_3^2$ form the 3 sides of a triangle ☆

but $|z_1|=|z_2|=|z_3|=1$

therefore the triangle is equilateral.

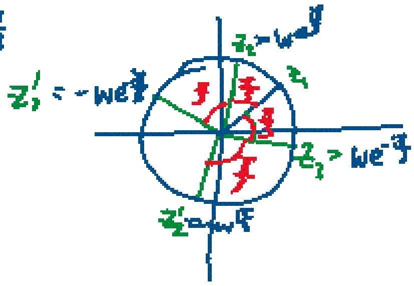


$\therefore z_2^2 = z_1^2 e^{\frac{2\pi}{3}i}$ $\quad z_3^2 = z_1^2 e^{-\frac{2\pi}{3}i}$

Let $z_1 = w$ where $|w|=1$ $\quad z_3^2 = w^2 e^{-2\frac{\pi}{3}i}$

$\therefore z_2^2 = w^2 e^{\frac{2\pi}{3}i}$

$z_2 = \pm w e^{\frac{i\pi}{3}}$ $\quad z_3 = \pm w e^{-\frac{i\pi}{3}}$



If $z_1 = w$, $z_2 = -we^{\frac{i\pi}{3}}$, $z_3 = -we^{-\frac{i\pi}{3}}$

then $z_1+z_2+z_3 = w - we^{\frac{i\pi}{3}} - we^{-\frac{i\pi}{3}}$

$= w(1-(e^{\frac{i\pi}{3}}+e^{-\frac{i\pi}{3}}))$

$= w(1-2\cos\frac{\pi}{3})$

$= 0$

but $z_1+z_2+z_3 \neq 0$

$\therefore$ There are only 3 cases

① $z_1 = w$ $\quad z_2 = we^{\frac{i\pi}{3}}$, $z_3 = we^{-\frac{i\pi}{3}}$

② $z_1 = w$ $\quad z_2 = -we^{\frac{i\pi}{3}}$ $\quad z_3 = we^{-\frac{i\pi}{3}}$

③ $z_1 = w$ $\quad z_2 = we^{\frac{i\pi}{3}}$ $\quad z_3 = -we^{-\frac{i\pi}{3}}$

from ☆

z_1^2, z_2^2, z_3^2 are unique since

if $z_1 = z_2$

then $z_1^2+z_2^2+z_3^2 = 2z_1^2+z_3^2 = 0$

$\therefore z_3^2 = 2z_1^2$

$|z_3|^2 = 2|z_1|^2$

$= 2$

which is contradiction

hence z_1^2, z_2^2, z_3^2 must form a triangle

Case ① $|z_1^n+z_2^n+z_3^n| = |w^n + w^n e^{\frac{in\pi}{3}} + w^n e^{-\frac{in\pi}{3}}|$

$= |w^n|\,|1+e^{\frac{in\pi}{3}}+e^{-\frac{in\pi}{3}}|$

$= |1+2\cos\frac{n\pi}{3}|$

Case ② $|z_1^n+z_2^n+z_3^n| = |w^n + w^n(e^{i\pi}e^{\frac{i\pi}{3}})^n + e^{-\frac{in\pi}{3}}|$

$= |w^n|\,|1+e^{\frac{4\pi i n}{3}}+e^{-\frac{in\pi}{3}}|$

$= |w^n|\,|1+e^{-\frac{2\pi i n}{3}}+e^{-\frac{in\pi}{3}}|$

$= |e^{-\frac{in\pi}{3}}|\,|e^{\frac{in\pi}{3}}+e^{-\frac{in\pi}{3}}+1|$

$= |1+2\cos n\frac{\pi}{3}|$

Case ③ $|z_1^n+z_2^n+z_3^n| = |w^n + w^n e^{\frac{i\pi n}{3}} + w^n e^{\frac{2\pi}{3}in}|$

$= |w^n e^{\frac{i\pi n}{3}}|\,|e^{-\frac{i\pi n}{3}}+1+e^{\frac{i\pi n}{3}}|$

$= |1+2\cos n\frac{\pi}{3}|$

$$\therefore |z_1^n+z_2^n+z_3^n| = |1+2\cos\frac{n\pi}{3}| = \begin{cases} |1+1|=2 & n=1 \\ |1-1|=0 & n=2 \\ |1-2|=1 & n=3 \\ |1-1|=0 & n=4 \\ |1+1|=2 & n=5 \\ |1+2|=3 & n=6 \end{cases}$$

$\therefore |z_1^n+z_2^n+z_3^n|$ only take 4 integer values 0, 1, 2, 3