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| Student Number: |  |  |
| Class:          |  |  |



# TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2014

# MATHEMATICS EXTENSION 2

### General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 11 - 16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

### Total Marks 100

### Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

### Section II: 90 Marks

- Attempt Question 11 - 16
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

**Section I**

**10 marks Attempt Questions 1–10**

**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

1. If  $z = 1 + 2i$  and  $w = 3 - i$ , what is the value of  $z - \bar{w}$ ?

(A)  $3i - 2$

(B)  $4 + 3i$

(C)  $i - 2$

(D)  $4 + i$

2. Which of the following is an expression for?  $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} dx$

(A)  $x + \frac{1}{2} \cos 2x + c$

(B)  $x - \frac{1}{2} \cos 2x + c$

(C)  $x + \frac{1}{2} \sin^2 x + c$

(D)  $x - \frac{1}{2} \sin^2 x + c$

3. The equation of the tangent to  $xy^3 + 2y = 4$  at the point (2, 1) is

(A)  $x + 8y = 10$

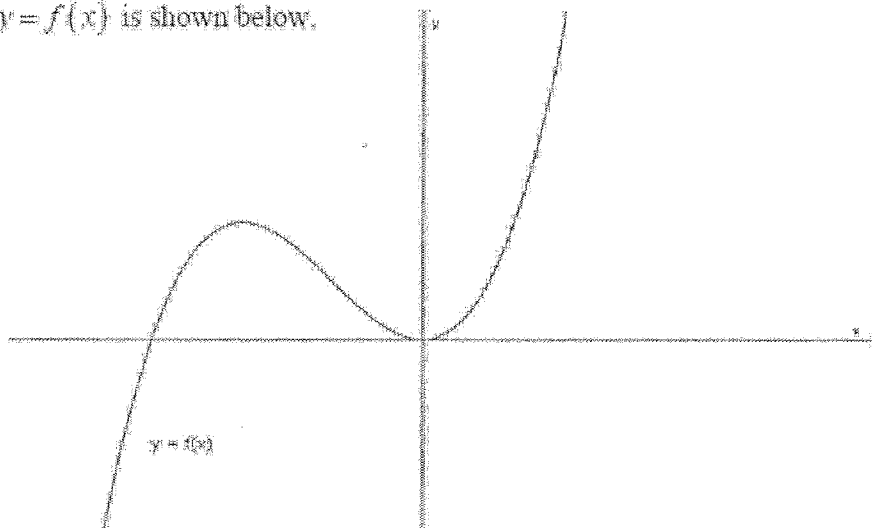
(B)  $x - 8y = 10$

(C)  $x + 8y = -10$

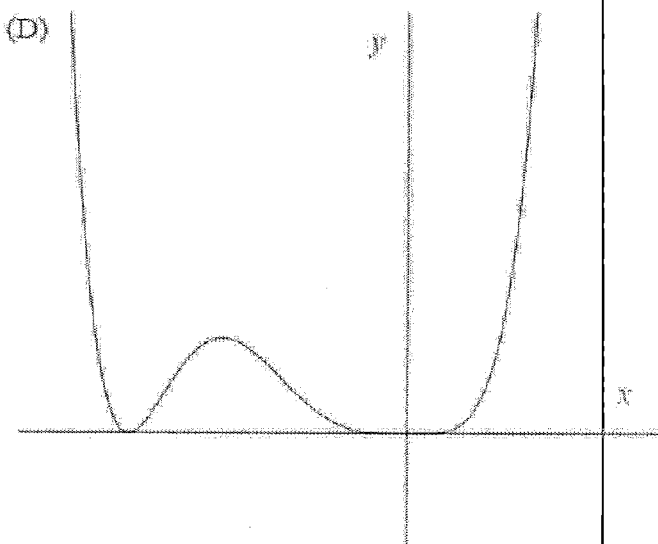
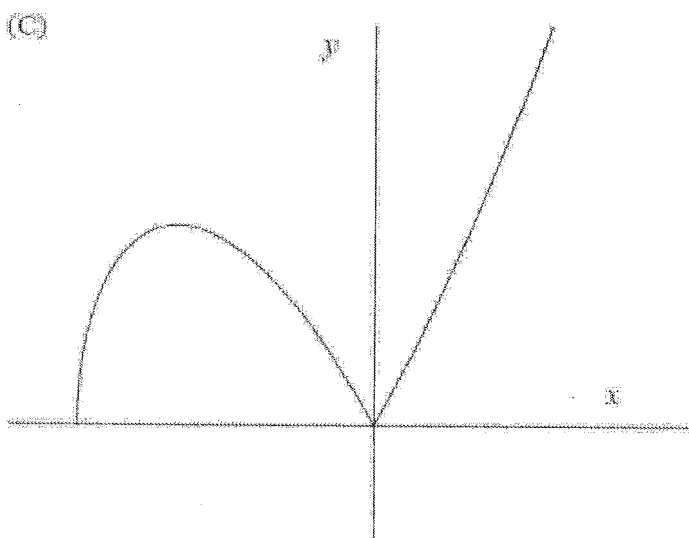
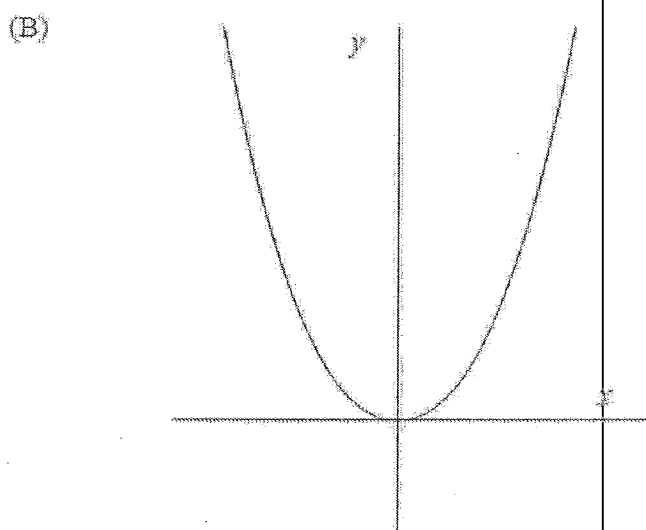
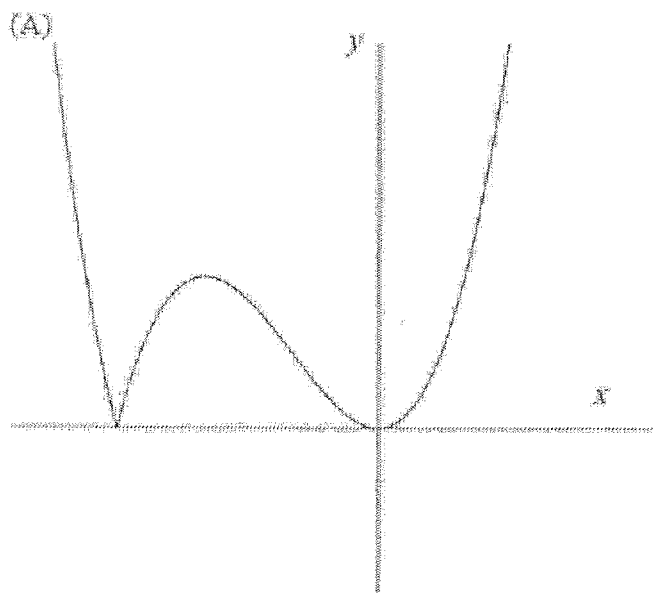
(D)  $x - 8y = -10$

4.

The graph of  $y = f(x)$  is shown below.



Which of the following graphs best represents  $y = f(|x|)$ ?



5. The point  $P(a \cos \theta, b \sin \theta)$  lies on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  where  $a > b > 0$

What is the equation of the normal at P?

(A)  $\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$

(B)  $\frac{ax}{\cos \theta} + \frac{by}{\sin \theta} = a^2 + b^2$

(C)  $\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$

(D)  $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$

6. What is the multiplicity of the root  $x=1$  of the equation  $3x^5 - 5x^4 + 5x - 3 = 0$

(A) 1

(B) 2

(C) 3

(D) 4

7. Without evaluating the integrals which one of the following will give an answer of zero?

(A)  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^3 \theta + 1}{\cos^2 \theta} d\theta$

(B)  $\int_{-1}^1 (x^2 - 1)(1 - x^2)^3 dx$

(C)  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \cos x dx$

(D)  $\int_{-2}^2 |x^2 - 4| dx$

8. The base of a solid is the circle  $x^2 + y^2 = 1$ . Every cross section of the solid taken perpendicular to the  $x$  axis is a right angled, isosceles triangle with its hypotenuse lying in the base of the solid. Which of the following is an expression for the volume  $V$  of the solid?

(A)  $\int_{-1}^1 (1 - x^2) dx$

(B)  $2 \int_{-1}^1 (1 - x^2) dx$

(C)  $4 \int_{-1}^1 (1 - x^2) dx$

(D)  $\frac{1}{2} \int_{-1}^1 (1 - x^2) dx$

9. A particle of mass  $m$  is moving horizontally in a straight line. Its motion is opposed by a force of magnitude  $mk(v + v^2)$  Newtons when its speed is  $v \text{ ms}^{-1}$  (where  $k$  is a positive constant). At time  $t$  seconds the particle has displacement  $x$  metres from a fixed point  $O$  on the line and velocity  $v \text{ ms}^{-1}$ . Which of the following is an expression for  $x$  in terms of  $v$ ?

(A)  $\frac{1}{k} \int \frac{1}{1 + v} dv$

(B)  $\frac{1}{k} \int \frac{1}{v(1 + v)} dv$

(C)  $-\frac{1}{k} \int \frac{1}{v(1 + v)} dv$

(D)  $-\frac{1}{k} \int \frac{1}{1 + v} dv$

10 Four digit numbers are formed from the digits 1, 2, 3 and 4. Each digit is used once only. The sum of all the numbers that can be formed is?

(A) 266,640

(B) 66,660

(C) 44,440

(D) 6,666

## Section II

90 marks Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question on a SEPARATE sheet of paper. Extra paper is available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

### Question 11

a) Find  $\int \frac{dx}{\sqrt{6x-x^2}}$  2

b) Show that  $\int_2^4 \frac{dx}{x\sqrt{x-1}} = \frac{\pi}{6}$  3

c) Find  $\int e^x \sin x \, dx$  3

d) Draw a diagram to illustrate the locus of the points  $z$  on the Argand diagram such that:

(i)  $|z - \bar{z}| \leq 1$  and  $|z - 1| \leq 2$  2

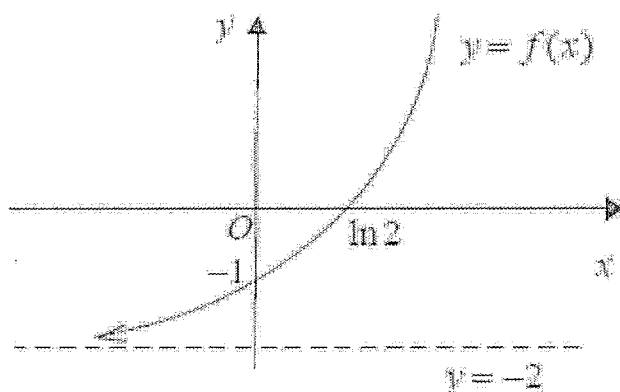
(ii)  $\text{Arg} \left\{ \frac{z-1}{z+1} \right\} = \frac{\pi}{4}$  2

e) Show by geometrical means or otherwise that if  $z_1$  and  $z_2$  are complex numbers such that 3

$|z_1| = |z_2|$ , then  $\frac{z_1 + z_2}{z_1 - z_2}$  is purely imaginary.

**Question 12 ( Start a new page)**

- (a) The diagram below shows the graph of  $f(x) = e^x - 2$



On separate diagrams sketch the following graphs, in each case showing the intercepts on the axes and the equations of the asymptotes.

(i)  $y = (f(x))^2$

(ii)  $y = \log_e f(x)$

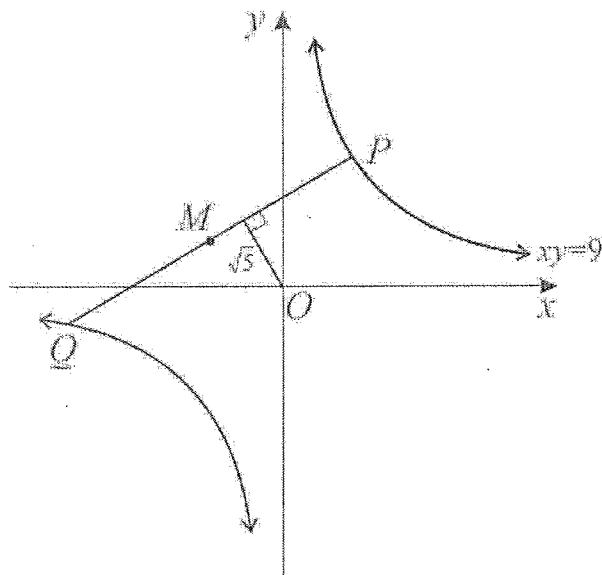
(iii)  $y = \frac{1}{f(x)}$

(iv)  $y^2 = |f(x)|$

- (b) (i) Show that  $4x^2 + 9y^2 + 16x + 18y - 11 = 0$  represents an ellipse.

(ii) Find the eccentricity and hence the coordinates of its foci and the equation of its directrices.

(c)



In the diagram above,  $P\left(3p, \frac{3}{p}\right)$  and  $Q\left(3q, \frac{3}{q}\right)$  are variable points on the rectangular hyperbola

$xy = 9$ . The perpendicular distance from the origin to the chord PQ is  $\sqrt{5}$  units. Let M be the midpoint of the chord PQ.

(i) Show that the chord PQ has the equation  $x + pqy = 3(p + q)$  2

(ii) Using the perpendicular distance formula, or otherwise, show that  $9(p + q)^2 = 5(1 + p^2q^2)$  1

(iii) Show that the locus of M has the Cartesian equation 3

$$y^2 = \frac{5x^2}{4x^2 - 5}$$

## Question 13 (Start a new page)

(a) (i) Use the substitution  $x=10\sqrt{2}\sin\theta$  to show that  $\int_{-10}^{10}\sqrt{200-x^2}\,dx=100+50\pi$

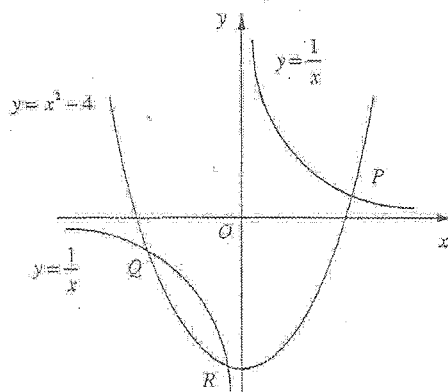
(ii) Use a geometrical argument to verify the result in part (i)

(iii) A mould for a model railway tunnel is made by rotating the region bounded by the curve  $y=\sqrt{200-x^2}$  and the  $x$ -axis between the lines  $x=-10$  and  $x=10$  through  $180^\circ$  about the line  $x=100$  (where all the measurements are in cm).

Use the method of cylindrical shells to show that the volume  $V\text{cm}^3$  of the tunnel is given by  $\pi\int_{-10}^{10}(100-x)\sqrt{200-x^2}\,dx$ .

Hence find the volume of the tunnel in  $\text{m}^3$  correct to 2 significant figures.

(b)



The curves  $y=x^2-4$  and  $y=\frac{1}{x}$  intersect at the points P, Q, R where  $x=\alpha$ ,  $x=\beta$ ,  $x=\gamma$

(i) Show that  $\alpha, \beta, \gamma$  are the roots of the equation  $x^3-4x-1=0$

(ii) Find a polynomial equation with integer coefficients which has roots  $\alpha^2, \beta^2, \gamma^2$ .

(iii) Find a polynomial equation with integer coefficients which has roots  $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ .

(iv) Hence find the numerical value of  $OP^2 + OQ^2 + OR^2$

**Question 14 (Start a new page)**

- (a) Figure 1 below shows a scale model of the volcano Mt Snaefellsjökull. The base of the model is elliptical in shape with the axes 60cm by 40cm reducing uniformly to a circle of radius 12cm at the top. The hollow core of the model has circular cross sections with a circle of radius 6cm at the base rising uniformly to a circle of radius 12cm at the top. The model is 24cm high. Figure 2 shows the top view of the cross sectional area of the volcano

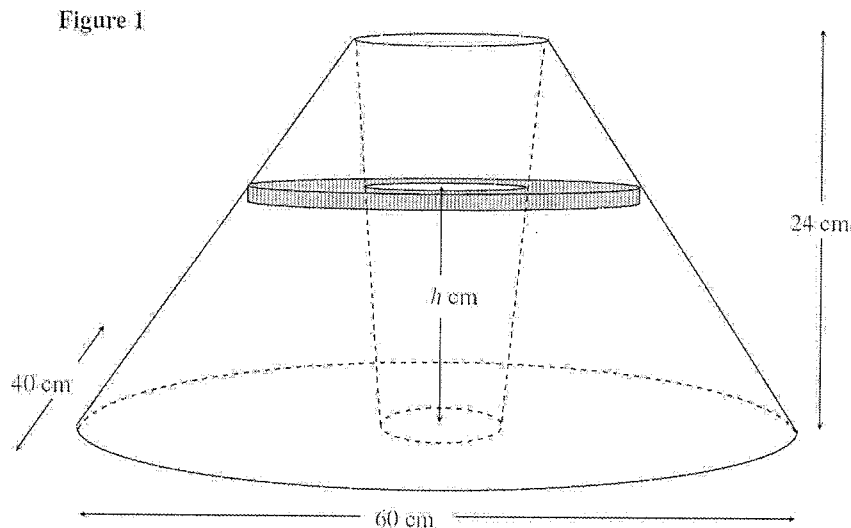
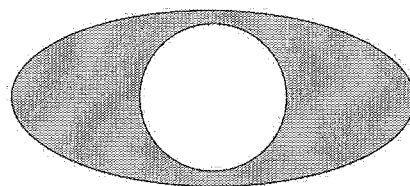


Figure 2



- (i) Show that at height  $h$ , the length of the semi-major axis is given by

$$a = 30 - \frac{3}{4}h$$

- (ii) Show that the cross sectional slice at height  $h$  is given by

$$A = \frac{\pi}{16}(9024 - 448h + 3h^2)$$

You can assume the area of an ellipse with semi-major axis  $a$  and semi-minor axis  $b$  is given by  $\pi ab$

- (iii) Find the volume of the scale model of Mt Snaefellsjökull

Question 14 continued on next page

**Question 14** continued

- (b) (i) Show that if  $n$  is an even positive integer, then

$$(1+x)^n + (1-x)^n = 2 \sum_{k=0}^{n/2} \binom{n}{2k} x^{2k}$$

2

- (ii) An alphabet consists of three letters A, B and C

- (I) Show that the number of words of 4 letters containing exactly 2B's is

$$\binom{4}{2} 2^2$$

1

- (II) Hence, or otherwise, show that if  $n$  is an even positive integer then the number of words of  $n$  letters with zero or an even number of B's is given by

$$\frac{1}{2}(3^n + 1)$$

2

- (c) Three men Bill, Garry and Jason observe a vertical tower.

Bill stands due North of the tower, and sees its top at an angle of elevation of  $\alpha^\circ$

Garry stands due East of the tower, and sees its top at an angle of elevation of  $\beta^\circ$

Jason stands on a line from Bill to Garry, exactly half way between them.

If Jason observes the top of the tower at an angle of elevation of  $\theta^\circ$ .

Show that  $\cot \theta^\circ = \frac{1}{2} \sqrt{\cot^2 \alpha^\circ + \cot^2 \beta^\circ}$

## Question 15 (Start a new page)

(a) Let  $I_n = \int_1^2 \left(1 - \frac{1}{x}\right)^n dx$  for  $n = 1, 2, 3, \dots$

(i) Show that  $\frac{1}{n+1} I_{n+1} = \frac{1}{n} I_n - \frac{1}{n(n+1)2^n}$  for  $n = 1, 2, 3, \dots$

(ii) Hence show that  $\frac{1}{n+1} I_{n+1} = I_1 - \sum_{r=1}^n \frac{1}{r(r+1)2^r}$

(i) Show that  $\sum_{r=1}^n \frac{1}{r(r+1)2^r} = (1 - \log_e 2) - \frac{1}{n+1} I_{n+1}$  and hence find the limiting sum of the series  $\frac{1}{1 \times 2 \times 2^1} + \frac{1}{2 \times 3 \times 2^2} + \frac{1}{3 \times 4 \times 2^3} + \dots$

(b) Let  $\alpha$  be the complex root of the polynomial  $z^7 = 1$  with the smallest positive argument.

Let  $\theta = \alpha + \alpha^2 + \alpha^4$  and  $\phi = \alpha^3 + \alpha^5 + \alpha^6$

(i) Show that  $\theta + \phi = -1$  and  $\theta\phi = 2$

(ii) Write a quadratic equation whose roots are  $\theta$  and  $\phi$

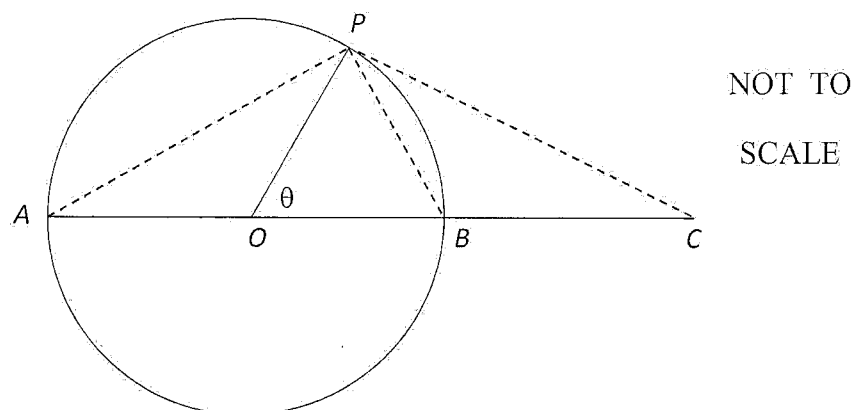
Hence show that  $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$  and  $\phi = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$

(iii) Show that  $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} - \cos \frac{\pi}{7} = -\frac{1}{2}$

**Question 16 (Start a new page)**

- (a) The diagram below shows a point P rotating in a circle of radius 1 metre, whose centre is at O.

AB is a diameter produced to C such that  $OC = 2$  metres.



The angular velocity of P about O is given by  $\dot{\theta} = \pi$  rad/sec. ( $\theta = \angle POB$ )

- |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |   |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---|
| (i) Find the angular velocity of P about A and about B.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | 2 |
| (ii) If $\angle PCO = \alpha$ , show that $\sin(\alpha + \theta) = 2 \sin \alpha$ .                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | 1 |
| (iii) Hence find the angular velocity of P about C at the instant when $\theta = \frac{\pi}{2}$ .                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 4 |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |   |
| (b) The circular bend on a bike track has a constant radius of 20 metres and is banked at a constant angle of $30^\circ$ to the horizontal. A bicycle rider can safely negotiate the bend if the maximum sideways thrust F, up or down the slope is at most one-tenth of the normal reaction N. By resolving the forces vertically and horizontally, show that the range of speeds V, correct to two decimal places and in metres per second, at which the bend can be safely negotiated, is $9.50 \leq V \leq 11.99$ . Take $g = 10 \text{ m/s}^2$ . | 8 |

# Multiple Choice Answers:

1) C 2) D 3) A 4) B 5) A 6) C 7) C 8) A 9) D 10) B

①  $1+2i-(3+i)$   
 $= -2+i$  (C)

$$\frac{dy}{d\theta} = \frac{b \cos \theta}{-a \sin \theta}$$

$$y - b \sin \theta = \frac{a \sin \theta}{b \cos \theta} (x - a \cos \theta)$$

②  $\int \frac{(\cos x + \sin x)(\cos^2 x - \cos x \sin x + \sin^2 x)}{\cos x + \sin x} dx$  where  $y = b^2 \cos \sin \theta$

$$= a \sin \theta x - a^2 \sin \theta \cos \theta$$

$$\therefore \frac{a x}{\cos \theta} - \frac{b y}{\sin \theta} = a^2 - b^2 (-\sin \theta \cos \theta)$$

$$= \int 1 - \cos x \sin x dx$$

$$= x - \frac{1}{2} \sin^2 x + C$$
 (D)

③  $xy^3 + 2y = u$

$$y^3 + x \times 3y^2 \frac{dy}{dx} + 2 \frac{dy}{dx} = 0$$

$$x=2 \quad y=1$$

$$1 + 6 \frac{dy}{dx} + 2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{1}{8}$$

$$y-1 = -\frac{1}{8}(x-2)$$

$$x+8y=10$$
 (A)

④ part B.

⑤  $\frac{dx}{d\theta} = -a \sin \theta \quad \frac{dy}{d\theta} = b \cos \theta$

⑥  $P'(x) = 15x^4 - 20x^3 + 5$

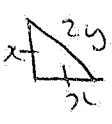
$$P''(x) = 60x^3 - 60x^2$$

$$P'(1) = 0 \quad P''(1) = 0$$

$$P'''(x) = 180x^2 - 60 \quad P'''(1) \neq 0$$

part C

⑦ A even B even C odd D even

⑧   $\frac{x}{2y} = \cos 45^\circ$

$$x = \frac{2y}{\sqrt{2}}$$

$$\text{area} = \frac{1}{2} \times \frac{2y}{\sqrt{2}} \times \frac{2y}{\sqrt{2}}$$

$$\therefore \int (1-x^2) dx$$
 A

⑨  $m \ddot{x} = -nk(v+v^2)$

$$x = -\frac{1}{k} \int \frac{1}{1+u}$$

⑩  $1000+100+10+1=1111$   
 3! ways of arranging 2,3,4  
 $3! (1111) (1+2+3+4)$   
 $= 6666$

## MATHEMATICS Extension 2: Question...!!...

| Suggested Solutions                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | Marks                      | Marker's Comments         |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------|---------------------------|
| <p>(a) <math>\int \frac{dx}{\sqrt{6x-x^2}} dx = \int \frac{dx}{\sqrt{9-(x^2-6x+9)}}</math></p> <p><math>= \int \frac{dx}{\sqrt{9-(x-3)^2}}</math></p> <p><math>= \sin^{-1} \frac{x-3}{3} + C</math></p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | <p>①</p> <p>①</p>          | 1 mark off for each error |
| <p>(b) let <math>u = \sqrt{x-1}</math></p> <p><math>\frac{du}{dx} = \frac{1}{2\sqrt{x-1}}</math>      when <math>x=2, u=1</math><br/>when <math>x=4, u=\sqrt{3}</math></p> <p>now <math>u^2 = x-1</math><br/><math>\therefore x = u^2 + 1</math></p> <p><math>\therefore 2 \int_1^{\sqrt{3}} \frac{du}{u^2+1} = 2 [\tan^{-1} u]_1^{\sqrt{3}}</math></p> <p><math>= 2 [\tan^{-1} \sqrt{3} - \tan^{-1} 1]</math></p> <p><math>= 2 (\pi/3 - \pi/4)</math></p> <p><math>= 2 \times \frac{\pi}{12}</math></p> <p><math>= \pi/6</math></p>                                                                                                                                                                                          | <p>①</p> <p>①</p> <p>①</p> |                           |
| <p>(c) let <math>u = \sin x</math>      <math>\frac{dv}{dx} = e^x</math><br/><math>\frac{du}{dx} = \cos x</math>      <math>v = e^x</math></p> <p><math>\therefore I = uv - \int v du</math></p> <p><math>I = e^x \sin x - \int e^x \cos x dx</math></p> <p>now let <math>u = \cos x</math>      <math>\frac{dv}{dx} = e^x</math><br/><math>\frac{du}{dx} = -\sin x</math>      <math>v = e^x</math></p> <p><math>I = e^x \sin x - [e^x \cos x + \int e^x \sin x dx]</math></p> <p><math>I = e^x \sin x - e^x \cos x - \int e^x \sin x dx</math></p> <p>but <math>I = \int e^x \sin x dx</math></p> <p><math>\therefore 2I = e^x \sin x - e^x \cos x</math></p> <p><math>I = \frac{1}{2} (e^x \sin x - e^x \cos x)</math></p> | <p>①</p> <p>①</p> <p>①</p> |                           |

## MATHEMATICS Extension 2: Question...11...

## Suggested Solutions

## Marks

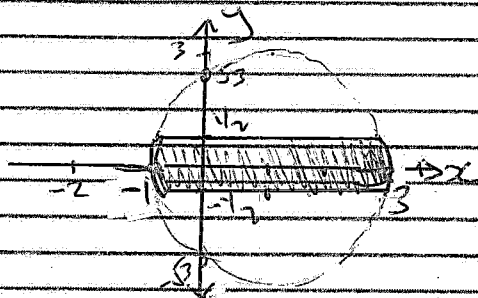
## Marker's Comments

(d)(i)  $|z - \bar{z}| \leq 1$  and  $|z - 1| \leq 2$   
 (1,0) radius 2.

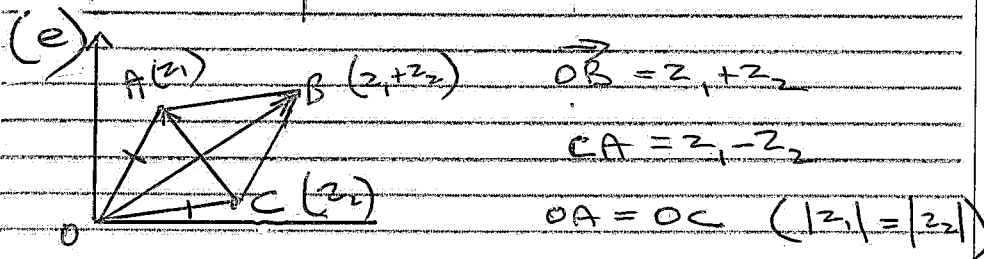
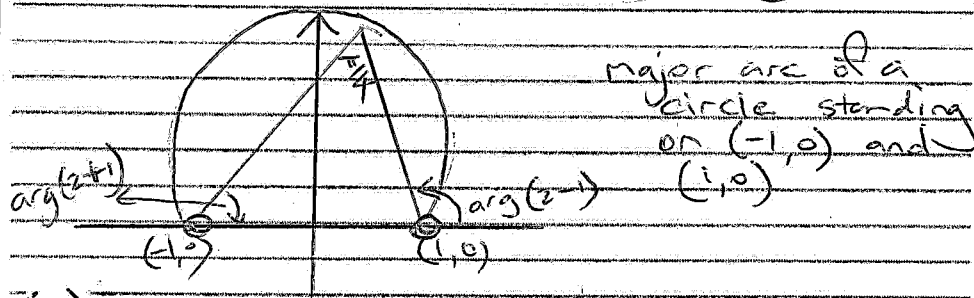
$$|x + yi - (x + yi)| \leq 1$$

$$|2iy| \leq 1$$

$$|y| \leq \frac{1}{2}$$



(ii)  $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$   
 $\arg(z-1) - \arg(z+1) = \frac{\pi}{4}$



OACB is a parallelogram  
 $OA = OC$  (as  $|z_1| = |z_2|$ )

$\therefore$  OACB is a rhombus (parn with a pair of adjacent sides equal.)  
 OB and AC intersect at  $90^\circ$  (diagonals of a rhombus intersect at  $90^\circ$ )

$$\therefore \arg(z_1 + z_2) - \arg(z_1 - z_2) = \pm \frac{\pi}{2}$$

$$\arg\left(\frac{z_1 + z_2}{z_1 - z_2}\right) = \pm \frac{\pi}{2}$$

$\therefore \frac{z_1 + z_2}{z_1 - z_2}$  is purely imaginary.

\* 1 off for each error.

1 for circle (lightly drawn)

1 for correct region shaded.  
 (most students forgot  $y \geq -\frac{1}{2}$ ).

1 for open circles

1 for major arc (and both lines and  $\frac{\pi}{4} \rightarrow$  lightly)

①

①

①

or could discuss rotation by  $90^\circ$ .

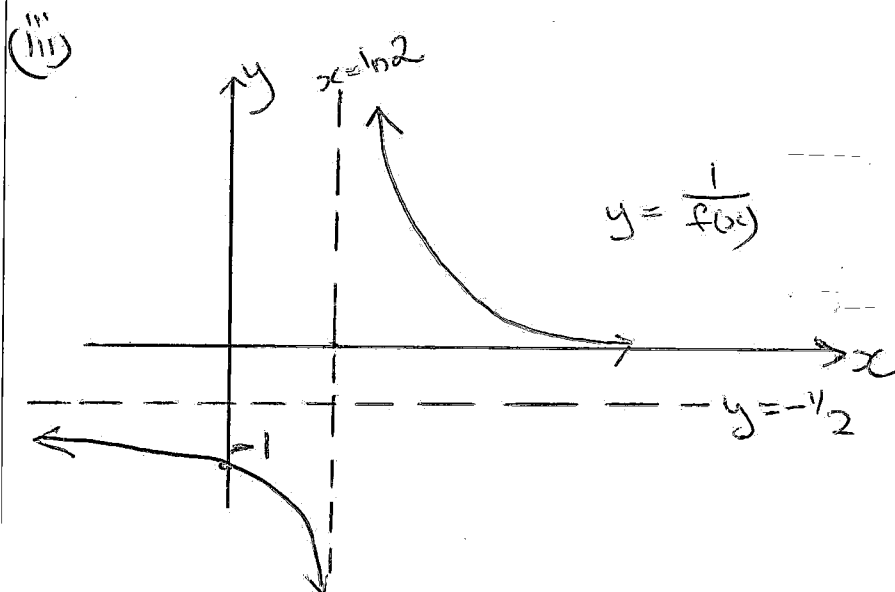
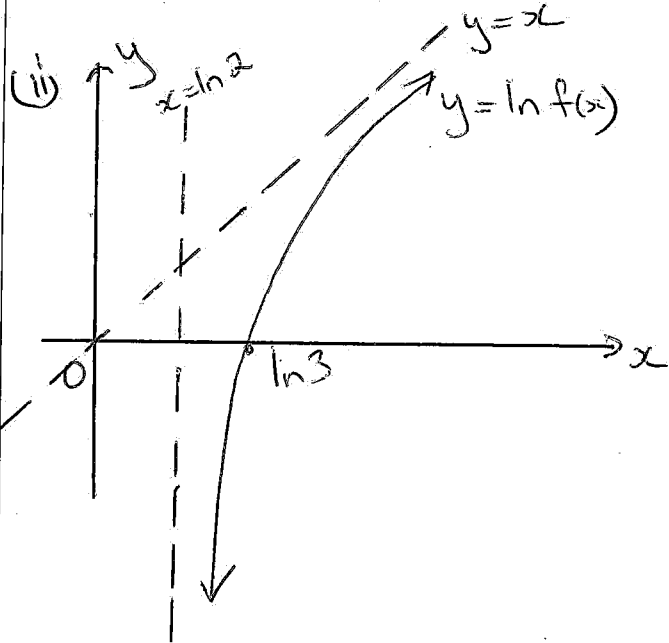
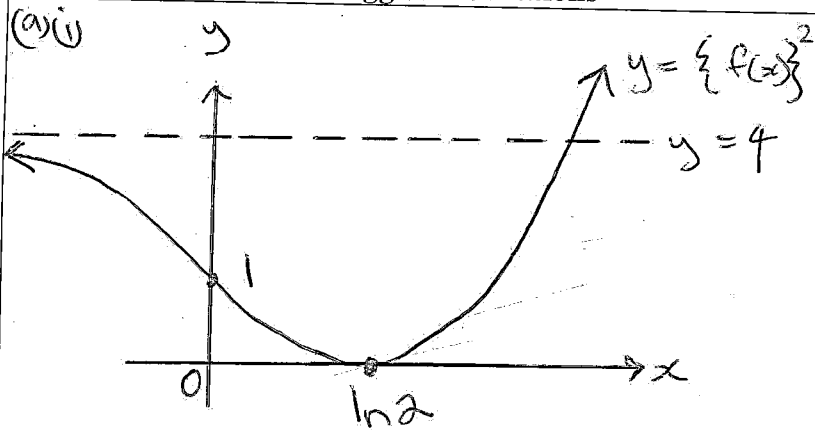
or it could have been done algebraically

# MATHEMATICS Extension 2: Question 12

## Suggested Solutions

## Marks Awarded

## Marker's Comments



-if you left off an intercept then = 0 mks

-a lot of students forgot the asymptote  $y = x$ ,  
-if you left off the x-intercept then you lost the mark.

both asymptotes

shape + intercept

\* as soon as you forgot one thing you lost the mark.

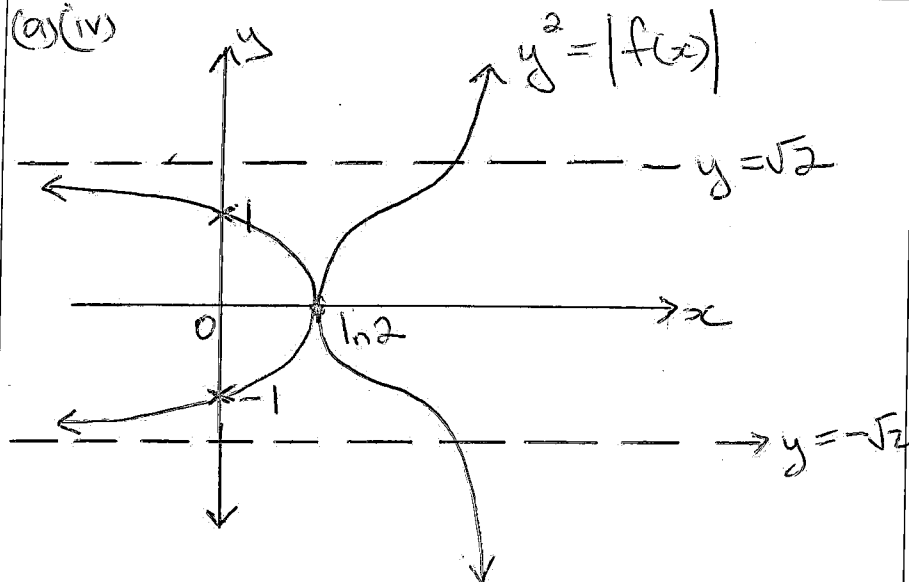
## MATHEMATICS Extension 2: Question... 12... cont.

## Suggested Solutions

## Marks Awarded

## Marker's Comments

(a)(iv)



$$\begin{aligned} \text{b(i)} \quad 4(x^2 + 4x + 4) + 9(y^2 + 2y + 1) &= 11 + 16 + 9 \\ 4(x+2)^2 + 9(y+1)^2 &= 36 \\ \frac{(x+2)^2}{9} + \frac{(y+1)^2}{4} &= 1 \end{aligned}$$

which is in the form  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$   
and so it is an ellipse.

$$\begin{aligned} \text{(ii)} \quad b^2 &= a^2(1 - e^2) \\ 4 &= 9(1 - e^2) \\ e^2 &= \frac{5}{9} \end{aligned}$$

$$e = \frac{\sqrt{5}}{3} \quad \text{as } (0 < e < 1)$$

$$\begin{aligned} \text{directrices} \quad \frac{a}{e} &= \frac{3}{\frac{\sqrt{5}}{3}} = \frac{9}{\sqrt{5}} \end{aligned}$$

$$\begin{aligned} \text{focus} \quad ae &= 3 \times \frac{\sqrt{5}}{3} = \sqrt{5} \end{aligned}$$

$$x = -2 \pm \frac{9}{\sqrt{5}}$$

$$\begin{aligned} &(-2 + \sqrt{5}, -1) \text{ and } \\ &(-2 - \sqrt{5}, -1) \end{aligned}$$

all 3 intercepts  
+ horizontal asymptote.

shape of both  
graphs.

①

①

①

had to get  
both correct to  
get the 2nd mark.

**MATHEMATICS Extension 2: Question...12**

**Suggested Solutions**

**Marks Awarded**

**Marker's Comments**

c) i) Equation of the chord PQ

-1

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{3}{p} - \frac{3}{q}}{3p - 3q} = \frac{-3(q/p)}{pq(p-q)} = -\frac{1}{pq} \quad p \neq q$$

Equation of PQ  $y - y_1 = m(x - x_1)$   
 $y - \frac{3}{p} = -\frac{1}{pq}(x - 3p)$

$$pqy - 3q = -x + 3p$$

$$x + pqy = 3(p+q)$$

(ii) For perpendicular distance

$$d = \frac{|ax+by+c|}{\sqrt{a^2+b^2}} \quad \text{Point } (0,0)$$

line  $x + pqy - 3(p+q) = 0$   
 $d = \sqrt{5}$

$$\sqrt{5} = \frac{|0(1) + 0(pq) - 3(p+q)|}{\sqrt{1 + (pq)^2}} \quad \text{square b.s.}$$

$$5 = \frac{9(p+q)^2}{1 + p^2q^2}$$

$$5(1 + p^2q^2) = 9(p+q)^2$$

(iii) Midpoint  $\left( \frac{3(p+q)}{2}, \frac{\frac{3}{p} + \frac{3}{q}}{2} \right)$

PQ  $x = \frac{3(p+q)}{2}, y = \frac{3(p+q)}{2pq}$

$p+q = \frac{2x}{3}, y = \frac{x}{pq}, pq = \frac{x}{y}$

sub into  $5(1 + p^2q^2) = 9(p+q)^2$  from part (ii)

$$5\left(1 + \frac{x^2}{y^2}\right) = 9\left(\frac{2x}{3}\right)^2$$

$$1 + \frac{x^2}{y^2} = \frac{4x^2}{5}$$

$$\frac{x^2}{y^2} = \frac{4x^2}{5} - 1$$

$$\therefore y^2 = \frac{5x^2}{4x^2 - 5}$$

(2)

① gradient

① Equation with correct working

(1)

Must show perpendicular distance formula and substitution showing zeros.

(3)

① Mid point

①  $pq = \frac{x}{y}$   
 (or similar substitution)

① correct answer with working

## Suggested Solutions

Marks

Marker's Comments

a) Let  $x = 10\sqrt{2} \sin \theta$

$$\therefore \frac{dx}{d\theta} = 10\sqrt{2} \cos \theta$$

$$dx = 10\sqrt{2} \cos \theta d\theta$$

When  $x = 10$ ,  $\sin \theta = \frac{1}{\sqrt{2}}$ ,  $\theta = \frac{\pi}{4}$

$x = -10$ ,  $\sin \theta = -\frac{1}{\sqrt{2}}$ ,  $\theta = -\frac{\pi}{4}$

$$\int_{-10}^{10} \sqrt{200 - x^2} dx = \int_{-\pi/4}^{\pi/4} \sqrt{200 - 200 \sin^2 \theta} 10\sqrt{2} \cos \theta d\theta$$

$$= 200 \int_{-\pi/4}^{\pi/4} \cos^2 \theta d\theta$$

$$= 100 \int_{-\pi/4}^{\pi/4} (1 + \cos 2\theta) d\theta$$

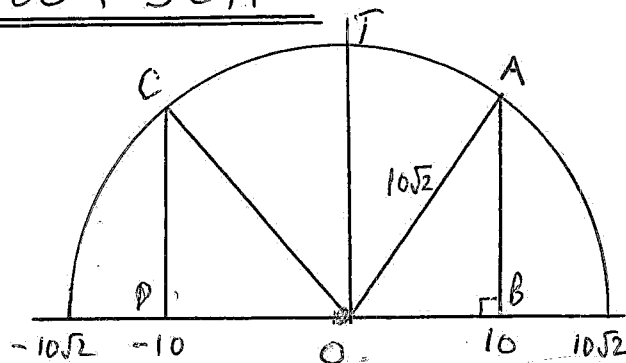
$$= 100 \left[ \theta + \frac{\sin 2\theta}{2} \right]_{-\pi/4}^{\pi/4}$$

$$= 100 \left[ \frac{\pi}{4} + \frac{1}{2} - \left( -\frac{\pi}{4} - \frac{1}{2} \right) \right]$$

$$= 100 \left( \frac{\pi}{2} + 1 \right)$$

$$= \underline{\underline{100 + 50\pi}}$$

b)



Generally well done.

Some people used even symmetry to reduce work at substitution.

A diagram was fairly essential to explain this answer.

## Suggested Solutions

Marks

Marker's Comments

Integral represents the area  $BACD$ .

i.e.  $\triangle ABO + \triangle OCD + \text{Sector } OCA$

By Pythagoras,  $AB = 10$ .

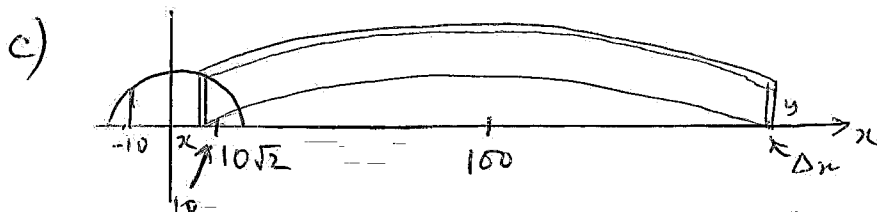
$$\therefore \text{Area } \triangle ABO = \text{Area } \triangle OCD = \frac{1}{2} \times 10 \times 10 = 50.$$

$\triangle OAB$  is isosceles and  $\frac{\pi}{2}$  at  $B$

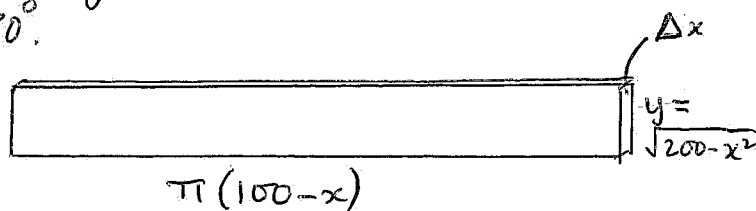
$$\therefore \angle AOC = 2\angle AOB = \frac{\pi}{2}$$

$$\therefore \text{Area sector is } \frac{(10\sqrt{2})^2}{2} \frac{\pi}{2} = 50\pi$$

$$\therefore \text{Integral equals } 50 + 50 + 50\pi = 100 + 50\pi$$



N.B. Only a half shell as rotated by  $180^\circ$ .



$$\Delta V \doteq \pi(100-x)y \Delta x$$

$$= \pi(100-x)\sqrt{200-x^2} \Delta x$$

$$\therefore V = \lim_{\Delta x \rightarrow 0} \sum_{x=-10}^{10} \pi(100-x)\sqrt{200-x^2} \Delta x$$

$$= \pi \int_{-10}^{10} 100\sqrt{200-x^2} dx - \pi \int_{-10}^{10} x\sqrt{200-x^2} dx$$

But the second integral is of

Only 1 mark so all issues had to be addressed.

Alternative version possible with rectangle + segment.

One mark for recognising the half cylinder ( $180^\circ$ )

for setup i.e. diagram(s)

$$\Delta V \doteq$$

$$V = \lim_{\Delta x \rightarrow 0} \sum \text{etc.}$$

Too many lost this mark.

| Suggested Solutions                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | Marks                      | Marker's Comments                                                                                                                                              |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>an odd function between symmetric limits, i.e. equals 0.</p> $\therefore V = 100\pi \int_{-10}^{10} \sqrt{200 - x^2} dx$ $= 100\pi (100 + 50\pi) \text{ using (i)}$ $= 80763.94... \text{ cm}^3$ $= \underline{\underline{0.081 \text{ m}^3}} \text{ (to 2 S.F.)}$                                                                                                                                                                                                                                                                                                                           | <p>1</p> <p>1</p>          | <p>Too many people lost this last mark. Highlight potential traps when reading question.</p>                                                                   |
| <p>b)i) Points of intersection of the two graphs will provide the roots <math>\alpha, \beta = \gamma</math>. (as the x values of intersections)</p> <p>Substitute <math>y = \frac{1}{x}</math> into <math>y = x^2 - 4</math></p> $\frac{1}{x} = x^2 - 4$ $1 = x^3 - 4x \quad (x \neq 0)$ <p>i.e. <u><math>x^3 - 4x - 1 = 0</math></u></p> <p>ii) We need <math>y = x^2 \Rightarrow x = \sqrt{y}</math></p> <p>Sub. into equation</p> $y\sqrt{y} - 4\sqrt{y} - 1 = 0$ $\sqrt{y}(y - 4) = 1$ $y(y^2 - 8y + 16) = 1 \quad (\text{Square both sides})$ <u><math>y^3 - 8y^2 + 16y - 1 = 0</math></u> | <p>1</p> <p>1</p> <p>1</p> | <p>Substitution alone was NOT enough. The mention of the word "intersection" or equivalent guaranteed the mark.</p> <p>No real need to change back to x's,</p> |

## 2014 TRIAL X2 MATHEMATICS: Question 13.

| Suggested Solutions                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | Marks                               | Marker's Comments                                                                                                                |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| <p>iii) From eqn in (ii),<br/> Substitute <math>z = \frac{1}{y} \Rightarrow y = \frac{1}{z}</math></p> $\therefore \frac{1}{z^3} - \frac{8}{z^2} + \frac{16}{z} - 1 = 0$ $\underline{\underline{z^3 - 16z^2 + 8z - 1 = 0}}$ <p>iv) The co.ordinates of P are given by <math>(\alpha, \frac{1}{\alpha})</math>, say.<br/> Then <math>OP^2 = \alpha^2 + \frac{1}{\alpha^2}</math> (Pythagoras)<br/> Similarly <math>OQ^2 = \beta^2 + \frac{1}{\beta^2}</math>, say<br/> <math>OR^2 = \gamma^2 + \frac{1}{\gamma^2}</math>, say.</p> $\therefore OP^2 + OQ^2 + OR^2 = (\alpha^2 + \beta^2 + \gamma^2) + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)$ $= 8 + 16$ <p>(Sum of roots in equations from (ii) &amp; (iii))</p> $\underline{\underline{= 24 \text{ (units)}}}$ | <p>1</p> <p>1</p> <p>1</p> <p>1</p> | <p>Many people started from (i) with <math>z = \frac{1}{\sqrt{x}}</math>.</p> <p>First mark for equivalent point as in (ii).</p> |

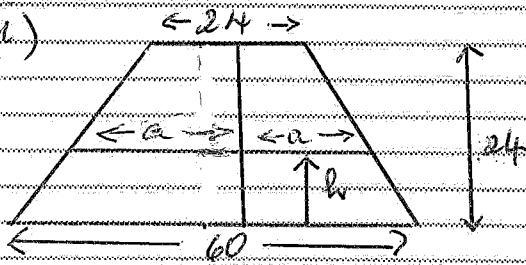
EXT 2  
MATHEMATICS: Question 14

Suggested Solutions

Marks

Marker's Comments

a) (i)



|   |    |    |
|---|----|----|
| h | 0  | 24 |
| a | 30 | 12 |

Linear Relationship

$$a = mh + b$$

$$m = \frac{30-12}{0-24} = -\frac{3}{4}$$

when  $h = 0$   $a = 30$   
 $a = -\frac{3}{4}h + 30$   
 $a = 30 - \frac{3}{4}h$

(ii) Similarly for minor axis

$$b = \left( \frac{12-20}{24-0} \right) h + 20$$

$$= 20 - \frac{1}{3}h$$

|   |    |    |
|---|----|----|
| h | 0  | 24 |
| b | 20 | 12 |

Inner Radius

$$R = \frac{12-6}{24-0}h + 6$$

$$= \frac{1}{4}h + 6$$

|   |   |    |
|---|---|----|
| h | 0 | 24 |
| R | 6 | 12 |

Area is  $\pi ab - \pi R^2$   
 $= \pi [ab - R^2]$   
 $= \pi \left[ \left( 20 - \frac{1}{3}h \right) \left( 30 - \frac{3}{4}h \right) - \left( 6 + \frac{1}{4}h \right)^2 \right]$   
 $= \pi \left[ \frac{60-h}{3} \left[ \frac{120-3h}{4} \right] - \left[ \frac{24+h}{4} \right]^2 \right]$   
 $= \frac{\pi}{16} [9024 - 448h + 3h^2]$

(iii)  $\delta V = A \delta h$   
 $V = \lim_{\delta h \rightarrow 0} \sum \frac{\pi}{16} [9024 - 448h + 3h^2] \delta h$   
 $= \frac{\pi}{16} \int_0^{24} [9024 - 448h + 3h^2] dh$   
 $= \frac{\pi}{16} \left[ 9024h - 224h^2 + h^3 \right]_0^{24}$   
 $= \frac{\pi}{16} [9024(24) - 224(24)^2 + (24)^3]$   
 $= 6336\pi$  Volume is  $6336\pi \text{ cm}^3$

②

① reasoning

① answer with correct working

Could also be done with similar triangles.

③

①  $b = 20 - \frac{h}{3}$

$R = \frac{1}{4}h + 6$

① correct answer with working

① Correct answer.

14

MATHEMATICS: Question.....

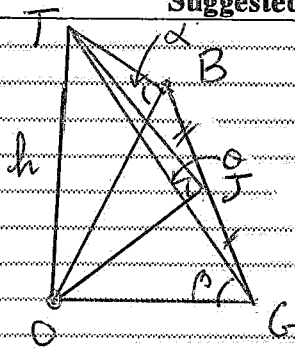
| Suggested Solutions                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | Marks    | Marker's Comments                                                                                                                                        |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|----------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>(b) (i) To show <math>(1+x)^n + (1-x)^n = 2 \sum_{k=0}^{n/2} \binom{n}{2k} x^{2k}</math></p> <p>LHS <math>= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n + \binom{n}{0} (-x)^0 + \binom{n}{1} (-x)^1 + \binom{n}{2} (-x)^2 + \dots + \binom{n}{n} (-x)^n</math></p> <p><math>\binom{n}{0} x^0 = \binom{n}{0} (-x)^0</math><br/> <math>\binom{n}{1} x^1 = -\binom{n}{1} (-x)^1</math><br/> <math>\binom{n}{2} x^2 = \binom{n}{2} (-x)^2</math><br/> <math>\dots</math><br/> <math>\binom{n}{n} x^n = -\binom{n}{n} (-x)^n</math></p> <p><math>(-1)^n = 1</math> when <math>n</math> is even</p> <p><math>= 2 \left[ \binom{n}{0} x^0 + \binom{n}{2} x^2 + \binom{n}{4} x^4 + \dots + \binom{n}{n} x^n \right]</math></p> <p><math>= 2 \left[ \binom{n}{2 \times 0} x^{2 \times 0} + \binom{n}{2 \times 1} x^{2 \times 1} + \binom{n}{2 \times 2} x^{2 \times 2} + \dots + \binom{n}{2 \times (n/2)} x^{2 \times (n/2)} \right] \times</math></p> <p><math>= 2 \left[ \sum_{k=0}^{n/2} \binom{n}{2k} x^{2k} \right]</math></p> | <p>②</p> | <p>① For expansions</p> <p>① Cancelling odd powers, adding even power and showing new series is only even powers and <math>\frac{n}{2}</math> terms.</p> |
| <p>(ii) 2 B's and 2 other letters (A or C)</p> <p>4 letter word.</p> <p>choose 2 places for 2 B's = <math>4C_2</math></p> <p>other spaces <math>2 \times 2</math> choices = <math>2^2</math></p> <p>Total is <math>4C_2 \times 2^2</math></p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | <p>①</p> | <p>Correct description</p>                                                                                                                               |
| <p>(iii) Even number of B's in even number of letters (A or C)</p> <p>zero B's no. of words = <math>1 \times 2^n</math></p> <p>2 B's " " = <math>nC_2 \times 2^{n-2}</math></p> <p>4 B's " " = <math>nC_4 \times 2^{n-4}</math></p> <p>etc</p> <p><math>n-2</math> B's " " = <math>nC_{n-2} \times 2^2</math></p> <p>all B's = <math>nC_n \times 2^0</math></p> <p><math>\therefore</math> Total = <math>nC_0 2^n + nC_2 2^{n-2} + nC_4 2^{n-4} + \dots + nC_n 2^0</math></p> <p>But <math>nC_r = nC_{n-r}</math></p> <p><math>\therefore</math> Total = <math>nC_n 2^0 + nC_{n-2} 2^2 + \dots + nC_2 2^{n-2} + nC_0 2^n</math></p> <p><math>= \sum_{k=0}^{n/2} nC_{2k} 2^{2k} = \frac{1}{2} \left[ (1+2)^n + (1-2)^n \right]</math> from (i) <math>x=2</math>.</p> <p><math>= \frac{1}{2} [3^n + (-1)^n]</math></p> <p><math>= \frac{1}{2} [3^n + 1]</math> as <math>(-1)^n = 1</math> as <math>n</math> is even</p>                                                                                                                                                | <p>②</p> | <p>① For full explanation of series</p> <p>① sub <math>x=2</math> into eqn in (i).</p>                                                                   |

**MATHEMATICS: Question 14..**

**Suggested Solutions**

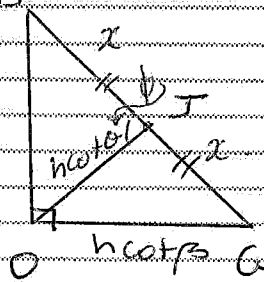
**Marks**

**Marker's Comments**



Let  $OT = h$

$h \cot \alpha$



$$\tan \alpha = \frac{OT}{OB}$$

$$h \cot \alpha = OB$$

$$\tan \theta = \frac{OT}{OJ}$$

$$\tan \beta = \frac{OT}{OG}$$

$$h \cot \beta = OG$$

$$OT = h \cot \theta$$

Many methods

① as  $BJ = JG$  and  $\angle BOG = 90^\circ$   
 $BG$  is a diameter of circle through points  $B, O$  and  $G$

$BJ = JG = OJ$  (radii of circle)

$$OT = \frac{1}{2} BG = h \cot \theta$$

By Pythagoras.  $OB^2 + OG^2 = BG^2$

$$BG^2 = h^2 \cot^2 \alpha + h^2 \cot^2 \beta$$

$$\therefore h \cot \theta = \frac{h}{2} \sqrt{\cot^2 \alpha + \cot^2 \beta}$$

$$\cot \theta = \frac{1}{2} \sqrt{\cot^2 \alpha + \cot^2 \beta}$$

② Alternative Let  $\angle BJO = \phi$  and  $BJ = JG = x$   
 in  $\Delta OJB$

$$h^2 \cot^2 \alpha = h^2 \cot^2 \theta + x^2 - 2h \cot \theta x \cos \phi$$

in  $\Delta OJG$

$$h^2 \cot^2 \beta = h^2 \cot^2 \theta + x^2 - 2h \cot \theta x \cos (180 - \phi)$$

ADD EQUATIONS

$$\cos (180 - \phi) = -\cos \phi$$

$$h^2 (\cot^2 \alpha + \cot^2 \beta) = 2h^2 \cot^2 \theta + 2x^2$$

$$\text{But } 4x^2 = h^2 \cot^2 \alpha + h^2 \cot^2 \beta \quad (\text{Pythagoras})$$

$$\therefore h^2 (\cot^2 \alpha + \cot^2 \beta) = 2h^2 \cot^2 \theta + \frac{1}{2} (h^2 \cot^2 \alpha + h^2 \cot^2 \beta)$$

$$\therefore 2 \cot^2 \theta = \frac{1}{2} (\cot^2 \alpha + \cot^2 \beta)$$

$$\cot \theta = \frac{1}{2} \sqrt{\cot^2 \alpha + \cot^2 \beta}$$

(4)

- ① stating  $OT = \frac{1}{2} BG$ .
- ① with reasons (1 proof)
- ① Pythagoras statement
- ① correct answer with working

② cos Rule statements

① Pythagoras statement

① answer with correct working

$$a_i) I_n = \int_1^2 \left(1 - \frac{1}{x}\right)^n dx \quad \text{for } n = 1, 2, 3, \dots$$

$$= \int_1^2 \frac{(x-1)^n}{x^n} dx$$

$$\text{Let } u = x^{-n} \quad u' = \frac{-n}{x^{n+1}}$$

$$v' = (x-1)^n \quad v = \frac{(x-1)^{n+1}}{n+1}$$

$$= uv - \int v du$$

$$= \frac{(x-1)^{n+1}}{(n+1)x^n} \Big|_1^2 - \int_1^2 \frac{(x-1)^{n+1}}{n+1} \cdot \frac{-n}{x^{n+1}} dx$$

$$= \frac{1}{(n+1)2^n} + \frac{n}{n+1} \int_1^2 \frac{(x-1)^{n+1}}{x^{n+1}} dx$$

$$I_n = \frac{1}{2^n(n+1)} + \frac{n}{n+1} I_{n+1}$$

$$\therefore \frac{I_{n+1}}{n+1} = \frac{I_n}{n} - \frac{1}{n(n+1)2^n} \quad \#$$

$$ii) \frac{I_n}{n} = \frac{I_{n-1}}{n-1} - \frac{1}{(n-1)n \cdot 2^{n-1}} \quad \left( \begin{array}{l} \text{from i} \\ \text{replace } n+1 \text{ by } n \end{array} \right)$$

$$\frac{I_{n-1}}{n-1} = \frac{I_{n-2}}{n-2} - \frac{1}{(n-2)(n-1)2^{n-2}}$$

$$\frac{I_2}{2} = \frac{I_1}{1} - \frac{1}{1 \cdot 2 \cdot 2^1}$$

$$\therefore \frac{I_{n+1}}{n+1} = \frac{I_1}{1} - \left[ \frac{1}{1 \cdot 2 \cdot 2} + \dots + \frac{1}{(n-2)(n-1)2^{n-2}} + \frac{1}{n(n-1)2^{n-1}} + \frac{1}{n(n+1)2^n} \right]$$

$$\frac{I_{n+1}}{n+1} = I_1 - \sum_{r=1}^n \frac{1}{r(r+1)2^r}$$

$$(iii) I_1 = \int_1^2 \left(1 - \frac{1}{x}\right) dx = x - \ln(x) \Big|_1^2 = 2 - \ln 2 - (1 - \ln 1) = 1 - \ln 2$$

$$iv) \sum_{r=1}^n \frac{1}{r(r+1)2^r} = I_1 - \frac{I_{n+1}}{n+1} = 1 - \ln 2 - \frac{I_{n+1}}{n+1}$$

Other alternatives

$$u = \left(1 - \frac{1}{x}\right)^{n+1} \quad dv = dx$$

$$du = (n+1)\left(1 - \frac{1}{x}\right)^n \left(\frac{1}{x^2}\right) dx, \quad v = x$$

many mistakes here

/ m

/ m

/ m

There are other alternatives

1 m  
must show  
at least 3 expansions  
there

/ m

/ m

$$0 < \int_1^2 \left(1 - \frac{1}{x}\right)^n dx < 1$$

$$0 < I_n < 1 \text{ and}$$

$$I_{n+1} < I_n$$

$$\therefore 0 < I_{n+1} < 1$$

$$0 < \frac{I_{n+1}}{n+1} < \frac{1}{n+1}$$

$$\text{As } n \rightarrow \infty \quad \frac{I_{n+1}}{n+1} \rightarrow 0$$

$$\text{Hence limiting sum } \frac{1}{1 \cdot 2 \cdot 2^1} + \frac{1}{2 \cdot 3 \cdot 2^2} + \frac{1}{3 \cdot 4 \cdot 2^3} + \dots$$

$$\approx \underline{1 - \ln 2}$$

$$b) \quad z^7 = 1$$

$$z^7 - 1 = (z-1)(1+z+z^2+\dots+z^6) = 0$$

Since  $\alpha$  is a complex root with smallest argument

$$\alpha^7 - 1 = (\alpha-1)(1+\alpha+\alpha^2+\dots+\alpha^6) = 0$$

$$\alpha \neq 1 \quad \therefore 1+\alpha+\alpha^2+\dots+\alpha^6 = 0$$

$$\theta + \phi = (\alpha + \alpha^2 + \alpha^4) + (\alpha^3 + \alpha^5 + \alpha^6)$$

$$= (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6) - 1$$

$$= 0 - 1$$

$$= -1$$

$$\theta \phi = (\alpha + \alpha^2 + \alpha^4)(\alpha^3 + \alpha^5 + \alpha^6)$$

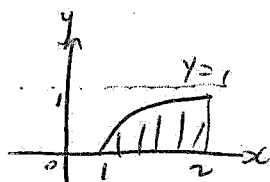
$$= \alpha^4 + \alpha^7 + \alpha^8 + \alpha^6 + \alpha^7 + \alpha^9 + \alpha^7 + \alpha^8 + \alpha^{10}$$

$$(\alpha^7 = 1) \Rightarrow \alpha^4 + \alpha^5 + 1 + \alpha^6 + 1 + (\alpha^7)\alpha^2 + 1 + (\alpha^7)\alpha + (\alpha^7)\alpha^3$$

$$\therefore \theta \phi = \alpha^4 + \alpha^5 + 1 + \alpha^6 + 1 + \alpha^2 + 1 + \alpha + \alpha^3$$

$$= (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6) + 2$$

$$= 0 + 2 = \underline{2}$$



$$0 < I_n < 1 \quad \left. \begin{array}{l} \text{Any} \\ 2 \\ \text{get} \\ 1 \text{ m} \end{array} \right\} \begin{array}{l} I_{n+1} < I_n \\ \frac{I_{n+1}}{n+1} \rightarrow 0 \end{array}$$

as  $n \rightarrow \infty$

All 3 of the above with correct answer  $1 - \ln 2$  will get 2 m

must mention  $\alpha$  is root  $\alpha \neq -1$

must show this line in terms of  $\alpha$

1 m

v. poorly done. alternative see later

1 m well done

1 m

$$(\text{from above}) \quad 1 + \alpha + \alpha^2 + \dots + \alpha^6 = 0$$

well done

Alternative to (i)

Let  $z = r \cos \theta$

$z^7 = r^7 \cos 7\theta$  (De Moivre's Th)

$r^7 \cos 7\theta = 1 = 1^7 \cos 0$

$\therefore r = 1, \theta = \frac{2n\pi}{7} \quad n \in \mathbb{J}$

Rts of  $z^7 = 1$  are

$\cos 0 = 1, \cos \frac{2\pi}{7}(\alpha), \cos \frac{4\pi}{7}(\alpha^2), \cos \frac{6\pi}{7}(\alpha^3)$

$\cos \frac{8\pi}{7}(\alpha^4), \cos \frac{10\pi}{7}(\alpha^5), \cos \frac{12\pi}{7}(\alpha^6)$

Sum of Rts  $= 1 + \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{8\pi}{7} + \cos \frac{10\pi}{7} + \cos \frac{12\pi}{7}$   
 $= 0$

$\therefore 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$

$\therefore (\alpha + \alpha^2 + \alpha^4) + (\alpha^3 + \alpha^5 + \alpha^6) = -1$   
 $\theta \neq \phi = -1 \quad \#$

(ii)  $x^2 + x + 2 = 0$

$x = \frac{-1 \pm i\sqrt{7}}{2}$

$\text{Im}(\theta) = \text{Im} \left[ \sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} + \sin \frac{8\pi}{7} \right] > 0$

$\therefore \theta = \frac{-1 + i\sqrt{7}}{2}, \phi = \frac{-1 - i\sqrt{7}}{2}$

(iii)  $\text{Re}(\theta) = \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{8\pi}{7} = -\frac{1}{2}$

but  $\cos \frac{8\pi}{7} = -\cos \frac{\pi}{7}$

since  $\cos(\pi + \theta) = -\cos \theta$

$\therefore \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} - \cos \frac{\pi}{7} = -\frac{1}{2}$

P.3

must

Prove  $\alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$   
are also roots

must relate

$\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$   
together as roots

1 m

1 m

1 m well done

1 m very few  
can explain why

$\theta = \frac{-1 + i\sqrt{7}}{2}$

1 m

must explain  
this

1 m

a)  $\theta = 2\phi$  (centre at centre twice  
angle at circumference)

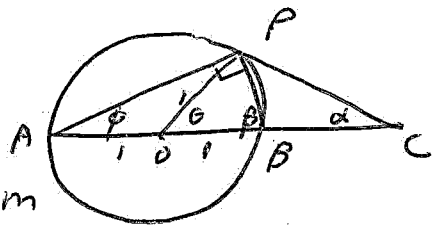
$$\dot{\theta} = 2\dot{\phi}$$

$$\dot{\phi} = \frac{\dot{\theta}}{2} = \frac{\pi}{2} \text{ rad/sec (angular vel about A)} \quad 1\text{m}$$

$$\angle APB = \frac{\pi}{2} \text{ (angle of semi-circle)}$$

$$\beta = \pi - \frac{\pi}{2} - \phi \text{ (angle sum of } \triangle APB)$$

$$\dot{\beta} = -\dot{\phi} = -\frac{\pi}{2} \text{ rad/sec (angular vel about B)} \quad 1\text{m}$$



must give reasons

ii) In  $\triangle OPC$

$$\angle OPC = \pi - (\theta + \alpha) \text{ (angle sum of triangle)}$$

$$\text{Using Sine Rule} \quad \frac{\sin \alpha}{OP} = \frac{\sin \angle OPC}{OC}$$

$$\frac{\sin \alpha}{1} = \frac{\sin(\pi - (\theta + \alpha))}{2} \quad 1\text{m}$$

$$\text{Since } \sin(\pi - A) = \sin A$$

$$\therefore 2 \sin \alpha = \sin(\theta + \alpha)$$

$$\text{iii) } (2 \cos \alpha) \dot{\alpha} = \cos(\theta + \alpha) [\dot{\theta} + \dot{\alpha}] \quad 1\text{m}$$

$$\text{when } \theta = \frac{\pi}{2}$$

$$(2 \cdot \frac{2}{\sqrt{5}}) \dot{\alpha} = \cos(\frac{\pi}{2} + \alpha) [\pi + \dot{\alpha}] \quad 1\text{m}$$

$$\frac{4}{\sqrt{5}} \dot{\alpha} = -\sin \alpha (\pi + \dot{\alpha}) \quad 1\text{m}$$

$$\frac{4}{\sqrt{5}} \dot{\alpha} = -\frac{1}{\sqrt{5}} (\pi + \dot{\alpha}) \quad 1\text{m}$$

$$5 \dot{\alpha} = -\pi$$

$$\dot{\alpha} = -\frac{\pi}{5} \text{ rad/sec} \quad 1\text{m}$$

Alternatively using chain rule

$$-\sin \theta \sin \alpha \frac{d\alpha}{d\theta} + \cos \alpha \cos \theta - \sin \alpha \sin \theta + \cos \theta \cos \alpha \frac{d\alpha}{d\theta} = 2 \cos \alpha \frac{d\alpha}{d\theta} \quad 1\text{m}$$

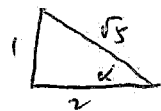
$$\frac{d\alpha}{d\theta} = \frac{-\cos(\theta + \alpha)}{\cos \theta \cos \alpha - \sin \theta \sin \alpha - 2 \cos \alpha} = \frac{-\cos(\frac{\pi}{2} + \alpha)}{\cos \frac{\pi}{2} \cos \alpha - \sin \frac{\pi}{2} \sin \alpha - 2 \cos \alpha} \quad 1\text{m}$$

$$\frac{d\alpha}{d\theta} = \frac{\sin \alpha}{-\sin \alpha - 2 \cos \alpha} = \frac{\frac{1}{\sqrt{5}}}{-\frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}}} = -\frac{1}{5} \quad 1\text{m}$$

$$\frac{d\alpha}{dt} = \frac{d\alpha}{d\theta} \times \frac{d\theta}{dt} = -\frac{1}{5} \times \pi = -\frac{\pi}{5} \text{ rad/sec} \quad 1\text{m}$$

must mention  $\triangle OPC$   
& sine rule

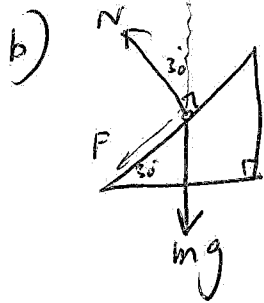
many thought  $\angle OPC = \frac{\pi}{2}$



1m for sub all below

$$\begin{cases} \sin \alpha = \frac{1}{\sqrt{5}} \\ \cos \alpha = \frac{2}{\sqrt{5}} \\ \theta = \pi \end{cases}$$

Q16

Case 1 max vel  $F \downarrow$ 

Vertically

$$N \cos 30^\circ = 10 \text{ m} + F \sin 30^\circ$$

$$N \frac{\sqrt{3}}{2} = 10 \text{ m} + \frac{F}{2}$$

$$N \left( \frac{\sqrt{3}}{2} - \frac{1}{20} \right) = 10 \text{ m} \quad (1)$$

Horizontally  $\frac{mv^2}{20} = F \cos 30^\circ + N \sin 30^\circ$

$$\frac{mv^2}{20} = F \frac{\sqrt{3}}{2} + \frac{N}{2}$$

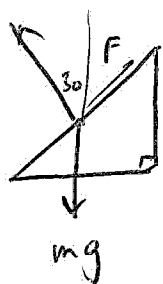
$$N \left[ \frac{1}{2} + \frac{\sqrt{3}}{20} \right] = \frac{mv^2}{20} \quad (2)$$

$$(2) \div (1)$$

$$\frac{\frac{mv^2}{20}}{10 \text{ m}} = \frac{N(10 + \sqrt{3})}{N(10\sqrt{3} - 1)}$$

$$v^2 = 200 \left( \frac{10 + \sqrt{3}}{10\sqrt{3} - 1} \right) = 143.77$$

$$v_{\text{MAX}} = 11.99 \text{ m/s} \quad (v > 0)$$

Case 2 min vel  $F \uparrow$ 

Replace  $F$  by  $-F$  in the above calculation

$$10 \text{ m} = N \left( \frac{\sqrt{3}}{2} + \frac{1}{20} \right)$$

$$\frac{mv^2}{20} = N \left( \frac{1}{2} - \frac{\sqrt{3}}{20} \right)$$

$$(2) \div (1)$$

$$v^2 = 200 \left( \frac{10 - \sqrt{3}}{10\sqrt{3} + 1} \right)$$

$$v^2 = 90.2589$$

$$v = 9.5 \text{ m/s} \quad (v > 0)$$

$$\therefore 9.5 \leq v \leq 11.99$$

1 m } max vel  $F \downarrow$   
min vel  $F \uparrow$

1 m diagram with angles & forces

1 m vertical forces

1 m horizontal forces

1 m for sub in  
numerical value

1 m

} 1 m must have  
numerical values

1 m