Student Number

Class

Gosford High School

Trial HSC 2025

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black or blue pen.
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- In Section II, show relevant mathematical reasoning and/ or calculations.

Total Marks

100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section
- Answer questions on the Multiple-choice answer sheet

Section II – 90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section
- Answer questions on booklets provided for each question

Section I
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Shade the best response on the multiple-choice answer sheet.

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1. Which of the following statements is a correct interpretation of '$\forall\ a \in \mathbb{Z},\ \exists\ b \in \mathbb{Z}$ such that $\frac{a}{b} \in \mathbb{Z}$' ?

   (A) Every integer is also a rational number

   (B) There exists a rational number that corresponds to each pair of integers

   (C) Every rational number is also an integer

   (D) Every integer has an integer factor

2. Which of the following is a unit vector in the opposite direction to $-\underset{\sim}{i}+2\underset{\sim}{j}-2\underset{\sim}{k}$

   (A) $\frac{1}{\sqrt{5}}\left(-\underset{\sim}{i}+2\underset{\sim}{j}-2\underset{\sim}{k}\right)$

   (B) $\frac{1}{\sqrt{5}}\left(\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}\right)$

   (C) $\frac{1}{3}\left(\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}\right)$

   (D) $\frac{1}{3}\left(-\underset{\sim}{i}+2\underset{\sim}{j}-2\underset{\sim}{k}\right)$

3. Which of the following is a primitive of $x\sin(x)$?

   (A) $\sin(x)-x\cos(x)$

   (B) $x\cos(x)-\sin(x)$

   (C) $-x\cos(x)-\sin(x)$

   (D) $x\cos(x)+\sin(x)$

4. What is the Cartesian equation of the curve defined by $\boldsymbol{r}(t)=\begin{pmatrix}\sin t\\ \cos t\\ \cos(2t)\end{pmatrix}$?

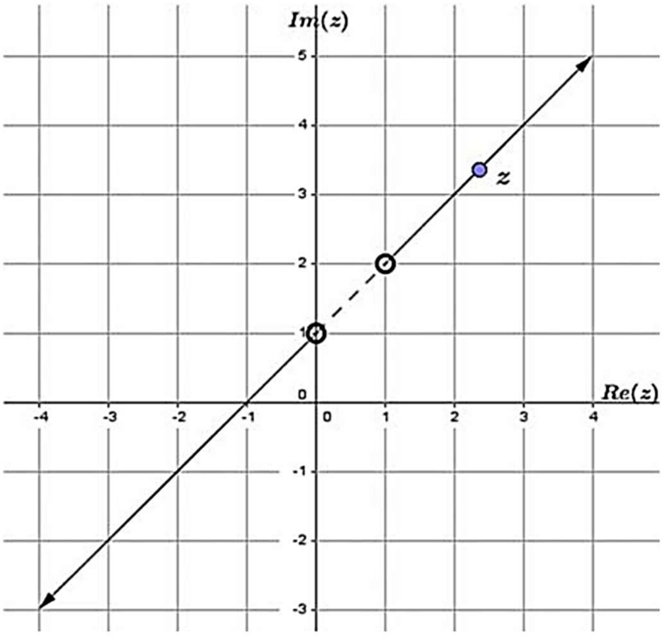
(A) $z=y^2-x^2$

(B) $z=1-2x^2$

(C) $x^2+y^2=1$

(D) $z=2y^2-1$

5. Which of the following defines the locus of the complex number z sketched in the diagram below?



(A) $\arg\left(\dfrac{z-(1+2i)}{z-i}\right)=\pi$

(B) $\arg\left(\dfrac{z-i}{z-(1+2i)}\right)=\pi$

(C) $\arg(z-i)=\arg(z-(1+2i))$

(D) $\arg(z+i)=\arg(z-(1+2i))$

6. Which of the following statements is the negation of '$\forall\ x,y \in \mathbb{R}$, $xy\leq 9 \Rightarrow x\leq 3$ or $y\leq 3$ '?

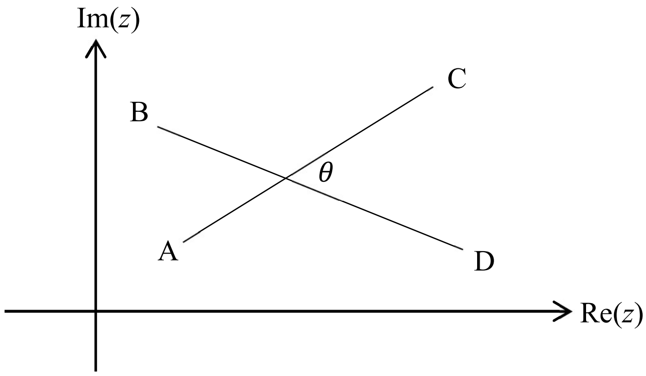
(A) $\forall\ x,y \in \mathbb{R}\ x\leq 3$ or $y\leq 3\ \Rightarrow xy\leq 9$

(B) $\forall\ x,y \in \mathbb{R}\ x>3$ and $y>3\ \Rightarrow xy>9$

(C) $\exists\ x,y \in \mathbb{R}$ with $x>3$ or $y>3$ such that $xy\leq 9$

(D) $\exists\ x,y \in \mathbb{R}$ with $x>3$ and $y>3$ such that $xy\leq 9$

7. The points A, B, C and D on the Argand diagram below represent the complex numbers a, b, c, and d, respectively. The angle θ is the acute angle between AC and BD. Which of the following is a correct expression for θ ?



(A) $\theta = \arg\left(\dfrac{c-d}{a-b}\right)$

(B) $\theta = \arg\left(\dfrac{a-b}{c-d}\right)$

(C) $\theta = \arg\left(\dfrac{c-a}{d-b}\right)$

(D) $\theta = \arg\left(\dfrac{d-b}{c-a}\right)$

8. If $\dfrac{dy}{dx} = (1 + \ln x)y$ and when $x = 1$, $y = 1$ then $y = ?$

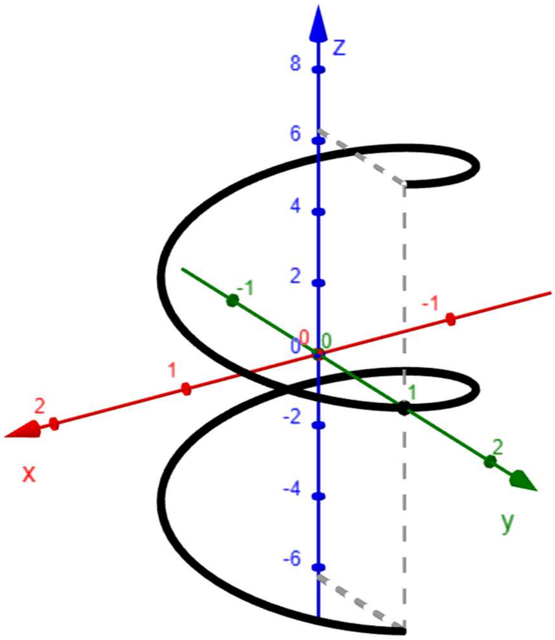
(A) $e^{x\ln x}$

(B) $1 + \ln x$

(C) $e^{2x + x\ln x - 2}$

(D) $e^{\frac{x^2-1}{x^2}}$

9. Which of the following could be the parametric representation of the curve below?



(A) $x = t^2$, $y = t$, $z = 1$

(B) $x = \sin t$, $y = \cos(2t)$, $z = t$

(C) $x = \sin t$, $y = \cos t$, $z = t$

(D) $x = \sin t$, $y = \cos t$, $z = 2t$

10. Consider the complex number z such that $|z - z_1| = \sqrt{5}$ where $z_1 = \dfrac{7+4i}{3-2i}$.
The largest possible value for $|z|$ is

(A) $2\sqrt{5}$

(B) $\sqrt{5}$

(C) $\sqrt{\dfrac{11}{5}}$

(D) $\sqrt{65}$

Section II begins on the next page.

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section.

Question 11 (16 marks) **Marks**

Answer in the booklet labelled Question 11

(a) Given that $z=\sqrt{3}-i$, find:

(i) $z-\bar{z}$ **1**

(ii) z^{-1} **1**

(b) Prove or disprove the statement 3^n+2 is prime for all integers $n\geq 0$ **2**

(c) Find $\int 3x^2\tan^{-1}(x^3)\,dx$ **3**

(d) Determine the square roots of $-15+8i$. Show all relevant working. **2**

(e) For two vectors $\boldsymbol{m}$ and $\boldsymbol{p}$, it is known that $|\boldsymbol{m}|=5$, $|\boldsymbol{p}|=3$, and $\boldsymbol{m}\cdot\boldsymbol{p}=9$. Find $|\boldsymbol{m}-\boldsymbol{p}|$. **2**

(f) Evaluate $\displaystyle\int_1^2 \frac{1}{x^2(x+2)}\,dx$ **3**

(g) Prove by contradiction that there are no integers a, b, such that $25a+15b=1$. **2**

Question 12 begins on the next page

Question 12 (16 Marks) **Marks**

Answer in the booklet labelled Question 12

(a) Represent z on an Argand diagram, where $\frac{\pi}{3} \leq \arg(z) \leq \frac{\pi}{2}$ **2**

(b) (i) Find values of A and B such that $A(\cos x + \sin x) + B(\cos x - \sin x) \equiv 4\sin x$ **1**

(ii) Hence, or otherwise, find $\int_0^{\frac{\pi}{4}} \frac{4\sin x}{\cos x + \sin x} dx$ **3**

(c) If $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary, find θ where $-\pi < \theta \leq \pi$ **2**

(d) Prove that $2^{2n} - 1$ is divisible by 3 for all positive integers n. **2**

(e) Find the point of intersection of the lines $\boldsymbol{r} = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{p} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ **3**

(f) Consider the sequence defined by the recurrence relation $t_n = 2t_{n-1} + 3$ for $n \geq 2$ where $t_1 = 3$.

Prove by mathematical induction that $t_n = 3 \times 2^n - 3$ for $n \geq 1$. **3**

Question 13 begins on the next page

Question 13 (15 Marks) **Marks**

Answer in the booklet labelled Question 13

(a) The point P $(2,3)$ represents the complex number $2+3i$ on the Argand diagram.
The line segment OP is rotated anticlockwise about the origin by 60° to the location of point Q.
Determine the coordinates of point Q. **2**

(b) Relative to a fixed origin *O*, the points *A, B* and *D* have position vectors $(2\mathbf{i}-\mathbf{j}+5\mathbf{k})$, $(5\mathbf{i}+2\mathbf{j}+10\mathbf{k})$ and $(-\mathbf{i}+\mathbf{j}+4\mathbf{k})$ respectively. The line *l* passes through the points *A* and *B*.

(i) Find the vector equation of line *l*. **1**

(ii) The points *A,B,C,D* form a parallelogram when joined in the given order.
Find the position vector of point C. **2**

(iii) Determine the area of the parallelogram ABCD, correct to 3 significant figures. **4**

(c) Evaluate $\int_0^{\frac{\pi}{3}} \cos^5 x \, dx$ **3**

(d) If $a,b>0$ and $a+b=1$ prove that the minimum value of $\frac{1}{a}+\frac{1}{b}$ is 4. **3**

Question 14 begins on the next page

Question 14 (16 Marks) **Marks**

Answer in the booklet labelled Question 14

(a) (i) Solve $z^5 + 1 = 0$ **1**

(ii) Hence express $z^5 + 1$ as a product of real linear and quadratic factors **2**

(iii) Hence deduce that $4\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{5}\right) = 1$ **2**

(b) Find $\int \frac{x^3}{x^2 + 2x + 5}\, dx$ **4**

(c) For positive a and b, show by mathematical induction that $\left(\frac{a+b}{2}\right)^n \leq \frac{a^n + b^n}{2}$ for all positive integers n. **3**

(d) Sketch the locus of w if $w = \frac{z-1}{z}$, given that $|z| = 2$ **4**

Question 15 begins on the next page

Question 15 (12 Marks) **Marks**

Answer in the booklet labelled Question 15

(a) By considering a suitable trigonometric substitution, or otherwise, find $\int \frac{9}{(9-x^2)^{\frac{3}{2}}}\,dx$ **4**

(b) For positive x, y, and z, prove that $x+y+z+\frac{1}{4x}+\frac{1}{4y}+\frac{1}{4z}\geq 3$ **3**

(c) If $I_n=\int_0^1 (1-x^2)^{\frac{n}{2}}\,dx$

(i) Show that $I_n=\frac{n}{n+1}I_{n-2}$ **3**

(ii) Hence evaluate I_5 **2**

Question 16 begins on the next page

Question 16 (15 Marks) **Marks**

Answer in the booklet labelled Question 16

(a) (i) Show that $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$ **2**

(ii) Explain why $\int_0^{\frac{\pi}{2}} e^{-\sin x}\, dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\, dx$ **2**

(iii) Show that $\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\, dx = \int_0^{\frac{\pi}{2}} e^{-\sin x}\, dx$ **2**

(iv) Hence show that $\int_0^{\pi} e^{-\sin x}\, dx < \frac{\pi}{e}(e-1)$ **2**

(b) In the Argand diagram, the points A, B and C represent the triangle ABC.
The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are represented by the complex numbers z_1 and z_2 respectively with $Arg(z_1) = \alpha \quad Arg(z_2) = \beta$ and $\angle BAC = \theta$.
By expressing z_1 and z_2 in Mod-Arg form, or otherwise, show that the area of triangle ABC is given by Area= $\left|\frac{1}{2}\, Im\left(z_1\,\overline{z_2}\right)\right|$ **3**

(c) By establishing a reduction formula for $I_n = \int (\sin^{-1}x)^n\, dx$, show that **4**

$$\int_0^1 (\sin^{-1}x)^3\, dx = \frac{\pi^3}{8} - 3\pi + 6$$

End of Exam

SOLUTIONS

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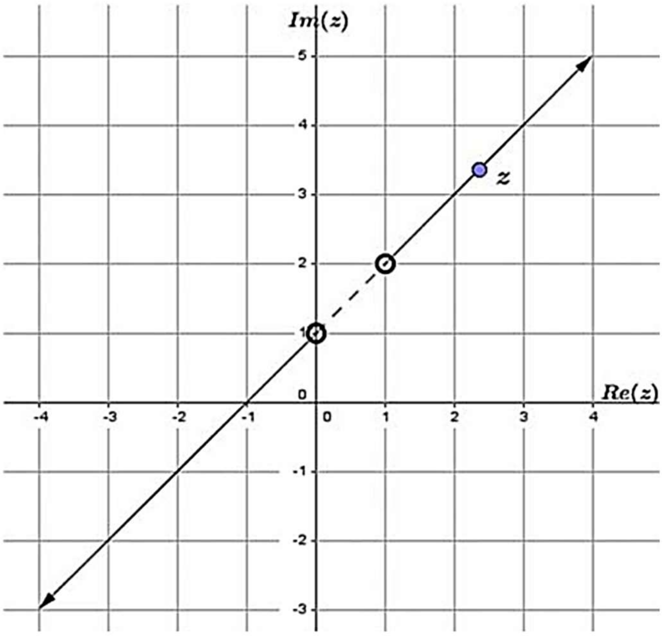

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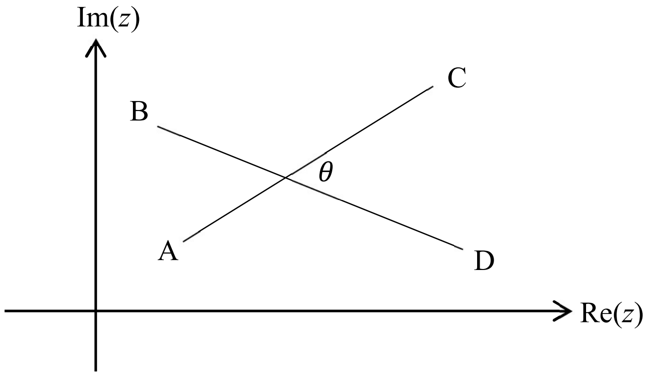
7. The points *A, B, C* and *D* on the Argand diagram below represent the complex numbers *a, b, c,* and *d,* respectively. The angle θ is the acute angle between *AC* and *BD*. Which of the following is a correct expression for θ ?



(A) $\theta = \arg\left(\dfrac{c-d}{a-b}\right)$

(B) $\theta = \arg\left(\dfrac{a-b}{c-d}\right)$

(C) $\theta = \arg\left(\dfrac{c-a}{d-b}\right)$

(D) $\theta = \arg\left(\dfrac{d-b}{c-a}\right)$

8. If $\dfrac{dy}{dx} = (1+\ln x)y$ and when $x=1,\ y=1$ then $y=?$

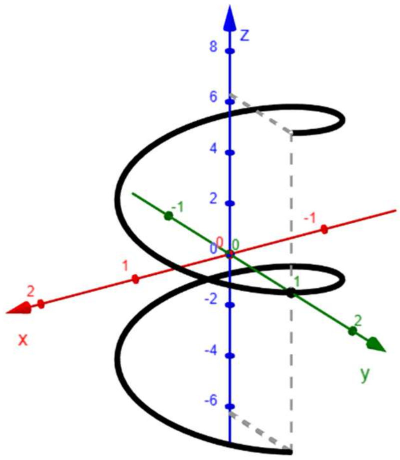
(A) $e^{x\ln x}$

(B) $1+\ln x$

(C) $e^{2x+x\ln x-2}$

(D) $e^{\frac{x^2-1}{x^2}}$

9. Which of the following could be the parametric representation of the curve below?



(A) $x = t^2$, $y = t$, $z = 1$

(B) $x = \sin t$, $y = \cos(2t)$, $z = t$

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10. Consider the complex number z such that $|z - z_1| = \sqrt{5}$ where $z_1 = \dfrac{7+4i}{3-2i}$.
The largest possible value for $|z|$ is

(A) $2\sqrt{5}$

(B) $\sqrt{5}$

(C) $\sqrt{\dfrac{11}{5}}$

(D) $\sqrt{65}$

Section II begins on the next page.

Section II
90 marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section.

Question 11 (16 marks) **Marks**

Answer in the booklet labelled Question 11

(a) Given that $z=\sqrt{3}-i$, find:

(i) $z-\bar{z}$ **1**

Sample solution

$$\begin{aligned} z-\bar{z} &= 2\ \mathrm{Im(z)} \\ &= -2 \end{aligned}$$

Marks	Criteria
1	Correct solution

(ii) z^{-1} **1**

Sample solution

$$\begin{aligned} z^{-1} &= \frac{1}{\sqrt{3}-i}\times\frac{\sqrt{3}+i}{\sqrt{3}+i} \\ &= \frac{\sqrt{3}+i}{3+1} \\ &= \frac{1}{4}\left(\sqrt{3}+i\right) \end{aligned}$$

Marks	Criteria
1	Correct solution

(b) Prove or disprove the statement 3^n+2 is prime for all integers $n\geq 0$ **2**

Sample solution

Let $n=5$ $3^5+2=243+2$ which is not prime. Hence disproved by counter example.

$$\begin{aligned} &= 245 \\ &= 5\times 49 \end{aligned}$$

Marks	Criteria
2	Correct solution, including identifying that the statement is disproven by counter-example
1	Counter example found but not identified as a proof by counter-example, or equivalent merit

(c) Find $\int 3x^2\tan^{-1}(x^3)\,dx$ **3**

Sample solution

Let $u = \tan^{-1}(x^3)$ $\quad v = x^3$

$$u' = 3x^2 \times \frac{1}{1+x^6} \qquad v' = 3x^2$$

$$= \frac{3x^2}{1+x^6}$$

$$\int 3x^2\tan^{-1}(x^3)\,dx = x^3\tan^{-1}(x^3) - \int \frac{3x^5}{1+x^6}\,dx$$

$$= x^3\tan^{-1}(x^3) - \frac{1}{2}\ln|1+x^6| + C$$

$$= x^3\tan^{-1}(x^3) - \frac{1}{2}\ln(1+x^6) + C$$

Marks	Criteria
3	Correct solution
2	Correctly applies integration by parts formula, but an error prevents arrival at correct solution, or equivalent merit.
1	Differentiates one factor and integrates the other factor with the intention of applying differentiation by parts, or equivalent merit.

(d) Determine the square roots of $-15 + 8i$. Show all relevant working. **2**

Sample solution

Let $(a+bi)^2 = -15 + 8i$

$a^2 - b^2 = -15$

$2ab = 8$

$ab = 4$

By inspection, $a = \pm 1$, $b = \pm 4$

Hence $\sqrt{-15+8i} = \pm(1+4i)$

Marks	Criteria
2	Correct solution
1	Obtains a pair of simultaneous equations, or equivalent merit.

(e) For two vectors $\boldsymbol{m}$ and $\boldsymbol{p}$, it is known that $|\boldsymbol{m}| = 5$, $|\boldsymbol{p}| = 3$, and $\boldsymbol{m} \cdot \boldsymbol{p} = 9$. Find $|\boldsymbol{m} - \boldsymbol{p}|$. **2**

Sample solution

Method 1

$$\left|\underset{\sim}{m} - \underset{\sim}{p}\right|^2 = \left(\underset{\sim}{m} - \underset{\sim}{p}\right)\cdot\left(\underset{\sim}{m} - \underset{\sim}{p}\right)$$
$$= \underset{\sim}{m}\cdot\underset{\sim}{m} - 2\underset{\sim}{m}\cdot\underset{\sim}{p} + \underset{\sim}{p}\cdot\underset{\sim}{p}$$
$$= |\underset{\sim}{m}|^2 - 2\underset{\sim}{m}\cdot\underset{\sim}{p} + |\underset{\sim}{p}|^2$$
$$= 5^2 - 2(9) + 3^2$$
$$= 16$$
$$\therefore \quad \left|\underset{\sim}{m} - \underset{\sim}{p}\right| = 4$$

Method 2

$$\underset{\sim}{m}\cdot\underset{\sim}{p} = |\underset{\sim}{m}|\,|\underset{\sim}{p}|\cos\theta$$
$$9 = 5 \times 3 \times \cos\theta$$
$$\cos\theta = \frac{3}{5}$$
$$\left|\underset{\sim}{m} - \underset{\sim}{p}\right|^2 = |\underset{\sim}{m}|^2 + |\underset{\sim}{p}|^2 - 2|\underset{\sim}{m}||\underset{\sim}{p}|\cos\theta$$
$$\left|\underset{\sim}{m} - \underset{\sim}{p}\right|^2 = 5^2 + 3^2 - 2(5)(3)\left(\frac{3}{5}\right)$$
$$\left|\underset{\sim}{m} - \underset{\sim}{p}\right|^2 = 16$$
$$\left|\underset{\sim}{m} - \underset{\sim}{p}\right| = 4$$

Marks	Criteria
2	Correct solution
1	Applies the relationship $\lvert\underset{\sim}{v}\rvert^2 = \underset{\sim}{v}\cdot\underset{\sim}{v}$ OR finds $\cos\theta$, or equivalent merit.

(f) Evaluate $\displaystyle\int_1^2 \frac{1}{x^2(x+2)}\,dx$ **3**

Sample solution

Consider $\quad \dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{x+2} \equiv \dfrac{1}{x^2(x+2)}$

$$A(x+2) + Bx(x+2) + Cx^2 \equiv 1$$

Let $x = 0$: $\quad 2A = 1 \ \Rightarrow A = \dfrac{1}{2}$

Let $x = -2$: $4C = 1 \ \Rightarrow C = \dfrac{1}{4}$

Let $x = -1$: $\dfrac{1}{2} - B + \dfrac{1}{4} = 1 \ \Rightarrow B = -\dfrac{1}{4}$

$$\int_1^2 \frac{1}{x^2(x+2)}\,dx = \int_1^2 \frac{1}{2}x^{-2} - \frac{1}{4}\cdot\frac{1}{x} + \frac{1}{4}\cdot\frac{1}{x+2}\,dx$$
$$= \left[-\frac{1}{2x} - \frac{1}{4}\ln|x| + \frac{1}{4}\ln|x+2|\right]_1^2$$
$$= \left[-\frac{1}{2x} + \frac{1}{4}\ln\left|\frac{x+2}{x}\right|\right]_1^2$$
$$= \left[-\frac{1}{4} + \frac{1}{4}\ln(2)\right] - \left[-\frac{1}{2} + \frac{1}{4}\ln(3)\right]$$
$$= \frac{1}{4} + \frac{1}{4}\ln\left(\frac{2}{3}\right)$$

Marks	Criteria
3	Correct solution
2	Obtains correct primitive function, or equivalent merit
1	Decomposes into partial fractions, or equivalent merit

(g) Prove by contradiction that there are no integers a, b, such that $25a + 15b = 1$. **2**

Sample solution

Negation: suppose $25a + 15b = 1$ where a and b are integers.

$25a = 1 - 15b$

If a is an integer then LHS is a multiple of 5. Hence RHS must also be a multiple of 5.

Let $-15b + 1 = 5M$ where $M \in \mathbb{Z} \Rightarrow b = \dfrac{5M - 1}{15}$

But $5M - 1$ cannot be divisible by 15 as $5M - 1$ is not divisible by 5. Which leaves $b \notin \mathbb{Z}$ contradicting the definition of b being an integer.

The negation is false by contradiction, hence the original statement must be true.

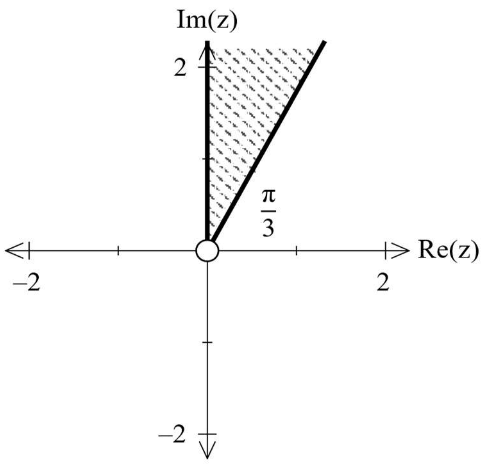
Marks	Criteria
2	Correct solution
1	Sets up appropriate conditions including both sides being multiples of 5 if a and b are integers, or equivalent merit.

Question 12 begins on the next page

Question 12 (16 Marks) **Marks**

(a) Represent z on an Argand diagram, where $\frac{\pi}{3} \leq \arg(z) \leq \frac{\pi}{2}$ **2**

Sample solution



Marks	Criteria
2	Correct solution
1	Progress towards correct solution

(b) (i) Find values of A and B such that $A(\cos x + \sin x) + B(\cos x - \sin x) \equiv 4\sin x$ **1**

Sample solution

$$A(\cos x + \sin x) + B(\cos x - \sin x) \equiv 4\sin x$$

$$(A + B)\cos x + (A - B)\sin x \equiv 4\sin x$$

Equating coefficients on LHS and RHS:

$$A + B = 0$$

$$A - B = 4$$

$$\therefore \quad A = 2,\ B = -2$$

Marks	Criteria
1	Correct solution

(ii) Hence, or otherwise, find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{4\sin x}{\cos x + \sin x}dx$ **3**

Sample solution

$$\int_0^{\frac{\pi}{4}} \frac{4\sin x}{\cos x + \sin x}dx = \int_0^{\frac{\pi}{4}} \frac{2(\cos x + \sin x)}{\cos x + \sin x} - \frac{2(\cos x - \sin x)}{\cos x + \sin x}dx$$

$$= \int_0^{\frac{\pi}{4}} 2 - 2\frac{-\sin x + \cos x}{\cos x + \sin x}\, dx$$

$$= \Big[2x - 2\ln|\cos x + \sin x|\Big]_0^{\frac{\pi}{4}}$$

$$= \left[2\left(\frac{\pi}{4}\right) - 2\ln\left|\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right|\right] - \left[2(0) - 2\ln|\cos(0) + \sin(0)|\right]$$

$$= \frac{\pi}{2} - 2\ln\left(\frac{2}{\sqrt{2}}\right) - 0 - \ln(1)$$

$$= \frac{\pi}{2} - \ln(2)$$

Marks	Criteria
3	Correct solution
2	Obtains correct primitive function
1	Applies result from part (a)

(c) If $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, find θ where $-\pi < \theta \leq \pi$ **2**

Sample solution

$$\frac{3+2i\sin\theta}{1-2i\sin\theta} \times \frac{1+2i\sin\theta}{1+2i\sin\theta} = \frac{3+8i\sin\theta - 4\sin^2\theta}{1+4\sin^2\theta}$$

Purely imaginary ⇒ Real part is zero.

$$3 - 4\sin^2\theta = 0$$

$$\sin\theta = \pm\frac{\sqrt{3}}{2}$$

$$\therefore \quad \theta = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$$

Marks	Criteria
2	Correct solution
1	Progress towards correct solution. I.e. realises the denominator and simplifies the numerator.

(d) Prove that $2^{2n} - 1$ is divisible by 3 for all positive integers n. **2**

Sample solution

$2^{2n} - 1 = (2^n - 1)(2^n + 1)$ by difference of two squares.

Now $2^n - 1$, 2^n and $2^n + 1$ are three consecutive integers, hence one of them must be divisible by 3.

2^n is not divisible by 3. Hence either $2^n - 1$ or $2^n + 1$ must be divisible by 3.

$\therefore (2^n - 1)(2^n + 1)$ is divisible by 3 and hence $2^{2n} - 1$ is divisible by 3.

Marks	Criteria
2	Correct solution
1	Progress towards correct solution

(e) Find the point of intersection of the lines $\boldsymbol{r} = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{p} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ **3**

Sample solution

x component: $0 - \lambda = -1 - \mu$ $\quad \therefore \mu - \lambda = -1$ $\quad$ (1)

y component: $3 + \lambda = 2 + 2\mu$ $\quad \therefore 2\mu - \lambda = 1$ $\quad$ (2)

z component: $-5 + 2\lambda = 3 - \mu$ $\quad \therefore \mu + 2\lambda = 8$ $\quad$ (3)

(2) $-$ (1) : $\quad \mu = 2$

$\therefore \quad \lambda = 3$

Check for consistency in (3) : $(2) + 2(3) = 8 = RHS$

$\therefore$ point of intersection occurs where $\lambda = 3$ and $\mu = 2$

Point of intersection is $\begin{pmatrix} -3 \\ 6 \\ 1 \end{pmatrix}$

Marks	Criteria
3	Correct solution
2	Finds the values of μ and λ, or makes an error in solving the simultaneous equations but goes onto find the point of intersection.
1	Obtains a pair of simultaneous equations that may lead to finding μ or λ

(f) Consider the sequence defined by the recurrence relation $t_n = 2t_{n-1} + 3$ for $n \geq 2$ where $t_1 = 3$.

Prove by mathematical induction that $t_n = 3 \times 2^n - 3$ for $n \geq 1$. **3**

Sample solution

For $n = 1$: $t_1 = 3 \times 2^1 - 3$

$= 3$ which is the given value for t_1. $\therefore$ true for $n = 1$

Assume true for $n = k$: $t_k = 3 \times 2^k - 3$ where k is an integer.

For $n = k + 1$, $R.T.P.$ $t_{k+1} = 3 \times 2^{k+1} - 3$

$$
\begin{aligned}
\mathrm{L}HS &= t_{k+1} \\
&= 2t_k + 3 \text{ by recurrence definition} \\
&= 2(3 \times 2^k - 3) + 3 \text{ by assumption} \\
&= 3 \times 2^{k+1} - 6 + 3 \\
&= 3 \times 2^{k+1} - 3 \\
&= RHS \text{ as required.}
\end{aligned}
$$

$\therefore$ true for integers $n \geq 1$ by process of mathematical induction.

Marks	Criteria
3	Correct solution
Allow 1 for each	Incorporates the assumption in the induction step
	Verifies the initial case and applies the recurrence definition in the induction step

Question 13 begins on the next page

Question 13 (15 Marks) **Marks**

(a) The point P $(2,3)$ represents the complex number $2+3i$ on the Argand diagram.
The line segment OP is rotated anticlockwise about the origin by 60° to the location of point Q.
Determine the coordinates of point Q. **2**

Sample solution

Rotating by $\frac{\pi}{3}$ anticlockwise about the origin is equivalent to multiplying by $\text{cis}\left(\frac{\pi}{3}\right)$.

$$(2+3i)\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)=(2+3i)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$=\left(1-\frac{3\sqrt{3}}{2}\right)+\left(\frac{3}{2}+\sqrt{3}\,i\right)$$

$$\therefore Q \text{ has coordinates } \left(\left(1-\frac{3\sqrt{3}}{2}\right),\left(\frac{3}{2}+\sqrt{3}\right)\right)$$

Marks	Criteria
2	Correct solution
1	Recognises that multiplying by $\text{cis}\left(\frac{\pi}{3}\right)$ generates the required rotation, or equivalent merit.

(b) Relative to a fixed origin O, the points A, B and D have position vectors $(2\mathbf{i}-\mathbf{j}+5\mathbf{k})$, $(5\mathbf{i}+2\mathbf{j}+10\mathbf{k})$ and $(-\mathbf{i}+\mathbf{j}+4\mathbf{k})$ respectively. The line l passes through the points A and B.

(i) Find the vector equation of line l. **1**

Sample solution

$$l=\begin{pmatrix}2\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}5-2\\2+1\\10-5\end{pmatrix}$$

$$\therefore l=\begin{pmatrix}2\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\3\\5\end{pmatrix}$$

Marks	Criteria
1	Correct solution

(ii) The points A,B,C,D form a parallelogram when joined in the given order.

Find the position vector of point C. **2**

Sample solution

$$\overrightarrow{DC} = \overrightarrow{AB}$$

$$\overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$$

Let C be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} x+1 \\ y-1 \\ z-4 \end{pmatrix} = \begin{pmatrix} 5-2 \\ 2+1 \\ 10-5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}$$

$$\therefore C \begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix}$$

Marks	Criteria
2	Correct solution
1	Attempts to equate equivalent vectors

(iii) Determine the area of the parallelogram ABCD, correct to 3 significant figures. **4**

Sample solution

Let θ be the angle between the tails of $\overrightarrow{DA}$ and $\overrightarrow{DC}$

$$\cos\theta = \frac{\overrightarrow{DA} \cdot \overrightarrow{DC}}{\left|\overrightarrow{DA}\right|\left|\overrightarrow{DC}\right|}$$

$$= \frac{\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}}{\sqrt{14} \cdot \sqrt{43}}$$

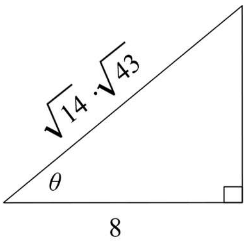
$$= \frac{8}{\sqrt{14} \cdot \sqrt{43}}$$

$$Area = 2 \times \frac{1}{2}\left|\overrightarrow{DA}\right|\left|\overrightarrow{DC}\right|\sin\theta$$

$$= \cancel{\sqrt{14}}\,\cancel{\sqrt{43}}\,\frac{\sqrt{538}}{\cancel{\sqrt{14}}\,\cancel{\sqrt{43}}}$$

$$= \sqrt{538} \text{ sq units}$$

$(\approx 23.19 \quad \text{sq units})$



$\sqrt{538}$ by Pythagoras' theorem

$$\therefore \sin\theta = \frac{\sqrt{538}}{\sqrt{14} \cdot \sqrt{43}}$$

Marks	Criteria
4	Correct solution
3	Significant progress towards solution
2	Finds the angle, or the cosine of the angle between two sides
1	Finds the length of side of the parallelogram

(c) Evaluate $\int_0^{\frac{\pi}{3}} \cos^5 x \, dx$ **3**

Sample solution

$$\int_0^{\frac{\pi}{3}} \cos^5 x \, dx = \int_0^{\frac{\pi}{3}} \cos x(\cos^4 x) dx$$

$$= \int_0^{\frac{\pi}{3}} \cos x(1 - \sin^2 x)^2 \, dx$$

$$= \int_0^{\frac{\pi}{3}} \cos x(1 - 2\sin^2 x + \sin^4 x) \, dx$$

$$= \int_0^{\frac{\pi}{3}} (\cos x - 2\cos x \sin^2 x + \cos x \sin^4 x) \, dx$$

$$= \left[\sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{3}}$$

$$= \left[\sin\left(\frac{\pi}{3}\right) - \frac{2}{3}\left(\sin\left(\frac{\pi}{3}\right)\right)^3 + \frac{1}{5}\left(\sin\left(\frac{\pi}{3}\right)\right)^5\right] - \left[\sin(0) - \frac{2}{3}(\sin(0))^2 + \frac{1}{5}(\sin(0))^5\right]$$

$$= \frac{\sqrt{3}}{2} - \frac{2}{3} \times \frac{3\sqrt{3}}{8} + \frac{1}{5} \times \frac{9\sqrt{3}}{32}$$

$$= \frac{49\sqrt{3}}{160}$$

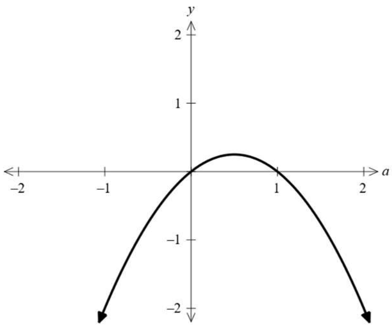
Marks	Criteria
3	Correct solution
2	Finds correct primitive function
1	Expresses the integral in a form that can be integrated

(d) If $a, b > 0$ and $a + b = 1$ prove that the minimum value of $\frac{1}{a} + \frac{1}{b}$ is 4. **3**

Sample solution

$$\frac{\frac{1}{a} + \frac{1}{b}}{2} \geq \sqrt{\frac{1}{ab}} \qquad \text{(AM/GM inequality)}$$

$$\therefore \quad \frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$$



Now, $\frac{2}{\sqrt{ab}}$ is minimised when ab is maximised for positive a, b.

Furthermore, $ab = a(1 - a)$ since $a + b = 1$. The maximun value of ab occurs when $a = \frac{1}{2}$ by the symmetry of the quadratic function $a(1 - a)$.

$$Max[ab] = \frac{1}{2}\left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{4}$$

Hence $Max\left[\sqrt{ab}\right] = \frac{1}{2}$

Hence $Min\left[\frac{2}{\sqrt{ab}}\right] = \frac{2}{\frac{1}{2}}$

$$= 4$$

$$\therefore \quad \frac{1}{a} + \frac{1}{b} \geq 4$$

i.e. $\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}} \geq 4$

$\therefore \frac{1}{a} + \frac{1}{b} \geq 4$ as required.

Marks	Criteria
3	Correct solution
Allow 1 each for:	Applies relationship $\frac{1}{x}$ is minimised when x is maximised, or equivalent merit
	Applies AM/GM inequality, or equivalent merit

Question 14 begins on the next page

Question 14 (16 Marks) — Marks

(a) (i) Solve $z^5 + 1 = 0$ **1**

Sample solution

$$z^5 = -1$$

$$z = -1,\ \text{cis}\left(\pi \pm \frac{2\pi}{5}\right),\ \text{cis}\left(\pi \pm \frac{4\pi}{5}\right)$$

$$= -1,\ \text{cis}\left(\pm \frac{\pi}{5}\right),\ \text{cis}\left(\pm \frac{3\pi}{5}\right)$$

Marks	Criteria
1	Correct solution

(ii) Hence express $z^5 + 1$ as a product of real linear and quadratic factors **2**

Sample solution

$$z^5 + 1 = (z-(-1))\left(z - \text{cis}\frac{\pi}{5}\right)\left(z - \text{cis}\left(-\frac{\pi}{5}\right)\right)\left(z - \text{cis}\frac{3\pi}{5}\right)\left(z - \text{cis}\left(-\frac{3\pi}{5}\right)\right)$$

$$= (z+1)\left(z^2 - \left(2\cos\frac{\pi}{5}\right)z + 1\right)\left(z^2 - \left(2\cos\frac{3\pi}{5}\right)z + 1\right)$$

Marks	Criteria
2	Correct solution
1	Expresses $z^5 + 1$ as a product of linear factors

(iii) Hence deduce that $4\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{5}\right) = 1$ **2**

Sample solution 1

Product of the roots of $z^5 + 1 = 0$ *is* $\frac{f}{a} = \frac{1}{1}$

$$-\text{cis}\left(\frac{\pi}{5}\right)\text{cis}\left(-\frac{\pi}{5}\right)\text{cis}\left(\frac{3\pi}{5}\right)\text{cis}\left(-\frac{3\pi}{5}\right) = 1$$

$$-2\cos\left(\frac{\pi}{5}\right) \times 2\cos\left(\frac{3\pi}{5}\right) = 1$$

$$-4\cos\left(\frac{\pi}{5}\right) \times \sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right) = 1$$

$$-4\cos\left(\frac{\pi}{5}\right) \times \sin\left(-\frac{\pi}{10}\right) = 1$$

$$\therefore \quad 4\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{10}\right) = 1 \text{ since } \sin(-\theta) = -\sin\theta \text{ (odd function)}$$

Sample solution 2

$z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$ and

$z^5 + 1 = (z+1)\left(z^2 - 2\cos\frac{\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5}z + 1\right)$ from (i)

Hence $z^4 - z^3 + z^2 - z + 1 = \left(z^2 - 2\cos\frac{\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5}z + 1\right)$

Equating the coefficients of z^2 on each side gives

$$1 = 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} + 1 + 1$$

$$-1 = 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} \quad (1)$$

Now $\cos\frac{3\pi}{5} = \sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right)$ by complementary angle result

$$= \sin\left(-\frac{\pi}{10}\right)$$

$$= -\sin\left(\frac{\pi}{10}\right)$$

And substituting this into (1) gives:

$$-1 = 4\cos\left(\frac{\pi}{5}\right) \times -\sin\left(\frac{\pi}{10}\right)$$

$$\therefore 4\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{5}\right) = 1 \text{ as required.}$$

Marks	Criteria
2	Correct solution
1	Significant progress towards solution

(b) Find $\displaystyle\int \frac{x^3}{x^2+2x+5}\,dx$ **4**

Sample solution

$$\int \frac{x^3}{x^2+2x+5}dx = \int \frac{x(x^2+2x+5)}{x^2+2x+5} + \frac{-2(x^2+2x+5)}{x^2+2x+5} + \frac{-x+10}{x^2+2x+5}\,dx$$

$$= \int x - 2 - \left(\frac{1}{2}\right)\frac{2x+2}{x^2+2x+5} + \frac{11}{x^2+2x+5}\,dx$$

$$= \int x - 2 - \left(\frac{1}{2}\right)\frac{2x+2}{x^2+2x+5} + 11\,\frac{1}{2^2+(x+1)^2}\,dx$$

$$= \frac{x^2}{2} - 2x - \frac{1}{2}\ln|x^2+2x+5| + \frac{11}{2}\arctan\left(\frac{x+1}{2}\right) + C$$

$$\left[= \frac{1}{2}\left(x^2 - 4x - \ln|x^2+2x+5| + 11\arctan\left(\frac{x+1}{2}\right)\right) + C\right]$$

Marks	Criteria
4	Correct solution
3	Finds the primitive including the $\ln\lvert f(x)\rvert$ or $\frac{1}{a}\arctan\left(\frac{f(x)}{a}\right)$
2	Expresses the integral in the form required to perform integration
1	Makes progress with expressing the integral in the required form to allow for integration

(c) For positive a and b, show by mathematical induction that $\left(\frac{a+b}{2}\right)^n \le \frac{a^n+b^n}{2}$ for all positive integers n. **3**

Sample solution

For $n = 1$: $\quad LHS = \left(\frac{a+b}{2}\right)^1$

$$= \frac{a+b}{2}$$

$$RHS = \frac{a^1+b^1}{2}$$

$$= LHS \quad \therefore \text{ true for } n = 1$$

Assume true for $n = k$: $\left(\frac{a+b}{2}\right)^k \le \frac{a^k+b^k}{2}$

For $n = k+1$, RTP $\left(\frac{a+b}{2}\right)^{k+1} \le \frac{a^{k+1}+b^{k+1}}{2}$

I.e. RTP $\left(\frac{a+b}{2}\right)^{k+1} - \frac{a^{k+1}+b^{k+1}}{2} \le 0$

$$LHS = \left(\frac{a+b}{2}\right)^{k+1} - \frac{a^{k+1}+b^{k+1}}{2}$$

$$= \left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}\right)^{k} - \frac{a^{k+1}+b^{k+1}}{2}$$

$$\le \left(\frac{a+b}{2}\right)\left(\frac{a^k+b^k}{2}\right) - \frac{a^{k+1}+b^{k+1}}{2} \quad \text{by assumption}$$

$$= \frac{a^{k+1}+b^{k+1}+ba^k+ab^k}{4} - \frac{2a^{k+1}+2b^{k+1}}{4}$$

$$= \frac{-a^{k+1}-b^{k+1}+ba^k+ab^k}{4}$$

$$= \frac{a^k(b-a)-b^k(b-a)}{4}$$

$$= \frac{(b-a)(a^k-b^k)}{4}$$

≤ 0 for all positive a and b.

Case 1 : $a = b \Rightarrow ((b-a)(a^k-b^k)) = 0,$

Case 2 : $a \ne b \Rightarrow (b-a)$ and (a^k-b^k) will have opposite signs and the product will be negative.

$\therefore$ true for positive integer values of n by process of mathematical induction.

Marks	Criteria
3	Correct solution
2	Substantial progress towards solution. Accept correct incorporation of assumption then expansion of appropriate brackets and collection of like terms.
1	Verifies initial case and incorporates the assumption

(d) Sketch the locus of w if $w = \dfrac{z-1}{z}$, given that $|z| = 2$ **4**

Sample solution 1

$$zw = z - 1$$

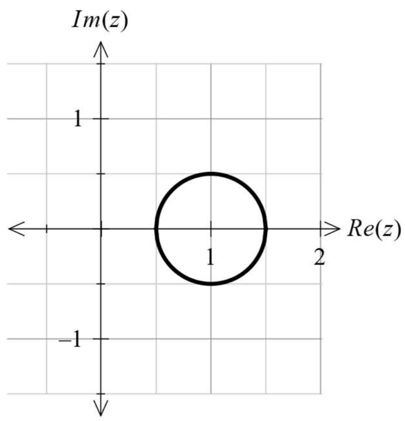
$$z(w-1) = -1$$

$$z = \frac{1}{1-w}$$

$$|z| = \left|\frac{1}{1-w}\right|$$

$$2 = \frac{1}{|1-w|}$$

$$\therefore \; |1-w| = \frac{1}{2}$$

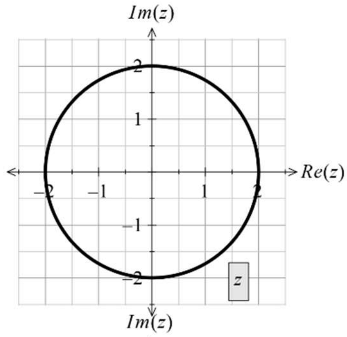


$|w-1| = \dfrac{1}{2}$ which represents a circle centred at $1 + 0i$ with radius $\dfrac{1}{2}$.

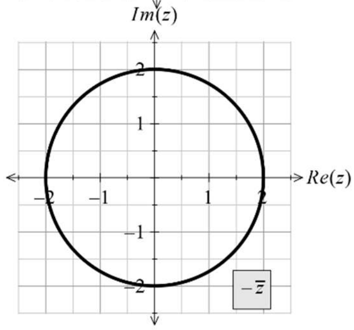
Sample solution 2

$$w = \frac{z-1}{z}$$
$$= 1 - \frac{1}{z}$$
$$= 1 - \frac{\overline{z}}{|z|^2}$$
$$= -\frac{\overline{z}}{4} + 1$$

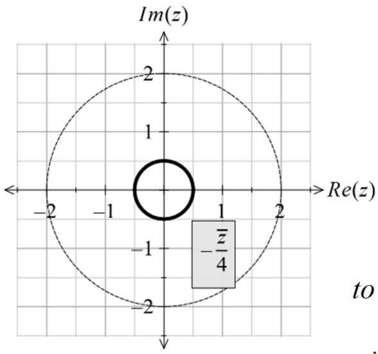
Locus of z *is a circle centred at the origin with radius 2*



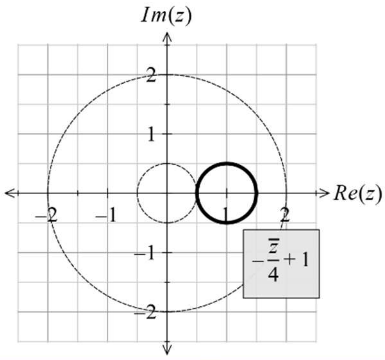
Locus of $\overline{z}$ *is the above circle reflected through the Real axis, i.e. the same circle, and the locus of* $-\left(\overline{z}\right)$ *will be this same circle rotated by* π *radians about the origin which still generates the same circle.*



Locus of $-\frac{\overline{z}}{4}$ *will have radius* $\frac{1}{4} \times 2 = \frac{1}{2}$



Locus of $-\frac{\overline{z}}{4} + 1$ *will now have 1 added to the real part of* $-\frac{\overline{z}}{4}$ *thus giving a circle with radius* $\frac{1}{2}$ *centred at* $1 + 0i$



Marks	Criteria
4	Correct solution, clearly indicating centre and radius of the circle, supported by appropriate calculations.
3	Significant progress towards solution
2	Expresses z in terms of w, or equivalent merit and incorporates |z|=2 in a meaningful way
1	Expresses z in terms of w, or equivalent merit, or incorporates |z|=2 in a meaningful way

Question 15 begins on the next page

Question 15 (12 Marks) **Marks**

(a) By considering a suitable trigonometric substitution, or otherwise, find $\displaystyle\int \frac{9}{(9-x^2)^{\frac{3}{2}}}\,dx$ **4**

Sample solution

Let $x = 3\sin\theta$

$\dfrac{dx}{d\theta} = 3\cos\theta$

$dx = 3\cos\theta \, d\theta$

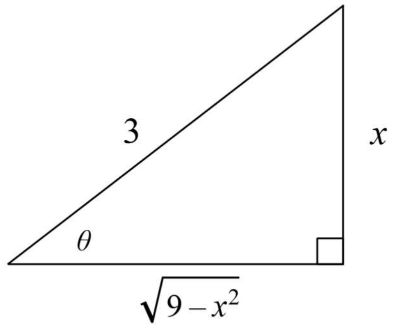
$$\int \frac{9}{(9-x^2)^{\frac{3}{2}}}\,dx = \int \frac{9 \times 3\cos\theta}{(9-9\sin^2\theta)^{\frac{3}{2}}}\,d\theta$$

$$= \int \frac{27\cos\theta}{(9\cos^2\theta)^{\frac{3}{2}}}\,d\theta$$

$$= \int \frac{27\cos\theta}{27\cos^3\theta}\,d\theta$$

$$= \int \sec^2\theta \, d\theta$$

$$= \tan\theta + C$$



If $x = 3\sin\theta$, $\sin\theta = \dfrac{x}{3}$ and by Pythagoras theorem,

$\tan\theta = \dfrac{x}{\sqrt{9-x^2}}$

$$\therefore \int \frac{9}{(9-x^2)^{\frac{3}{2}}}\,dx = \frac{x}{\sqrt{9-x^2}} + C$$

Marks	Criteria
4	Correct solution
3	Completes integration in terms of θ, or equivalent merit
2	Partially simplifies the integral but makes no further valid progress.
1	Completes a valid substitution with change of variable

(b) For positive x, y, and z, prove that $x+y+z+\frac{1}{4x}+\frac{1}{4y}+\frac{1}{4z}\geq 3$ **3**

Sample solution

$$\left(\sqrt{a}-\frac{1}{2\sqrt{a}}\right)^2 \geq 0 \text{ for all real } a>0$$

$$a-1+\frac{1}{4a}\geq 0$$

$$\therefore \quad a+\frac{1}{4a}\geq 1$$

This must also be true for x, y and z, since x, y, and z are all positive.

$$\therefore \quad x+\frac{1}{4x}\geq 1 \quad ①$$

$$\therefore \quad y+\frac{1}{4y}\geq 1 \quad ②$$

$$\therefore \quad z+\frac{1}{4z}\geq 1 \quad ③$$

① + ② + ③ gives:

$$x+\frac{1}{4x}+y+\frac{1}{4y}+z+\frac{1}{4z}\geq 1+1+1$$

$\therefore x+y+z+\frac{1}{4x}+\frac{1}{4y}+\frac{1}{4z}\geq 3$ for positive x, y, z.

Marks	Criteria
3	Correct solution
2	Substantial progress towards correct solution
1	Notes perfect square is always non-negative, or equivalent merit

(c) If $I_n = \int_0^1 (1-x^2)^{\frac{n}{2}}\, dx$

(i) Show that $I_n = \dfrac{n}{n+1} I_{n-2}$ **3**

Sample solution

Let $u = (1-x^2)^{\frac{n}{2}}$ $\quad v = x$

$u' = -nx(1-x^2)^{\frac{n-2}{2}}$ $\quad v' = 1$

$$I_n = \left[x(1-x)^{\frac{n}{2}}\right]_0^1 - \int_0^1 -nx^2(1-x^2)^{\frac{n-2}{2}}\, dx$$

$$= 0 + n\int_0^1 \left(1-(1-x^2)\right)(1-x^2)^{\frac{n-2}{2}}\, dx$$

$$= n\int_0^1 (1-x^2)^{\frac{n-2}{2}} - (1-x^2)^{\frac{n}{2}}\, dx$$

$$= n(I_{n-2} - I_n)$$

$$I_n(1+n) = nI_{n-2}$$

$\therefore \quad I_n = \dfrac{n}{n+1} I_{n-2}$ as required

Marks	Criteria
3	Correct solution
2	Significant progress towards solution
1	Correct substitution into IBP formula

(ii) Hence evaluate I_5 **2**

Sample solution

$$I_5 = \frac{5}{6} I_3$$

$$= \frac{5}{6} \times \frac{3}{4} I_1$$

$$= \frac{5}{6} \times \frac{3}{4} \times \int_0^1 (1-x^2)^{\frac{1}{2}}\, dx$$

$= \dfrac{5}{6} \times \dfrac{3}{4} \times \dfrac{1}{4}\pi(1)^2$ since $\sqrt{1-x^2}$ from $x = 0$ to $x = 1$ forms a quadrant with radius 1.

$$= \frac{5\pi}{32}$$

Marks	Criteria
2	Correct solution
1	Evaluates I_1 or expresses I_5 in terms of I_1, or equivalent merit

Question 16 begins on the next page

Question 16 (15 Marks) **Marks**

(a) (i) Show that $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$ **2**

Sample solution

Method 1

Consider the graphs of $y = \sin x$ and $y = \frac{2}{\pi}x$ for $0 < x < \frac{\pi}{2}$. The graphs have points of intersection at $x = 0$ and $x = 1$ (bounds) and since $\sin x$ is concave down and $\frac{2}{\pi}x$ is linear over this domain, there are no further points of intersection. Hence $\sin x$ must always be above or below $\frac{2x}{\pi}$ over this domain.

Consider $x = \frac{\pi}{4}$:

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\quad \frac{2 \times \frac{\pi}{4}}{\pi} = \frac{1}{2}$ which is less than $\frac{1}{\sqrt{2}}$. Hence $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$

Marks	Criteria
2	Correct solution supported by appropriate justification
1	Progress towards solution

(ii) Explain why $\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\,dx$ **2**

Sample solution

$\sin x > \frac{2x}{\pi}$ from part(i)

$\therefore\ e^{\sin x} > e^{\frac{2x}{\pi}}$

$\frac{1}{e^{\sin x}} < \frac{1}{e^{\frac{2x}{\pi}}}$ since both $e^{\sin x}$ and $e^{\frac{2x}{\pi}}$ are positive for all real x.

$\therefore e^{-\sin x} < e^{-\frac{2x}{\pi}}$

Hence $\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\,dx$ as required.

Marks	Criteria
2	Correct solution supported by appropriate justification
1	Progress towards solution

(iii) Show that $\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx = \int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx$ **2**

Sample solution

$$\sin x = \sin(\pi - x)$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx = \int_0^{\frac{\pi}{2}} e^{-\sin(\pi - x)}\,dx \quad (1)$$

Now, let $u = \pi - x$ — Bounds

$$\frac{du}{dx} = -1 \qquad x = \frac{\pi}{2} \to u = \frac{\pi}{2}$$

$$dx = -1\,du \qquad x = 0 \to u = \pi$$

Hence

$$\int_0^{\frac{\pi}{2}} e^{-\sin(\pi - x)}\,dx = \int_{\pi}^{\frac{\pi}{2}} -e^{-\sin u}\,du$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^{-\sin u}\,du$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx$$ since the variable used is inconsequential (2)

Hence from (1) and (2), $\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx = \int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx$ as required.

Marks	Criteria
2	Correct solution with all links clearly communicated
1	Progress towards solution

(iv) Hence show that $\int_0^{\pi} e^{-\sin x}\,dx < \frac{\pi}{e}(e-1)$ **2**

Sample solution

$$\int_0^{\pi} e^{-\sin x}dx = \int_0^{\frac{\pi}{2}} e^{-\sin x}dx + \int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}dx$$

$$= \int_0^{\frac{\pi}{2}} e^{-\sin x}dx + \int_0^{\frac{\pi}{2}} e^{-\sin x}dx \text{ from (iii}$$

$$= 2\int_0^{\frac{\pi}{2}} e^{-\sin x}dx$$

$$< 2\int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}dx \text{ from (ii)}$$

$$= -\frac{2\pi}{2}\int_0^{\frac{\pi}{2}} -\frac{2}{\pi}e^{-\frac{2x}{\pi}}dx$$

$$= -\pi\left[e^{-\frac{2x}{\pi}}\right]_0^{\frac{\pi}{2}}$$

$$= -\pi(e^{-1}-e^{0})$$

$$= -\pi\left(\frac{1}{e}-1\right)$$

$$= -\frac{\pi}{e}(1-e)$$

$$= \frac{\pi}{e}(e-1)$$

Hence

$$\int_0^{\pi} e^{-\sin x}dx < \frac{\pi}{e}(e-1) \text{ as required}$$

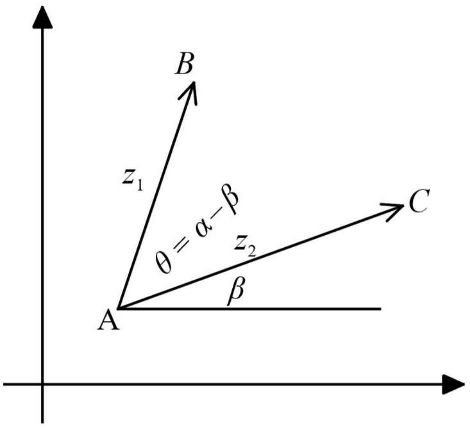
Marks	Criteria
2	Correct solution with all appropriate links clearly communicated
1	Progress towards solution

(b) In the Argand diagram, the points A, B and C represent the triangle ABC.

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are represented by the complex numbers z_1 and z_2 respectively with $Arg(z_1)=\alpha$ $Arg(z_2)=\beta$ and $\angle BAC=\theta$.

By expressing z_1 and z_2 in Mod-Arg form, or otherwise, show that the area of triangle ABC is given by Area= $\left|\frac{1}{2}\,Im\left(z_1\,\overline{z_2}\right)\right|$ **3**

Sample solution



$$z_1 = |z_1|\text{cis}\alpha$$

$$z_2 = |z_2|\text{cis}\beta$$

$$z_1\,\overline{z}_2 = |z_1|\text{cis}\alpha|z_2|\text{cis}(-\beta)$$

$$= |z_1||z_2|\text{cis}(\alpha-\beta) \quad ①$$

$$\therefore\ Im\left(z_1\,\overline{z}_2\right) = |z_1||z_2|\sin(\alpha-\beta)$$

$Area = \frac{1}{2}|z_1||z_2|\sin|\alpha-\beta|$ from area of a non right triangle and that the angle between the two sides of a triangle will always be acute or obtuse

$= \frac{1}{2}|z_1||z_2||\sin(\alpha-\beta)|$ since sine of an angle in the first or second quads will be pos

$$= \frac{1}{2}|\ |z_1|\ |z_2|\ \sin(\alpha-\beta)\ |$$

$= \left|\frac{1}{2}\,Im\left(z_1\overline{z}_2\right)\right|$ as required.

Marks	Criteria
3	Correct solution
2	Finds an expression for area of the triangle in terms of α and β or finds an expression for $Im\left(z_1\overline{z}_2\right)$
1	Finds a simplified expression for $z_1\overline{z}_2$ in terms of α and β

(c) By establishing a reduction formula for $I_n = \int (\sin^{-1}x)^n \, dx$, show that **4**

$$\int_0^1 (\sin^{-1}x)^3 \, dx = \frac{\pi^3}{8} - 3\pi + 6$$

Sample solution

$u = (\sin^{-1}x)^n$ $\quad v = x$

$u' = n(\sin^{-1}x)^{n-1} \cdot \frac{1}{\sqrt{1-x^2}}$ $\quad v' = 1$

$$I_n = \int (\sin^{-1}x)^n \, dx$$

$$= x(\sin^{-1}x)^n - n\int \frac{x}{\sqrt{1-x^2}}(\sin^{-1}x)^{n-1} dx$$

$u = (\sin^{-1}x)^{n-1}$ $\quad v = (1-x^2)^{\frac{1}{2}}$

$u' = (n-1)(\sin^{-1}x)^{n-2} \cdot \frac{1}{\sqrt{1-x^2}}$ $\quad v' = x(1-x^2)^{-\frac{1}{2}}$

$$= x(\sin^{-1}x)^n - n\left[(\sin^{-1}x)^{n-1}\sqrt{1-x^2} - (n-1)\int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}(\sin^{-1}x)^{n-2} dx\right] + C$$

$$= x(\sin^{-1}x)^n - n\sqrt{1-x^2}(\sin^{-1}x)^{n-1} + n(n-1)I_{n-2} + C$$

$$I_1 = \int_0^1 \sin^{-1}x dx$$

$$= \left[x\sin^{-1}x\right]_0^1 + \frac{1}{2}\int_0^1 -\frac{2x}{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} + \left[\sqrt{1-x^2}\right]_0^1$$

$$= \frac{\pi}{2} + (0-1)$$

$$= \frac{\pi}{2} - 1$$

$$I_3 = \left[x(\sin^{-1}x)^3 - 3\sqrt{1-x^2}(\sin^{-1}x)^2\right]_0^1 - 3 \times 2I_1$$

$$= \left(1 \times \left(\frac{\pi}{2}\right)^3 - 3(0)(0)\right) - (0 \times 0 - 3 \times 1 \times 0) - 6\left(\frac{\pi}{2} - 1\right)$$

$$= \frac{\pi^3}{8} - 3\pi + 6$$

Marks	Criteria
4	Correct solution
3	Establishes formula for I_n and correctly evaluates I_1 or, Makes an error with formula for I_1 but goes on to follow correct process for finding I_3.
2	Establishes formula for I_n or correctly evaluates I_1
1	Successfully completes one application of IBP somewhere in the problem.

End of Exam