International Grammar School

**2008**
**TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

# Mathematics
## Extension 2

**General Instructions**

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.

Total marks (**120**)

- Attempt Questions 1-8
- All questions are of equal value

**Total Marks – 120**
**Attempt Questions 1-8**
**All Questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

**Question 1 (15 marks)** Use a SEPARATE answer booklet **Marks**

a) Find $\displaystyle\int \frac{dx}{\sqrt{16-9x^2}}$ **2**

b) Find $\displaystyle\int 5\cos x\ \sin^2 x\ dx$ **2**

c) Evaluate $\displaystyle\int_1^e x \ln x\ dx$ **3**

d) Evaluate $\displaystyle\int_2^3 \frac{dx}{x^2-1}$ **4**

e) Using the substitution $t = tan\dfrac{\theta}{2}$ or otherwise find **4**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$$

**End of Question 1**

**Question 2 (15 marks)** Use a SEPARATE writing booklet. **Marks**

a) Let $A = 3 + 4i$ and $B = 2 - 2i$. Find in the form $x + iy$ ($x$ and $y$ real).

i) $\dfrac{A}{B}$ **2**

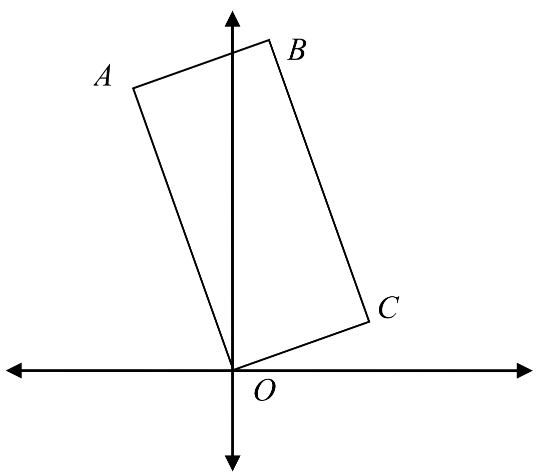
ii) $\sqrt{A}$ **3**

iii) $A - \overline{B}$ **1**

b) i) Write $1 + \sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$ **2**

ii) Hence write $\left(1 + \sqrt{3}i\right)^6$ showing that it is real. **2**

c)




The points $OABC$ are the vertices of a rectangle on the Argand diagram with $|OA| = 2|OC|$. If $OC$ represents the complex number $p + iq$, write down the complex numbers represented by:

i) $OA$ **1**

ii) $OB$ **1**

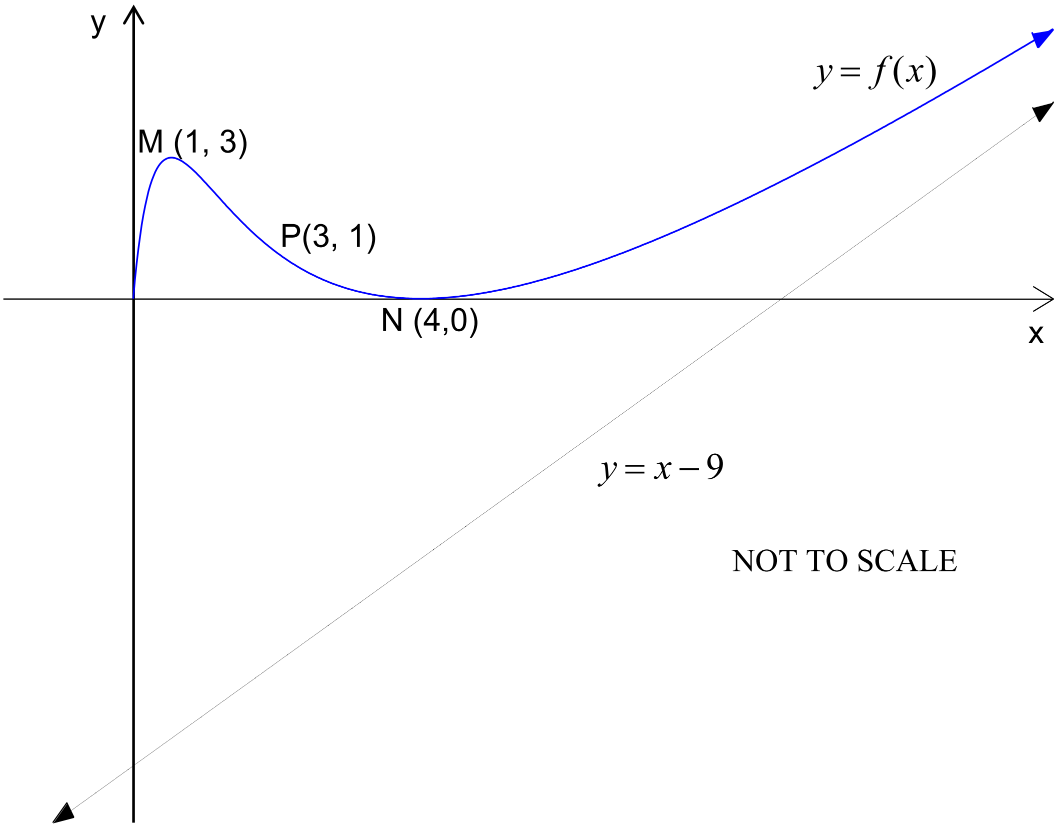
iii) $BC$ **1**

iv) $AC$ **2**

**End of Question 2**

**Question 3 (15 marks)** Use a SEPARATE writing booklet. **Marks**

a)




The diagram shows the graph of $y = f(x)$ for $x \geq 0$.
M(1, 3) and N(4, 0) are stationary points of $y = f(x)$ and P(3, 1) is a point of inflexion of $y = f(x)$. The line $y = x - 9$ is an asymptote as $x \to \infty$. Draw separate one third page sketches showing any special features for the following:

i) $f'(x)$ **2**

ii) $\dfrac{1}{f(x)}$ **2**

iii) $-(f(x))^2$ **2**

**Question 3 continues on the next page**

**Question 3 continued** **Marks**

b) Determine the gradient of the tangent to the curve $x^2 + 2xy - y^2 = 17$ at the point $(3,\ 2)$ **2**

c) The zeros of $x^3 - 3x^2 - 2x + 4$ are $\alpha$, $\beta$ and $\gamma$

i) Find a cubic polynomial whose zeros are $\alpha^2, \beta^2$ and $\gamma^2$ **2**

ii) Hence or otherwise find the value of $\alpha^2 + \beta^2 + \gamma^2$ **1**

iii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$ **2**

d) The equation $P(x) = x^3 + 3x^2 - 24x + k = 0$ has a double root. Find the possible values of $k$. **2**

**End of Question 3**

**Question 4 (15 marks)** Use a SEPARATE writing booklet. **Marks**

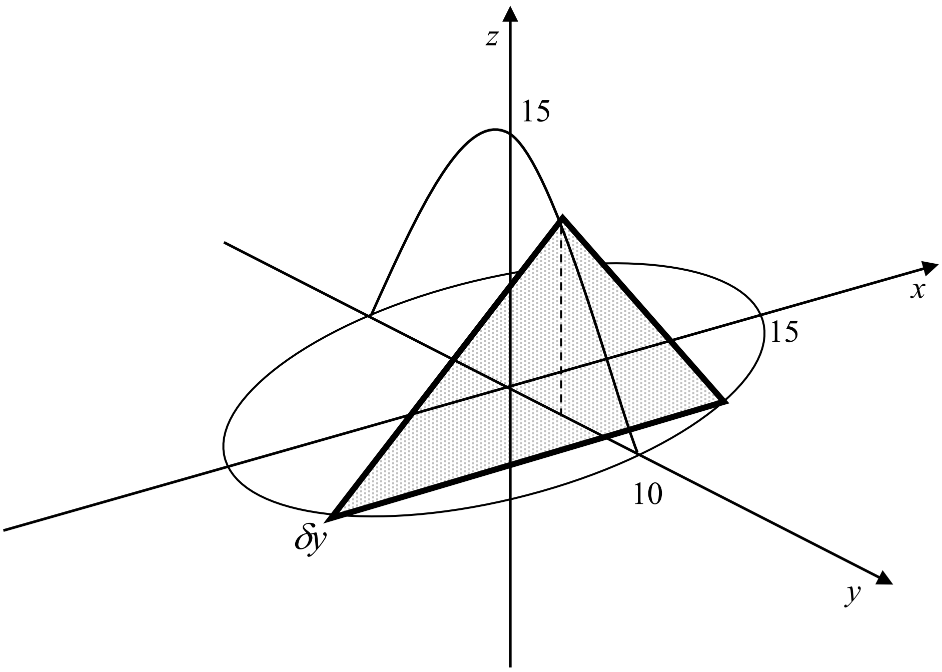
a) i) Show that a reduction formula for $I_n = \int (ln\,x)^n dx$ **3**

is $I_n = x(ln\,x)^n - nI_{n-1}$

ii) Hence evaluate $\int_1^{e^4} (ln\,x)^3\,dx$ **4**

b) The arc of the curve $y = 6x - x^2 - 8$ where $y \geq 0$ is rotated about the line $x = 1$. By applying the technique of cylindrical shells determine the exact volume of the solid formed **4**

c)




A cake is made with base in the shape of an ellipse, with semi-major axis 15 cm and semi-minor axis 10 cm. Slices of the cake parallel to the major axis of the base are isosceles triangles, whose vertices trace out a semi-elliptical path with the same semi-major axis and semi-minor axis lengths as in the diagram below.

i) Show that the volume of a 'typical' triangular slice is given by:

$$V_{slice} \approx xz\delta y \ \ cm^3$$ **1**

ii) Find the exact volume of the cake in $cm^3$. **3**

**End of Question 4**

**Question 5 (15 marks)** Use a SEPARATE writing booklet. **Marks**

a) The line $y = mx + a$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at two points which have x coordinates $x_1$ and $x_2$. **3**

i) Express $x_2$ in terms of *m, a, b* and $x_1$. **1**

ii) Hence or otherwise show that the line is a tangent to the ellipse at the point where $\frac{-a^3 m}{b^2 + a^2 m^2}$.

b) A parabola has parametric equations $x = 2at$ and $y = at^2$.

i) Find the equation of the normal to the parabola at the point where $t = p$. **2**

ii) Hence show that, through the point $(x_1, y_1)$, it is possible to draw up to three normals to the parabola. **2**

c) Given the complex number $z = cos\,\theta + i\,sin\,\theta$

i) Use De Moivres Theorem and the binomial expansion find an expression for $cos\,4\theta$ in terms of $cos\,\theta$ **3**

ii) Also, using $z^n + \frac{1}{z^n} = 2\,cos\,n\theta$ determine an expansion for $cos^4\,\theta$ in terms of $cos\,n\theta$ **2**

iii) Hence evaluate $\int_0^{\frac{\pi}{2}} cos^4\theta\, d\theta$ **2**

**End of Question 5**

**Question 6 (15 marks)** Use a SEPARATE writing booklet. **Marks**

a) On a suitably labelled Argand diagram sketch the region determined by
$[\text{Re}(z)]^2 + \text{Im}z < 0$ **2**

(b) Consider the function

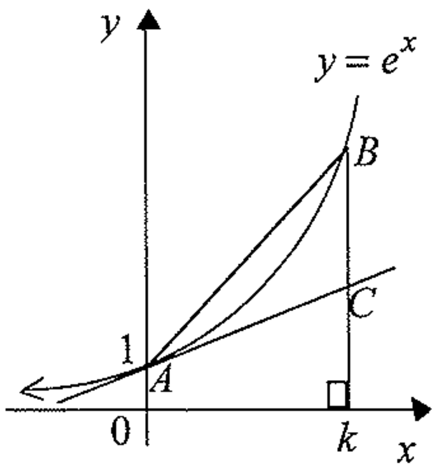
$$f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0 \\ 1 \quad , & x = 0 \end{cases}$$

(i) Use differentiation to show that $e^{-x} + x - 1 \geq 0$ for all values of $x$. Hence show that $f(x)$ is an increasing function for $x \neq 0$ **3**

(ii) Show that $f(x)$ is continuous at $x = 0$. **2**

(iii) Sketch the graph of $y = f(x)$. **1**

(c)




The curve $y = e^x$ cuts the y axis at A. B is a second point on the curve such that $x = k$, where $k > 0$. The tangent to the curve $y = e^x$ at A cuts the vertical line $x = k$ at the point C.

(i) By considering areas, show that $\frac{1}{2}k(k+2) < e^k - 1 < \frac{1}{2}k(1+e^k)$. Hence deduce that $2.5 < e < 3$. **4**

(ii) Show that the curve $y = e^x$ bisects the area of $\Delta ABC$ for some value of $k$ such that $2 < k < 3$. Taking $k = 2.7$ as a first approximation, apply Newton's method once to obtain a second approximation. Give your answer to one decimal place. **3**

**End of Question 6**

**Question 7 (15 marks)** Use a SEPARATE writing booklet. **Marks**

a) Given that $z^5 - 1 = 0$

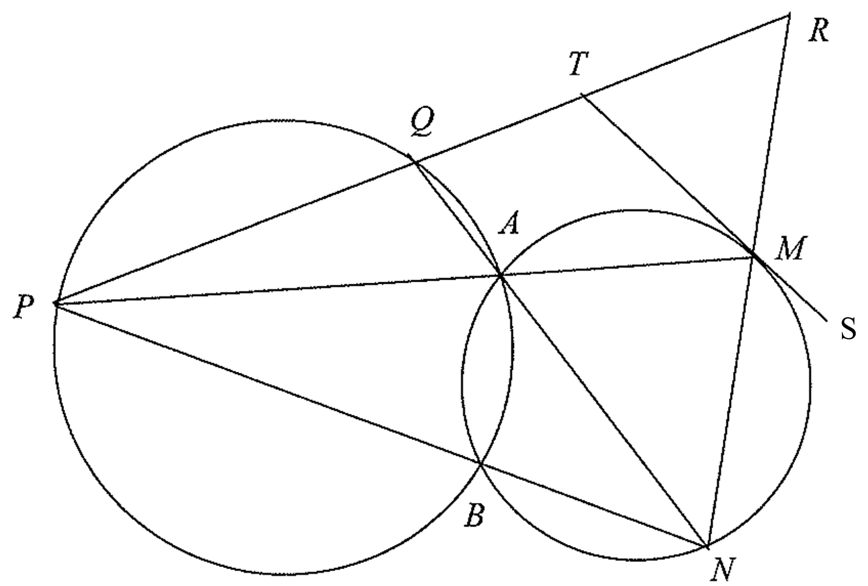
i) Solve for Z over the complex field in the form $\cos\theta + i\sin\theta$. **3**

i) Hence express $z^5 - 1$ as the product of linear and quadratic factors. **2**

iii) Write down the complex roots of $z^4 + z^3 + z^2 + z + 1 = 0$. **1**

iv) Without evaluating, show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ **2**

b)




In the diagram, the two circles intersect at A and B. P is a point on one circle. PA and PB produced meet the other circle at M and N respectively. NA produced meets the first circle at Q. PQ and NM produced meet at R. The tangent at M to the second circle meets PR at T.
Copy or trace the diagram into your answer booklet.

(i) Show that QAMR is a cyclic quadrilateral. **3**

(ii) Show that TM=TR. **4**

**End of Question 7**

**Question 8 (15 marks)** Use a SEPARATE writing booklet **Marks**

a) i) Show that for all values of $x$ and $y$: **1**

$$\sin(x+y)-\sin(x-y)=2\cos x\sin y$$

ii) Use mathematical induction to show that for all positive integers $n$: **4**

$$\cos x+\cos 2x+\cos 3x+\cdots \cos nx=\frac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

iii) Hence show that: **4**

$$\cos 2x+\cos 4x+\cos 6x+\cdots +\cos 16x = 8.\cos 9x.\cos 4x.\cos 2x.\cos x$$

b) Show that a relationship between the coefficients of $p(x)=x^3+ax^2+bx+c=0$ is $2a^3-9ab-27c=0$, if the roots are three consecutive terms of an arithmetic series. **4**

c) Solve the differential equation $\frac{dy}{dx}=2y$ for $y$ given that when $x = 1$, $y = 1$ **2**

**End of Examination**

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

CORRECTED SOLNS. IGS TRIAL 2008

**Question 1** **Trial HSC Examination- Mathematics 2008**
**Extension 2**

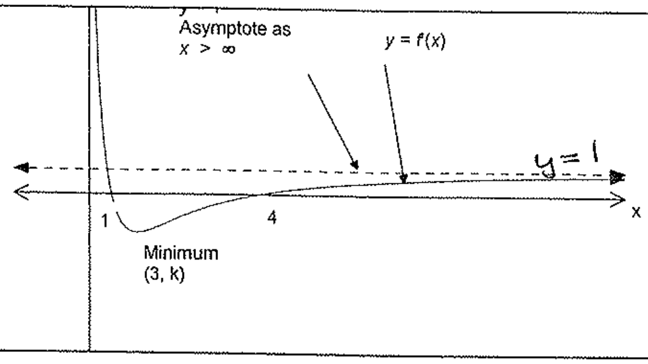
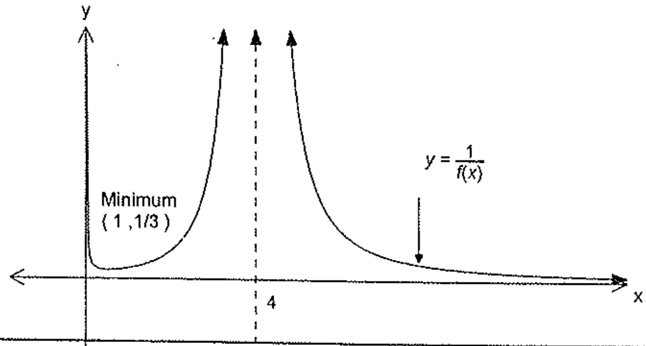
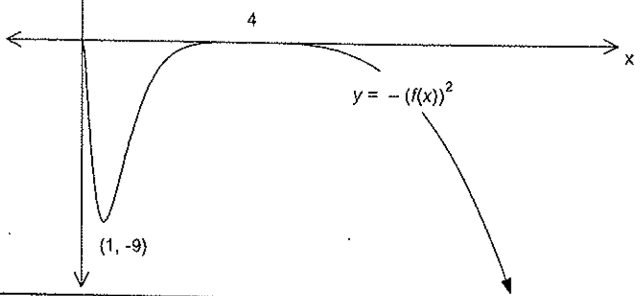
| Part | Solution | Marks | Comment |
|---|---|---|---|
| (a) | $\displaystyle\int\frac{dx}{\sqrt{16-9x^2}}=\frac{1}{3}\int\frac{dx}{\sqrt{\frac{16}{9}-x^2}}$ | 2 | 1 for rearranging |
| | $\displaystyle=\frac{1}{3}sin^{-1}\frac{x}{4/3}+c$ | | 1 for inv trig integral |
| | $\displaystyle=\frac{1}{3}sin^{-1}\frac{3x}{4}+c$ | | |
| (b) | $\displaystyle\int 5\cos x\ sin^2 x\ dx=\frac{5}{3}sin^3 x+c$ | 2 | 2 for solution<br>1 if simple error made |
| (c) | $\displaystyle\int_1^e x\ln x\,dx=\left[\frac{x^2}{2}ln\,x\right]_1^e-\int_1^e\frac{x^2}{2}\frac{1}{x}dx$ | 3 | 1 for breakup into parts |
| | $\displaystyle=\left[\frac{x^2}{2}ln\,x\right]_1^e-\int_1^e\frac{x}{2}dx$ | | |
| | $\displaystyle=\left[\frac{x^2}{2}ln\,x-\frac{x^2}{4}\right]_1^e$ | | 1 for integral |
| | $\displaystyle=\left[\frac{x^2}{4}(2\,ln\,x-1)\right]_1^e$ | | 1 for final answer |
| | $\displaystyle=\frac{e^2}{4}(2-1)-\frac{1}{4}(-1)$ | | |
| | $\displaystyle=\frac{e^2}{4}+\frac{1}{4}$ | | |
| (d) | Let $\displaystyle\frac{A}{x+1}+\frac{B}{x-1}=\frac{1}{x^2-1}$ | 4 | |
| | $A(x-1)+B(x+1)=1$ | | 1 value of B |
| | When $x=1$ $2B=1$ $\rightarrow B=\frac{1}{2}$ | | 1 value of A |
| | When $x=-1$ $-2A=1$ $\rightarrow A=-\frac{1}{2}$ | | |
| | $\displaystyle\int_2^3\frac{dx}{x^2-1}=\frac{1}{2}\int_2^3\left(\frac{-1}{x+1}+\frac{1}{x-1}\right)dx$ | | |
| | $\displaystyle=\frac{1}{2}\left[ln(x-1)-ln(x+1)\right]_2^3$ | | 1 integral |
| | $\displaystyle=\frac{1}{2}\left[ln\left(\frac{x-1}{x+1}\right)\right]_2^3$ | | |
| | $\displaystyle=\frac{1}{2}\left[ln\frac{1}{2}-ln\frac{1}{3}\right]_2^3$ | | 1 for answer |
| | $\displaystyle=\frac{1}{2}ln\frac{3}{2}$ | | |

**Question 1** **Trial HSC Examination- Mathematics Extension 2** **2008**

| Part | Solution | Marks | Comment |
|---|---|---|---|
| (e) | If $t=tan\frac{\theta}{2}$ | 4 | |
| | $\frac{dt}{d\theta}=\frac{1}{2}sec^2\frac{\theta}{2}$ | | |
| | $2cos^2\frac{\theta}{2}dt=d\theta$ | | |
| | $cos^2\frac{\theta}{2}=\frac{1}{1+t^2}$ | | 1 for d$\theta$ |
| | $d\theta=\frac{2}{1+t^2}dt$ | | |
| | Limits $\theta=\frac{\pi}{2}\longrightarrow t=1$ | | |
| | $\theta=0\longrightarrow t=0$ | | 1 for correct statement of integral including limits |
| | $\int_0^{\frac{\pi}{2}}\frac{d\theta}{2+\cos\theta}=\int_0^1\frac{1}{2+\frac{1-t^2}{1+t^2}}.\frac{2}{1+t^2}dt$ | | |
| | $=\int_0^1\frac{1+t^2}{\left(2+2t^2+1-t^2\right)}.\frac{2}{1+t^2}dt$ | | |
| | $=\int_0^1\frac{2}{\left(3+t^2\right)}dt$ | | 1 for completing integral |
| | $=2\left[\frac{1}{\sqrt{3}}tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$ | | |
| | $=\frac{2}{\sqrt{3}}\left(tan^{-1}\frac{1}{\sqrt{3}}-tan^{-1}0\right)$ | | |
| | $=\frac{2}{\sqrt{3}}.\frac{\pi}{6}$ | | 1 for result |
| | $=\frac{\pi}{3\sqrt{3}}$ | | |

| Question 2 | Trial HSC Examination- Mathematics Extension 2 | 2008 | |
|---|---|---|---|
| **Part** | **Solution** | **Marks** | **Comment** |
| (a) i) | $\frac{a}{b}=\frac{3+4i}{2-2i}\times\frac{2+2i}{2+2i}$<br>$=\frac{6+6i+8i-8}{4+4}$<br>$=\frac{-2+14i}{8}$<br>$=-\frac{1}{4}+\frac{7}{4}i$ | 2 | 1 for multiplying by conjugate.<br>1 for correct answer |
| ii) | Let $\sqrt{A}=x+iy$<br>$\therefore A=x^2-y^2+2xyi$<br>$\therefore 3+4i=x^2-y^2+2xyi$<br>$\therefore 3=x^2-y^2$ ................(1)<br>$\therefore 4=2xy$ ................(2)<br>$(1)^2+(2)^2$<br>$x^4+2x^2y^2+y^4=25$<br>$(x^2+y^2)^2=25$<br>$x^2+y^2=5$ ................(3)<br>(1) + (3) $2x^2=8$<br>$x=\pm 2$<br>$y=\pm 1$<br>$\sqrt{A}=\pm(2+i)$ | 3 | 1 for squaring and equating real and imaginary<br>1 for eliminating y (or x)<br>1 for final solution |
| iii) | $A-\overline{B}=3+4i-(2+2i)$<br>$=1+2i$ | 1 | 1 for answer |
| (b) i) | $1+\sqrt{3}i=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$<br>$\cos\theta=\frac{1}{2}$ and $\sin\theta=\frac{\sqrt{3}}{2}$<br>$\therefore \theta=\frac{\pi}{3}$<br>$\therefore 1+\sqrt{3}i=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ | 2 | 1 value of $\theta$<br>1 for result |
| ii) | $(1+\sqrt{3}i)^6=2^6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6$<br>$=64(\cos 2\pi+i\sin 2\pi)$ by De Moivres Theorem<br>$=64(1+0i)$<br>$=64$<br>Which is ~~totally~~ real. | 2 | 1 Use of De Moivre Thm<br>1 for answer |

| Question 2 | Trial HSC Examination- Mathematics Extension 2 | 2008 | |
|---|---|---|---|
| **Part** | **Solution** | **Marks** | **Comment** |
| (c) i) | $OA = 2iOC$<br>$= 2i(p+iq)$<br>$= -2q+2pi$ | 1 | 1 for answer |
| ii) | $OB = OC+OA$<br>$= (p+iq)+(-2q+2pi)$<br>$= (p-2q)+(2p+q)i$ | 1 | 1 for answer |
| iii) | $BC = -OA$<br>$= 2q-2pi$ | 1 | 1 for answer |
| iv) | $AC = AB+BC$<br>$= OC-OA$<br>$= (p+iq)-(-2q+2pi)$<br>$= (p+2q)+(q-2p)i$ | 2 | 1 for sum of vectors<br><br>1 for answer |

<table>
<tr><th>Question 3</th><th colspan="2">Trial HSC Examination- Mathematics Extension 2</th><th>2008</th></tr>
<tr><th>Part</th><th>Solution</th><th>Marks</th><th>Comment</th></tr>
<tr><td>(a)<br>i)</td><td>
</td><td>2</td><td>1 for basic shape<br><br>1 for asymptote</td></tr>
<tr><td>ii)</td><td>
</td><td>2</td><td>1 for basic shape<br><br>1 for discontinuity</td></tr>
<tr><td>iii)</td><td>
</td><td>2</td><td>1 for shape<br><br>1 for below axis</td></tr>
<tr><td>(b)</td><td>

$x^2+2xy-y^2=17$

$2x+2y+2x\dfrac{dy}{dx}-2y\dfrac{dy}{dx}=0$

$\dfrac{dy}{dx}(2x-2y)=-(2x+2y)$

$\dfrac{dy}{dx}=\dfrac{-(x+y)}{(x-y)}$

$=\dfrac{x+y}{y-x}$

At (3,2)

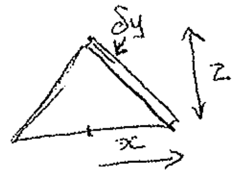
Gradient of tangent $=\dfrac{3+2}{2-3}=-5$

</td><td>2</td><td>1 for implicit differentiation<br><br>1 for derivative</td></tr>
</table>

| Question 3 | Trial HSC Examination- Mathematics Extension 2 | 2008 | |
|---|---|---|---|
| **Part** | **Solution** | **Marks** | **Comment** |
| (c) i) | For $x^3-3x^2-2x+4=0$<br>$x=\alpha,\ \beta$ and $\gamma$<br>Let $X=x^2$<br>$\sqrt{X}=x$<br>$X\sqrt{X}-3X-2\sqrt{X}+4=0$<br>$\sqrt{X}(X-2)=3X-4$<br>Squaring $X\left(X^2-4X+4\right)=9X^2-24X+16$<br>$X^3-4X^2+4X=9X^2-24X+16$<br>$\therefore$ Required polynomial is $x^3-13x^2+28x-16=0$ | 2 | Any method okay.<br><br>1 mark for partial solution or complete solution with simple error. |
| ii) | As above has roots $\alpha^2,\ \beta^2$ and $\gamma^2$<br>$\alpha^2+\beta^2+\gamma^2=\dfrac{-b}{a}=13$ | 1 | 1 for answer |
| iii) | As $\alpha,\ \beta$ and $\gamma$ are roots of $x^3-3x^2-2x+4=0$<br>Then $\alpha^3-3\alpha^2-2\alpha+4=0$<br>$\beta^3-3\beta^2-2\beta+4=0$<br>$\gamma^3-3\gamma^2-2\gamma+4=0$<br>Adding<br>$\left(\alpha^3+\beta^3+\gamma^3\right)-3\left(\alpha^2+\beta^2+\gamma^2\right)-2(\alpha+\beta+\gamma)+12=0$<br>$\left(\alpha^3+\beta^3+\gamma^3\right)-3(13)-2(3)+12=0$<br>$\left(\alpha^3+\beta^3+\gamma^3\right)=\ 33$ | 2 | Any method okay.<br><br>1 mark for partial solution or complete solution with simple error. |
| (d) | $P(x)=x^3+3x^2-24x+k$<br>$P'(x)=3x^2+6x-24$<br>$=3(x-2)(x+4)$<br>If $P'(x)=0 \quad x=2, \quad x=-4$<br>If $x=2$ is a double zero,<br>$P(2)=(2)^3+3(2)^2-24(2)+k=0$<br>$k=28$<br>$P(-4)=(-4)^3+3(-4)^2-24(-4)+k=0$<br>$k=-80$ | 2 | 1 possible zeros<br><br>1 values of k |

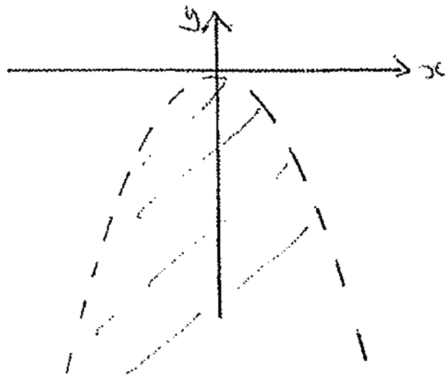
| Question 4 | Trial HSC Examination- Mathematics Extension 2 | 2008 |
|---|---|---|

| Part | Solution | Marks | Comment |
|---|---|---|---|
| (a) i) | $I_n = \int (\ln x)^n\,dx = x(\ln x)^n - \int x.n(\ln x)^{n-1}.\frac{1}{x}$<br>$= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$<br>$= x(\ln x)^n - nI_{n-1}$ | 3 | 1 for use of Int by parts<br>1 for simplifying<br>1 for result in terms of $I_n$ |
| ii) | Consider $\int (\ln x)^3\,dx = I_3$<br>$\therefore I_3 = x(\ln x)^3 - 3I_2$<br>Now $I_2 = x(\ln x)^2 - 2I_1$<br>and $I_1 = x(\ln x) - 1I_0$<br>$= x(\ln x) - x$<br>$\therefore I_3 = x(\ln x)^3 - 3\left(x(\ln x)^2 - 2\left(x(\ln x) - x\right)\right)$<br>$= x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6x$<br>$\therefore \int_1^{e^4} (\ln x)^3 dx = \left(e^4.64 - 3e^4.16 + 6e^4.4 - 6e^4\right) - (-6)$<br>$= 34e^4 + 6$ | 4 | 1 for $I_3$<br>1 for $I_2$<br>1 full expression including $I_1$<br>1 sub and evaluate |
| (b) | $y = 6x - x^2 - 8$<br>$y = 0 \quad x = 2, 4$<br>For shells about Y axis<br>$V = 2\pi \int_a^b xy\,dx$<br>About $x = 1$<br>$V = 2\pi \int_a^b (x-1)\,y\,dx$<br>$= 2\pi \int_2^4 (x-1)(6x - x^2 - 8)\,dx$<br>$= 2\pi \int_2^4 (7x^2 - x^3 - 14x + 8)\,dx$<br>$= 2\pi \left[\frac{7x^3}{3} - \frac{x^4}{4} - 7x^2 + 8x\right]_2^4$<br>$= 2\pi\left[\left(\frac{448}{3} - 64 - 112 + 32\right) - \left(\frac{56}{3} - 4 - 28 + 16\right)\right]$<br>$= \frac{16\pi}{3}$ units$^3$ | 4 | 4 marks for full solution<br>3 marks if simple error made<br>2 marks if major error or 2 simple errors<br>1 mark if start made using correct formula or from scratch with correct method. |

| Question 4 | Trial HSC Examination- Mathematics Extension 2 | 2008 | |
|---|---|---|---|
| **Part** | **Solution** | **Marks** | **Comment** |
| (c) | <br><br>$V_{slice} \approx Ah\delta y$<br>$\approx \frac{1}{2} 2x . z . \delta y$<br>$\approx xz\delta y$ | 4 | 1 dimensions of slice |
| | $V_{solid} = \int_{-10}^{10} xz\,dy$<br>$\frac{x^2}{15^2} + \frac{y^2}{10^2} = 1$<br>$\therefore x = \sqrt{15^2\left(1 - \frac{y^2}{10^2}\right)}$<br>and $\frac{z^2}{15^2} + \frac{y^2}{10^2} = 1$<br>$\therefore z = \sqrt{15^2\left(1 - \frac{y^2}{10^2}\right)}$<br>$\therefore V_{solid} = 450\int_0^{10}\left(1 - \frac{y^2}{10^2}\right)dy$<br>$= 450\left[y - \frac{y^3}{300}\right]_0^{10}$<br>$= 3000$ cms | | 1 $x = f(y)$ $z = f(y)$<br>1 correct integrand and limits<br>1 correct exact volume |

| Question 5 | Trial HSC Examination- Mathematics Extension 2 | 2008 | |
|---|---|---|---|
| **Part** | **Solution** | **Marks** | **Comment** |
| (a) i) | Solving simultaneously<br>$y = mx + a$ ............(1)<br>$b^2x^2 + a^2y^2 = a^2b^2$ .....(2)<br>sub (1) into (2)<br>$b^2x^2 + a^2(mx+a)^2 = a^2b^2$<br>$b^2x^2 + a^2m^2x^2 + 2a^3mx + a^4 = a^2b^2$<br>$(b^2 + a^2m^2)x^2 + 2a^3mx + a^4 - a^2b^2 = 0$<br>If $x = x_1$ and $x_2$ are the roots then<br>$x_1 + x_2 = \dfrac{-2a^3m}{b^2 + a^2m^2}$<br>$\therefore x_2 = \dfrac{-2a^3m}{b^2 + a^2m^2} - x_1$ | 3 | 1 for sub into equation<br><br>1 for simplify<br><br>1 for expression for $x_2$ |
| ii) | For a tangent $x_1 = x_2 = x$<br>$x_1 + x_2 = 2x = \dfrac{-2a^3m}{b^2 + a^2m^2}$<br>$\therefore x = \dfrac{-a^3m}{b^2 + a^2m^2}$ | 1 | 1 for answer |
| (b) i) | $x = 2at$ and $y = at^2$<br>Grad of tangent $= \dfrac{dy}{dx} = t$<br>Grad of normal $= = \dfrac{-1}{t}$<br>At $t = p$ $m = \dfrac{-1}{p}$ [point $(2ap, ap^2)$]<br>Equation of normal<br>$y - ap^2 = \dfrac{-1}{p}(x - 2ap)$<br>$py - ap^3 = -x + 2ap$<br>$x + py - ap^3 - 2ap = 0$ | 2 | 1 for gradient of normal<br><br>1 for equation of normal |
| ii) | Normal passes through $(x_1, y_1)$ then<br>$x_1 + py_1 - ap^3 - 2ap = 0$<br>To find intersection with the parabola, this equation must be solved for $p$.<br>As the equation is a cubic in $p$, there can be from 1 to 3 values for $p$.<br>$\therefore$ Up to three normals can be drawn | 2 | 2 for any reasonable explanation |

| Question 5 | Trial HSC Examination- Mathematics Extension 2 | 2008 | |
|---|---|---|---|
| **Part** | **Solution** | **Marks** | **Comment** |
| (c) i) | $z=\cos\theta+i\sin\theta=c+is$<br>$z^4=(c+is)^4$<br>$=c^4+4c^3(is)+6c^2(-s^2)+4c(-is^3)+s^4$<br>$=c^4-6c^2s^2+s^4+i(4c^3s-4cs^3)$<br>By De Moivres Thm<br>$z^4=\cos 4\theta+i\sin 4\theta$<br>Equating real parts<br>$\cos 4\theta=c^4-6c^2(1-c^2)+(1-c^2)^2$<br>$=c^4-6c^2+6c^4+1-2c^2+c^4$<br>$=8c^4-8c^2+1$<br>$=8\cos^4\theta-8\cos^2+1$ | 3 | 1 for expanding<br>1 for De Moivre<br>1 for solution |
| (ii) | $\left(z+\frac{1}{z}\right)^4=(2\cos\theta)^4$<br>$=16\cos^4\theta$<br>and $\left(z+\frac{1}{z}\right)^4=z^4+4z^3\frac{1}{z}+6z^2\frac{1}{z^2}+4z\frac{1}{z^3}+\frac{1}{z^4}$<br>$=z^4+\frac{1}{z^4}+4\left(z^2+\frac{1}{z^2}\right)+6$<br>$16\cos^4\theta=2\cos 4\theta+8\cos 2\theta+6$<br>$\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$ | 2 | 1 for expansion<br>1 for expression |
| iii) | $\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta=\int_0^{\frac{\pi}{2}}\left(\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}\right)d\theta$<br>$=\left[\frac{1}{32}\sin 4\theta+\frac{1}{4}\sin 2\theta+\frac{3}{8}\theta\right]_0^{\frac{\pi}{2}}$<br>$=\left[\left(\frac{3}{8}.\frac{\pi}{2}\right)-(0)\right]$<br>$=\frac{3\pi}{16}$ | 2 | 1 for integral<br>1 for evaluating |

| Question 6 | Trial HSC Examination-Mathematics Extension 2 | | 2008 |
|---|---|---|---|
| **Part** | **Solution** | **Marks** | **Comment** |
| (a) | $[\mathrm{Re}(z)]^2 + \mathrm{Im}\, z < 0$<br>$x^2 + y < 0$<br>$y < -x^2$<br> | 2 | 1 for correct Cartesian equation<br><br>1 mark for correct region |
| b) i) | i. Consider the function $g(x) = e^{-x} + x - 1$.<br>$g(0) = 0$ and $g'(x) = -e^{-x} + 1 \Rightarrow g'(0) = 0$<br>Also $g''(x) = e^{-x} > 0$ for all $x$<br>$\therefore g(x)$ has a minimum value of 0 when $x = 0$.<br>$\therefore e^{-x} + x - 1 \geq 0$ for all $x$, with equality only if $x = 0$.<br>For $x \neq 0$, $f'(x) = \dfrac{d}{dx}\left(\dfrac{e^x - 1}{x}\right)$<br>$= \dfrac{e^x . x - (e^x - 1) . 1}{x^2}$<br>$= \dfrac{1 + xe^x - e^x}{x^2}$<br>$= \dfrac{e^x}{x^2}\left(e^{-x} + x - 1\right)$<br>$> 0$<br>Hence $f(x)$ is an increasing function for $x \neq 0$. | 3 | 1 shows by differentiation that $e^{-x} + x + 1$ has min value of 0 when x=0<br><br>1 finds $f'(x)$<br><br>1 rearranges $f'(x)$ as a product of $e^{-x} + x + 1$ and deduces $f'(x)>0$ |
| ii) | ii. Let $h(x) = e^x$. Then $h'(x) = e^x$.<br>$\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = \lim\limits_{x \to 0} \dfrac{e^x - e^0}{x - 0}$<br>$= h'(0)$<br>$= 1$<br>$= f(0)$<br>$\therefore f(x)$ is continuous at $x = 0$. | 2 | 1 Expresses the limiting value of $f(x)$ as $x \to 0$ as the derivative of $e^x$ at x=0<br><br>1 Evaluates this derivative to show limiting value is 1 |

| Question 6 | Trial HSC Examination- Mathematics Extension 2 | 2008 | |
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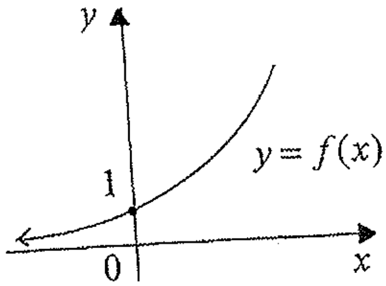
| **Part** | **Solution** | **Marks** | **Comment** |
| iii) | iii.   | 1 | Correct shape curve with y intercept of 1 and asymptote |
| (c) i) | i. $\frac{dy}{dx}=e^x=1$ at $x=0$<br>Hence tangent $AC$ has equation $y=x+1$.<br>Let $D(k,0)$<br>$\therefore C(k,k+1)$. Also $B(k,e^k)$. Then<br>$Area\ \ AODC<\int_0^k e^x\,dx<Area\ \ AODB$<br>$\frac{1}{2}k(k+2)<\left[e^x\right]_0^k<\frac{1}{2}k(1+e^k)$<br>$\therefore\ \frac{1}{2}k(k+2)<e^k-1<\frac{1}{2}k(1+e^k)$<br>For $k=1$, $1\cdot5<e-1<0\cdot5+\frac{1}{2}e$<br>Hence $2\cdot5<e$ and $\frac{1}{2}e<1\cdot5$<br>$\therefore 2\cdot5<e<3$ | 4 | 1 for equation of tangent AC and coords of C<br><br>1 for lower bound for $e^k-1$ using area AODC<br><br>1 for upper bound using area AODB<br><br>1 for using k=1 and rearranging |
| ii) | ii. Area of $\Delta ABC$ is bisected if<br>$(e^k-1)-\frac{1}{2}k(k+2)=\frac{1}{2}k(1+e^k)-(e^k-1)$<br>$(4-k)e^k-k^2-3k-4=0$<br><br>Let $f(k)=(4-k)e^k-k^2-3k-4$<br>Then $f(2)\approx0\cdot78>0$, $f(3)\approx-1\cdot9<0$<br>and $f(k)$ is continuous.<br>Hence $f(k)=0$, and the area is bisected, for some $k$ such that $2<k<3$<br><br>$f(k)=(4-k)e^k-k^2-3k-4$<br>$f'(k)=\left\{-e^k+(4-k)e^k\right\}-2k-3$<br>$=(3-k)e^k-2k-3$<br>Taking $k_0=2\cdot7$, $k_1=2\cdot7-\frac{f(2\cdot7)}{f'(2\cdot7)}$<br>$\therefore k_1\approx2\cdot7-\frac{-0\cdot04635}{3\cdot9361}$<br>$\approx2\cdot688$<br>Hence second approximation is $2\cdot7$ (to one decimal place). | 3 | 1 for equation if triangle area bisected<br><br>1 for f(k)=0 and establishes existence of root k 2<k<3<br><br>1 for using Newton's method for 2nd approximation |

| Question 7 | Trial HSC Examination- Mathematics Extension 2 | 2008 | |
|---|---|---|---|
| **Part** | **Solution** | **Marks** | **Comment** |
| (a) i) | $z^5=1$<br>By De Moivres Thm<br>$\cos 5\theta + i\sin 5\theta = 1$<br>Equating real and imaginary<br>$\cos 5\theta = 1 \quad \sin 5\theta = 0$<br>$\therefore\ 5\theta = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ 8\pi$<br>$\theta = 0,\ \frac{2\pi}{5},\ \frac{4\pi}{5},\ \frac{6\pi}{5},\ \frac{8\pi}{5}$<br>$z_1 = cis0 = 1$<br>$z_2 = cis\frac{2\pi}{5}$<br>$z_3 = cis\frac{4\pi}{5}$<br>$z_4 = cis\frac{6\pi}{5} = cis\frac{-4\pi}{5} = \overline{z}_3$<br>$z_5 = cis\frac{8\pi}{5} = cis\frac{-2\pi}{5} = \overline{z}_2$ | 3 | 1 values of $\theta$<br><br>2 for values of $z_1$ - $z_5$ |
| ii) | $z^5-1=(z-z_1)(z-z_2)(z-z_5)(z-z_3)(z-z_4)$<br>$=(z-z_1)(z^2-(z_2+z_5)z+z_2z_5)(z^2-(z_3+z_4)z+z_3z_4)=$<br>$=(z-z_1)\left(z^2-2\cos\frac{2\pi}{5}z+1\right)\left(z^2-2\cos\frac{4\pi}{5}z+1\right)$ | 2 | 1 factors<br><br>1 in quadratics |
| iii) | $z^5-1=0$<br>$(z-1)(z^4+z^3+z^2+z+1)=0$<br>Roots are $z_2,\ z_3,\ z_4,\ z_5$ from above. | 1 | |
| iv) | Sum of roots of $z^5-1=0$ is zero.<br>$\therefore z_1+z_2+z_3+z_4+z_5=0$<br>$z_1+z_2+\overline{z}_2+z_3+\overline{z}_3=0$<br>$1+2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=0$<br>$2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=-1$<br>$\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=\frac{-1}{2}$ | 2 | 1 for conjugates<br><br>1 for answer |

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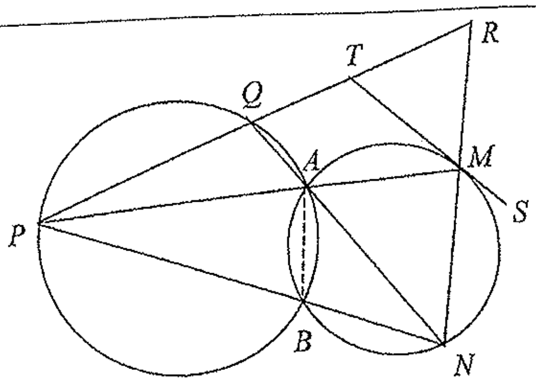
| (b) |  | | |
| i) | $\angle RMA = \angle ABN$ (*exterior angle of cyclic quad. ABNM is equal to interior opposite angle*)<br><br>Similarly<br>$\angle ABN = \angle AQP$ in cyclic quadrilateral $ABPQ$.<br><br>Hence quadrilateral $QAMR$ is cyclic.<br>(*exterior angle AQP is equal to interior opposite angle RMA*) | 3 | 2 marks for using properties of cyclic quads to deduce equal angles<br><br>1 for applying test to QAMR |
| | ii. Produce $TM$ to $S$. Then<br>$\angle TMR = \angle SMN$ (*vertically opposite angles are equal*)<br>$\angle SMN = \angle MAN$ (*angle between tangent and chord drawn to point of contact is equal to angle subtended by that chord in the alternate segment*)<br>$\angle MAN = \angle PAQ$ (*vertically opposite angles are equal*)<br>$\angle PAQ = \angle TRM$ (*exterior angle of cyclic quad. QAMR is equal to interior opposite angle*)<br>Hence in $\Delta TMR$, $\angle TMR = \angle TRM$ and hence $TM = TR$ (*sides opposite equal angles are equal*) | 4 | 1 alt seg theorem<br><br>1 vert opp angles<br><br>1 equality of angles<br><br>1 deduces TMR has equal angles so equal sides |

| Question 8 | Trial HSC Examination- Mathematics Extension 2 | 2008 |
|---|---|---|

| Part | Solution | Marks | Comment |
|---|---|---|---|
| (a) | $\sin(x+y)-\sin(x-y)=\sin x\cos y+\cos x\sin y-(\sin x\cos y-\cos x\sin y)$<br>$=2\cos x\sin y$ | 1 | 1 for answer |
| a) ii) | If $n=1$ LHS $=\cos x$<br>RHS $=\dfrac{\sin\left(\frac{3x}{2}\right)-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$<br>Using i) above RHS $=\dfrac{2\cos x\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$<br>$=\cos x=$ LHS<br>$\therefore$ true for $n=1$<br>Assume true for $n=k$<br>i.e $\cos x+\cos 2x+\cos 3x......+\cos kx=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$<br>When $n=k+1$<br>$\cos x+\cos 2x+\cos 3x......+\cos kx+\cos(k+1)x=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}+\cos(k+1)x$<br>$=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)+\cos(k+1)x.2\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$<br>$=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)+\sin\left((k+1)x+\frac{x}{2}\right)-\sin\left((k+1)x-\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$ using i) above<br>$=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)+\sin\left(k+\frac{3}{2}\right)x-\sin\left(k+\frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}$<br>$=\dfrac{\sin\left((k+1)+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$<br>$\therefore$ True for $n=k+1$<br>$\therefore$ Since true for $n=1$, by induction is true for all positive integral values of $k\geq 1$ | 4 | 1 for n =1 case<br>1 for using i)<br>1 for simplifying<br>1 for stating k+1 case |

| Question 8 | Trial HSC Examination- Mathematics Extension 2 | 2008 | |
|---|---|---|---|
| **Part** | **Solution** | **Marks** | **Comment** |
| (a)<br>(ii) | $\cos 2x+\cos 4x+\cos 6x\ldots\ldots+\cos 16x=\cos(2x)+\cos 2(2x)+\cos 3(2x)\ldots\ldots+\cos 8(2x)$<br>$=\dfrac{\sin\left(8+\frac{1}{2}\right)2x-\sin\left(\frac{2x}{2}\right)}{2\sin\left(\frac{2x}{2}\right)}$<br>$=\dfrac{\sin 17x-\sin x}{2\sin x}$<br>$=\dfrac{\sin(9+8)x-\sin(9-8)x}{2\sin x}$<br>$=\dfrac{2\cos 9x\sin 8x}{2\sin x}$ Using i) above<br>$=\dfrac{2\cos 9x.2\sin 4x\cos 4x}{2\sin x}$ Using double angle on $\sin 8x$<br>$=\dfrac{4\cos 9x.2\sin 2x\cos 2x\cos 4x}{2\sin x}$ Using double angle on $\sin 4x$<br>$=\dfrac{8\cos 9x.2\sin x\cos x\cos 2x\cos 4x}{2\sin x}$ Using double angle on $\sin 2x$<br>$=\dfrac{8\cos 9x.\cancel{2\sin x}\cos x\cos 2x\cos 4x}{\cancel{2\sin x}}$<br>$=8\cos 9x\cos 4x\cos 2x\cos x$ | 4 | 1 for sub into expression<br>1 for breaking up 17x<br>2 for completing simplification |
| (b) | Let roots be $\alpha-d$ , $\alpha$ and $\alpha+d$<br>$\therefore$ Sum of the roots $=(\alpha-d)+\alpha+(\alpha+d)=-a$<br>$\therefore 3\alpha=-a$<br>$\alpha=\dfrac{-a}{3}$<br>$\therefore$ Sum of the roots 2 at a time $=(\alpha-d)\alpha+(\alpha-d)(\alpha+d)+(\alpha+d)\alpha=b$<br>$\alpha^2-\alpha d+\alpha^2-d^2+\alpha^2+\alpha d=b$<br>$3\alpha^2-d^2=b$<br>$d^2=3\alpha^2-b$<br>$d^2=3\left(\dfrac{-a}{3}\right)^2-b=\dfrac{a^2}{3}-b$<br>$\therefore$ Product of the roots $=\alpha(\alpha-d)(\alpha+d)=c$<br>$\alpha^3-\alpha d^2=c$<br>$\left(\dfrac{-a}{3}\right)^3-\left(\dfrac{-a}{3}\right)\left(\dfrac{a^2}{3}-b\right)=c$<br>$\dfrac{-a^3}{27}+\dfrac{a^3}{9}-\dfrac{ab}{3}=c$<br>$\dfrac{2a^3}{27}-\dfrac{ab}{3}=c$<br>$\therefore 2a^3-9ab-27c=0$ | 4 | 1 each for expressions for sums & products = 3 marks<br>1 for substitution and simplifying |

| Question 8 | Trial HSC Examination- Mathematics Extension 2 | 2008 |
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| Part | Solution | Marks | Comment |
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| (c) | $\frac{dy}{dx} = 2y$<br>$\therefore \frac{dx}{dy} = \frac{1}{2y}$<br>$\therefore x = \frac{1}{2}\ln y + c$<br>When $x = 1, y = 1$<br>$\therefore c = 1$<br>$\therefore x = \frac{1}{2}\ln y + 1$<br>$\frac{1}{2}\ln y = x - 1$<br>$\ln y = 2x - 2$<br>$y = e^{2x-2}$ | 2 | 1 for expression for $x$<br><br>1 for result |