HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2025

STUDENT NUMBER:__________________ **STUDENT NAME:**______________________ **TEACHER:**_____________

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided separately
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination room

Total marks – 100

Section I Pages 3 – 5

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 6 – 11

90 marks

Attempt Questions 11 – 16

Start each question in a new writing booklet

Write your student number on every writing booklet

Question	***1-10***	***11***	***12***	***13***	***14***	***15***	***16***	***Total***
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/15*	*/15*	*/100*

This assessment task constitutes 30% of the Higher School Certificate Course School Assessment

Outcomes assessed: MEX12-1, MEX12-2, MEX12-3, MEX12-4, MEX12-5, MEX12-7, MEX12-8.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section.

Use the Objective Response answer sheet for Questions 1 – 10

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1. Given $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, the expression that equals $\left(\bar{z}\right)^{-1}$ is

(A) $\frac{1}{2}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

(B) $2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

(C) $\frac{1}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(D) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

2. Which of the following is a primitive of $\frac{\sec^2 x}{\tan^4 x}$?

(A) $\frac{1}{3}\cot^3 x$

(B) $-\frac{1}{3}\cot^3 x$

(C) $\frac{1}{5}\cot^5 x$

(D) $-\frac{1}{5}\cot^5 x$

3. Consider the statement:

'Any flying mammal is a bat'

The negation of the statement is:

(A) Any flying mammal is not a bat.

(B) All flying mammals are not bats.

(C) Any non-flying mammal is not a bat.

(D) All non-flying mammals are not bats.

4. Which of the following is the vector equation of the line segment joining

the points $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$

(A) $\underset{\sim}{r} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}, \ 0 \le \lambda \le 1$

(B) $\underset{\sim}{r} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}, \ 0 \le \lambda \le 1$

(C) $\underset{\sim}{r} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 6 \\ 1 \end{bmatrix}, \ 0 \le \lambda \le 1$

(D) $\underset{\sim}{r} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 6 \\ -1 \end{bmatrix}, \ 0 \le \lambda \le 1$

5. One of the complex solutions to $z^5 = -a$, where a is a positive real constant, is $a^{\frac{1}{5}}\text{cis}\left(\frac{\pi}{5}\right)$.

One of the other solutions is a real number and is equal to

(A) $a^{\frac{1}{5}}\text{cis}\left(\frac{3\pi}{5}\right)$

(B) $-a^{\frac{1}{5}}$

(C) $a^{\frac{1}{5}}$

(D) $a^{\frac{1}{5}}\text{cis}\left(\frac{9\pi}{5}\right)$

6. If $\frac{4}{(2x-1)(3-x)} = \frac{A}{2x-1} + \frac{B}{3-x}$, then A and B have values of:

(A) $A = \frac{-8}{5}, \ B = \frac{4}{5}$

(B) $A = \frac{8}{5}, \ B = \frac{-4}{5}$

(C) $A = \frac{-8}{5}, \ B = \frac{-4}{5}$

(D) $A = \frac{8}{5}, \ B = \frac{4}{5}$

7. If ω is the complex cube root of unity, then the value of $\left(1-3\omega+\omega^2\right)\left(1+\omega-8\omega^2\right)$ is:

(A) $36\omega^3$

(B) -36

(C) $-36\omega^3$

(D) 36

8. The sign on a fence reads

"If you are not authorised, you may not enter."

Which of the following statements is logically equivalent to the one above?

(A) If you may enter, then you are authorised.

(B) If you are authorised, then you may enter.

(C) If you are not authorised, then you may enter.

(D) If you may not enter, then you are not authorised.

9. P, Q and R are three collinear points with position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively, where Q lies between P and R.

If $\left|\overrightarrow{QR}\right| = \frac{1}{2}\left|\overrightarrow{PQ}\right|$, then $\underset{\sim}{r}$ is equal to

(A) $\frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

(B) $\frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$

(C) $\frac{1}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

(D) $\frac{3}{2}\underset{\sim}{q} - \frac{3}{2}\underset{\sim}{p}$

10. Suppose that $x + \frac{1}{x} = -1$. What is the value of $x^{2025} + \frac{1}{x^{2025}}$?

(A) 1

(B) 2

(C) $\frac{2\pi}{3}$

(D) $\frac{4\pi}{3}$

END OF SECTION

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **START A NEW WRITING BOOKLET**

(a) Find $\int \frac{dx}{\sqrt{14+6x-x^2}}$. **3**

(b) (i) Express $-1+\sqrt{3}\,i$ in modulus-argument form. **2**

(ii) Hence evaluate $\left(-1+\sqrt{3}\,i\right)^{-6}$. **2**

(iii) State whether $\left(-1+\sqrt{3}\,i\right)^{n}$, $n \in Z$, is purely imaginary. Justify. **2**

(c) Express $e^{1-\frac{i\pi}{6}}$ in Cartesian form **1**

(d) Consider the following statements about a polynomial $P(x)$. **2**

(i) If $P(x)$ is even, then $P'(x)$ is odd.

(ii) If $P'(x)$ is even, then $P(x)$ is odd.

Indicate whether each of these statements is true or false.
Give reasons for your answers.

(e) Use the substitution of $t = \tan\frac{x}{2}$ to find $\int \frac{15}{17+8\cos x}\,dx$. **3**

End of Question 11

Question 12 (15 marks) **START A NEW WRITING BOOKLET**

(a) (i) Find $\int \frac{2x-5}{x^2+2x+2}\,dx$ **3**

(ii) Find $\int x^3\sqrt{4-x^2}\,dx$ **3**

(b) The vertices of a triangle ABC are defined by the position vectors

$$\overrightarrow{OA} = \begin{bmatrix} 1 \\ -6 \\ -2 \end{bmatrix} \quad \overrightarrow{OB} = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} \text{ and } \quad OC = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$$

(i) Show that $\cos\angle BAC = \frac{1}{2}$ **3**

(ii) Hence, find the exact area of triangle ABC **1**

(c) Consider the point $P(2,1,-6)$ and the line L with equation **3**

$$\underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Find the shortest distance from the point P to the line.

(d) Prove by contradiction that if a and b are both positive integers then **2**

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

End of Question 12

Question 13 (15 marks) **START A NEW WRITING BOOKLET**

(a) The line L_1 has vector equation $\underset{\sim}{r_1} = 6\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k} + \lambda\left(3\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)$, where λ is a scalar parameter.

The line L_2 has vector equation $\underset{\sim}{r_2} = 7\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k} + \mu\left(2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$, where μ is a scalar parameter.

Determine whether the two lines:

(i) are parallel. **1**

(ii) intersect and if so where do they intersect. **3**

(iii) are skew. **1**

(iv) are perpendicular. **1**

(b) Find $\int \sec^3 x \, dx$. **3**

(c) Sketch the region where the inequalities $-\frac{\pi}{2} \le \arg(z - 1 - 2i) \le \frac{\pi}{4}$ and $|z| \le \sqrt{5}$ both hold. **3**

(d) Let P be the proposition: if $k+1$ is divisible by 3, then k^3+1 is divisible by 3, $k \in Z^+$. **3**

Write down the converse of the proposition P and state, with reasons, whether this converse is true or false.

End of Question 13

Question 14 (15 marks) **START A NEW WRITING BOOKLET**

(a) Given that $a_{n+1} = \sqrt{2a_n}$ and $a_1 = 1$, prove by mathematical induction that for all integers $n \geq 1$ **3**

$$a_n = 2^{(1-2^{(1-n)})}$$

(b) Find $\int \cos^5 x\,dx$ **3**

(c) Find the points of intersection of the line $\underset{\sim}{r} = t(4\underset{\sim}{i} - 2\underset{\sim}{j} - 5\underset{\sim}{k})$ and the sphere which has equation $x^2 + y^2 + z^2 = 90$. **3**

(d) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$, where n is an integer, and $n \geq 3$.

(i) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$ **3**

(ii) Hence evaluate I_5 **3**

End of Question 14

Question 15 (15 marks) **START A NEW WRITING BOOKLET**

(a) Find $\displaystyle\int \frac{x}{(2x+1)(x-3)^2}\,dx$ **3**

(b) (i) Find an expression for $\cot 2\theta$ in terms of $\tan\theta$. **1**

(ii) Show that $\tan\theta$ and $-\cot\theta$ satisfy the equation **2**

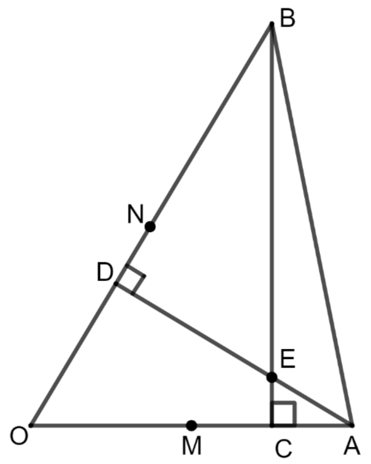
$$x^2 + 2x\cot 2\theta - 1 = 0.$$

(iii) Hence, or otherwise, find the exact value of $\tan\dfrac{\pi}{8}$. **2**

$$x^2 + 2x\cot 2\theta - 1 = 0.$$

(iv) Hence find the exact value of $\tan\dfrac{\pi}{16} - \cot\dfrac{\pi}{16}$. **2**

(c)



NOT TO SCALE

In $\Delta OAB, BC$ is the altitude from B to OA and AD is the altitude from B to OA. M and N are the midpoints of OA and OB respectively. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

(i) Use vector methods to show that $\left|\overrightarrow{OM}\right|\left|\overrightarrow{OC}\right| = \left|\overrightarrow{ON}\right|\left|\overrightarrow{OD}\right|$. **3**

(ii) A line drawn from O passes through E to meet the side BA at point F. By letting $\overrightarrow{OE} = \underset{\sim}{e}$, prove that $\overrightarrow{OF} \perp \overrightarrow{BA}$. **2**

End of Question 15

Question 16 (15 marks) **START A NEW WRITING BOOKLET**

(a) Using mathematical induction, prove that for $n \geq 2$, where n is an integer **3**

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}>\frac{2n}{n+1} .$$

(b) For positive real numbers a, b and c, where $a \neq b \neq c$

(i) Prove **2**

$$a^3+b^3>ab(a+b)$$

(ii) Hence or otherwise, prove that **2**

$$2(a^3+b^3+c^3)>a^2(b+c)+b^2(a+c)+c^2(a+b)$$

(c) The function $F(p)$ is defined as

$$F(p)=\lim_{t\to\infty}\int_0^t x^{p-1}e^{-x}dx \text{, for } p>0$$

(i) Show that $F(1)=1$ **3**

(ii) Use integration by parts to show $F(p+1)=p\times F(p)$ **3**

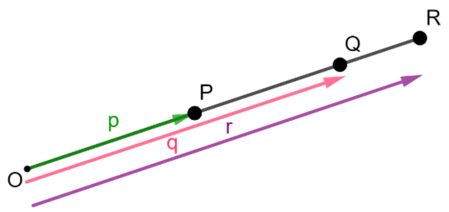
(iii) Hence find $F(n)$ for integers $n \geq 1$. **2**

END OF ASSESSMENT

Section I Multiple Choice

Question	Answers
1	**C**
2	**B**
3	**B**
4	**C**
5	**B**
6	**D**
7	**D**
8	**A**
9	**A**
10	**B**

1.	$z=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ $\overline{z}=2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$ $\therefore\left(\overline{z}\right)^{-1}=\left[2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)\right]^{-1}$ by De Moivre's theorem $=\frac{1}{2}\left[\cos\left(-\frac{\pi}{3}\right)-i\sin\left(-\frac{\pi}{3}\right)\right]$ $\therefore\left(\overline{z}\right)^{-1}=\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ **(C)**	97% answered correctly.
2.	$\int\frac{\sec^2 x}{\tan^4 x}\,dx=\int\sec^2 x\left(\tan x\right)^{-4}\,dx$ $=\frac{\left(\tan x\right)^{-3}}{-3}+C$ $=-\frac{1}{3}\cot^3 x+C$ **(B)**	82% answered correctly.
3.	'Any flying mammal is a bat. Key Words/Phrases – Negation; Any – All; Is – Is not or Are not $\therefore$ The negation of the statement is **(B)**	50% answered correctly. Most errors occur where 'any' was not changed to 'all'.

4.	$\begin{bmatrix}4\\2\\3\end{bmatrix}-\begin{bmatrix}5\\-4\\2\end{bmatrix}=\begin{bmatrix}-1\\6\\1\end{bmatrix}$ direction vector $\therefore \underset{\sim}{r}=\begin{bmatrix}4\\2\\3\end{bmatrix}+\lambda\begin{bmatrix}-1\\6\\1\end{bmatrix},\quad 0\le\lambda\le1$ **(C)**	94% answered correctly.
5.	$z^5=-a$ has roots separated by an argument of $\frac{2\pi}{5}$. So the real root is $a^{\frac{1}{5}}\text{cis}\left(\frac{5\pi}{5}\right)=a^{\frac{1}{5}}\text{cis}\pi=-a^{\frac{1}{5}}$ **(B)**	70.6% answered correctly. Most errors occur where positive answer was obtained.
6.	$4\equiv A(3-x)+B(2x-1)$ Let $x=3,\quad 4=5B\quad\Rightarrow B=\frac{4}{5}$ $x=\frac{1}{2},\quad 4=\frac{5}{2}A\quad\Rightarrow A=\frac{8}{5}$ **(D)**	94% answered correctly.
7.	since $w^3=1$ then $w^3-1=0$ therefore $(w-1)(w^2+w+1)=0$ $(1-3\omega+\omega^2)(1+\omega-8\omega^2)=(-\omega-3\omega)(-\omega^2-8\omega^2)$ $=-4\omega(-9\omega^2)$ $=36\omega^3$ $=36$ **(D)**	67.6% answered correctly. Most errors occur where negative answer was obtained.
8.	The contrapositive is: "If you may enter, then you are authorized" **(A)**	85.3% answered correctly. Most errors occur where the converse was chosen.
9.	 $\overrightarrow{OR}=\overrightarrow{OQ}+\overrightarrow{QR}$ $=\underset{\sim}{q}+\frac{1}{2}\overrightarrow{PQ}$ $=\underset{\sim}{q}+\frac{1}{2}(\underset{\sim}{q}-\underset{\sim}{p})$ $\therefore \underset{\sim}{r}=\frac{3}{2}\underset{\sim}{q}-\frac{1}{2}\underset{\sim}{p}$ **(A)**	91.2% answered correctly.

10.	$x+\frac{1}{x}=-1$ $x^2+1=-x$ $x^2+x+1=0$ $(x-1)(x^2+x+1)=0$ $x^3-1=0$ $\therefore x^3=1$ (i.e. x is a non-real roof of 1) Now $\therefore x^{2025}=(x^3)^{675}=1$ $\therefore \frac{1}{x^{2025}}=\frac{1}{1}$ $x^{2025}+\frac{1}{x^{2025}}=1+1=2$ **(B)**	52.9% answered correctly. Most errors occur where students forgot to add the reciprocal.

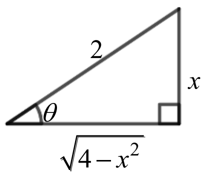
SECTION 2

11.	(a) $\int \frac{dx}{\sqrt{14+6x-x^2}}=\int \frac{dx}{\sqrt{14-(x^2-6x)}}$ $=\int \frac{dx}{\sqrt{14+9-(x^2-6x+9)}}$ $=\int \frac{dx}{\sqrt{23-(x-3)^2}}$ ① where $a=\sqrt{23}$, $f(x)=x-3$ $f'(x)=1$ $= \sin^{-1}\frac{(x-3)}{\sqrt{23}}+C$ ① ①	Mostly done well, otherwise, due to careless arithmetic, factorizing and minor errors.
11.	(b)(i) $-1+\sqrt{3}\,i=\left\|-1+\sqrt{3}\,i\right\|\text{cis}\left(\tan^{-1}\frac{\sqrt{3}}{(-1)}\right)$ $=\sqrt{(-1)^2+(\sqrt{3})^2}\,\text{cis}\left(\frac{2\pi}{3}\right)$ $=2\text{cis}\left(\frac{2\pi}{3}\right)$ ① ①	Mostly done well. Errors for obtaining $\tan^{-1}\frac{\sqrt{3}}{(-1)}$ value.

11.	(b) (ii) $\left(-1+\sqrt{3}\,i\right)^{-6}=\left(2\text{cis}\dfrac{2\pi}{3}\right)^{-6}$ ① $=(2)^{-6}\,\text{cis}\left[-6\left(\dfrac{2\pi}{3}\right)\right]$ (de Moivre's theorem) $=\dfrac{1}{64}\text{cis}\left[-4\pi\right]$ $=\dfrac{1}{64}\left[\cos(-4\pi)+i\sin(-4\pi)\right]$ $=\dfrac{1}{64}(1+0i)$ $=\dfrac{1}{64}$ ①	Mostly done well.
11.	(b)(iii) $\left(-1+\sqrt{3}\,i\right)^{n}=\left(2\text{cis}\dfrac{2\pi}{3}\right)^{n}$ $=(2)^{n}\,\text{cis}\left[n\left(\dfrac{2\pi}{3}\right)\right]$ (de Moivre's Theorem) $=2^{n}\text{cis}\left(\dfrac{2n\pi}{3}\right)$ ① For purely imaginary, $\dfrac{2n\pi}{3}=\dfrac{(2k+1)\pi}{2},\ k\in Z$ $\dfrac{2n}{3}=\dfrac{(2k+1)}{2}$ $\dfrac{4n}{3}=2k+1$ $2k=\dfrac{4n}{3}-1$ $\therefore k=\dfrac{1}{2}\left(\dfrac{4n-3}{3}\right)$ $\therefore k=\dfrac{4n-3}{6}$ $\therefore 6k=4n-3$ but $6k\neq 4n-3,\ \forall n\in Z$ ① $2(3k)\neq 2(2n-1)-1$ (ie. An even $\neq$ an even minus an odd) Hence it is not possible for $\left(-1+\sqrt{3}\,i\right)^{n}$ to be purely imaginary	Poorly done.

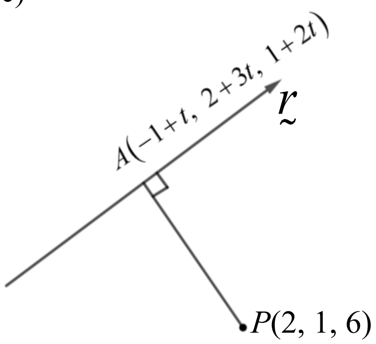
11.	(c) $e^{1-\frac{i\pi}{6}} = e^1 \times e^{-\frac{i\pi}{6}}$ $= e\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ $= e\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$ $= e\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$ $= \frac{e}{2}\left(\sqrt{3} - i\right)$	Mostly done well.
11.	(d) (i) If $P(x)$ is even, i.e. $P(x) = a_n x^{2n} + a_{n-1}x^{2n-2} + a_{n-2}x^{2n-4} + ... + a_2 x^2 + a_0$ $P'(x) = 2na_n x^{2n-1} + (2n-2)a_{n-1}x^{2n-3} + ... + 2a_2 x^1$ where the powers of each term of $P'(x)$ are odd. Hence the statement is true. ① (ii) If $P'(x)$ is even, i.e. $P'(x) = a_n x^{2n} + a_{n-1}x^{2n-2} + a_{n-2}x^{2n-4} + ... + a_2 x^2 + a_1$ $P(x) = \frac{a_n x^{2n+1}}{2n+1} + \frac{a_{n-1}x^{2n-1}}{2n-1} + ... + \frac{a_3 x^3}{3} + a_1 x + C$ where the powers of each term of the function $P(x)$ are odd, but the constant C is not an odd term. Hence the statement is false. ①	Poorly done. 1) Some students did not read carefully and had $P(x)$ as a trig. function, or an inverse trig function. $P(x)$ is stated to be a polynomial, which means **all** polynomials. Many did not produce $P(x) = a_n x^{2n} + a_{n-1}x^{2n-2} + ... + a_0$ but chose only one term. 2) Many instead of showing mathematically, uses words and did not demonstrate the key features required of the answer. 3) Few students showed very poor understanding of odd and even functions. 5) For (ii), many students forgot the existence of constant C which makes $P(x)$ neither odd nor even.

11.	(e) $\displaystyle\int \frac{15}{17+8\cos x}\,dx$ Let $t=\tan\frac{x}{2}$ $\tan^{-1}t=\frac{x}{2}$ $x=2\tan^{-1}t$ $\frac{dx}{dt}=\frac{2}{1+t^2}$ $dx=\frac{2}{1+t^2}\,dt$ and $\cos x=\frac{1-t^2}{1+t^2}$ $\displaystyle=15\int \frac{1}{17+8\left(\frac{1-t^2}{1+t^2}\right)}\left(\frac{2}{1+t^2}\right)dt$ ① $\displaystyle=30\int \frac{1}{\frac{17(1+t^2)+8(1-t^2)}{1+t^2}}\left(\frac{1}{1+t^2}\right)dt$ $\displaystyle=30\int \frac{1}{17+17t^2+8-8t^2}\,dt$ $\displaystyle=30\int \frac{1}{25+9t^2}\,dt$ where $a=5,\ f(t)=3t$ $f'(t)=3$ $\displaystyle=\frac{30}{15}\int \frac{15}{25+9t^2}\,dt$ $\displaystyle=2\int \frac{15}{25+9t^2}\,dt$ ① $=2\tan^{-1}\left(\frac{3t}{5}\right)+C$ $=2\tan^{-1}\left(\frac{3}{5}\tan\frac{x}{2}\right)+C$ ①	Mostly done well.

12.	(a)(i) $\displaystyle\int \frac{2x-5}{x^2+2x+2}\,dx = \int \frac{2x+2-5-2}{x^2+2x+2}\,dx$ ① $\displaystyle= \int \frac{(2x+2)-7}{x^2+2x+2}\,dx$ $\displaystyle= \int \frac{2x+2}{x^2+2x+2}\,dx - \int \frac{7}{x^2+2x+2}\,dx$ ① $\displaystyle= \ln\left(x^2+2x+2\right) - \int \frac{7}{x^2+2x+1+1}\,dx$ $\displaystyle= \ln\left(x^2+2x+2\right) - \int \frac{7}{(x+1)^2+1}\,dx$ $\displaystyle= \ln\left(x^2+2x+2\right) - 7\tan^{-1}(x+1) + C$ ①	Very well done
12.	(a)(ii) $\displaystyle\int x^3\sqrt{4-x^2}\,dx$ let $x = 2\sin\theta$  $dx = 2\cos\theta\,d\theta$ $\displaystyle= \int 8\sin^3\theta\sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta$ $\displaystyle= 16\int \sin^3\theta\sqrt{4\cos^2\theta}\;\cos\theta\,d\theta$ $\displaystyle= 32\int \sin^3\theta\cos^2\theta\,d\theta$ ① $\displaystyle= 32\int \sin\theta\left(1-\cos^2\theta\right)\cos^2\theta\,d\theta$ $\displaystyle= 32\int \sin\theta\cos^2\theta - \sin\theta\cos^4\theta\,d\theta$ $\displaystyle= 32\left(-\frac{\cos^3\theta}{3}\right) + 32\left(\frac{\cos^5\theta}{5}\right) + C$ $\displaystyle= \frac{32\cos^5\theta}{5} - \frac{32\cos^3\theta}{3} + C$ ① $\displaystyle= \frac{32}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5 - \frac{32}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C$ $\displaystyle= \frac{1}{5}\left(\sqrt{4-x^2}\right)^5 - \frac{4}{3}\left(\sqrt{4-x^2}\right)^3 + C$ ①	Mostly well done. Quite a few students did not put it back in terms of x

12.	(b)(i) $\overrightarrow{AB}=\begin{bmatrix}0-1\\-4-{}^{-}6\\-1-{}^{-}2\end{bmatrix}=\begin{bmatrix}-1\\2\\1\end{bmatrix}$ $\overrightarrow{AC}=\begin{bmatrix}3-1\\-4-{}^{-}6\\2-{}^{-}2\end{bmatrix}=\begin{bmatrix}2\\2\\4\end{bmatrix}$ ① $\cos\angle CAB=\dfrac{\overrightarrow{AB}\bullet\overrightarrow{AC}}{\lvert\overrightarrow{AB}\rvert\lvert\overrightarrow{AC}\rvert}$ $=\dfrac{-1\times2+2\times2+1\times4}{\sqrt{(-1)^2+(2)^2+(1)^2}\times\sqrt{(2)^2+(2)^2+(4)^2}}$ ① $=\dfrac{6}{\sqrt{6}\times\sqrt{24}}$ $=\dfrac{6}{\sqrt{144}}$ $=\dfrac{6}{12}$ $=\dfrac{1}{2}$ (as required) ① $\therefore\angle CAB=\dfrac{\pi}{3}$	Mostly well done
12.	(b)(ii) $A=\dfrac{1}{2}\lvert\overrightarrow{AB}\rvert\times\lvert\overrightarrow{AC}\rvert\times\sin\dfrac{\pi}{3}$ $=\dfrac{1}{2}\sqrt{6}\times\sqrt{24}\times\dfrac{\sqrt{3}}{2}$ $=\dfrac{1}{2}\times12\times\dfrac{\sqrt{3}}{2}$ $=3\sqrt{3}$ units^2 ①	Well done

12. (c)



$\begin{bmatrix}1\\3\\2\end{bmatrix}$ is a direction vector

$$\underset{\sim}{r}=\begin{bmatrix}-1\\2\\1\end{bmatrix}+t\begin{bmatrix}1\\3\\2\end{bmatrix}$$

$$\overrightarrow{PA}=\begin{bmatrix}-1+t-2\\2+3t-1\\1+2t-{}^{-}6\end{bmatrix}=\begin{bmatrix}t-3\\3t+1\\2t+7\end{bmatrix} \quad ①$$

Since $\overrightarrow{PA}\perp\underset{\sim}{r}$, $\quad \overrightarrow{PA}\bullet\begin{bmatrix}1\\3\\2\end{bmatrix}=0$

$$\begin{bmatrix}t-3\\3t+1\\2t+7\end{bmatrix}\bullet\begin{bmatrix}1\\3\\2\end{bmatrix}=0$$

$$(t-3)\times1+(3t+1)\times3+(2t+7)\times2=0$$

$$14t+14=0$$

$$\therefore t=-1 \quad ①$$

$$\therefore \overrightarrow{PA}=\begin{bmatrix}-4\\-2\\5\end{bmatrix}$$

$$\therefore \left|\overrightarrow{PA}\right|=\sqrt{(-4)^2+(-2)^2+(5)^2}$$

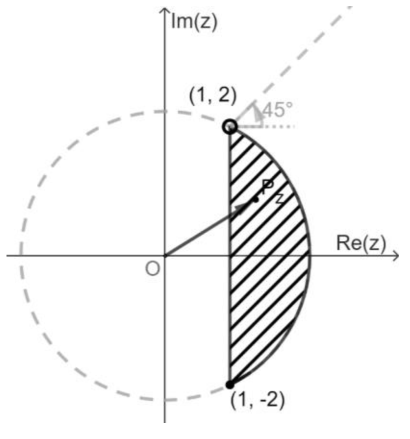
$$=\sqrt{45}$$

$$=3\sqrt{5}\text{ units} \quad ①$$

Mostly well done. There were a few students who did not know the correct method.

12.	(d) RTP by contradiction, if *a*, *b* > 0 integers $$\frac{a}{b}+\frac{b}{a}\geq 2$$ Assume $\frac{a}{b}+\frac{b}{a}<2$ then $\frac{a^2+b^2}{ab}<2$ $a^2+b^2<2ab$ $a^2-2ab+b^2<0$ ① $(a-b)^2<0$ But $n^2\geq 0$ for all real n $\therefore (a-b)^2<0$ is a contradiction ① $\therefore$ original assumption is false. $\therefore \frac{a}{b}+\frac{b}{a}\geq 2$	Very well done
13.	(a)(i) $$\underset{\sim}{r_1}=6\underset{\sim}{i}+5\underset{\sim}{j}+7\underset{\sim}{k}+\lambda\left(3\underset{\sim}{i}+4\underset{\sim}{j}+5\underset{\sim}{k}\right)$$ $$\underset{\sim}{r_2}=7\underset{\sim}{i}+3\underset{\sim}{j}+4\underset{\sim}{k}+\mu\left(2\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right)$$ The ratio of direction numbers $\frac{3}{2},\frac{4}{1},\frac{5}{1}$ and since $\frac{3}{2}\neq\frac{4}{1}\neq\frac{5}{1}$, ① $\therefore$ the two lines are not parallel.	Mostly done well.

<table>
<tr><td>13.</td><td>(a)(ii)
Equating x, y and z components: $6+3\lambda=7+2\mu$ — ①
$5+4\lambda=3+\mu$ — ②
$7+5\lambda=4+\mu$ — ③
③−② $2+\lambda=1$
$\therefore \lambda=-1$ — ④ ①
Sub ④ into ② $5+4(-1)=3+\mu$
$1=3+\mu$
$\therefore \mu=-2$ — ⑤
Sub ④ and ⑤ into ① LHS $=6+3(-1)$
$=3$
RHS $=7+2(-2)$
$=3$
$\therefore$ LHS = RHS ①
$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}-\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$
$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$
①
Hence there is a unique pair of λ and μ values, $\therefore$ the two lines intersect at a point (3, 1, 2)</td><td>Mostly done well.</td></tr>
<tr><td>13.</td><td>(a)(iii)
Since the lines intersect at a point, the lines are not skew. ①</td><td>Mostly done well.</td></tr>
<tr><td>13.</td><td>(a)(iv)
$\left(3\underset{\sim}{i}+4\underset{\sim}{j}+5\underset{\sim}{k}\right)\bullet\left(2\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right)=6+4+5$
$\neq 0$
Hence the two lines are not perpendicular to each other. ①</td><td>Mostly done well.</td></tr>
</table>

13.	(b) Let $I=\int \sec^3 x\,dx$ $=\int \sec x\sec^2 x\,dx$ $=uv-\int u'v\,dx$ where $u=\sec x$ and $v'=\sec^2 x$ $u'=\sec x\tan x$ $v=\tan x$ $=\sec x\tan x-\int \sec x\tan^2 x\,dx$ $=\sec x\tan x-\int \sec x\left(\sec^2 x-1\right)dx$ $=\sec x\tan x-\int \sec^3 x-\sec x\,dx$ $=\sec x\tan x-\int \sec^3 x\,dx+\int \sec x\,dx$ $I=\sec x\tan x-I+\int \sec x\times\frac{\sec x+\tan x}{\sec x+\tan x}\,dx$ $2I=\sec x\tan x+\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\,dx$ $2I=\sec x\tan x+\ln\left\|\sec x+\tan x\right\|+C$ $\therefore I=\frac{1}{2}\sec x\tan x+\frac{1}{2}\ln\left\|\sec x+\tan x\right\|+K$	Poorly done.
13.	(c) $\left\{z:-\frac{\pi}{2}\le\arg(z-1-2i)\le\frac{\pi}{4}\right\}\cap\left\{z:\|z\|\le\sqrt{5}\right\}$ $\|z\|\le\sqrt{5}$  	Poorly done. Interesting that many students did not realized that point (1, 2) is on the circle.

13.	(d) Original statement: 'If $k+1$ is divisible by 3, then k^3+1 is divisible by 3, $k \in Z^+$.' The Converse statement: 'If k^3+1 is divisible by 3, then $k+1$ is divisible by 3, $k \in Z^+$.' The contrapositive of converse statement: 'If $k+1$ is not divisible by 3, then k^3+1 is not divisible by 3, $k \in Z^+$.' Let the integer $k + 1 = 3j + 1$, $j \in Z$. $k = 3j$, $k^3+1=(3j)^3+1$ $=27j^3+1$ $=3(9j^3)+1$ i.e. not divisible by 3 Let the integer $k + 1 = 3j + 2$, $j \in Z$. $k=3j+1$, $k^3+1=(3j+1)^3+1$ $=27j^3+3(9j^2)+3(3j)+1+1$ $=27j^3+27j^2+9j+2$ $=3(9j^3+9j^2+3j)+2$ i.e. not divisible by 3 $\therefore$ true for contrapositive of converse statement Hence true for converse statement. Hence if k^3+1 is divisible by 3, then $k+1$ is divisible by 3, $k \in Z^+$.' *OR* Using proof by contradiction, assume k^3+1 is divisible by 3, but $k+1$ is not divisible by 3. Let $k^3+1=3p$, $p \in Z$ $3p=(k+1)(k^2-k+1)$ and since $k+1$ is not divisible by 3, hence $k^2-k+1=3q$, $q \in Z$ $3q=(k+1)^2-3k$ This means $(k+1)^2$ is divisible by 3, and hence $k+1$ is divisible by 3. This is a contradiction.	Poorly done. Students who used the contrapositive of converse, obtained answer with ease.

14.	(a) Step 1: Show for $n = 1$ $a_1 = 2^{1-2^{1-1}}$ $= 2^{1-1}$ $= 2^0$ $= 1$ which agrees with recursive definition. ① $\therefore$ true for $n = 1$. Step 2: Assume true for $n = k$ for $k \geq 1$ $k = 1,\quad a_1 = 1$ $k = 2,\quad a_2 = 2^{1-2^{(1-2)}}$ $k = 3,\quad a_3 = 2^{1-2^{(1-3)}}$ $\vdots \qquad \vdots$ $k = k,\quad a_k = 2^{1-2^{(1-k)}}$ Step 3: RTP for $n = k + 1$, $a_{k+1} = 2^{1-2^{-k}}$ ① LHS $= a_{k+1}$ $= \sqrt{2a_k}$ $= \sqrt{2 \times 2^{1-2^{(1-k)}}}$ by assumption $= \sqrt{2 \times 2^{\left(1-2^{(1-k)}\right)}}$ $= \sqrt{2^{1+\left(1-2^{(1-k)}\right)}}$ $= \sqrt{2^{\left(2-2^{(1-k)}\right)}}$ $= \sqrt{2^{2\left(1-2^{-k}\right)}}$ $= 2^{\left(1-2^{-k}\right)}$ ① $\therefore$ LHS = RHS $\therefore$ true for $n = k + 1$ if true for $n = k$. Step 4: Since true for $n = 1$, then it is true for $n = 1 + 1 = 2$, $n = 2 + 1 = 3$, and so on. Hence it is true for all positive integers of n.	For the most well done, except there are quite a few who are not writing out the sequence in step 2 to show the recurrence relation. Also a few students also did n=2 in step one which is not needed in a first degree recurrence induction.
14.	(b) $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx$ $= \int \left(1 - \sin^2 x\right)^2 \cos x\, dx$ ① $= \int \left(1 - 2\sin^2 x + \sin^4 x\right) \cos x\, dx$ $= \int \left(\cos x - 2\sin^2 x \cos x + \sin^4 x \cos x\right) dx$ ① $= \sin x - \dfrac{2\sin^3 x}{3} + \dfrac{\sin^5 x}{5} + C$ ①	Generally well done if it was correctly separated in the beginning

<table>
<tr>
<td>14.</td>
<td>(c) $x=4t$
$y=-2t$
$z=-5t$
$$x^2+y^2+z^2=90$$
$$(4t)^2+(-2t)^2+(-5t)^2=90 \quad ①$$
$$16t^2+4t^2+25t^2=90$$
$$45t^2=90$$
$$t^2=2$$
$$\therefore t=\pm\sqrt{2} \quad ①$$
$\therefore$ The points are $\left(-4\sqrt{2},\ 2\sqrt{2},\ 5\sqrt{2}\right)$ and
$\left(4\sqrt{2},\ -2\sqrt{2},\ -5\sqrt{2}\right)$ ①</td>
<td>Well done</td>
</tr>
<tr>
<td>14.</td>
<td>(d)(i) $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx$ and $I_{n-1}=\int_0^{\frac{\pi}{4}}\tan^{n-1} x\,dx$
$$=\int_0^{\frac{\pi}{4}}\tan^{n-2}x.\tan^2 x\,dx$$
$$=\int_0^{\frac{\pi}{4}}\tan^{n-2}x.\left(\sec^2 x-1\right)dx \quad ①$$
$$=\int_0^{\frac{\pi}{4}}\sec^2 x\tan^{n-2}x\,dx-\int_0^{\frac{\pi}{4}}\tan^{n-2}x\,dx$$
$$=\left[\frac{\tan^{n-1}x}{n-1}\right]_0^{\frac{\pi}{4}}-I_{n-2} \quad ①$$
$$\therefore I_n+I_{n-2}=\frac{1}{n-1}\left[\tan^{n-1}x\right]_0^{\frac{\pi}{4}}$$
$$\therefore I_n+I_{n-2}=\frac{1}{n-1}\left[\left(\tan\frac{\pi}{4}\right)^{n-1}-0\right]_0^{\frac{\pi}{4}} \quad ①$$
$$\therefore I_n+I_{n-2}=\frac{1}{n-1} \quad \text{(as required)}$$</td>
<td>This was well done if the correct separation was made in the integrand. A few students did not do this correctly. They need to learn the correct method.</td>
</tr>
</table>

14.	(d)(ii) $I_5 + I_3 = \frac{1}{5-1} = \frac{1}{4}$ $I_3 + I_1 = \frac{1}{3-1} = \frac{1}{2}$ ① $I_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$ $= \left[-\ln\lvert\cos x\rvert\right]_0^{\frac{\pi}{4}}$ $= \left[-\ln\frac{1}{\sqrt{2}} + 0\right]$ $= \frac{1}{2}\ln 2$ $I_3 = \frac{1}{2} - I_1$ $\therefore I_3 = \frac{1}{2} - \frac{1}{2}\ln 2$ ① $I_5 = \frac{1}{4} - I_3$ $= \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\ln 2\right)$ $= \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\ln 2$ $\therefore I_5 = \frac{1}{2}\ln 2 - \frac{1}{4}$ or equivalent. ①	For the most this was well done if students did not try to do it all in one go. There were a few sign mistakes at times in the calculation.

15.	(a) $\displaystyle\int \frac{x}{(2x+1)(x-3)^2}\,dx$ $\displaystyle=\int \frac{A}{2x+1}+\frac{B}{(x-3)^2}+\frac{C}{x-3}\,dx$ where $\displaystyle\frac{x}{(2x+1)(x-3)^2}\equiv\frac{A}{2x+1}+\frac{B}{(x-3)^2}+\frac{C}{x-3}$ $x=A(x-3)^2+B(2x+1)+C(x-3)(2x+1)$ When $x=3,\ 3=7B$ $B=\frac{3}{7}$ When $x=-\frac{1}{2},\ -\frac{1}{2}=A\left(-\frac{7}{2}\right)^2$ $-\frac{1}{2}=A\frac{49}{4}$ $-\frac{2}{49}=A$ When $x=0,\ 0=-\frac{2}{49}(-3)^2+\frac{3}{7}-3C$ $0=-\frac{18}{49}+\frac{3}{7}-3C$ $3C=\frac{3}{49}$ $C=\frac{1}{49}$ $\displaystyle=\frac{1}{49}\int -\frac{2}{2x+1}+\frac{21}{(x-3)^2}+\frac{1}{x-3}\,dx$ $\displaystyle=\frac{1}{49}\int -\frac{2}{2x+1}+21(x-3)^{-2}+\frac{1}{x-3}\,dx$ $\displaystyle=\frac{1}{49}\left[-\ln\|2x+1\|-\frac{21}{x-3}+\ln\|x-3\|\right]+C$ $\displaystyle=\frac{1}{49}\left[\ln\left\|\frac{x-3}{2x+1}\right\|-\frac{21}{x-3}\right]+C$ $\displaystyle=\frac{1}{49}\ln\left\|\frac{x-3}{2x+1}\right\|-\frac{3}{7(x-3)}+C$	Surprisingly poorly done. Students who used $\displaystyle\frac{x}{(2x+1)(x-3)^2}\equiv\frac{A}{2x+1}+\frac{B}{(x-3)^2}+\frac{C}{x-3}$ obtained the fractions with eased and integrated with ease. Students who used $\displaystyle\frac{x}{(2x+1)(x-3)^2}\equiv\frac{A}{2x+1}+\frac{Bx+C}{(x-3)^2}$ Had more problems in obtaining the fractions with arithmetic errors and integration was not as straight forward.
15.	(b)(i) $\displaystyle\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$ $\displaystyle\therefore \cot 2\theta=\frac{1-\tan^2\theta}{2\tan\theta}$ ①	Mostly done well

15.

(b)(ii) Method 1: Using $\cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta}$

$$\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$$

$$2\tan\theta\cot 2\theta = 1-\tan^2\theta \quad ①$$

$$\tan^2\theta + 2\tan\theta\cot 2\theta - 1 = 0$$

Let $x = \tan\theta$, $\therefore x^2 + 2x\cot 2\theta - 1 = 0$

Hence $x = \tan\theta$ is a solution of this quadratic equation.

Product the two roots is -1, i.e. $x\tan\theta = -1$

$$x = -\frac{1}{\tan\theta}$$

$$\therefore x = -\cot\theta \quad ①$$

Method 2: Using α and β roots.

Let the roots of $x^2 + 2x\cot 2\theta - 1 = 0$ be α and β

$\sum \alpha: \quad \therefore \alpha + \beta = -\dfrac{b}{a}$

$$\alpha + \beta = -2\cot 2\theta$$
$$= \frac{\tan^2\theta - 1}{\tan\theta}$$
$$= \tan\theta - \cot\theta$$
$$= \tan\theta + (-\cot\theta)$$

$\sum \alpha\beta: \quad \therefore \alpha\beta = \dfrac{c}{a}$

$$\alpha\beta = -1$$

$$\therefore \beta = -\frac{1}{\alpha}$$

Hence the two roots are $\tan\theta$ and $-\cot\theta$.

Method 3: Using Substitutions

When $x = \tan\theta$, LHS $= x^2 + 2x\cot 2\theta - 1$

$$= \tan^2\theta + 2\tan\theta\cot 2\theta - 1$$
$$= \tan^2\theta + 2\tan\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$$
$$= \tan^2\theta + (1-\tan^2\theta) - 1$$
$$= 0$$

$\therefore$ LHS = RHS

$\therefore x = \tan\theta$ satisfies the equation.

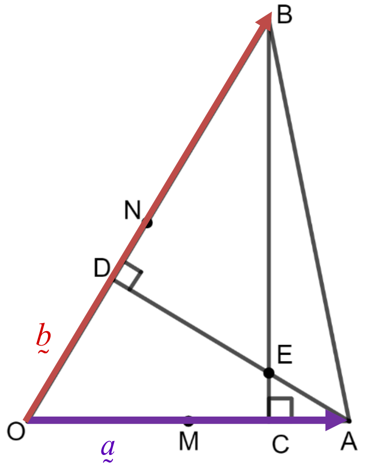
When $x = -\cot\theta$,

LHS $= x^2 + 2x\cot 2\theta - 1$

$$= (-\cot\theta)^2 + 2(-\cot\theta)\cot 2\theta - 1$$
$$= \cot^2\theta + 2(-\cot\theta)\frac{1}{2}\cot\theta(1-\tan^2\theta) - 1$$
$$= \cot^2\theta - \cot^2\theta(1-\tan^2\theta) - 1$$

	$=\cot^2\theta-\cot^2\theta+\cot^2\theta\tan^2\theta-1$ $=\cot^2\theta-\cot^2\theta+1-1$ $=0$ $\therefore$ LHS = RHS $\therefore x=-\cot\theta$ satisfies the equation.	
15.	(b)(iii) $x^2+2x\cot 2\theta-1=0$ $x^2+2x\cot\frac{\pi}{4}-1=0$ i.e. $\theta=\frac{\pi}{8}$ ① Hence $\tan\frac{\pi}{8}$ and $-\cot\frac{\pi}{8}$ are the solutions of $x^2+2x\cot\frac{\pi}{4}-1=0$ $x^2+2x-1=0$ $x^2+2x=1$ $x^2+2x+1=1+1$ $(x+1)^2=2$ $x+1=\pm\sqrt{2}$ $\therefore x=-\sqrt{2}-1<0$ or $x=\sqrt{2}-1>0$ $\therefore \tan\frac{\pi}{8}=\sqrt{2}-1$ ①	Poorly done.
15.	(b)(iv) $\tan\frac{\pi}{16}$ and $-\cot\frac{\pi}{16}$ are the roots of $x^2+2x\cot\frac{\pi}{8}-1=0$ $\sum\alpha$: $\therefore \tan\frac{\pi}{16}+\left(-\cot\frac{\pi}{16}\right)=-\frac{b}{a}$ $\therefore \tan\frac{\pi}{16}-\cot\frac{\pi}{16}=-2\cot\frac{\pi}{8}$ ① $=2\left(-\cot\frac{\pi}{8}\right)$ $=2\left(-\sqrt{2}-1\right)$ [from (iii)] $=-2\sqrt{2}-2$ ①	Poorly done Students did not use part (ii) finding to solve this problem using sum of two roots.

<table>
<tr>
<td>15.</td>
<td>
(c)(i)



$\overrightarrow{OM} = \frac{1}{2}\underset{\sim}{a}$ and $\overrightarrow{OC} = \lambda \underset{\sim}{a}$ for some scalar λ.

$\therefore \left|\overrightarrow{OM}\right|\left|\overrightarrow{OC}\right| = \frac{1}{2}\lambda \underset{\sim}{a} \bullet \underset{\sim}{a}$ ①

Similarly, $\overrightarrow{ON} = \frac{1}{2}\underset{\sim}{b}$ and $\overrightarrow{OD} = \mu \underset{\sim}{b}$ for some scalar μ.

$\therefore \left|\overrightarrow{ON}\right|\left|\overrightarrow{OD}\right| = \frac{1}{2}\mu \underset{\sim}{b} \bullet \underset{\sim}{b}$

Given $\overrightarrow{AD} \perp \overrightarrow{OB}$, $\therefore \left(\mu \underset{\sim}{b} - \underset{\sim}{a}\right) \bullet \underset{\sim}{b} = 0$ ①

$\mu \underset{\sim}{b} \bullet \underset{\sim}{b} - \underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

$\mu \underset{\sim}{b} \bullet \underset{\sim}{b} = \underset{\sim}{a} \bullet \underset{\sim}{b}$

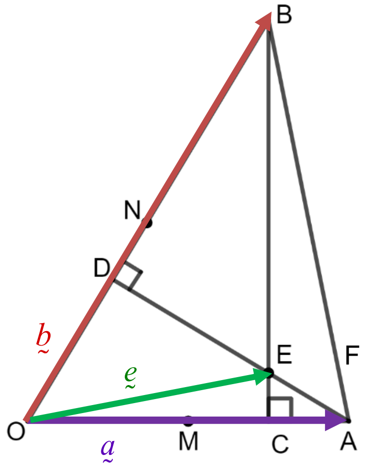
Similarly, $\overrightarrow{BC} \perp \overrightarrow{OA}$, $\therefore \lambda \underset{\sim}{a} \bullet \underset{\sim}{a} = \underset{\sim}{b} \bullet \underset{\sim}{a}$

Since $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{a}$, $\therefore \lambda \underset{\sim}{a} \bullet \underset{\sim}{a} = \mu \underset{\sim}{b} \bullet \underset{\sim}{b}$

$\therefore \frac{1}{2}\lambda \underset{\sim}{a} \bullet \underset{\sim}{a} = \frac{1}{2}\mu \underset{\sim}{b} \bullet \underset{\sim}{b}$ ①

$\therefore \left|\overrightarrow{OM}\right|\left|\overrightarrow{OC}\right| = \left|\overrightarrow{ON}\right|\left|\overrightarrow{OD}\right|$

(as required)
</td>
<td>Very poorly done.</td>
</tr>
</table>

15.	(c)(ii)  Let $\overrightarrow{OE}=\underset{\sim}{e}$ From part (i), $\overrightarrow{AD}\perp\overrightarrow{OB}$ i.e. $\overrightarrow{AE}\perp\overrightarrow{OB}$ $\therefore(\underset{\sim}{e}-\underset{\sim}{a})\bullet\underset{\sim}{b}=0$ ① $\underset{\sim}{e}\bullet\underset{\sim}{b}-\underset{\sim}{a}\bullet\underset{\sim}{b}=0$ $\therefore\underset{\sim}{e}\bullet\underset{\sim}{b}=\underset{\sim}{a}\bullet\underset{\sim}{b}$ — ① Similarly, $\overrightarrow{BC}\perp\overrightarrow{OA}$ i.e. $\overrightarrow{BE}\perp\overrightarrow{OA}$ $\therefore(\underset{\sim}{e}-\underset{\sim}{b})\bullet\underset{\sim}{a}=0$ $\underset{\sim}{e}\bullet\underset{\sim}{a}-\underset{\sim}{b}\bullet\underset{\sim}{a}=0$ $\therefore\underset{\sim}{e}\bullet\underset{\sim}{a}=\underset{\sim}{b}\bullet\underset{\sim}{a}$ — ② Since $\underset{\sim}{a}\bullet\underset{\sim}{b}=\underset{\sim}{b}\bullet\underset{\sim}{a}$, $\therefore\underset{\sim}{e}\bullet\underset{\sim}{a}=\underset{\sim}{e}\bullet\underset{\sim}{b}$ $\underset{\sim}{e}\bullet\underset{\sim}{a}-\underset{\sim}{e}\bullet\underset{\sim}{b}=0$ $\underset{\sim}{e}\bullet(\underset{\sim}{a}-\underset{\sim}{b})=0$ ① $\therefore\overrightarrow{OE}\perp\overrightarrow{BA}$ Since $\overrightarrow{OF}=k\overrightarrow{OE}$, $\therefore\overrightarrow{OF}\perp\overrightarrow{BA}$ (as required).	Very poorly done.

16.

(a) RTP: $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}>\frac{2n}{n+1}$ for $n\geq 2,\quad n\in Z^{+}$

Step 1: Prove for $n=2$,

$$\text{LHS }=1+\frac{1}{2}=\frac{3}{2}$$

$$\begin{aligned}\text{RHS}&=\frac{2\times 2}{2+1}\\&=\frac{4}{3}\end{aligned}$$

LHS > RHS for $n=2$

Hence true for $n=2$.

Step 2: Assume true for $n=k,\ k\in Z^{+},\quad k\geq 2$ ①

i.e. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}>\frac{2k}{k+1}$

Step 3: Prove for $n=k+1$,

i.e. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\frac{1}{k+1}>\frac{2(k+1)}{(k+1)+1}=\frac{2k+2}{k+2}$

Proof: $\text{LHS}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\frac{1}{k+1}$

$$\begin{aligned}&>\frac{2k}{k+1}+\frac{1}{k+1}\quad\text{(from assumption)}\quad ①\\
&=\frac{2k+1}{k+1}\\
&=\frac{2k+1}{k+1}\times\frac{k+2}{k+2}\\
&=\frac{(2k+1)(k+2)}{(k+1)(k+2)}\\
&=\frac{2k^2+5k+2}{(k+1)(k+2)}\\
&=\frac{2k^2+5k-k+2+k}{(k+1)(k+2)}\\
&=\frac{2k^2+4k+2+k}{(k+1)(k+2)}\\
&=\frac{2(k^2+2k+1)+k}{(k+1)(k+2)}\\
&=\frac{2(k+1)^2+k}{(k+1)(k+2)}\\
&=\frac{2(k+1)^2}{(k+1)(k+2)}+\frac{k}{(k+1)(k+2)}\quad ①\end{aligned}$$

Average mark for this was 2 out of 3, due to students not dealing with the k+1 becoming k+2 in the denominator.
There were quite a few fudges of this , but some dealt with it beautifully in other ways than what this solution shows.

	$= \dfrac{2(k+1)}{(k+2)} + \dfrac{k}{(k+1)(k+2)}$ $= \dfrac{2k+2}{k+2} + \dfrac{k}{(k+1)(k+2)}$ $> \dfrac{2k+2}{k+2}$ as $\dfrac{k}{(k+1)(k+2)} > 0$ for $k \ge 2$ $\therefore$ true for $n = k + 1$ if true for $n = k$. Step 4: Since true for $n = 2$, then it is true for $n = 2 + 1 = 3$, $n = 3 + 1 = 4$, and so on. Hence it is true for all positive integers of n.	
16.	(b)(i) RTP: $a^3 + b^3 > ab(a+b)$ Method 1: Using $(a-b)^2 > 0$ $(a-b)^2 > 0$ since $a \neq b$ $a^2 - 2ab + b^2 > 0$ $a^2 - ab + b^2 > ab$ — ① ① Now $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ $> (a+b)ab$ using ① (as required) ① Method 2: Factorising $a^3 + b^3$ $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ $= (a+b)(a^2 - ab - ab + b^2 + ab)$ ① $= (a+b)(a^2 - 2ab + b^2 + ab)$ $= (a+b)((a-b)^2 + ab)$ ① $> (a+b)ab$ since $(a-b)^2 > 0$, $a \neq b$ (as required)	This part was reasonably well done, but would like to see better responses if a question similar to this was presented again to students. Please have a careful look at these two methods and compare yours to them if yours needs improving.
16.	(b)(ii) Similarly from ① $a^3 + c^3 > ac(a+c)$ $b^3 + c^3 > bc(b+c)$ with $a^3 + b^3 > ab(a+b)$ Adding inequalities, $2(a^3 + b^3 + c^3) > ab(a+b) + ac(a+c) + bc(b+c)$ ① $= a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2$ $= a^2(b+c) + b^2(a+c) + c^2(a+b)$ $\therefore 2(a^3 + b^3 + c^3) > a^2(b+c) + b^2(a+c) + c^2(a+b)$ ① (as required)	This part was very well done

16.	(c)(i) $F(p)=\lim\limits_{t\to\infty}\int_0^t x^{p-1}e^{-x}\,dx$ $F(1)=\lim\limits_{t\to\infty}\int_0^t x^{1-1}e^{-x}\,dx$ ① $=\lim\limits_{t\to\infty}\int_0^t e^{-x}\,dx$ $=\lim\limits_{t\to\infty}\left[-e^{-x}\right]_0^t$ ① $=\lim\limits_{t\to\infty}\left[-e^{-t}-\left(-e^{0}\right)\right]$ $=\lim\limits_{t\to\infty}\left[-\frac{1}{e^t}+1\right]$ As $t\to\infty$, $-\frac{1}{e^t}\to 0$ ① $=0+1$ $=1$	For the most well done, however the question says to "show." Full reasons and steps need to be shown. A lot missed this explanation step
16.	(c)(ii) $F(p+1)=\lim\limits_{t\to\infty}\int_0^t x^{(p+1)-1}e^{-x}\,dx$ $=\lim\limits_{t\to\infty}\int_0^t x^{p}e^{-x}\,dx$ $=\lim\limits_{t\to\infty}\left\{uv-\int_0^t u'v\,dx\right\}$ where $u=x^p$ and $v=e^{-x}$ $u'=px^{p-1}$ $v'=-e^{-x}$ $=\lim\limits_{t\to\infty}\left\{\left[-x^pe^{-x}\right]_0^t+p\int_0^t x^{p-1}e^{-x}\,dx\right\}$ ① $=\lim\limits_{t\to\infty}\left\{\left[-t^pe^{-t}+0^pe^0\right]+p\int_0^t x^{p-1}e^{-x}\,dx\right\}$ $=\lim\limits_{t\to\infty}\left(-t^p\times\frac{1}{e^t}\right)+\lim\limits_{t\to\infty}p\int_0^t x^{p-1}e^{-x}\,dx$ ① As $t\to\infty$, $-\frac{1}{e^t}\to 0$ $=0+pF(p)$ ① $=pF(p)$ (as required)	This question said "show" and use "integration by parts" In integration you must show the substitution of boundaries. Again, a lot missed this explanation step, but was not penalised if you were already in part (i)

16.	(c)(iii) $F(p+1)=pF(p)$ $\therefore F(p)=(p-1)F(p-1)$ $\therefore F(n)=(n-1)F(n-1)$ ① $=(n-1)\left[(n-2)F(n-2)\right]$ $=(n-1)(n-2)F(n-2)$ $=(n-1)(n-2)(n-3)F(n-3)$ and so on ① $=(n-1)(n-2)(n-3)\ldots 1\times F(1)$ $=(n-1)(n-2)(n-3)\ldots 1\times 1$ from (i)_ $=(n-1)!$	Reasonably well done if students recognized the pattern properly. Quite a few started with n instead of (n-1)