JRAHS Ext 2 Trial 2002

QUESTION 1 (15 MARKS) Start a new page

a) (i) Given f(x) is an odd function, show that $\int_{-a}^{a} f(x)dx = 0$ using $x = -t$.

(ii) Hence, evaluate $\int_{-2}^{2} x^4(1+\sin^3 x)dx$.

b) (i) Let $I_n = \int_1^e x(\ln x)^n\, dx, n = 0,1,2,3,\ldots..$

Using integration by parts, show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}, n = 1,2,3,\ldots..$

(ii) The area bounded by the curve $y = \sqrt{x}(\ln x)^2, x \ge 1$, the x-axis and the line x = e is rotated about the x-axis through 2π radians.
Find the exact volume of the solid of revolution so formed.

c) (i) Sketch the curve $f(x) = \frac{x^2 - x - 6}{x-1}$.

(ii) Hence, sketch the graph of $y^2 = f(x)$.

QUESTION 2 (15 MARKS) Start a new page

a) Find the following:

(i) $\int \frac{e^{-x}}{1+e^x} dx$.

(ii) $\int \frac{x}{\sqrt{1-2x-x^2}} dx$.

b) Evaluate in exact form:

(i) $\int_0^{\frac{\pi}{6}} \frac{d\theta}{9-8\cos^2\theta}$ using the substitution $t = \tan\theta$.

(ii) $\int_0^{\frac{\pi}{6}} \sec^3 2\theta d\theta$.

c) Using the method of cylindrical shells, find the volume generated by revolving the area bounded by the lines $x = \pm 2$ and the hyperbola $\frac{y^2}{9} - \frac{x^2}{4} = 1$ about the y-axis.

QUESTION 3 (15 MARKS) Start a new page

a) Let α, β, γ be the roots of the cubic equation $x^3 + px^2 + q = 0$, where p and q are real. The equation $x^3 + ax^2 + bx + c = 0$ has roots α^2, β^2 and γ^2. Find a, b, c as functions of p and q.

b) (i) Find the complex square roots of 5 - 12i, expressing your answer in the form $a+bi$, where a and b are real.

(ii) Hence, solve the equation: $z^2 + 4z - 1 + 12i = 0$

c) Given that $z = \cos\theta + i\sin\theta$,

(i) Show that $z^n + z^{-n} = 2\cos n\theta$ using De Moivre's Theorem.

(ii) Hence, solve the equation: $2z^4 - z^3 + 3z^2 - z + 2 = 0$.

QUESTION 4 (15 MARKS) Start a new page

a) The sequence U_n, is defined such that

$U_{n+2} = 4U_{n+1} - U_n$, $n \geq 1$ and $U_1 = 2, U_2 = 4$.

Prove by mathematical induction that:

$U_n = (2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}$.

b) $P(a\cos\theta, b\sin\theta)$ and $Q(a\sec\theta, b\tan\theta)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, respectively as shown.

M and N are the feet of the perpendicular from P and Q respectively to the x-axis. $0<\theta<\frac{\pi}{2}$, and QP meets the x-axis at K. A is the point $(a,0)$.

(i) Given $\Delta KPM \parallel\!| \Delta KQN$, show that $\frac{KM}{KN}=\cos\theta$.

(ii) Hence, show that K has coordinates $(-a,0)$.

(iii) **Show** that the tangent to the ellipse at P has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$, **and** deduce it passes through N.

(iv) **Given** that the tangent to the hyperbola at Q has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$, show that the tangent passes through M.

(v) Show that the tangents PN, QM and the common tangent at A are concurrent. Find the point of concurrence.

QUESTION 5 (15 MARKS) Start a new page

a) (i) If $\omega=i-1$, evaluate the following points $\overline{\omega}$, $i\omega$, $\frac{1}{\omega}$

(ii) Hence, indicate ω, $\overline{\omega}$, $i\omega$, $\frac{1}{\omega}$ on the Argand diagram.

b) Sketch the region R in the Argand diagram consisting of the points z for which:

$$|\arg z|<\frac{\pi}{3} \quad , \quad z+\overline{z}<4 \quad , \quad |z|>2$$

c) On the Argand diagram, P represents the complex number z, and R the number $\frac{1}{z}$. A square $PQRS$ is drawn in the plane with PR as a diagonal. If P lies on the circle $|z|=2$,

(i) Prove that Q will lie on the ellipse whose equation has the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$;

(ii) Hence, specify the numerical values for a and b.

QUESTION 6 (15 MARKS) Start a new page

a) The base of a solid is the region bounded by the line $y = 2x$ and the parabola $y = 4x - x^2$ and the x-axis. Cross-sections parallel to the x-axis are right-angled isosceles triangles with the hypotenuse in the base of the solid.

(i) Show that the volume of the solid is given by:
$V = \frac{1}{16}\int_0^4 (4 + 2\sqrt{4-y} - y)^2\,dy$

(ii) Hence, find the exact volume of the solid.

b) A particle of mass, m kg, moves in a straight line with velocity v metres/second, under a constant force, P Newtons, and a resistance, R Newtons . Initially, the particle has a speed v_o metres/second. If $R = 5 + 3v$ and $P = 10$.

(i) Show that velocity, $v = \frac{5}{3}(1 - e^{-\frac{3t}{m}}) + v_o e^{-\frac{3t}{m}}$.

(ii) Find the terminal velocity of the particle.

(iii) When the particle accelerates from v_o to v_1 , show that the distance travelled, x metres, is given by :
$$x = \frac{m}{9}\left[3(v_0 - v_1) + 5\ln\left(\frac{5-3v_0}{5-3v_1}\right)\right].$$

QUESTION 7 (15 MARKS) Start a new page

a) With respect to the x and y axis, the line $x = 1$ is a directrix, and the point (2,0) is a focus of a conic of eccentricity $\sqrt{2}$.

Find the equation of the conic, and sketch the curve indicating its asymptotes, foci, and directrices.

b) A circular cone of semi vertical angle θ is fixed with its vertex upwards as shown. A particle P, of mass $2m$ kg, is attached to the vertex V by a light inextensible string of length $2a$ metres.

The particle P rotates with uniform angular velocity ω radians/second in a horizontal circle on the outside surface of the cone and in contact with it.

(i) Find the tension (T) in the string, in Newtons.

(ii) Find the normal force (N) on P, in Newtons.

(iii) Show that, for the particle to remain in uniform circular motion on the surface of the cone, then $\omega < \left[\dfrac{g}{2a\cos\theta}\right]^{\frac{1}{2}}$ where g is acceleration due to gravity.

c) A car of mass, m kg, with speed v metres/second travels around a circular track of radius R metres, inclined at angle θ to the horizontal and g is the acceleration due to gravity.

(i) Show that if there is a tendency for the car to slip that

$$\tan\theta = \frac{v^2}{gR}.$$

(ii) If the speed of the car is now halved, prove that the sideways frictional force F, on the wheels, exerted by the track is given:

$$F = \frac{3mgv^2}{4\sqrt{v^4 + g^2R^2}}\,.$$

QUESTION 8 (15 MARKS) Start a new page

a) Two particles of mass 4kg and 6kg are attached at either end of a light inextensible string of length 7 metres, which passes through a small vertical frictionless ring *R*. The heavier particle *A* hangs vertically at a distance of 4 metres below the ring while the other particle *B* describes a horizontal circle whose centre is *O*. Let θ be the acute angle which particle *B* makes with the vertical.

Find:

(i) The distance *OR* and the radius *OB*, of the horizontal circle.

(iii) The angular velocity of *B* about *O* in revolutions/minute to 2 decimal places (use g = 9.8 m/s^2).

b) In the diagram below, *EC* and *ED* are perpendicular to *BA* and *AC* at *G* and *H* respectively. The lines *AC* and *BD* meet at *I*. Let $\angle ECA = \alpha$.

(i) Copy the diagram, then show that *FGCH* is a cyclic quadrilateral.

(ii) Prove ΔBCD is isosceles.

(iii) Prove $\Delta CID \;|||\; \Delta CDA$.

(iv) Given that $\Delta CIB \;|||\; \Delta CBA$ and $AB + AD = 2BC$, prove that $2CI = BD$.

SOLUTIONS TO 4UNIT TRIAL 2002

Question 1 (15 marks)

a) (i) $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$ - - - - - ①

Consider $\int_{-a}^{0} f(x)\,dx$

Let $x = -t$

$dx = -dt$

When $x = 0$ $t = 0$

when $x = -a$ $t = a$

$= \int_{a}^{0} -f(-t)\,dt$

$= -\int_{a}^{0} -f(t)\,dt$ ← since $-f(t) = f(-t)$ as $f(x)$ is odd

$= \int_{a}^{0} f(t)\,dt.$

$= -\int_{0}^{a} f(t)\,dt$

But $-\int_{0}^{a} f(t)\,dt \Longleftrightarrow -\int_{0}^{a} f(x)\,dx$.

$\therefore$ From ①: $\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$= 0$ as required.

(ii) $\int_{-2}^{2} x^4(1+\sin^3 x)\,dx$

$= \int_{-2}^{2} x^4 + \int_{-2}^{2} x^4 \sin^3 x\,dx$

$= \left[\frac{x^5}{5}\right]_{-2}^{2} + 0$ *

$= \frac{64}{5}$

*

Consider $\int_{-2}^{2} x^4 \sin^3 x\,dx$

Let $f(x) = x^4 \sin^3 x$

$f(-x) = (-x)^4 \sin^3(-x)$

$= x^4 . (-\sin x)^3$

$= -x^4 \sin^3 x$

$= -f(x).$

$\therefore f(x)$ is odd

$\therefore \int_{-2}^{2} x^4 \sin^3 x\,dx = 0$

b) (i) $I_n = \int_1^e x(\ln x)^n \, dx \quad , \quad n = 0, 1, 2, 3, \ldots$

$$= \left[\frac{x^2}{2}(\ln x)^n\right]_1^e - \int_1^e \frac{x^2}{2} \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} \, dx$$

$$= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\ln x)^{n-1} \, dx$$

$$= \frac{e^2}{2} - \frac{n}{2} \cdot I_{n-1}$$

(ii) $V = \pi \int_1^e y^2 \, dx = \pi \int_1^e x(\ln x)^4 \, dx = \pi \cdot I_4$

Now $I_4 = \frac{e^2}{2} - 2 I_3$

$$= \frac{e^2}{2} - 2\left[\frac{e^2}{2} - \frac{3}{2} I_2\right]$$

$$= -\frac{e^2}{2} + 3 I_2$$

$$= -\frac{e^2}{2} + 3\left[\frac{e^2}{2} - I_1\right]$$

$$= e^2 - 3 I_1$$

$$= e^2 - 3\left[\frac{e^2}{2} - \frac{1}{2} \cdot I_0\right]$$

$$= -\frac{e^2}{2} + \frac{3}{2} I_0 \longleftarrow *$$

$$* \; I_0 = \int_1^e x \, dx = \left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{2}.$$

$$= -\frac{e^2}{2} + \frac{3}{2}\left[\frac{e^2}{2} - \frac{1}{2}\right]$$

$$= \frac{e^2}{4} - \frac{3}{4}$$

$\therefore V = \pi\left(\frac{e^2}{4} - \frac{3}{4}\right)$

$$= \frac{\pi}{4}(e^2 - 3) \; u^3$$

c) (i) $f(x) = \frac{x^2 - x - 6}{x - 1} = \frac{(x-3)(x+2)}{x-1}$

VA: $x = 1$

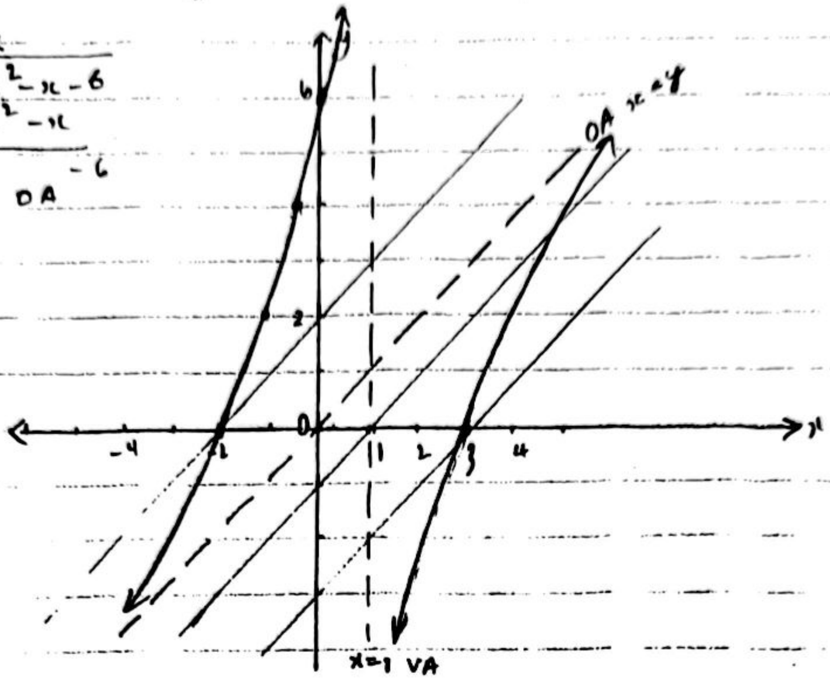
OA: $x - 1 \overline{\smash{)}\, x^2 - x - 6}$, quotient x

$\quad x^2 - x$

$\quad -6$

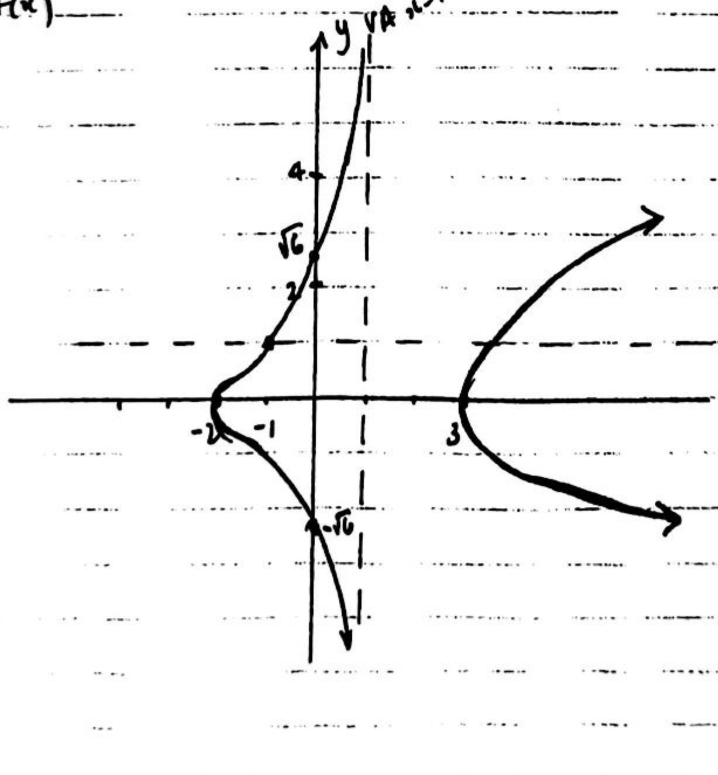
$\therefore y = x$ is OA

y int. $(0, 6)$



(ii) $y^2 = f(x)$

$y = \pm\sqrt{f(x)}$



Question 2

a) (i) $\displaystyle\int \frac{e^{-x}}{1+e^{x}}\,dx$

$$= \int \frac{1}{e^{x}(1+e^{x})}\,dx$$

$$= \int \left(\frac{1}{e^{x}} - \frac{1}{1+e^{x}}\right) dx \quad *$$

$$= \int e^{-x} - \left(\frac{1+e^{x}-e^{x}}{1+e^{x}}\right) dx$$

$$= \int e^{-x} - \left(1 - \frac{e^{x}}{1+e^{x}}\right) dx$$

$$= \int e^{-x} - 1 + \frac{e^{x}}{1+e^{x}}\,dx$$

$$= \underline{-e^{-x} - x + \ln(1+e^{x}) + c}$$

* Alternative answer

$$\int \frac{1}{1+e^{x}}\,dx$$

$$= \int \frac{e^{-x}}{e^{-x}+1}\,dx$$

$$= -\ln(e^{-x}+1)$$

$\therefore$ Answer could be:

$$\underline{\underline{-e^{-x} - \ln(e^{-x}+1) + c}}$$

(ii) $\displaystyle\int \frac{x}{\sqrt{1-2x-x^2}}\,dx$

$$= \int \frac{x}{\sqrt{2-(x+1)^2}}\,dx$$

$$= \int \frac{\sqrt{2}\cos\theta - 1}{\sqrt{2-2\cos^2\theta}} \cdot -\sqrt{2}\sin\theta\,d\theta$$

$$= \int \frac{\sqrt{2}\cos\theta - 1}{\sqrt{2(1-\cos^2\theta)}} \cdot -\sqrt{2}\sin\theta\,d\theta$$

$$= \int \frac{1-\sqrt{2}\cos\theta}{\cancel{\sqrt{2}\sin\theta}} \cdot \cancel{\sqrt{2}\sin\theta}\,d\theta$$

$$= \int 1 - \sqrt{2}\cos\theta\,d\theta$$

$$= \theta - \sqrt{2}\sin\theta + c$$

$$= \cos^{-1}\frac{x+1}{\sqrt{2}} - \cancel{\sqrt{2}} \cdot \frac{\sqrt{1-2x-x^2}}{\cancel{\sqrt{2}}} + c$$

$$= \underline{\cos^{-1}\frac{x+1}{\sqrt{2}} - \sqrt{1-2x-x^2} + c}$$

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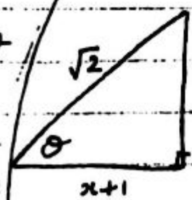


$$\frac{x+1}{\sqrt{2}} = \cos\theta$$

$$x+1 = \sqrt{2}\cos\theta$$

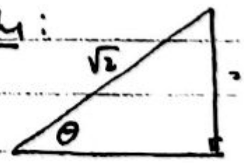
$$dx = -\sqrt{2}\sin\theta\,d$$

By pythagoras, third side is

$$= \sqrt{1-2x-x^2}$$

$$\therefore \sin\theta = \frac{\sqrt{1-2x-x^2}}{\sqrt{2}}.$$

* Alternatively:



$$\frac{x+1}{\sqrt{2}} = \sin\theta$$

$$x+1 = \sqrt{2}\sin\theta$$

$$dx = \sqrt{2}\cos\theta\,d$$

$$\therefore \cos\theta = \frac{\sqrt{1-2x-x^2}}{\sqrt{2}}$$

* Alternative answer:

(5)

b)

Evaluate exactly:

(i) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{d\theta}{9-8\cos^2\theta}$ using $t = \tan\theta$

$dt = \sec^2\theta \; d\theta$

$= (1+\tan^2\theta)\, d\theta$

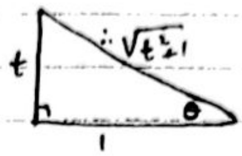
$\therefore d\theta = \dfrac{dt}{1+\tan^2\theta} = \dfrac{dt}{1+t^2}$

$\theta = 0,\ t = 0$

$\theta = \frac{\pi}{6},\ t = \tan\frac{\pi}{6}$

$t = \frac{1}{\sqrt{3}}$

Also:



$\tan\theta = t$

$\cos\theta = \dfrac{1}{\sqrt{t^2+1}}$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{dt}{1+t^2}}{9 - \frac{8}{t^2+1}}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{\cancel{1+t^2}} \cdot \frac{\cancel{t^2+1}}{9(t^2+1)-8}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{9t^2+1}$$

$$= \frac{1}{9}\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{\frac{1}{9}+t^2}$$

$$= \frac{1}{9}\left[3 \cdot \tan^{-1} 3t\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{1}{3}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 0\right)$$

$$= \frac{1}{3}\cdot\frac{\pi}{3}$$

$$= \frac{\pi}{9}$$

(ii) $\displaystyle\int_0^{\frac{\pi}{6}} \sec^3 2\theta \; d\theta$

$u = \sec 2\theta$

$du = 2\sec 2\theta \cdot \tan 2\theta$

$dv = \sec^2 2\theta$

$v = \dfrac{\tan 2\theta}{2}$

**

$$= \int_0^{\frac{\pi}{6}} \sec^2 2\theta \cdot \sec 2\theta \; d\theta$$

$$= \left[\frac{\tan 2\theta}{2} \cdot \sec 2\theta\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sec 2\theta \cdot \tan^2 2\theta \, d\theta$$

$$= \left[\frac{\tan 2\theta \cdot \sec 2\theta}{2}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sec 2\theta\left[\sec^2 2\theta - 1\right] d\theta$$

$$= \quad '' \quad - \int_0^{\frac{\pi}{6}} \sec^3 2\theta \; d\theta + \int_0^{\frac{\pi}{6}} \sec 2\theta \; d\theta$$

$$\therefore\ 2\int_0^{\frac{\pi}{6}} \sec^3 2\theta \; d\theta = \left[\frac{\tan 2\theta \cdot \sec 2\theta}{2}\right]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \sec 2\theta \; d\theta$$

$$= \frac{\sec\frac{\pi}{3}\tan\frac{\pi}{3}}{2} + \int_0^{\frac{\pi}{6}} \sec 2\theta \, d\theta$$

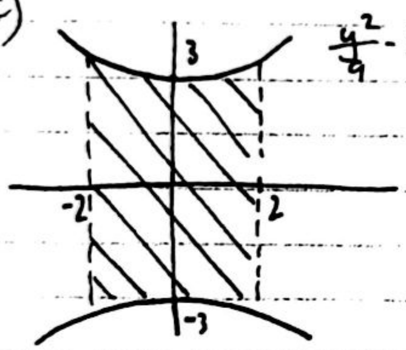
$$= \frac{2}{2}\cdot\sqrt{3} + \frac{1}{2}\ln\left[\sec 2\theta + \tan 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \sqrt{3} + \frac{1}{2}\ln\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - 0$$

$$= \sqrt{3} + \frac{1}{2}\ln\left(2+\sqrt{3}\right)$$

$$\therefore \int_0^{\frac{\pi}{6}} \sec^3 2\theta \, d\theta = \frac{1}{4}\left[2\sqrt{3} + \ln\left(2+\sqrt{3}\right)\right]$$

c)



$\frac{y^2}{9} - \frac{x^2}{4} = 1$

$$V = 2\pi \int_a^b (\text{shell radius})\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx$$

$$= 2\pi \int_0^2 x \cdot 2y \, dx$$

$$= 4\pi \int_0^2 xy \, dx$$

$$= \frac{3}{2}\cdot 4\pi \int_0^2 x\sqrt{4+x^2} \, dx \quad *$$

$$= \frac{6\pi}{2}\int_4^8 \sqrt{u} \, du \quad **$$

$$= \frac{2}{3}\cdot 3\pi \left[u^{\frac{3}{2}}\right]_4^8$$

$$= 2\pi\left(8^{3/2} - 4^{3/2}\right)$$

$$= 2\pi\left(16\sqrt{2} - 8\right)$$

$$= 16\pi\left(2\sqrt{2} - 1\right)$$

Now $\frac{y^2}{9} - \frac{x^2}{4} = 1$

$$y^2 = 9\left(1 + \frac{x^2}{4}\right)$$
$$= \frac{9}{4}\left(4 + x^2\right)$$
$$y = \frac{3}{2}\sqrt{4+x^2}$$

Also: Let $u = 4 + x^2$

$du = 2x \, dx$

$\frac{du}{2} = x \, dx$

If $x = 0$ $u = 4$

$x = 2$ $u = 8$

Question 3

a) α, β, γ are roots of $x^3 + px^2 + q = 0$
$\alpha^2, \beta^2, \gamma^2$ " " " $x^3 + ax^2 + bx + c = 0$

b)

Method 1

α, β, γ are the roots of the eq^n $x^3 + px^2 + q = 0$ where $\alpha^3 + p\alpha^2 + q = 0$
$\alpha^2, \beta^2, \gamma^2$ satisfy the required eq^n where $y = \alpha^2$
$\therefore \alpha = \sqrt{y}$

$$\therefore \sqrt{y}^3 + p\sqrt{y}^2 + q = 0$$
$$y^{\frac{3}{2}} + py + q = 0$$
$$y^{\frac{3}{2}} = -py - q$$
$$y^3 = p^2y^2 + 2pqy + q^2$$
$$y^3 - p^2y^2 - 2pqy - q^2 = 0$$

This is the same as a polynomial in x:
$$\text{ie } x^3 - p^2x^2 - 2pqx - q^2 = 0$$
$$\therefore \left.\begin{array}{l} a = -p^2 \\ b = -2pq \\ c = -q^2 \end{array}\right\}$$

Alternate Method

$$\left.\begin{array}{l} \alpha + \beta + \gamma = -p \\ \alpha\beta + \beta\gamma + \gamma\alpha = 0 \\ \alpha\beta\gamma = -q \end{array}\right\} \text{ using } x^3 + px^2 + q = 0$$

$$\left.\begin{array}{l} \alpha^2 + \beta^2 + \gamma^2 = -a \\ \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = b \\ \alpha^2\beta^2\gamma^2 = -c \end{array}\right\} \text{ using } x^3 + ax^2 + bx + c = 0$$

c)

Now: $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
$$(-p)^2 = -a + 2 \times 0$$
$$\therefore \underline{a = -p^2} \quad --- (1)$$

Also: $(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha^2\beta\gamma + 2\alpha\beta^2\gamma + 2\alpha\beta\gamma^2$
$$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$
$$0^2 = b + 2 \times -q \times -p$$

And : $(\alpha\beta\gamma)^2 = \alpha^2\beta^2\gamma^2$

$(-q)^2 = -c$

$\therefore \underline{c = -q^2}$ ---- (3)

(i) $\sqrt{5-12i} = a+bi$ where a and b are real.

$5-12i = a^2+2abi-b^2$

$\left.\begin{aligned} \therefore 5 &= a^2-b^2 \\ -6 &= ab \end{aligned}\right\}$

$\therefore b = \frac{-6}{a} \Longrightarrow a^2 - \frac{36}{a^2} = 5$

$a^4-5a^2-36=0$

$(a^2-9)(a^2+4)=0$

$a = \pm 3$ only (as a must be real)

$\therefore$ If $a=3 \Longrightarrow b=-2$

$a=-3 \Longrightarrow b=2$

$\therefore \underline{\sqrt{5-12i} = \pm(3-2i)}$

(ii) $z^2+4z-1+12i=0$

$z = \frac{-4 \pm \sqrt{16-4.1.(12i-1)}}{2}$

$= \frac{-4 \pm \sqrt{16-48i+4}}{2}$

$= \frac{-4 \pm 2\sqrt{5-12i}}{2}$

$= -2 \pm \sqrt{5-12i}$

$\therefore z = -2+3-2i = \underline{1-2i}$ from (i) above.

OR $= -2-3+2i = \underline{-5+2i}$

(i) $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta+i\sin\theta)^{-n} = \cos(-n\theta)+i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$\therefore z^n+z^{-n} = \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta \cancel{-i\sin n\theta}$

$= \underline{2\cos n\theta}$

(ii) $2z^4 - z^3 + 3z^2 - z + 2 = 0$

$2z^4 + 2 - z^3 - z + 3z^2 = 0$

$2(z^4 + 1) - (z^3 + z) + 3z^2 = 0$

$2(z^2 + z^{-2}) - (z + z^{-1}) + 3 = 0 \qquad \div z^2$

$2(2\cos 2\theta) - 2\cos\theta + 3 = 0$

$4\cos 2\theta - 2\cos\theta + 3 = 0$

$4(2\cos^2\theta - 1) - 2\cos\theta + 3 = 0$

$8\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta - 1)(4\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -\frac{1}{4}$.

$\therefore$ If $\cos\theta = \frac{1}{2}$ $\sin\theta = \pm\frac{\sqrt{3}}{2}$ $\therefore$ $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

If $\cos\theta = -\frac{1}{4}$ $\sin\theta = \pm\frac{\sqrt{15}}{4}$ $\therefore$ $z = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$

$\therefore$ Roots are $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, $-\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$

Question 4

a) $u_{n+2} = 4u_{n+1} - u_n \qquad u_1 = 2\ ;\ u_2 = 4$

$u_n = (2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}$

Step 1 : Show true for $n=1$: $u_1 = 1 + 1 = 2$ ✓

" " " $n = 2$: $u_2 = (2+\sqrt{3}) + (2-\sqrt{3}) = 4$ ✓

Now $u_3 = (2+\sqrt{3})^2 + (2-\sqrt{3})^2$

$= 4 + 4\sqrt{3} + 3 + 4 - 4\sqrt{3} + 3$ * See over

$= 14$

Also: $4u_2 - u_1 = 4 \times 4 - 2 = 16 - 2 = 14$

$\therefore u_3 = 4u_2 - u_1$.

$\therefore$ Statement is true for $n=1, n=2, n=3$.

Step 2 : Assume statement is true for $n=k$, $n=k+1$

ie $u_k = (2+\sqrt{3})^{k-1} + (2-\sqrt{3})^{k-1}$

$u_{k+1} = (2+\sqrt{3})^k + (2-\sqrt{3})^k$

Now prove true for $n = k+2$

ie Prove: $u_{k+2} = (2+\sqrt{3})^{k+1} + (2-\sqrt{3})^{k+1}$

Now $4u_{k+1} - u_k$

$$= 4\left[(2+\sqrt{3})^k + (2-\sqrt{3})^k\right] - \left[(2+\sqrt{3})^{k-1} + (2-\sqrt{3})^{k-1}\right] \quad \text{(by assumption)}$$

$$= (2+\sqrt{3})^{k-1}\left[4(2+\sqrt{3}) - 1\right] + (2-\sqrt{3})^{k-1}\left[4(2-\sqrt{3}) - 1\right]$$

$$= (2+\sqrt{3})^{k-1}\left[7+4\sqrt{3}\right] + (2-\sqrt{3})^{k-1}\left[7-4\sqrt{3}\right]$$

$$= (2+\sqrt{3})^{k-1}\left[4+4\sqrt{3}+3\right] + (2-\sqrt{3})^{k-1}\left[4-4\sqrt{3}+3\right]$$

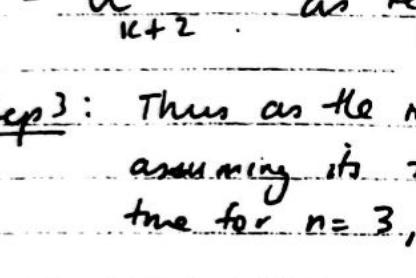
$$= (2+\sqrt{3})^{k-1}(2+\sqrt{3})^2 + (2-\sqrt{3})^{k-1}(2-\sqrt{3})^2 \quad \text{* from previous page}$$

$$= (2+\sqrt{3})^{k+1} + (2-\sqrt{3})^{k+1}$$

$= u_{k+2}$ as required

Step 3: Thus as the result is true for $n=1$, $n=2$ and $n=k+2$ assuming its true for $n=k$ and $n=k+1$ then it is true for $n=3$, $n=4$ etc and all positive integer n.

b) (i)



By similar triangles:

$$\frac{KM}{KN} = \frac{b\sin\theta}{b\tan\theta}$$

$$\therefore \frac{KM}{KN} = \cos\theta$$

(ii) Let $K = (k, 0)$

Using result above: $KM = a\cos\theta - k$

$KN = a\sec\theta - k$

$$\therefore \frac{a\cos\theta - k}{a\sec\theta - k} = \cos\theta$$

$$a\cos\theta - k = (a\sec\theta - k).\cos\theta$$

$$= a - k\cos\theta$$

$$a\cos\theta - a = k - k\cos\theta$$

$$a(\cos\theta - 1) = k(1-\cos\theta)$$

$$\therefore \underline{\underline{a = -k}}$$

$\therefore K$ is point $(-a, 0)$ as required

(iii) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Now $x = a\cos\theta$
$y = b\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$\therefore \frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$$

Eq^n of tangent at P is given by:

$$y = \frac{-b\cos\theta}{a\sin\theta}x + B \quad (B \text{ is constant})$$

ie $b\cos\theta\, x + a\sin\theta\, y = C$ (C is constant)

Since P lies on this tangent, it satisfies eq^n above:

ie $ab\cos^2\theta + ab\sin^2\theta = C$

$ab(\cos^2\theta + \sin^2\theta) = C$

$C = ab$

$\therefore$ tangent at P has eq^n:

$b\cos\theta\, x + a\sin\theta\, y = ab$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad (\div ab)$$

as required.

Also, N is point $(a\sec\theta, 0)$.
If the tangent passes through N, its coord. satisfy eq^n of tangent: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$$LHS = \frac{\cancel{a}\sec\theta\cos\theta}{\cancel{a}} + 0$$

$$= \frac{1}{\cos\theta}\cdot\cos\theta$$

$$= 1 = RHS$$

$\therefore$ N lies on the tangent at P.

(iv) M has coord $(a\cos\theta, 0)$ and it lies on the tangent it satisfies the equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

$LHS = \cancel{a}\cos\theta\cdot\frac{\sec\theta}{\cancel{a}} - 0$

$= \cos\theta\cdot\frac{1}{\cos\theta}$

$= 1$

(v) Common tangent at A has equation $\underline{x = a}$

Required to find the point of intersection of $x = a$ with the tangent at PN

ie $\left.\begin{aligned} & x = a \\ & \frac{x\cos\theta}{a} + y\frac{\sin\theta}{b} = 1 \end{aligned}\right\}$

ie $\cos\theta + y\frac{\sin\theta}{b} = 1$

$$y = \frac{b(1-\cos\theta)}{\sin\theta}$$

$\therefore$ PN meets common tangent at $\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$

Find pt of intersection of $x = a$ with the tangent at QM

ie $\left.\begin{aligned} & x = a \\ & \frac{x\sec\theta}{a} - y\frac{\tan\theta}{b} = 1 \end{aligned}\right\}$

ie $\sec\theta - y\frac{\tan\theta}{b} = 1$

$1 - y\frac{\sin\theta}{b} = \cos\theta$

$$y = b\frac{(1-\cos\theta)}{\sin\theta}$$

$\therefore$ QM meets common tangent at $\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$

$\therefore$ PN, QM and tangent at A are concurrent at

$$\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$$

Question 5

a) (i) $\omega = i - 1$

$\overline{\omega} = -1 - i$

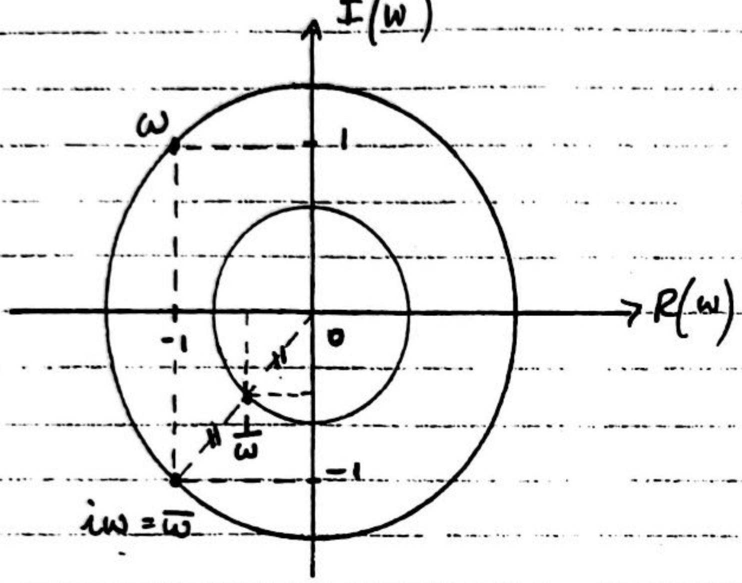
$i\omega = i(i-1)$

$= -i + i^2$

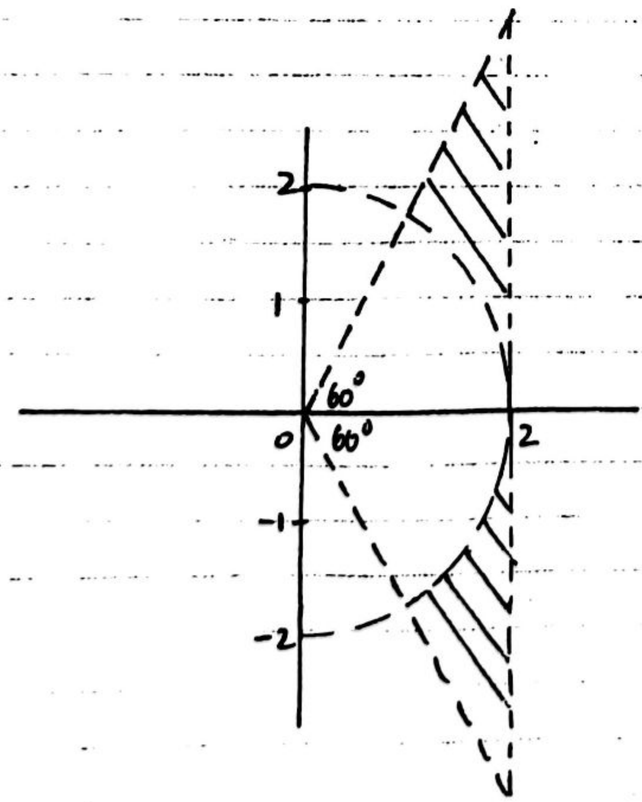
$= -i - 1 \quad (= \overline{\omega})$

$\frac{1}{\omega} = \frac{1}{-1+i} = \frac{-1-i}{1+1} = \frac{-1-i}{2} \quad \left(= \frac{\overline{\omega}}{2}\right)$

(ii)



b)



$|z| > 2$

$x^2 + y^2 > 2$

$z + \overline{z} < 4$

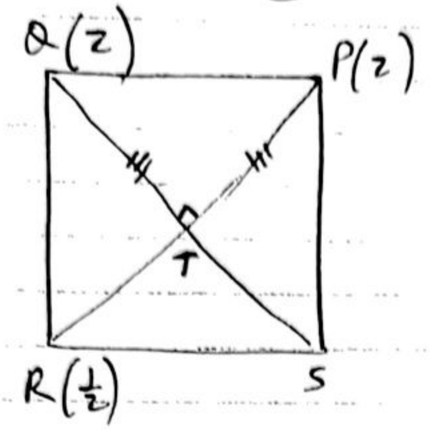
$x + \cancel{yi} + x - \cancel{yi} < 4$

$2x < 4$

$x < 2$

(14)

c)



We have $P(z)$, $R\left(\frac{1}{z}\right)$ and $PQRS$ is a square. Let Q be point $Q(z)$. Since TQ is obtained from TP by a rotation of $\frac{\pi}{2}$, where T is point $\frac{1}{2}\left(z+\frac{1}{z}\right)$:

$$Z - \frac{1}{2}\left(z+\frac{1}{z}\right) = i\left[z - \frac{1}{2}\left(z+\frac{1}{z}\right)\right]$$

$$\therefore 2Z = (1+i)z + (1-i)\frac{\bar{z}}{4}$$

By letting $z = 2(\cos\theta + i\sin\theta)$, the eq^n above yeilds:

$$2(x+yi) = 2(1+i)(\cos\theta + i\sin\theta) + \frac{1}{2}(1-i)(\cos\theta - i\sin\theta)$$

$$2(x+yi) = \frac{5}{2}(\cos\theta - \sin\theta) + \frac{3i}{2}(\cos\theta + i\sin\theta)$$

Hence:

$$\left.\begin{aligned} \frac{4}{5}x &= \cos\theta - \sin\theta \\ \frac{4}{3}y &= \cos\theta + \sin\theta \end{aligned}\right]$$

$$\left(\frac{4x}{5}\right)^2 + \left(\frac{4y}{3}\right)^2 = \cos^2\theta - \cancel{2\cos\theta\sin\theta} + \sin^2\theta + \cos^2\theta + \cancel{2\cos\theta\sin\theta} + \sin^2\theta.$$

$$\frac{16x^2}{25} + \frac{16y^2}{9} = 2$$

$$\frac{x^2}{\frac{25}{8}} + \frac{y^2}{\frac{9}{8}} = 1 \quad \text{which is an ellipse}$$

<u>Alternatively:</u>

$P(z)$ has coordinates $(2\cos\theta, 2\sin\theta)$

$\therefore OP = z = 2(\cos\theta + i\sin\theta)$

$R\left(\frac{1}{z}\right)$ has coordinates $\left(\frac{1}{2}\cos\theta, -\frac{1}{2}\sin\theta\right)$

$\therefore OR = \frac{1}{z} = \frac{1}{2}\cos\theta - \frac{i}{2}\sin\theta.$

$$RP = 2\cos\theta + 2\sin\theta - \left[\frac{1}{2}\cos\theta - \frac{i}{2}\sin\theta\right]$$
$$= \frac{3}{2}\cos\theta + \frac{5}{2}i\sin\theta.$$

$$RQ = \frac{1}{2}\left[3\cos\theta + 5i\sin\theta\right]\left[\frac{1+i}{2}\right]$$
$$= \frac{1}{4}\left[3\cos\theta - 5\sin\theta + i(3\cos\theta + 5\sin\theta)\right]$$

$$OQ = \frac{5}{4}(\cos\theta - \sin\theta) + \frac{3i}{4}(\cos\theta + \sin\theta)$$

$$\therefore \left.\begin{aligned} x &= \frac{5}{4}(\cos\theta - \sin\theta) \\ y &= \frac{3}{4}(\cos\theta + \sin\theta) \end{aligned}\right]$$

$$\left.\begin{aligned} \frac{4x}{5} &= \cos\theta - \sin\theta \\ \frac{4y}{3} &= \cos\theta + \sin\theta \end{aligned}\right]$$

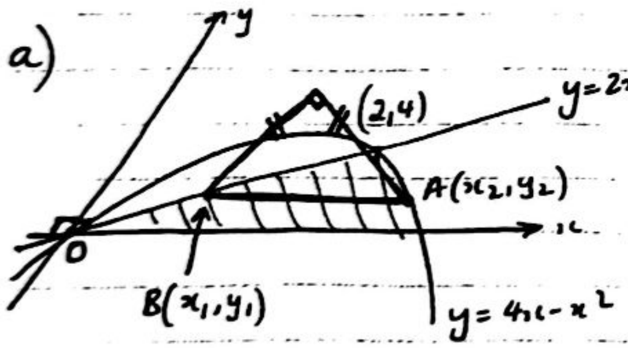
$$\Longrightarrow \frac{x^2}{\frac{25}{8}} + \frac{y^2}{\frac{9}{8}} = 1$$

(ii) Eq^n of ellipse is $\frac{x^2}{\frac{25}{8}} + \frac{y^2}{9/8} = 1$ where $a = \frac{5}{\sqrt{8}}$

$b = \frac{3}{\sqrt{8}}$

Question 6

a)



$$\left.\begin{aligned} y &= 2x \\ y &= 4x - x^2 \end{aligned}\right]$$

Solving gives $(0,0)$ & $(2,4)$ as pts of intersection

Let AB be base of a typical right-angled isosceles triangle

Let $AB = x_2 - x_1$

Now $y = 4x_2 - x_2^{\,2}$ at $A(x_2, y_2)$

$= 4 - (x_2 - 2)^2$

$(x_2 - 2)^2 = 4 - y$

$x_2 = 2 \pm \sqrt{4-y}$

$\therefore x_2 = 2 + \sqrt{4-y}$ (since at $x = 2, y = 4$)

Also, $y = 2x_1$ at $B(x_1, y_1)$

$\therefore x_1 = \frac{y}{2}$

$\therefore AB = 2 + \sqrt{4-y} - \frac{y}{2}$

(16)

Let 2 equal sides of isosceles triangle be 'a'

By pythagoras' th:

$$2a^2 = \left(2 + \sqrt{4-y} - \frac{y}{2}\right)^2$$

But area of required triangle is $\frac{1}{2}a^2$

$$\therefore A(y) = \frac{1}{4}\left(2 + \sqrt{4-y} - \frac{y}{2}\right)^2$$

$$= \frac{1}{16}\left(4 + 2\sqrt{4-y} - y\right)^2$$

$$V = \int_a^b A(y)\,dy = \frac{1}{16}\int_0^4 \left(4 + 2\sqrt{4-y} - y\right)^2 dy$$

as required.

ii) $V = \frac{1}{16}\int_0^4 \left(4-y + 2\sqrt{4-y}\right)^2 dy$

$$= \frac{1}{16}\int_0^4 \left[(4-y)^2 + 4(4-y)^{3/2} + 4(4-y)\right] dy$$

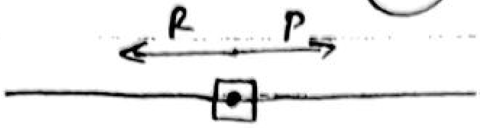
$$= \frac{1}{16}\left[\frac{(4-y)^3}{-3} + \frac{2.4}{5}\frac{(4-y)^{\frac{5}{2}}}{-1} + \frac{4(4-y)^2}{2\times -1}\right]_0^4$$

$$= -\frac{1}{16}\left[\frac{(4-y)^3}{3} + \frac{8}{5}(4-y)^{\frac{5}{2}} + 2(4-y)^2\right]_0^4$$

$$= +\frac{1}{16}\left[\frac{64}{3} + \frac{8}{5}.32 + 32\right]$$

$$= 6\frac{8}{15}\,u^3$$

(17)

b) 

When $t=0$, $v = V_0$ m/s.

$R = 5+3v$

$P = 10$

(i) $$m\ddot{x} = P - R$$
$$= 10 - (5+3v)$$
$$= 5 - 3v$$
$$\ddot{x} = \frac{5-3v}{m}$$

Now $\ddot{x} = \frac{dv}{dt} \implies \frac{dv}{dt} = \frac{5-3v}{m}$

$$\frac{dt}{dv} = \frac{m}{5-3v}$$

$$t = \int \frac{m}{5-3v}\, dv$$

$$t = \frac{m}{-3}\ln(5-3v) + c$$

When $t=0$, $v=v_0 \implies 0 = -\frac{m}{3}\ln(5-3v_0) + c$

$$c = \frac{m}{3}\ln(5-3v_0)$$

$$\therefore t = \frac{m}{3}\ln(5-3v_0) - \frac{m}{3}\ln(5-3v)$$

$$\therefore t = \frac{m}{3}\ln\left(\frac{5-3v_0}{5-3v}\right)$$

Now $$\frac{3t}{m} = \ln\left(\frac{5-3v_0}{5-3v}\right)$$

$$e^{\frac{3t}{m}} = \frac{5-3v_0}{5-3v}$$

$$(5-3v)e^{\frac{3t}{m}} = 5-3v_0$$

$$5e^{\frac{3t}{m}} - 3ve^{\frac{3t}{m}} = 5-3v_0$$

$$ve^{\frac{3t}{m}} = \frac{5}{3}e^{\frac{3t}{m}} + v_0 - \frac{5}{3}$$

$$V = \frac{5}{3} + V_0 e^{-\frac{3t}{m}} - \frac{5}{3}e^{-\frac{3t}{m}}$$

$$\therefore V = \frac{5}{3}\left(1 - e^{-\frac{3t}{m}}\right) + V_0 e^{-\frac{3t}{m}}$$

(ii) as $t \to \infty$, $v \to \frac{5}{3}$

$\therefore$ terminal velocity is $\frac{5}{3}$ m/s.

(iii) $\ddot{x} = \frac{5-3v}{m}$

taking $\ddot{x} = v.\frac{dv}{dx}$ gives:

$$v.\frac{dv}{dx} = \frac{5-3v}{m}$$

$$\frac{dv}{dx} = \frac{5-3v}{mv}$$

$$\frac{dx}{dv} = \frac{mv}{5-3v}$$

$$x = \int_{v_0}^{v_1} \frac{mv}{5-3v}\, dv$$

Now $u = 5-3v$

$du = -3\, dv$

$-\frac{du}{3} = dv$

also: $\frac{5-u}{3} = v$.

When $v = v_0$ $u = 5-3v_0$

$v = v_1$ $u = 5-3v_1$

$$\therefore x = \frac{-m}{3\times 3}\int_{5-3v_0}^{5-3v_1} \frac{5-u}{u}\, du$$

$$= \frac{m}{9}\int_{5-3v_0}^{5-3v_1} \frac{u-5}{u}\, du$$

$$= \frac{m}{9}\int_{5-3v_0}^{5-3v_1} 1 - \frac{5}{u}\, du$$

$$= \frac{m}{9}\Big[u - 5\ln u\Big]_{5-3v_0}^{5-3v_1}$$

$$= \frac{m}{9}\Big[5-3v_1 - 5\ln(5-3v_1) - (5-3v_0) + 5\ln(5-3v_0)\Big]$$

$$= \frac{m}{9}\Big[\cancel{5}-3v_1 - 5\ln(5-3v_1) - \cancel{5} + 3v_0 + 5\ln(5-3v_0)\Big]$$

$$= \frac{m}{9}\Big[3(v_0-v_1) + 5\ln\Big(\frac{5-3v_0}{5-3v_1}\Big)\Big]$$

Question 7

a) Let $P(x,y)$ be a point on the conic, and let S be pt $(2,0)$. M is the foot of the $\perp$ from P to the directrix.

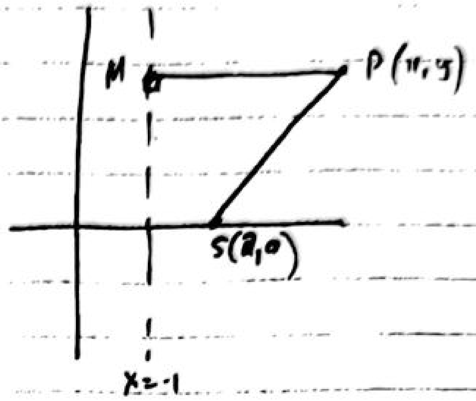
Then: $PS^2 = e^2 . PM^2$

$PS^2 = \sqrt{2}^2 . PM^2$ (since eccentricity is $\sqrt{2}$)

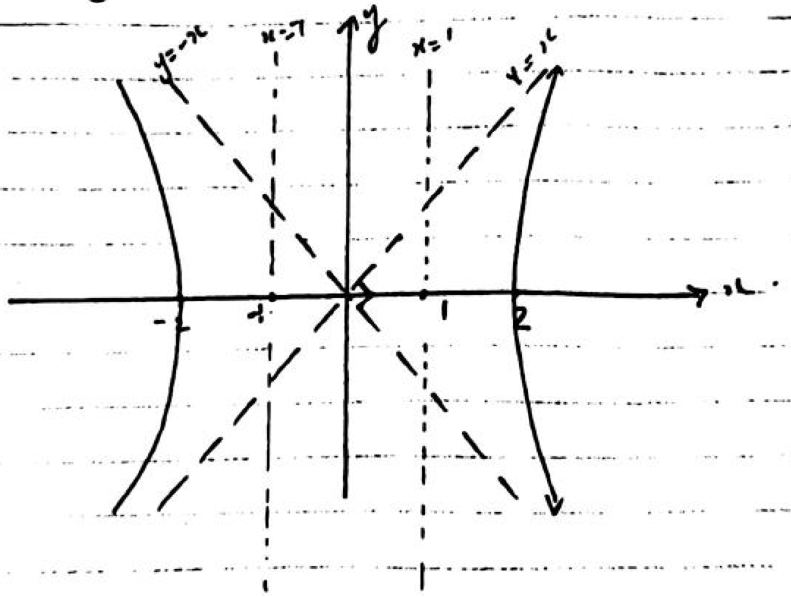
$(x-2)^2 + y^2 = 2(x-1)^2$

$x^2 - 4x + 4 + y^2 = 2x^2 - 4x + 2$

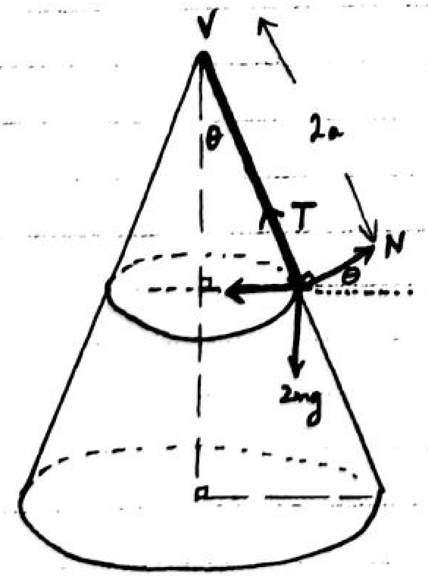
$\underline{x^2 - y^2 = 2}$



This is an eq^n of a hyperbola with asymptotes $y = \pm x$. Since the lines are perpendicular to each other, the conic is a rectangular hyperbola.



b)



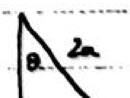
Forces acting on P are:

① Centripital force directed towards the central axis of the cone:

$= 2mr\omega^2$

$= 2m \cdot 2a \cdot \omega^2 \cdot \sin\theta$

$= 4ma\omega^2 \sin\theta$



$\frac{r}{2a} = \sin\theta$

$r = 2a.\sin\theta$

② Weight $= 2mg$

③ Normal reaction N, which is $\perp$ to VP

20

Resolving forces at P (vertically)

$$T\cos\theta + N\sin\theta = 2mg \quad \text{--- (1)}$$

Resolving forces at P (horizontally)

$$T\sin\theta - N\cos\theta = 4ma\omega^2\sin\theta \quad \text{--- (2)}$$

(i) (1) × $\cos\theta$ + (2) × $\sin\theta$:

$$\left.\begin{aligned} T\cos^2\theta + N\sin\theta\cos\theta &= 2mg\cos\theta \\ T\sin^2\theta - N\sin\theta\cos\theta &= 4ma\omega^2\sin^2\theta \end{aligned}\right]$$

$$T(\cos^2\theta + \sin^2\theta) = 2m(g\cos\theta + 2a\omega^2\sin^2\theta)$$

$$\therefore T = 2m(g\cos\theta + 2a\omega^2\sin^2\theta) \text{ Newtons}$$

(ii) (1) × $\sin\theta$ − (2) × $\cos\theta$:

$$\left.\begin{aligned} T\sin\theta\cos\theta + N\sin^2\theta &= 2mg\sin\theta \\ T\sin\theta\cos\theta - N\cos^2\theta &= 4ma\omega^2\sin\theta\cos\theta \end{aligned}\right]$$

$$N(\cos^2\theta + \sin^2\theta) = 2m(g\sin\theta - 2a\omega^2\sin\theta\cos\theta)$$

$$\therefore N = 2m(g\sin\theta - 2a\omega^2\sin\theta\cos\theta) \text{ Newtons}$$

(iii) For the particle to remain in contact with the surface of the cone, then $T > 0$ & $N > 0$ for all values of ω. Since T is always positive, need to only consider N:

$$\left[\begin{aligned} \text{ie } g\sin\theta &> 2a\omega^2\sin\theta\cos\theta \\ g\sin\theta - 2a\omega^2\sin\theta\cos\theta &> 0 \end{aligned}\right]$$

$N > 0$ $\qquad \sin\theta(g - 2a\omega^2\cos\theta) > 0$ ✓

$$\text{ie } g - 2a\omega^2\cos\theta > 0$$

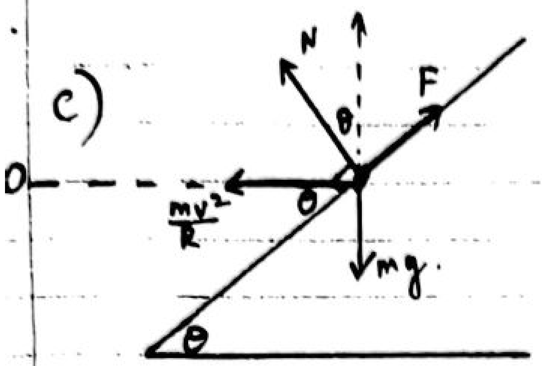
$$g > 2a\omega^2\cos\theta$$

$$\frac{g}{2a\cos\theta} > \omega^2$$

$$\omega^2 < \frac{g}{2a\cos\theta}$$

$$\omega < \left(\frac{g}{2a\cos\theta}\right)^{\frac{1}{2}}$$

c)



(i) If there is no lateral force then, $F=0$

ie $\frac{\cancel{m}v^2}{R}\cos\theta = \cancel{m}g\sin\theta$

$$\frac{v^2}{Rg} = \frac{\sin\theta}{\cos\theta}$$

$$\frac{v^2}{Rg} = \tan\theta.$$

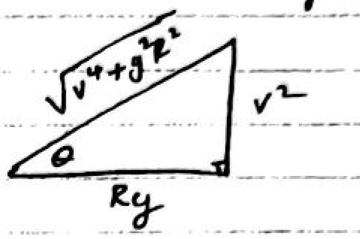
(ii) $F = mg\sin\theta - \frac{mv^2}{R}\cos\theta$

$$= mg\frac{v^2}{\sqrt{v^4+g^2R^2}} - \frac{m}{\cancel{R}}\left(\frac{v}{2}\right)^2 \cdot \frac{\cancel{R}g}{\sqrt{v^4+g^2R^2}}$$

$$= \frac{mv^2g}{\sqrt{v^4+g^2R^2}} - \frac{mv^2g}{4\sqrt{v^4+g^2R^2}}$$

$$F = \frac{3mv^2g}{4\sqrt{v^4+g^2R^2}}.$$

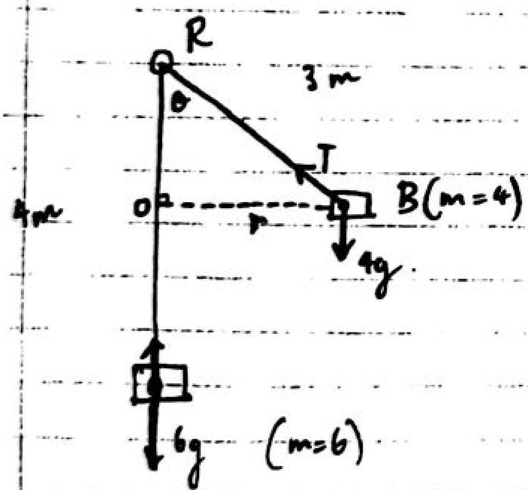
Now since $\tan\theta = \frac{v^2}{Rg}$.

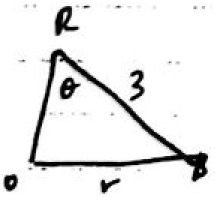


$$\sin\theta = \frac{v^2}{\sqrt{v^4+g^2R^2}}$$

$$\cos\theta = \frac{Rg}{\sqrt{v^4+g^2R^2}}$$

Question 8





$$\frac{r}{3} = \sin\theta$$

$$r = 3\sin\theta$$

As the ring is frictionless, the tensions in AR and BR are equal.

Resolving forces at B gives:

Vertically: $T\cos\theta = mg.$

ie $T\cos\theta = 4g$ --- ①

Horizontally. $T\sin\theta = mr\omega^2$

ie $T\cancel{\sin\theta} = 4 \times 3\cancel{\sin\theta}.\omega^2$

$T = 12\omega^2$ --- ②

Resolving forces at A gives (equilibrium)

$T = 6g$ --- ③

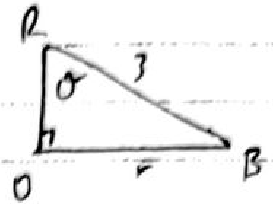
(i) From ③ $T = 6g$. Substituting into ① gives:

$$6g / \cos\theta = 4g$$

$$\cos\theta = \frac{2}{3}$$

$$\frac{OR}{3} = \frac{2}{3}$$

$$\underline{\underline{OR = 2m}}$$



By pythagoras, $OB^2 + OR^2 = RB^2$

$$OB^2 + 4 = 9$$

$$\underline{\underline{OB = \sqrt{5}\ m}}$$

(ii) From ②: $T = 12\omega^2$ & $T = 6g$

$$6g = 12\omega^2$$

$$\sqrt{\frac{9.8}{2}} = \omega$$

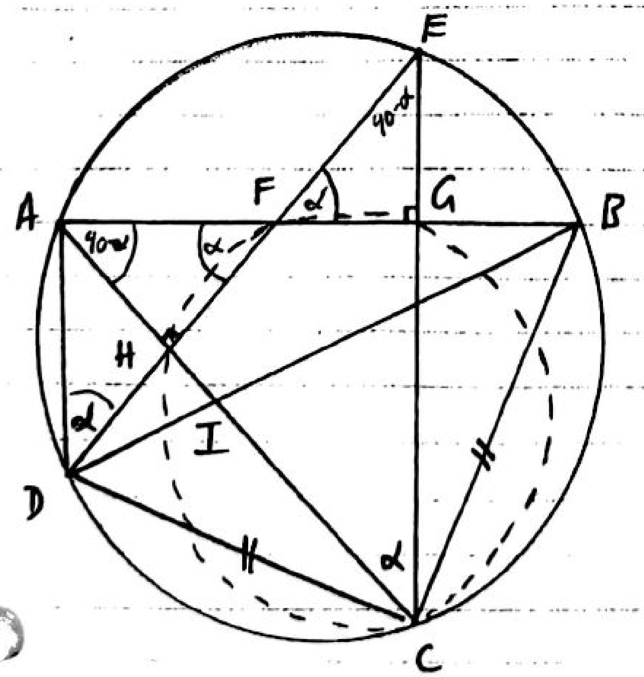
$$\therefore \omega = \sqrt{4.9} \text{ rads/sec}$$

$$= \sqrt{4.9} \cdot \frac{1}{2\pi} \text{ revolutions/sec}$$

$$= \frac{\sqrt{4.9} \times 60}{2\pi} \text{ revolutions/minute}$$

$$\doteqdot \underline{\underline{21.14 \text{ revolutions/minute}}} \quad (2dp)$$

b)



(i) FGCH is cyclic (opposite angles of quad. FGCH are supplementary)

(ii) Prove ΔBCD is isosceles.

$\angle EFB = \alpha$ (ext. angle of cyclic quad FGCH is equal to the interior opposite $\angle GCH$)
$\angle AFH = \alpha$ (vertically opposite angles)
$\angle ADE = \alpha$ (angles in same segment AE of circle (lge) are equal)
$\angle FAH = 90 - \alpha$ (angle sum of ΔAFH is $180°$)
$\angle FEG = 90 - \alpha$ (" " " ΔFEG " ")
$\therefore BC = DC$ (angles subtended to circumference by chords BC & CD are equal)
$\therefore$ ΔBCD is isosceles (2 equal sides BC & CD)

(iii) Prove $\Delta CID ||| \Delta CDA$

$\angle DBC = \angle DAC$ ($= \angle DEC$) $= 90 - \alpha$ (angles in same segment are equal).
$\angle CDI = \angle DBC = 90 - \alpha$ (base angles of isosceles ΔBCD
$\therefore \angle CDI = \angle DAC = 90 - \alpha$ ($\angle DAC$ is equivalent to $\angle DAH$)
$\angle ACD$ is common.
$\therefore$ $\Delta CID ||| \Delta CDA$ (equiangular).

(iv) Similarly $\Delta CIB ||| \Delta CBA$ (equiangular)

Now $\dfrac{CI}{DC} = \dfrac{DI}{AD}$ (corresponding sides of similar Δ's CID & CDA)

$$AD = \frac{DC \cdot DI}{CI} \quad \text{--- (1)}$$

Also, $\dfrac{CI}{BC} = \dfrac{BI}{AB}$ (corresponding sides of similar Δ's CIB & CBA)

$$AB = \frac{BC \cdot BI}{CI} \quad \text{--- (2)}$$

(1) + (2): $$AB + AD = \frac{BC \cdot BI}{CI} + \frac{DC \cdot DI}{CI}$$

$$2BC = \frac{BC \cdot BI}{CI} + \frac{BC \cdot DI}{CI} \qquad (BC = CD)$$

$$2\cancel{BC} = \cancel{BC}\left(\frac{BI + DI}{CI}\right)$$

$2CI = BD$ as required (as $BI + DI = BD$)