Girraween High School

2025

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time: 10 minutes
- Working time: 3 Hours
- Write using a black pen
- NESA approved calculators may be used
- Laminated reference sheets are provided
- Answer multiple-choice questions by completely colouring in the appropriate circle on your multiple-choice answer sheet.
- Answer questions in the writing booklet provided and show all relevant mathematical reasoning and/or calculations.

Total Marks: 100

Section 1 (Pages 1 – 5) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

Section 2 (Pages 6-10) **90 marks**

- Attempt Q11 - Q16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1 – 10

Question 1

The angle between the vectors $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$ is

A) 27^o

B) 68^o

C) 112^o

D) 133^o

Question 2

The statement "For all positive integers n, there exists a positive integer p such that $p = n^2$" is represented in mathematical symbols by:

A) $\in\ n \ni Z^+ \ni p \in Z^+ such\ that\ p = n^2$

B) $\ni\ n \in Z \ni p \in Z\ such\ that\ p = n^2$

C) $\forall\, n \in Z^+ \ni p \in Z^+ such\ that\ p = n^2$

D) $\forall\, n \in Z \ni p \in Z\ such\ that\ p = n^2$

__Examination continues next page…__

Question 3

The contrapositive of "If the sum of the digits of an integer is a multiple of 3, the integer is a multiple of 3" is

A) "If an integer is a multiple of 3 the sum of its digits will be a multiple of 3".

B) "If an integer is not a multiple of 3, the sum of its digits will NOT be a multiple of 3".

C) "If the sum of the digits of an integer is NOT a multiple of 3, the number will NOT be a multiple of 3".

D) "If the sum of the digits of an integer is NOT a multiple of 3, the integer is a multiple of 3".

Question 4

The converse of "If the sum of the digits of an integer is a multiple of 3, the integer is a multiple of 3" is

A) "If an integer is a multiple of 3 the sum of its digits will be a multiple of 3".

B) "If an integer is not a multiple of 3, the sum of its digits will NOT be a multiple of 3".

C) "If the sum of the digits of an integer is NOT a multiple of 3, the number will NOT be a multiple of 3".

D) "If the sum of the digits of an integer is NOT a multiple of 3, the integer is a multiple of 3".

Examination continues next page…

Question 5

$$\int \frac{6}{(2x-1)(x+1)}\,dx =$$

A) $\ln\left|\frac{x+1}{2x-1}\right| + C$

B) $\ln\left|\frac{2x-1}{x+1}\right| + C$

C) $2\ln\left|\frac{x+1}{2x-1}\right| + C$

D) $2\ln\left|\frac{2x-1}{x+1}\right| + C$

Question 6

$i^{2025} =$

A) i

B) $-i$

C) 1

D) -1

Examination continues next page…

Question 7

It is given for the complex number z that $|z-2-i| = 1$. The maximum modulus for z is

A) 1

B) $\sqrt{5}$

C) $\sqrt{5}-1$

D) $\sqrt{5}+1$

Question 8

Two of the roots of a monic polynomial of degree 3 with all real coefficients are 4 and $2+i$.
The polynomial is

A) $z^3 - 6z^2 + 18z - 40$

B) $z^3 + 8z^2 - 21z + 20$

C) $z^3 - 8z^2 + 21z - 20$

D) $z^3 + 6z^2 + 18z - 40$

Question 9

A particle is moving with simple harmonic motion with a period of 12 seconds and an amplitude of 5 metres. If it starts at the centre of motion and is moving to the right, when will it first be 5 metres to the left of the centre of motion?

A) After 3 seconds

B) After 6 seconds

C) After 9 seconds

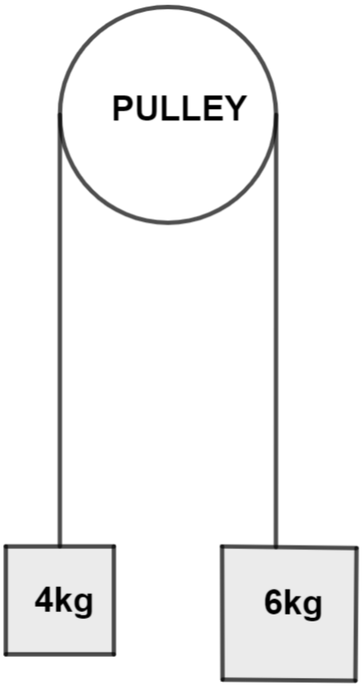
D) After 12 seconds

Examination continues next page…

Question 10

Two weights of 4kg and 6 kg respectively are attached to a pulley by the same string (see diagram).

The acceleration due to gravity is g metres per second squared.



The acceleration of the 6kg mass is

A) $10g\ m/s^2$

B) $5g\ m/s^2$

C) $\frac{g}{10} m/s^2$

D) $\frac{g}{5} m/s^2$

Examination continues next page…

Section II

90 marks
Attempt Questions 11 - 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Start a new writing page**

a) z and w are the numbers $1 - i$ and $1 - i\sqrt{3}$ respectively.

(i) Show that $\frac{z}{w} = \frac{(1+\sqrt{3})+i(\sqrt{3}-1)}{4}$ 1

(ii) Express both z and w in exponential form 2

(iii) Hence show that $cos\,\frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$ 2

b) (i) By letting $(x + iy)^2 = -5 - 12i, x, y$ real find the two square roots of $-5 - 12i$ 2

(ii) Hence solve the equation $z^2 + (2 + i)z + (2 + 4i) = 0$ 2

c) Shade the region on an Argand diagram where $|z + 2 - 2i| \geq 2$ and $\frac{\pi}{2} \leq Arg\, z < \frac{3\pi}{4}$ 4

d) Find the six sixth roots of $64i$. You may leave your answer in modulus argument form. 2

Examination continues next page…

Question 12 (15 marks) **Start a new writing page**

a) (i) Find the values of A, B and C if $\frac{A}{(x+1)^2} + \frac{B}{(x+1)} + \frac{C}{(x-3)} = \frac{-13x-25}{(x+1)^2(x-3)}$ 3

(ii) Hence find $\int \frac{-13x-25}{(x+1)^2(x-3)}dx$ 2

b) Find $\int_1^4 \frac{1}{2+\sqrt{x}}\, dx$ 4

c) Find $\int (\ln{(x)})^2 dx$ 3

d) Use the substitution $t = tan\left(\frac{x}{2}\right)$ to find $\int \frac{1}{2+\cos x}.dx$ 3

Examination continues next page…

Question 13 (15 marks) **Start a new writing page**

a) A particle moves so that its displacement at time t seconds is given by

$x = 4\cos\frac{t}{2} - 4\sqrt{3}\sin\frac{t}{2}$

(i) Show that the particle is moving in simple harmonic motion. 1

(ii) By expressing x in the form $A\cos\left(\frac{t}{2} + \alpha\right)$ or otherwise, find the period and amplitude of the motion. 2

b) A particle moves in simple harmonic motion so that its acceleration at time t is given by $\ddot{x} = -9(x + 2)$. If the particle passes through $x = 1$ with a speed of $12m/s$, show that $v^2 = 189 - 9x^2 - 36x$ and state the period, amplitude and centre of motion. 3

c) A car with a mass of $1000kg$ starts from rest at a traffic light and drives on to a horizontal freeway. The car's engine provides a total force of 16000 Newtons and the combined resistance from friction from the freeway's surface and air resistance is $10v^2$ Newtons.

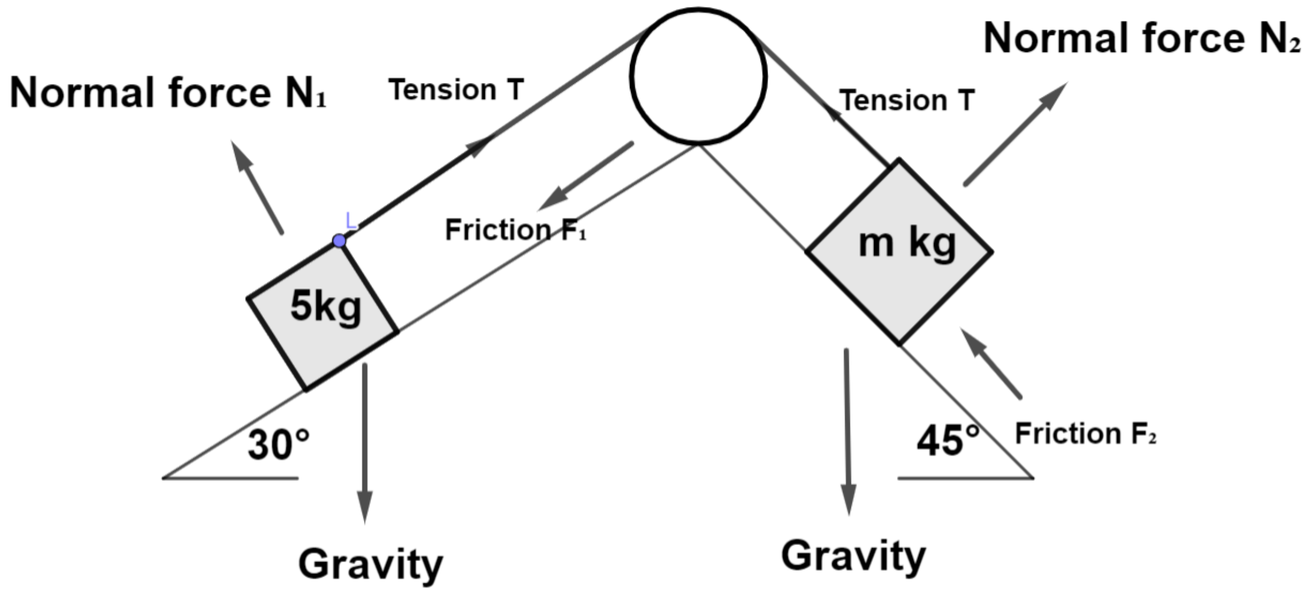
(i) Show that the car's acceleration is given by $\ddot{x} = 16 - 0.01v^2$ and find the speed the car can not exceed on the freeway (i.e. the speed it can never reach). 2

(ii) Show that $x = 50ln\left(\frac{1600}{1600-v^2}\right)$ and find the distance the car has gone when it achieves a speed of $30m/s$. 3

Examination continues next page…

Question 13 (continued)

d) Two square weights, one with mass $5kg$ and one with mass $m\ kg$ where $m > 5$ are attached by a single rope running over a smooth pulley. The $5kg$ mass is on a slope of 30^o to the horizontal and the $m\ kg$ mass is on a slope of 45^o to the horizontal. The two weights are stationary as they are being held in place by friction but are on the point of slipping with the $m\ kg$ weight being about to fall down its slope, resulting in the $5kg$ weight being pulled UP its slope in response. Apart from friction, the weights are being held in place by gravity (with force of 10 times each object's mass) and by normal forces pushing out from the plane. Friction in each case is 0.6 times the normal force (note that friction and the normal force are different at each object but the tension in the rope is the same) *(see diagram)*.



(i) By resolving forces both parallel and perpendicular to the plane at the $5kg$ weight, show that the tension in the rope is equal to $(15\sqrt{3} + 25)$ Newtons *(you may treat tension as parallel to the plane)*. 2

(ii) By resolving forces both parallel and perpendicular to the plane at the $m\ kg$ weight, find the mass (m) of the second weight in kilograms. 2

Examination continues next page…

Question 14 (15 marks) **Start a new writing page**

a) If $I_N = \int_0^{\frac{\pi}{2}} sin^N x \, dx$

(i) Show that $I_N = \frac{N-1}{N} I_{N-2}$ 2

(ii) Find the exact value of I_6 2

b) (i) State the centre and radius of the sphere S $(x-2)^2 + (y+1)^2 + (z-4)^2 = 14$ 2

(ii) By finding the perpendicular distance from the line L_1 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$ 2

to the centre of the sphere S, show that the line L_1 intersects with the circle in two places.

(iii) Find the two points of intersection of the line L and the sphere S. 3

(iv) Show that the line L_1 is perpendicular to the line L_2 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ 1

(v) Find the point of intersection of the lines L_1 and L_2 3

Examination continues next page…

Question 15 (15 marks) **Start a new writing page**

a) Prove $n! > 3^n$ for all positive integers $n \geq 8$ using the method of mathematical induction. 3

b) (i) Prove for all real numbers a and b that $a^2 + b^2 \geq 2ab$ 1

2

(ii) Hence prove $a^2 + b^2 + c^2 \geq ab + ac + bc$ for all real numbers a, b, c. 1

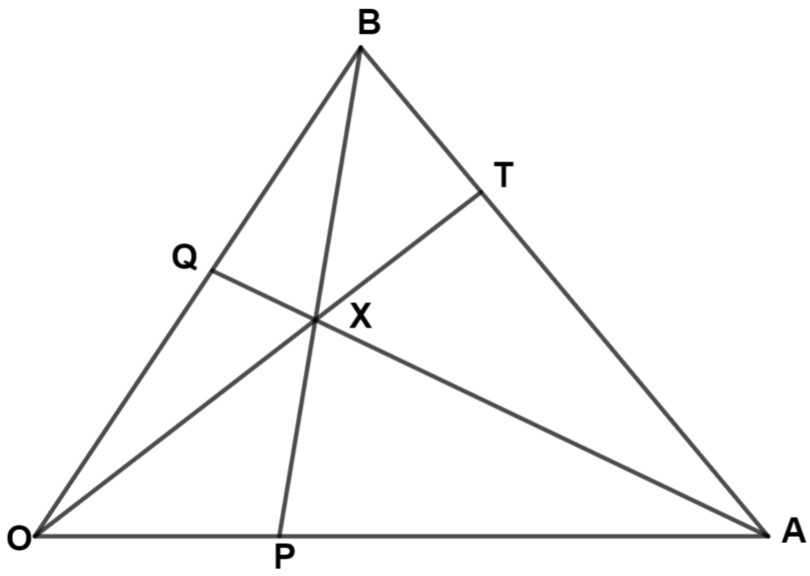
(iii) Hence prove $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{a+b+c}{abc}$ for all real numbers a, b, c. 1

c) Prove by contradiction that $\sqrt{29}$ is irrational. 2

Question 15 continues next page…

Question 15 (continued)

d) In triangle OAB, the vector $\overrightarrow{OA} = \underset{\sim}{a}$ and the vector $\overrightarrow{OB} = \underset{\sim}{b}$. P is on OA so that $\overrightarrow{OP} = \frac{1}{3}\overrightarrow{OA}$ and Q is on OB so that $\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OB}$. X is at the intersection of AQ and BP. $\overrightarrow{PX} = \lambda\,\overrightarrow{PB}$ and $\overrightarrow{QX} = \mu\,\overrightarrow{QA}$. T is on BA so that O, X and T are collinear. *(see diagram).*



(i) Show that $\overrightarrow{OX} = \frac{1}{3}(1-\lambda)\,\underset{\sim}{a} + \lambda\,\underset{\sim}{b}$ 2

(ii) $\overrightarrow{OX}$ can also be shown to be $\mu\,\underset{\sim}{a} + \frac{1}{2}(1-\mu)\,\underset{\sim}{b}$ *(Do NOT prove this!)* 2

By finding λ and μ, show that $\overrightarrow{OX} = \frac{1}{5}\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$

(iii) Given that T is collinear with O and X and is on the line AB, express $\overrightarrow{OT}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. 3

Examination continues next page…

Question 16 (10 marks) **Start a new writing page**

a) Sketch the curve $\binom{x}{y} = \binom{t\cos t}{t\sin t}$ from $t = 0$ to $t = 2\pi$ 2

b) (i) State the solutions to $z^9 - 1 = 0$ *(leave your solutions in modulus/argument form).* 1

(ii) Hence using that $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$, state the solutions to 1
$z^6 + z^3 + 1 = 0$.

(iii) Show that if $z = \cos\theta + i\sin\theta$, $z^n + \frac{1}{z^n} = 2\cos n\theta$. 1

(iv) Express $z^6 + z^3 + 1$ in terms of real quadratic factors. 2

(v) Hence show that

$\cos 3\theta + \frac{1}{2} = 4(\cos\theta - cos\frac{2\pi}{9})(\cos\theta - cos\frac{4\pi}{9})(\cos\theta + cos\frac{\pi}{9})$ 2

c) A projectile with mass $m\ kg$ is launched vertically from the ground with a velocity of v_0 metres per second where $v_0 < 200$ metres per second. It experiences gravity of $10m$ Newtons and air resistance of $0.05mv$ Newtons against its direction of motion, where v is the current velocity. If the projectile lands in the same place 5 seconds after launch, find its initial velocity v_0. 6

End of Examination.

2025 Year 12 Trial

Mathematics Extension 2

Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

1.	A ○	B ●	C ○	D ○
2.	A ○	B ○	C ●	D ○
3.	A ○	B ●	C ○	D ○
4.	A ●	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ●
6.	A ●	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ●
8.	A ○	B ○	C ●	D ○
9.	A ○	B ○	C ●	D ○
10.	A ○	B ○	C ○	D ●

Mathematics Assessment Solutions

Girraween High School

❑ Year 11	❑ Extension 1	Year 2025
☑ Year 12	☑ Extension 2	Task Number 4 TRIAL

Suggested solutions	Marker's Feedback

Multiple Choice:

(1) $\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$

$= \dfrac{\begin{pmatrix}1\\3\\-2\end{pmatrix} \cdot \begin{pmatrix}-3\\4\\1\end{pmatrix}}{\sqrt{14} \times \sqrt{26}}$

$= \dfrac{7}{\sqrt{364}}$

$\theta = 68^\circ$ (B)

(2) $\forall\, n \in \mathbb{Z}^+\ \exists\, p \in \mathbb{Z}^+$ such that $p = n^2$. (C)

(3) Contrapositive:
"If an integer is not a multiple of 3 its digits will not add to a multiple of 3". (B)

(4) Converse: "If an integer is a multiple of 3 its digits will add to a multiple of 3". (A)

(5) $\displaystyle\int \frac{6}{(2x-1)(x+1)}\,dx$

$= \displaystyle\int \frac{4}{(2x-1)} - \frac{2}{(x+1)}\,dx$

$= 2\ln(2x-1) - 2\ln(x+1) + C$

$= 2\ln\left|\dfrac{2x-1}{x+1}\right| + C$ (D)

Note:

$\dfrac{A}{2x-1} + \dfrac{B}{x+1} = \dfrac{6}{(2x-1)(x+1)}$

$A(x+1) + B(2x-1) = 6 \quad (1)$

Sub. $x=-1$ in (1):

$-3B = 6 \Rightarrow B = -2$.

Sub. $x = \frac{1}{2}$ in (1):

$\dfrac{3A}{2} = 6 \Rightarrow A = 4$.

Suggested solutions	Marker's Feedback

Multiple Choice (continued):

(6) i^{2025}

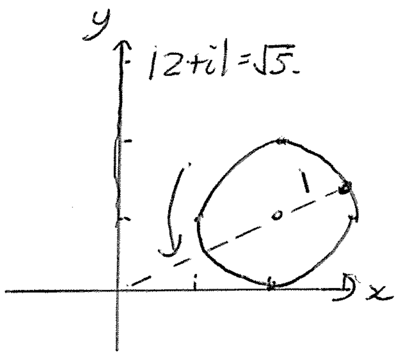
$= i^{2024} \times i$

$= [i^4]^{506} \times i$ (A)

$= 1 \times i$

$= \underline{i}$.

(7) $|z-2-i| = 1$.



Maximum modulus

$= \underline{\sqrt{5}+1}$.

(D)

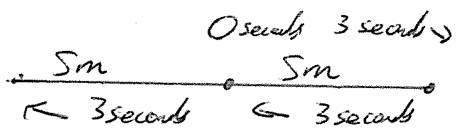
(8) By conjugate root theorem, other root is $2-i$.

$\therefore$ Polynomial $= (z-4)(z-2-i)(z-2+i)$

$= (z-4)(z^2-4z+5)$

$= z^3-8z^2+21z-20$ (C)

(9)



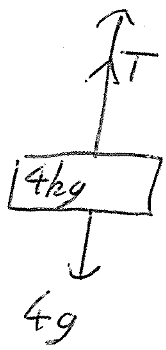
Will be 5m LEFT after $3+3+3 = 9$ seconds. (C)

Suggested solutions	Marker's Feedback

Multiple Choice Continued):

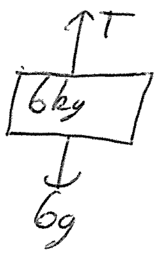
Q.(10)

At 4kg mass:



$T - 4g = 4a$. (1)

At 6kg mass:



$6g - T = 6a$ (2)

(1) + (2) $2g = 10a$

$\frac{g}{5} = a$. D

Suggested solutions	Marker's Feedback

Q.(11)(a)

(i) $\dfrac{z}{w} = \dfrac{1-i}{1-i\sqrt{3}} \times \dfrac{(1+i\sqrt{3})}{(1+i\sqrt{3})}$ $\quad$ 1 to here.

Well done.

$= \dfrac{1+i\sqrt{3}-i+\sqrt{3}}{4}$

$= \dfrac{(1+\sqrt{3})+i(\sqrt{3}-1)}{4}$

(ii) $z = \sqrt{2}e^{-\frac{\pi}{4}i}$ $\quad$ 1 $\qquad$ 2

$w = 2e^{-\frac{\pi i}{3}}$ $\quad$ 1

(iii) $\dfrac{z}{w} = \dfrac{1}{\sqrt{2}}e^{\frac{\pi i}{12}}$

$= \dfrac{1}{\sqrt{2}}\left[\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right]$ $\quad$ 1

Equating real parts with (i):

$\dfrac{1}{\sqrt{2}}\cos\dfrac{\pi}{12} = \dfrac{1+\sqrt{3}}{4} \times \sqrt{2}$ $\qquad$ 2

$\cos\dfrac{\pi}{12} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$ $\quad$ 1

(b)(i) $(x+iy)^2 = -5-12i$

$(x^2-y^2)+2ixy = -5-12i$

Equating reals, $x^2-y^2 = -5$ (1)

Equating imaginaries, $2xy = -12$

$y = -\dfrac{6}{x}$ (2) $\quad$ 1

Sub. (2) in (1): $x^2 - \left(-\dfrac{6}{x}\right)^2 = -5$

$x^2 - \dfrac{36}{x^2} = -5$

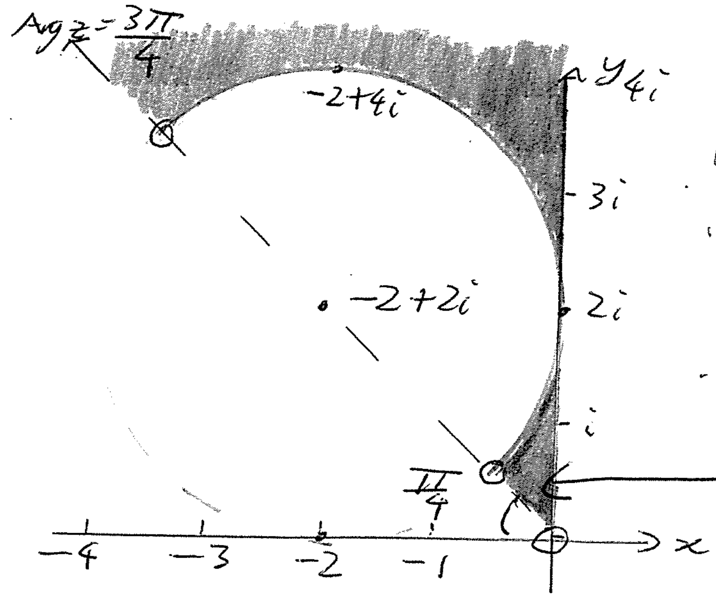
$x^4 + 5x^2 - 36 = 0$

$(x^2-4)(x^2+9) = 0$ $\qquad$ 2

As x is real, $x = \pm 2$. $\quad$ 1

As $y = -\dfrac{6}{x}$, $y = \mp 3$.

$\therefore \sqrt{-5-12i} = \pm(2-3i)$

Suggested solutions	Marker's Feedback
Q.(1)(b)(ii) $z^2+(2+i)z+(2+4i)=0$	
$z=\dfrac{-2-i\pm\sqrt{(2+i)^2-4\times1\times(2+4i)}}{2\times1}$ 1	
$=\dfrac{-2-i\pm\sqrt{-5-12i}}{2}$	
$=\dfrac{-2-i\pm(2-3i)}{2}$ 1 $\underline{2}$	
$z=\underline{-2i}$ or $z=\underline{-2+i}$.	
(c)  $\underline{4}$	
(d) $\sqrt[6]{64i}$	
$=\sqrt[6]{64\operatorname{cis}\frac{\pi}{2}}$ +1 for 1 answer, 1 for all answers	
$=2\operatorname{cis}\frac{\pi}{12}, 2\operatorname{cis}\frac{5\pi}{12}, 2\operatorname{cis}\frac{3\pi}{4},$ $\underline{2}$	
$2\operatorname{cis}\frac{13\pi}{12}, 2\operatorname{cis}\frac{17\pi}{12}, 2\operatorname{cis}\frac{7\pi}{4}.$	

Suggested solutions	Marker's Feedback
(12)(a)(i)	
$\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{C}{x-3}=\frac{-13x-25}{(x+1)^2(x-3)}$	
$\therefore A(x-3)+B(x+1)(x-3)+C(x+1)^2=-13x-25 \quad (1)$	
Sub. $x=-1$ in (1): $-4A=-13\times-1-25 \Rightarrow \underline{A=3}$	
Sub. $x=3$ in (1): $C(3+1)^2=-13\times3-25$.	
$16C=-64 \Rightarrow \underline{C=-4}$.	
Sub. $x=0, A=3, C=-4$ in (1):	1 for each of A, B, C $\underline{=3}$
$3\times-3-3B+-4=-25$.	Check for carry over error.
$-3B=-12 \Rightarrow \underline{B=4}$.	
$\underline{A=3,\ B=4,\ C=-4.}$	
(ii) Hence $\int\frac{-13x-25}{(x+1)^2(x-3)}\,dx$	
$=\int\frac{3}{(x+1)^2}+\frac{4}{x+1}-\frac{4}{x-3}\,dx$	
$=\underline{\frac{-3}{x+1}+4\ln\lvert x+1\rvert-4\ln\lvert x-3\rvert}$. $\underline{1}$	

Suggested solutions	Marker's Feedback
Q.(12)(b) $\int_1^4 \frac{1}{2+\sqrt{x}} \cdot dx$	
Let $x=u^2$. $dx = 2u.du$	
$= \int_{u=1}^{u=2} \frac{1}{2+u} \cdot 2u.du$ 1 for change of limits, 1 for change of variable with correct du.	
$= \int_1^2 \frac{2u+4}{u+2} - \frac{4}{u+2} \cdot du$ 1 for this step. Note: Can also be done using $x=(u-2)^2$ etc.	
$= \int_1^2 2 - \frac{4}{u+2} \cdot du$ $\underline{4}$	
$= \left[2u - 4\ln\lvert u+2\rvert\right]_1^2$	
$= \underline{2 - 4\ln 4 + 4\ln 3}$ 1 for answer.	
Note: Can simplify to $2 - 8\ln 2 + 4\ln 3$ etc. or $2 + 4\ln\left(\frac{3}{4}\right)$.	
(c) $\int [\ln x]^2 . dx$ $u=[\ln x]^2$ $v=x$; $u' = \frac{2\ln x}{x}$ $v'=1$	
By $\int uv'.dx = uv - \int vu'.dx$	
$\int [\ln x]^2 . dx = x[\ln x]^2 - 2\int \ln x . dx$ (1) $\underline{1}$ mark	
Taking $\int (\ln x).dx$ out of (1) $u=\ln x$ $v=x$; $u' = \frac{1}{x}$ $v'=1$ $\underline{3}$	
By $\int uv'.dx = uv - \int vu'.dx$	
$\int \ln x . dx = x\ln x - \int 1.dx = \underline{x\ln x - x}$ (2) 1 mark	
Sub. (2) in (1):	
$\int [\ln x]^2 . dx = x(\ln x)^2 - 2x\ln x + 2x + C$. 1 mark	

Suggested solutions	Marker's Feedback

Q.(12)(d) $\int \frac{1}{2+\cos x} \, dx$

Let $t = \tan\left(\frac{x}{2}\right)$.

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1+t^2}{2}$$

$$\therefore dx = \frac{dx}{dt} \cdot dt = \frac{2}{1+t^2} \cdot dt.$$ 1 to show $dx = \frac{dx}{dt} \cdot dt$

$$\therefore \int \frac{1}{2+\cos x} \, dx$$

$$= \int \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \cdot dt$$

$$= \int \frac{1}{2+2t^2+1-t^2} \cdot dt$$

$\underline{3}$

$$= \int \frac{1}{t^2+3} \cdot dt$$ 1 mark

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C.$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left[\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right] + C.$$ 1 mark

Suggested solutions	Marker's Feedback

Q.(13)(a)(i) Particle is moving with SHM if $\ddot{x} = -n^2 x$

$$x = 4\cos\frac{t}{2} - 4\sqrt{3}\sin\frac{t}{2}$$

$$\dot{x} = -2\sin\frac{t}{2} - 2\sqrt{3}\cos\frac{t}{2}$$

$$\ddot{x} = -\cos\left(\frac{t}{2}\right) + \sqrt{3}\sin\frac{t}{2}$$

$$\ddot{x} = -\frac{1}{4}\left[4\cos\frac{t}{2} - 4\sqrt{3}\sin\frac{t}{2}\right].$$

$$\ddot{x} = -\frac{1}{4}x.$$ 1 to here.

Particle is moving with SHM.

(ii) $4\cos\frac{t}{2} - 4\sqrt{3}\sin\frac{t}{2} \equiv A\cos\left(\frac{t}{2}+\alpha\right)$

$= A\cos\frac{t}{2}\cos\alpha \quad A\sin\frac{t}{2}\sin\alpha.$

Equating parts

$4\cos\frac{t}{2} = A\cos\frac{t}{2}\cos\alpha.$ | $-4\sqrt{3}\sin\frac{t}{2} = -A\sin\frac{t}{2}\sin\alpha$

$A\cos\alpha = 4 \quad (1)$ | $4\sqrt{3} = A\sin\alpha \quad (2)$

$(1)^2+(2)^2: A^2[\cos^2\alpha + \sin^2\alpha] = 4^2 + (4\sqrt{3})^2.$

$\underline{A = 8}$

1 for each of A & α.

$\underline{2}$

Sub. $A = 8$ in (1): $\cos\alpha = \frac{1}{2}.$

Sub. $A = 8$ in (2): $\sin\alpha = \frac{\sqrt{3}}{2} \Rightarrow \alpha = \frac{\pi}{3}.$

$\therefore \underline{x = 8\cos\left(\frac{t}{2} + \frac{\pi}{3}\right).}$

$\underline{\text{Period} = \frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi \text{ seconds.}}$

$\underline{\text{Amplitude} = 8\,m.}$

Suggested solutions	Marker's Feedback
Q(13)(b) $\ddot{x} = -9(x+2)$	
$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x - 18$ 1 to here.	
$\frac{d}{dx}(v^2) = -18x - 36.$	
$v^2 = \int -18x - 36 \, dx$	
$v^2 = -9x^2 - 36x + C$	
As $\lvert v \rvert = 12$ m/s when $x = 1$,	
$v^2 = 144$ when $x = 1$.	
$144 = -9(1)^2 - 36(1) + C$	
$189 = C$	
$\underline{v^2 = 189 - 9x^2 - 36x.}$	
By $\ddot{x} = -n^2 x$, $n^2 = 9$, $n = 3$.	
$\underline{\text{Period} = \frac{2\pi}{3} \text{ seconds.}}$ 1 mark	
Amplitude & centre of motion:	
Finding where $v = 0$:	
$189 - 9x^2 - 36x = 0$	
$x^2 + 4x - 21 = 0$	
$(x+7)(x-3) = 0$	
$x = -7$ or $x = 3$.	
Amplitude = 5m. Centre of motion is $\underline{x = -2.}$] 1 mark	
Note: $v^2 = 189 - 9x^2 - 36x$ can also be rearranged as	
$v^2 = 9\left[25 - (x+2)^2\right]$	
From $v^2 = n^2(a^2 - (x-c)^2)$ → They CAN do this this time (see note below)	
	3

(Was allowed in 2020 HSC although formula does not appear on reference sheet).

Suggested solutions	Marker's Feedback
Q.(13)(c)	

$\longrightarrow 16000 \quad 10v^2 \longleftarrow$

(i) By $F = ma$

$1000\ddot{x} = 16000 - 10v^2$ 1 mark

$\ddot{x} = 16 - 0{\cdot}01v^2$

Max. v is when $\ddot{x} \to 0$

i.e. $16 - 0{\cdot}01v^2 \to 0$

$v^2 \to 1600$

$v \to \underline{40\text{m/s.}}$ 1 mark

2

The car's maximum speed is 40m/s [= 144km/h]

(ii) $\ddot{x} = v.\dfrac{dv}{dx} = 16 - 0{\cdot}01v^2$

$\dfrac{dv}{dx} = \dfrac{16 - 0{\cdot}01v^2}{v}$

$\dfrac{dx}{dv} = \dfrac{v}{16 - 0{\cdot}01v^2}$

$= \dfrac{100v}{1600 - v^2}$

$x = -50\displaystyle\int \frac{-2v}{1600 - v^2}\,dv$

3

$= -50\ln(1600 - v^2) + C$ 1 mark

[note: absolute value not needed as $v < 40$].

As $v = 0$ when $x = 0$,

$0 = -50\ln 1600 + C \Rightarrow C = 50\ln 1600$

$x = 50\ln 1600 - 50\ln(1600 - v^2)$

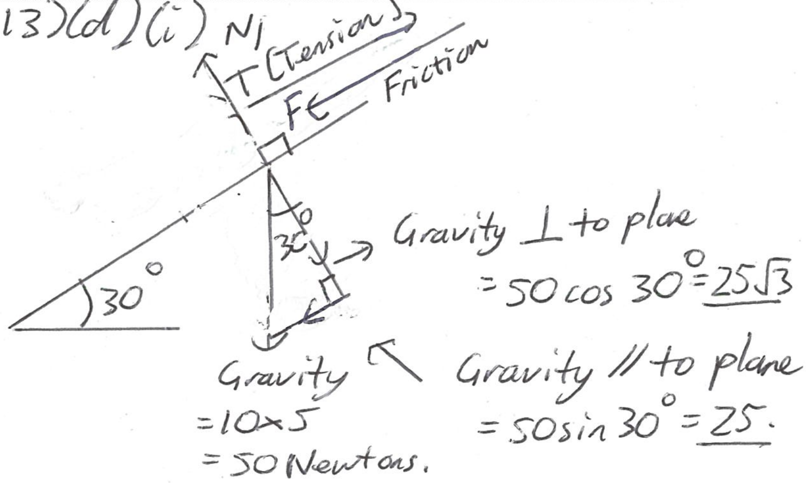
$x = 50\ln\left(\dfrac{1600}{1600 - v^2}\right)$ ⊛ Finding x when $v = 30$ 1 mark

$x = 50\ln\left(\dfrac{1600}{1600 - 30^2}\right) = \underline{41{\cdot}3\text{m}}$ 1 mark.

Marker's Feedback: Note: As we are finding distance from when $v=0$ to $v=V$, we can also use $x = -50\displaystyle\int_0^V \frac{-2v}{1600 - v^2}\,dv$

Suggested solutions	Marker's Feedback

(13)(d)(i)



Resolving ⊥ to plane:

$N_1 = 25\sqrt{3}$.

Note: $F = 0{\cdot}6N = 15\sqrt{3}$.] 1 mark

2

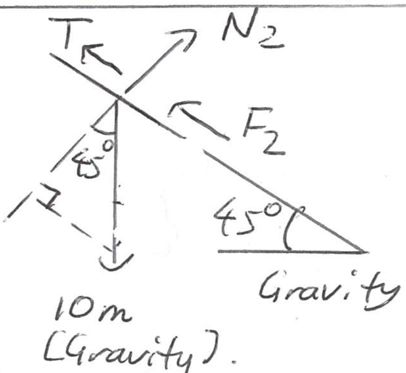
Resolving // to plane:

$F_1 + 25 = T$ 1 mark,

$15\sqrt{3} + 25 = T$

$\therefore T = (15\sqrt{3} + 25)$ Newtons.

(ii)



Gravity ⊥ to plane $= 10m\cos 45°$.

$\therefore$ Resolving ⊥ to plane: $N_2 = 10m\cos 45°$

$\therefore F_2 = 0{\cdot}6N_2 = 6m\cos 45°$

$= 3m\sqrt{2}$. 1 for F_2 (1 mark).

Resolving // to plane:

$F_2 + T = 10m\sin 45°$.

$3m\sqrt{2} + (15\sqrt{3} + 25) = 5m\sqrt{2}$

2

$15\sqrt{3} + 25 = 2m\sqrt{2}$

$18{\cdot}02 = m$. The mass is 18·02kg.

Suggested solutions	Marker's Feedback

Q.(14)(a) (i) $I_N = \int_0^{\frac{\pi}{2}} \sin^N x . dx$

$u = \sin^{N-1} x \quad | \quad v = -\cos x$

$u' = (n-1)\sin^{N-2} x \cos x \quad | \quad v' = \sin x$

By $\int u v' . dx = uv - \int v u' . dx$

$I_N = \left[-\cos x \sin^{N-1} x\right]_0^{\frac{\pi}{2}} + (N-1)\int_0^{\frac{\pi}{2}} \cos^2 x \sin^{N-2} x . dx$ 1 mark

$= 0 + (N-1)\int_0^{\frac{\pi}{2}} \cos^2 x \sin^{N-2} x . dx$

(as $\cos\frac{\pi}{2} = \sin 0 = 0$)

$= 2$

$I_N = (N-1)\int_0^{\frac{\pi}{2}} \sin^{N-2} x . dx - (N-1)\int_0^{\frac{\pi}{2}} \sin^N x . dx$ 1 mark

i.e. $I_N = (N-1) I_{N-2} - (N-1) I_N$

$I_N = (N-1) I_{N-2}$

$I_N = \frac{N-1}{N} I_{N-2}.$

(ii) $I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x . dx = \int_0^{\frac{\pi}{2}} 1 . dx = \frac{\pi}{2}.$ 1 for I_0

$I_2 = \frac{1}{2} I_0 = \frac{\pi}{4}.$

$I_4 = \frac{3}{4} I_2 = \frac{3}{4} \times \frac{\pi}{4} = \frac{3\pi}{16}$ $= 2$

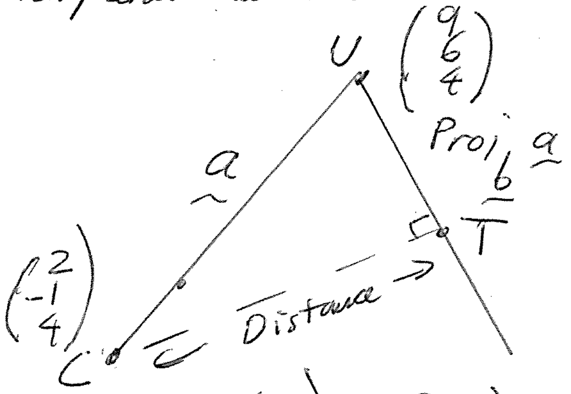
$I_6 = \frac{5}{6} I_4 = \frac{5}{6} \times \frac{3\pi}{16} = \frac{5\pi}{32}.$ 1 for I_6

Suggested solutions	Marker's Feedback

Q.(14)(b) Sphere: Centre $= \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ $r = \sqrt{14}$ units. $\underline{2}$

(i) 1mark 1 mark

(ii) Perpendicular distance:



$\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 9 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} -7 \\ -7 \\ 0 \end{pmatrix}.$

$\underset{\sim}{b}$ = direction vector of line $= \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

Perpendicular distance = CT

$= \left| \underset{\sim}{a} - \text{Proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} \right|$

Noting $\text{Proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}\, \underset{\sim}{b}$

$\text{Proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} = \dfrac{\begin{pmatrix} -7 \\ -7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}}{42} \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

$= -\dfrac{3}{2} \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -7\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$ 1 for $\text{Proj}_{\underset{\sim}{b}}\, \underset{\sim}{a}$

$\therefore \underset{\sim}{a} - \text{Proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} = \begin{pmatrix} -7 \\ -7 \\ 0 \end{pmatrix} - \begin{pmatrix} -6 \\ -7\frac{1}{2} \\ -1\frac{1}{2} \end{pmatrix}$ $\underline{2}$

$= \begin{pmatrix} -1 \\ \frac{1}{2} \\ 1\frac{1}{2} \end{pmatrix}$

& $\perp$ distance $= \left| \underset{\sim}{a} - \text{Proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} \right| = \sqrt{(-1)^2 + (\frac{1}{2})^2 + (1\frac{1}{2})^2}$

$= \sqrt{3\frac{1}{2}}$

$\doteqdot 1{\cdot}87$ units.

Suggested solutions	Marker's Feedback
(14)(b)(ii) (Continued). Note that as the perpendicular distance is less than the sphere's radius, the line must pass THROUGH the sphere, therefore hitting it in 2 places.	1 for answer $= 1.87$ with this explanation.
Note: ALTERNATIVE PERPENDICULAR DISTANCE METHOD (was asked in guided fashion in 2024 NESA paper).	OR
All points on the line L have equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9+4\lambda \\ 6+5\lambda \\ 4+\lambda \end{pmatrix}$ & so the vector from C to any point on L is given by $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 9+4\lambda \\ 6+5\lambda \\ 4+\lambda \end{pmatrix} = \begin{pmatrix} -7-4\lambda \\ -7-5\lambda \\ -\lambda \end{pmatrix}$	
& the magnitude of this vector $= \sqrt{(-7-4\lambda)^2+(-7-5\lambda)^2+(-\lambda)^2}$	
$\therefore (\text{Distance from } C \text{ to } L)^2 = (-7-4\lambda)^2+(-7-5\lambda)^2+(-\lambda)^2$ $= 42\lambda^2+126\lambda+98$	1 for this expression
Finding the value of λ for minimum distance, $\lambda = \frac{-126}{84} = -\frac{3}{2}$.	
& $(\text{minimum distance})^2 = 42\left(-\frac{3}{2}\right)^2+126\left(-\frac{3}{2}\right)+98$ $= \underline{3.5}$.	
So perpendicular distance $= \sqrt{3.5}$ $\underline{= 1.87 \text{ units}}$ &	1 for $\sqrt{3.5}$ & why it's inside circle.

Suggested solutions	Marker's Feedback

(14)(b)(iii) Points of intersection of line & sphere;

Line L: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9+4\lambda \\ 6+5\lambda \\ 4+\lambda \end{pmatrix}$

Sub. into $(x-2)^2+(y+1)^2+(z-4)^2=14$.

$(7+4\lambda)^2+(7+5\lambda)^2+(\lambda)^2 = 14$. 1 mark.

$42\lambda^2+126\lambda+98 = 14$

$42\lambda^2+126\lambda+84 = 0$

$\lambda^2+3\lambda+2 = 0$

$(\lambda+1)(\lambda+2) = 0$

$\lambda=-1$ or $\lambda=-2$. 1 mark.

If $\lambda=-1$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$

If $\lambda=-2$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$ 1 mark

3

(iv) $L_1 \perp$ to L_2 if dot product of direction vectors $=0$.

$\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -8+5+3=0$

$L_1 \perp L_2$. 1 mark.

Suggested solutions	Marker's Feedback

(14)(b)(v) intersection of L_1 & L_2:

$$\begin{pmatrix}9\\6\\4\end{pmatrix}+\lambda\begin{pmatrix}4\\5\\1\end{pmatrix}=\begin{pmatrix}2\\-1\\4\end{pmatrix}+\mu\begin{pmatrix}-2\\1\\3\end{pmatrix}$$

$x: \ 9+4\lambda = 2-2\mu \Rightarrow 4\lambda + 2\mu = -7 \ (1)$

$y: \ 6+5\lambda = -1+\mu \Rightarrow 5\lambda - \mu = -7 \ (2)$

$z: \ 4+\lambda = 4+3\mu \Rightarrow \lambda - 3\mu = 0$

$\quad$ or $\lambda = 3\mu \ (3)$

1 mark.

Sub. (3) in (1): $12\mu + 2\mu = -7 \Rightarrow \mu = \frac{-1}{2}.$

From (3), as $\lambda = 3\mu$, $\lambda = -\frac{3}{2}.$

Check to see if $\lambda = -\frac{3}{2}$, $= -\frac{1}{2}$ works

in (2): LHS $= 6+5\times -\frac{3}{2} = -\frac{3}{2}.$ RHS $= -1-\frac{1}{2} = -\frac{3}{2}.$

$\lambda = -\frac{3}{2}$, $\mu = -\frac{1}{2}$ works in (2):

1 for testing in unused equation (in case shown).

$\therefore$ Point of intersection

$$= \begin{pmatrix}9\\6\\4\end{pmatrix} - \frac{3}{2}\begin{pmatrix}4\\5\\1\end{pmatrix} = \begin{pmatrix}3\\-\frac{3}{2}\\\frac{5}{2}\end{pmatrix}$$

3

1 for answer

or
$$= \begin{pmatrix}2\\-1\\4\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-2\\1\\3\end{pmatrix} = \begin{pmatrix}3\\-\frac{3}{2}\\\frac{5}{2}\end{pmatrix}$$

Suggested solutions	Marker's Feedback

Q. (15)(a)

Show true for $n=$

LHS $= 7! = 5040$ RHS $= 3^7 = 2187$

LHS $>$ RHS

True for $n=7$.

1 for BOTH Steps 1 & 2.

Assume true for $n=k$, $k \geqslant 7$.

i.e. $k! > 3^k$.

Prove true for $n=k+1$, $k \geqslant 7$.

i.e. RTP $(k+1)! > 3^{k+1}$.

LHS

$= (k+1)!$

$= (k+1) \times k!$

$> (k+1) \times 3^k$ [by assumption] 1 for using assumption

$> 3 \times 3^k$ [as $k \geqslant 7$, $k+1 > 3$]. 1 for $k+1>3$.

$= 3^{k+1}$

3

$\therefore (k+1)! > (k+1) \times 3^k > 3^{k+1}$

$(k+1)! > 3^{k+1}$.

If it is true for $n=k$ it will be true for $n=k+1$.

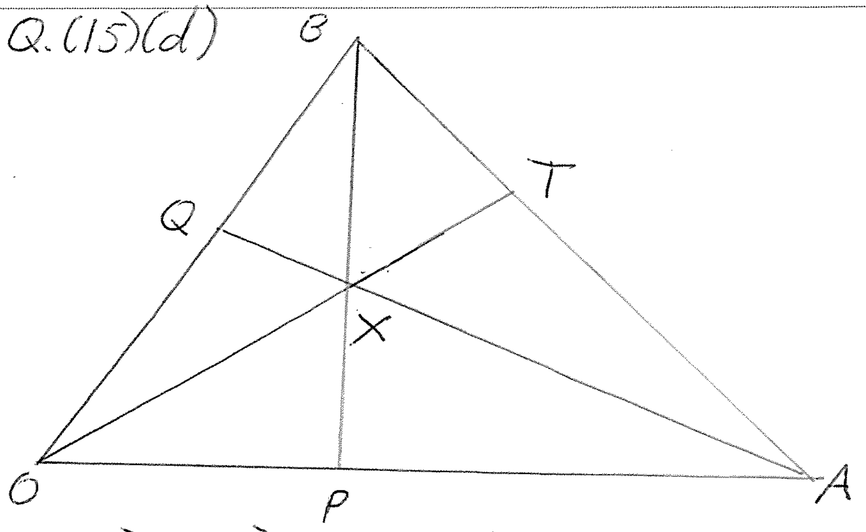
Hence as it is true for $n=7$ it will be true $\forall\ n \in \mathbb{Z}^+ \geqslant 7$ by the principle of mathematical induction.

Suggested solutions	Marker's Feedback
Q. (15)(b)(i) If a, b real $(a-b)^2 \geq 0$ [as $a-b$ real]. $\therefore a^2-2ab+b^2 \geq 0$ $\underline{1}$ mark. $\underline{a^2+b^2 \geq 2ab.}$ (ii) Hence for a, b, c real $a^2+b^2 \geq 2ab$ (1) $a^2+c^2 \geq 2ac$ (2) $b^2+c^2 \geq 2bc$ (3) $\underline{1}$ mark (1) + (2) + (3) $2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$ $\underline{a^2+b^2+c^2 \geq ab+ac+bc.}$ (iii) Using (ii) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}$ $= \frac{c+b+a}{abc}$ $= \frac{a+b+c}{abc}$ $\underline{1}$ mark. $\therefore \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{a+b+c}{abc}.$	

Suggested solutions	Marker's Feedback
Q.(15)(c) Let $\sqrt{29} = \frac{p}{q}$, $p, q \in \mathbb{Z}^+$ & p & q have no common factors. $\therefore \frac{p^2}{q^2} = 29$ $p^2 = 29q^2 \ldots (1)$ As p, q have no common factors & 29 is prime, as 29 is a factor of p^2 29 is a factor of p. → 1 mark to here. $\therefore p = 29k$, $k \in \mathbb{Z}^+$. $p^2 = 841k^2$, (2) Sub. (2) in (1): $841k^2 = 29q^2$ $29k^2 = q^2$. As 29 is a factor of q^2, 29 is a factor of q [as 29 is prime]. $\therefore$ 29 is a factor of both p & q. → 1 mark to here. But p & q have no common factors. $\therefore$ By contradiction, $\sqrt{29}$ must be irrational. $\underline{2}$	

Suggested solutions	Marker's Feedback

Q.(15)(d)



(i) $\vec{OX} = \vec{OP} + \vec{PX}$

$= \frac{1}{3}\underset{\sim}{a} + \lambda \vec{PB}$ 1 mark.

Noting $\vec{PB} = \vec{PO} + \vec{OB} = -\frac{1}{3}\underset{\sim}{a} + \underset{\sim}{b}$ (2) 2

$\vec{OX} = \frac{1}{3}\underset{\sim}{a} + \lambda(-\frac{1}{3}\underset{\sim}{a} + \underset{\sim}{b})$ 1 mark

$\vec{OX} = \frac{1}{3}(1-\lambda)\underset{\sim}{a} + \lambda \underset{\sim}{b}$ as required.

(ii) As $\vec{OX}$ also $= \mu \underset{\sim}{a} + \frac{1}{2}(1-\mu)\underset{\sim}{b}$

Equating (i) & (ii)

$\underset{\sim}{a}: \frac{1}{3}(1-\lambda) = \mu \quad \times 6$

$2 - 2\lambda = 6\mu$ (1)

$\underset{\sim}{b}: \lambda \quad \times 2 \quad = \frac{1}{2}(1-\mu) \quad \times 2$

$2\lambda = 1 - \mu$. (2)

1 mark for the 2 equations

Sub. (2) in (1):

$2 - (1-\mu) = 6\mu$

$\mu = \frac{1}{5}$.

Sub. $\mu = \frac{1}{5}$ in (1):

$2 - 2\lambda = \frac{6}{5}$.

$\lambda = \frac{2}{5}$.

1 mark for λ & μ

$\lambda = \frac{2}{5}$, $\mu = \frac{1}{5}$.

$\vec{OX} = \frac{1}{3}(1-\frac{2}{5})\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$

[using (i)]

$\vec{OX} = \frac{1}{5}\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$.

1 mark for finish

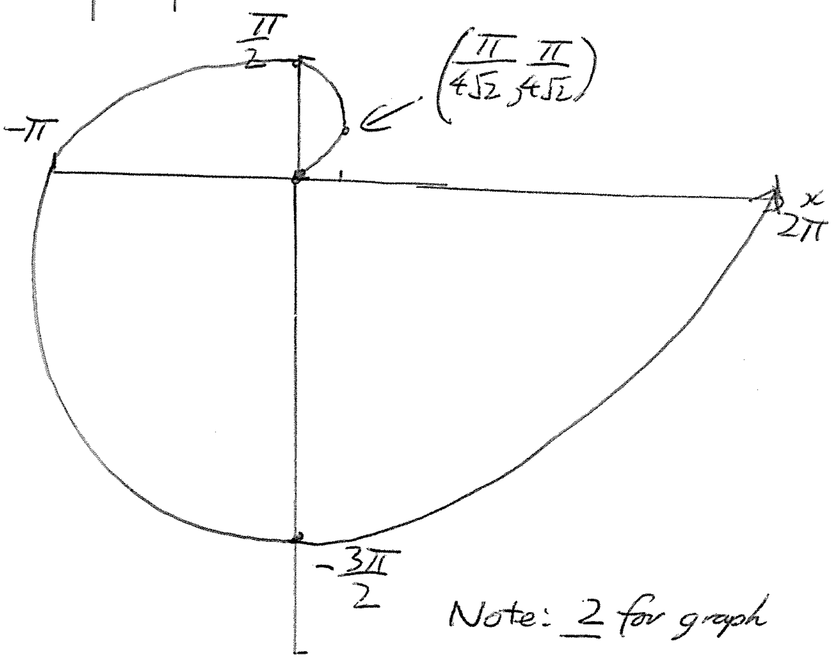
3

Suggested solutions	Marker's Feedback
(d) Q.(15)(~~b~~)(iii) If T is collinear with O & X then $\vec{OT} = k\,\vec{OX}$ $\vec{OT} = k\left(\frac{1}{5}\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}\right)$ (1) $\vec{OT}$ also $= \vec{OB} + \vec{BT}$ $= \underset{\sim}{b} + h\,\vec{BA}$ $= \underset{\sim}{b} + h(\underset{\sim}{a} - \underset{\sim}{b})$ $= h\underset{\sim}{a} + (1-h)\underset{\sim}{b}$ (2) 1 mark for 2 equations Equating parts, in (1) & (2) $\underset{\sim}{a}: \frac{1}{5}k\underset{\sim}{a} = h\underset{\sim}{a} \Rightarrow \frac{1}{5}k = h$. (3) + $\underset{\sim}{b}: \frac{2}{5}k\underset{\sim}{b} = (1-h)\underset{\sim}{b} \Rightarrow \frac{2}{5}k = 1-h$ (4) 1 mark for equating $\underset{\sim}{a}$ & $\underset{\sim}{b}$ parts. (3)+(4): $\frac{3}{5}k = 1$. $k = \frac{5}{3}$. ∴ Using (1): $\vec{OT} = \frac{5}{3}\left(\frac{1}{5}\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}\right)$ 1 mark $\underline{= 3}$ $= \frac{1}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b}$. [Note: Check: Can also use $\vec{OT} = ha + (1-h)\underset{\sim}{b}$ As $k = \frac{5}{3}$, $h = \frac{1}{5}k = \frac{1}{3}$. $= \frac{1}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b}$.	

Suggested solutions	Marker's Feedback

Q.(16)(a)

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	$\frac{\pi}{4\sqrt{2}}$	0	$-\pi$	0	2π
y	0	$\frac{\pi}{4\sqrt{2}}$	$\frac{\pi}{2}$	0	$-\frac{3\pi}{2}$	0



Note: 2 for graph with $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

[$t=\frac{\pi}{4}$ was just a check for the spiral shape].

1 for table without graph.

Suggested solutions	Marker's Feedback
Q(16)(b)(i) Solutions to $z^9 - 1 = 0$ are $z = cis\frac{2k\pi}{9}$, $k = 0, 1, 2, 3, 4, 5, 6, 7, 8$. → Satisfactory to list like this $\underline{1}$ or i.e. $z = cis\frac{2\pi}{9}$, $cis\frac{4\pi}{9}$, $cis\frac{2\pi}{3}$, $cis\frac{8\pi}{9}$, $cis\frac{10\pi}{9}$, $cis\frac{4\pi}{3}$, $cis\frac{14\pi}{9}$, $cis\frac{16\pi}{9}$, 1. using principal argument. (ii) As $z^9 - 1 = (z^3-1)(z^6+z^3+1)$ & the solutions to $z^3 - 1 = 0$ are $cis\frac{2\pi}{3}$, $cis\frac{4\pi}{3}$ & 1. the solutions to $z^6 + z^3 + 1 = 0$ are $z = cis\frac{2\pi}{9}$, $cis\frac{4\pi}{9}$, $cis\frac{8\pi}{9}$, $cis\frac{10\pi}{9}$, $cis\frac{14\pi}{9}$, $cis\frac{16\pi}{9}$. $\underline{1}$ (iii) If $z = \cos\theta + i\sin\theta$. $z^n = \cos n\theta + i\sin n\theta$ (by DeMoivre). $z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$ $= \cos n\theta - i\sin n\theta$ (as cos even, sin odd). Hence, $z^n + z^{-n}$ $= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ $= \underline{2\cos n\theta}$.	 Need both for $\underline{1}$

Suggested solutions	Marker's Feedback

Q.(16)(b)(iv) z^6+z^3+1

$=\left(z-\operatorname{cis}\frac{2\pi}{9}\right)\left(z-\operatorname{cis}\frac{16\pi}{9}\right)\left(z-\operatorname{cis}\frac{4\pi}{9}\right)\left(z-\operatorname{cis}\frac{14\pi}{9}\right)$

$\left(z-\operatorname{cis}\frac{8\pi}{9}\right)\left(z-\operatorname{cis}\frac{10\pi}{9}\right)$ 1 mark

Noting $\left(z-\operatorname{cis}\frac{2\pi}{9}\right)\left(z-\operatorname{cis}\frac{16\pi}{9}\right)$

$=\left(z-\operatorname{cis}\frac{2\pi}{9}\right)\left(z-\operatorname{cis}\left(-\frac{2\pi}{9}\right)\right)$

$=\left(z^2-z\left(\operatorname{cis}\frac{2\pi}{9}+\operatorname{cis}\frac{-2\pi}{9}\right)+\operatorname{cis}\frac{2\pi}{9}\times\operatorname{cis}\left(-\frac{2\pi}{9}\right)\right)$

$=\left(z^2-2z\cos\frac{2\pi}{9}+1\right)$ [using (iii)] 1

Total to have 2 marks to have.

Similarly,

$\left(z-\operatorname{cis}\frac{4\pi}{9}\right)\left(z-\operatorname{cis}\frac{14\pi}{9}\right)=z^2-2z\cos\frac{4\pi}{9}+1.$ $=2$

$\left(z-\operatorname{cis}\frac{8\pi}{9}\right)\left(z-\operatorname{cis}\frac{10\pi}{9}\right)=z^2-2z\cos\frac{8\pi}{9}+1.$

Hence

z^6+z^3+1

$=\left(z^2-2z\cos\frac{2\pi}{9}+1\right)\left(z^2-2z\cos\frac{4\pi}{9}+1\right)$

$\left(z^2-2z\cos\frac{8\pi}{9}+1\right)$

(v) Dividing (iv) by z^3:

$\frac{(z^6+z^3+1)}{z^3}=\left(\frac{z^2-2z\cos\frac{2\pi}{9}+1}{z}\right)\left(\frac{z^2-2z\cos\frac{4\pi}{9}+1}{z}\right)\left(\frac{z^2-2z\cos\frac{8\pi}{9}+1}{z}\right)$

$z^3+\frac{1}{z^3}+1=\left(z+\frac{1}{z}-2\cos\frac{2\pi}{9}\right)\left(z+\frac{1}{z}-2\cos\frac{4\pi}{9}\right)\left(z+\frac{1}{z}-2\cos\frac{8\pi}{9}\right)$ 1 to have.

As $z^n+\frac{1}{z^n}=2\cos n\theta.$

$2\cos3\theta+1=\left(2\cos\theta\cdot2\cos\frac{2\pi}{9}\right)\left(2\cos\theta-2\cos\frac{4\pi}{9}\right)\left(2\cos\theta-2\cos\frac{8\pi}{9}\right)$ $\frac{2}{2 \text{ to have}}$

$2\cos3\theta+1=8\left(\cos\theta-\cos\frac{2\pi}{9}\right)\left(\cos\theta-\cos\frac{4\pi}{9}\right)\left(\cos\theta-\cos\frac{8\pi}{9}\right)$ $\div2$

$\div2$

$\cos3\theta+\frac{1}{2}=4\left(\cos\theta-\cos\frac{2\pi}{9}\right)\left(\cos\theta-\cos\frac{4\pi}{9}\right)\left(\cos\theta+\cos\frac{\pi}{9}\right)$ & as $\cos\frac{8\pi}{9}=-\cos\frac{\pi}{9}.$

as required.

Suggested solutions	Marker's Feedback

Q.(16)(c) [Note: Two alternative solutions: 1st: if done in 1 hit, 2ND: if restating the question when the projectile starts to fall].

1st way: $\downarrow$ Gravity: 10m $\quad \downarrow$ 0·05

By F = ma (with up as positive),

$$m\ddot{x} = -10m - 0{\cdot}05$$

$$\ddot{x} = -10 - 0{\cdot}05v. \qquad 1$$

[Note: On the way down, v is negative $\therefore -0{\cdot}05v$ would be pushing UP so $\ddot{x} = -10 - 0{\cdot}05v$ would still be valid on the way down].

$$\therefore \frac{dv}{dt} = -10 - 0{\cdot}05v$$

$$\frac{dt}{dv} = \frac{-1}{10 + 0{\cdot}05v}$$

$$t = \int \frac{-20}{200+v}\,dv$$

$$= -20\ln(200+v) + C$$

When $t = 0$, $v = v_0$: (*)

$$\therefore \quad = -20\ln(200+v_0) + C$$

$$C = 20\ln(200+v_0).$$

$$\therefore t = -20\ln\left(\frac{200+v}{200+v_0}\right) \qquad 1$$

$$e^{-\frac{t}{20}} = \frac{200+v}{200+v_0}$$

$$(200+v_0)e^{-\frac{t}{20}} - 200 = v \qquad 1$$

[(*) Note: Could also use a definite integral & do $\int_{v_0}^{v} \frac{-20}{200+v}\,dv$]

Suggested solutions	Marker's Feedback

$$x = \int (200+v_0)e^{-\frac{t}{20}} - 200 \, dv$$

$$= -(4000+20v_0)e^{-\frac{t}{20}} - 200t + C$$

As $x=0$ when $t=0$,

$$0 = -(4000+20v_0)e^{0} + C$$

$$x = -(4000+20v_0)e^{-\frac{t}{20}} - 200t + (4000+20v_0) \quad \underline{1}$$

As projectile is at $x=0$ when $t=5$,

$$0 = -(4000+20v_0)e^{-\frac{1}{4}} - 1000 + (4000+20v_0) \quad \underline{1}$$

$$4000e^{-\frac{1}{4}} - 3000 = 20v_0\left(1-e^{-\frac{1}{4}}\right)$$

$$26.04\ldots = v_0$$

Initial velocity is $\underline{26.04 \text{ m/s}}$. $\underline{\text{ANSWER}}$ $\underline{1}$

Suggested solutions	Marker's Feedback
Alternative Solution from starting again at the top: [Noting 3 to here already = marks].	
1^{st} find maximum height: is x when $v=0$	
x when going up was	
$-(4000+20v_0)e^{-\frac{t}{20}}-200t+(4000+20v_0)$.	
Time when $v=0$ was	
$t=-20\ln\left(\frac{200}{200+v_0}\right) \Rightarrow e^{-\frac{t}{20}}=\frac{200}{200+v_0}$.	
So $x=-(4000+20v_0)\times\frac{200}{200+v_0}$	1 mark.
$-200\times -20\ln\left(\frac{200}{200+v_0}\right)+(4000+20v_0)$	
$x=4000\ln\left(\frac{200}{200+v_0}\right)+20v_0$.	
Now starting at top: down positive.	
$t=0$. $\downarrow$ 10m $\quad \ddot{x}=10-0.05v$.	
$0.05mv \uparrow \quad \frac{dv}{dt}=10-0.05v$	
$\frac{dt}{dv}=\frac{1}{10-0.05v}$	
$=\frac{20}{200-v}$.	
$t=-20\int\frac{-1}{200-v}\,dv$ [Don't need absolute value as $v<200$).	
$=-20\ln(200-v)+C$	
As $v=0$ when $t=0$ [starting at top).	
$=-20\ln(200)+C \Rightarrow C=20\ln 200$	
$t=-20\ln\left(\frac{200-v}{200}\right)$.	

Suggested solutions	Marker's Feedback
Alternative Solution Starting Again at Top: [continued].	

$$e^{-\frac{t}{20}} = \frac{200-v}{200}$$

$$v = 200 - 200e^{-\frac{t}{20}}$$

$$x = \int 200 - 200e^{-\frac{t}{20}}\,dt$$

$$x = 200t + 4000e^{-\frac{t}{20}} + C$$

As $x=0$ when $t=0$

$$0 = 4000 + C \Rightarrow C = -4000$$

$$\underline{x = 200t + 4000e^{-\frac{t}{20}} - 4000.}$$

As projectile takes 5 seconds to land, it will fall from top height of

$$4000\ln\left(\frac{200}{200+v_0}\right) + 20v_0$$

in $5 + 20\ln\left(\frac{200}{200+v_0}\right)$ seconds [5 seconds − time to top].

$\therefore$ By $x = 200t + 4000e^{-\frac{t}{20}} - 4000$

$$4000\ln\left(\frac{200}{200+v_0}\right) + 20v_0 = 200\left(5 + 20\ln\left(\frac{200}{200+v_0}\right)\right) + 4000 \times \frac{e^{-\frac{1}{4}}(200+v_0)}{200} - 4000$$

1 mark

[Note: If $t = 5 + 20\ln\left(\frac{200}{200+v_0}\right)$, $e^{-\frac{t}{20}} = e^{-\frac{1}{4}} \times e^{\ln\left(\frac{200+v_0}{200}\right)} = \frac{e^{-\frac{1}{4}}(200+v_0)}{200}$].

$$20v_0 = 20e^{-\frac{1}{4}}(200+v_0) - 3000$$

$$20v_0(1-e^{-\frac{1}{4}}) = 4000e^{-\frac{1}{4}} - 3000$$

$$v_0 = \frac{4000e^{-\frac{1}{4}} - 3000}{20(1-e^{-\frac{1}{4}})}$$

$$= 26.04\text{m/s}$$ as required. 1 mark

END OF SOLUTIONS!!!