

Student Number:	
Class:	

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2016

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black pen.
- Board approved calculators & templates may be used
- A Reference Sheet is provided.
- In Questions 11-16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 90 Marks

- Attempt Question 11-16
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section I

10 Marks

Attempt Questions 1 - 10. Mark your answers on the sheet provided.
Allow about 15 minutes for this section.

1. The complex equation $|z - 3 + 2i| = |z + 3|$ can be expressed in Cartesian form as:
(A) $y = 3x - 1$ (B) $y = x^2 + 1$ (C) $y = 3 - x$ (D) $y = 1 - 3x$
2. A particle moves along a straight line with an acceleration of $\frac{2}{v} \text{ m/s}^2$, where $v \text{ m/s}$ is the velocity at any instant. The initial velocity of the particle is -1 m/s .
The particle will move to the
(A) Left increasing in speed (B) Left, stop, then move to the right
(C) Right increasing in speed (D) Right, stop, then move to the left
3. Using an appropriate substitution, then $\int e^{2x} \sqrt{e^x - 1} dx$ is equivalent to:
(A) $\int (u^3 - u) du$ (B) $\int (u^3 + u) du$ (C) $\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$ (D) $\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$
4. When factorised, $x^2 + 6x + 1$ is equal to:
(A) $(x + 3 + 2\sqrt{2})(x + 3 - 2\sqrt{2})$ (B) $(x - 3 - 2\sqrt{2})(x - 3 - 2\sqrt{2})$
(C) $(x + 3 + 2\sqrt{2}i)(x + 3 - 2\sqrt{2}i)$ (D) $(x - 3 - 2\sqrt{2}i)(x - 3 - 2\sqrt{2}i)$
5. The range of the function $f(x) = e^{\tan x}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is:
(A) $y \in \mathbb{R}$ (B) $y \geq 0$ (C) $y > 0$ (D) $0 < y \leq 1$
6. The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when:
(A) $y = 0$ (B) $y = \pm 3$ (C) $y = \pm\sqrt{3}$ (D) $y = \frac{1}{2}$
7. The number of non zero ^{positive} integer solutions of the equation $w + x + y + z = 20$ is:
(A) 969 (B) 1140 (C) 1771 (D) 4845

8. The position of a moving object is given by the Cartesian coordinates $(3t, e^t)$. Its acceleration is:
- (A) constant in both magnitude and direction (B) constant in magnitude only
(C) constant in direction only (D) constant in neither magnitude nor direction
9. The region bounded by the parabolas $y = x^2$ and $y = 6x - x^2$ is rotated around the x -axis so that a vertical line segment cut off by the curves generates a ring. The value of x for which the ring of largest area is obtained is:
- (A) 4 (B) 3 (C) 2.5 (D) 2
10. The value of $\lim_{h \rightarrow 0} \frac{1}{h} \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{4}+h} \frac{\sin x}{x} dx \right)$ is:
- (A) 0 (B) 1 (C) $\frac{1}{\pi\sqrt{2}}$ (D) $\frac{2\sqrt{2}}{\pi}$

END OF SECTION I

Section II 90 marks

Attempt Questions 11-16

Allow about 2 hours 45 minutes for this section

Question 11 (15 marks)

(a) Let $z = 1 + i$ and $w = 3 + 2i$

(i) Express $\frac{z}{w}$ in simplest form with a real denominator. 1

(ii) Write $\bar{z}$ in modulus-argument form. 1

(b) Simplify $(1 + 2\omega + 3\omega^2)(1 + 3\omega + 2\omega^2)$, where ω is a complex cube root of unity. 3

(c) (i) Prove for any two real numbers a and b that $a^2 + b^2 > 2ab$, where $a \neq b$. 1

(ii) Hence, or otherwise, prove if $a + b + c = 1$ then $(1 - a)(1 - b)(1 - c) > 8abc$ where a , b and c are positive numbers and $a \neq b \neq c$. 2

(d) Evaluate $\int_0^1 \frac{dx}{(x+2)(x+3)}$. 3

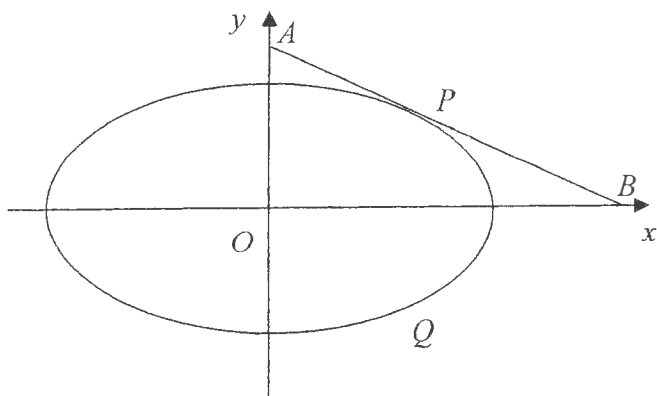
(e) $P(3\cos\theta, 2\sin\theta)$ is a point on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ for $0 < \theta < \frac{\pi}{2}$.

The tangent at P meets the y -axis at A and the x -axis at B , as shown in the diagram below.

Find the minimum length of AB .

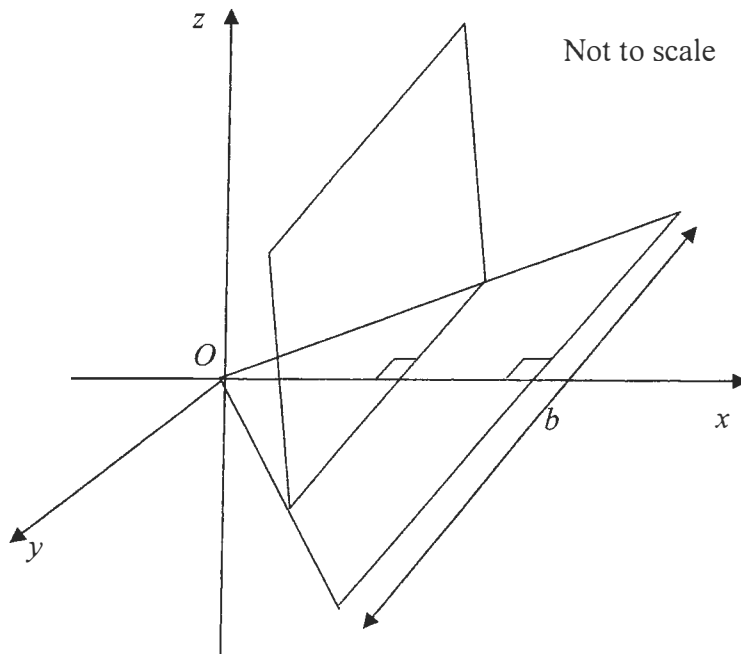
4

You may assume the equation of the tangent at P is $\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$

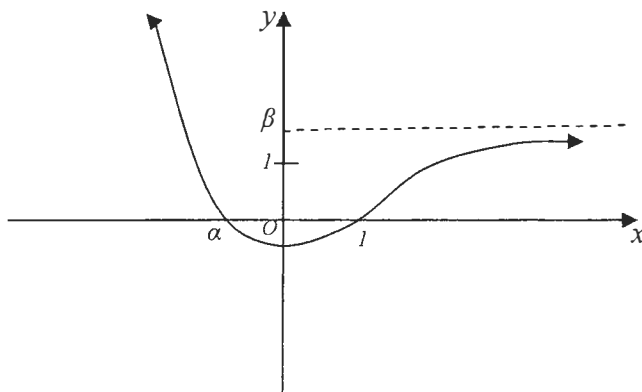


Question 12 (15 marks) Start a new page

- (a) The base of a solid is in the shape of an equilateral triangle of side b , with one vertex at the origin and its altitude along the positive x -axis as shown in the diagram below. Each cross-sectional plane perpendicular to the x -axis is a square with one side in the base of the solid.



- (i) Calculate the side length of a square cross-section x units from the origin. 1
 (ii) Find the volume of the solid. 3
- (b) Consider the function $y = \frac{3}{5}x^{\frac{5}{3}} - 3x^{\frac{2}{3}}$.
- (i) Find the x -coordinate(s) of any stationary points and describe their nature. 3
 (ii) Describe the concavity of the function for $x < 0$. 2
 (iii) Sketch the graph, showing all important features. 3
- (c) The diagram below shows the graph of the function $y = f(x)$.



Copy the diagram and neatly sketch the graph of $y^2 = f(x)$ on the same set of axes, showing all important features.

3

Question 13 (15 marks) Start a new page

- (a) A sequence is defined by the formula:

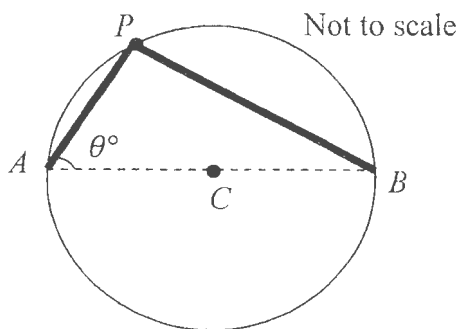
$$U_0 = 0 \text{ and } U_n = \sqrt{U_{n-1} + 2} \text{ for } n = 1, 2, 3 \dots$$

Prove by mathematical induction that $U_n = 2\cos\left[\frac{\pi}{2^{n+1}}\right]$ for $n = 0, 1, 2, 3 \dots$

4

- (b) A 50 kg boy (P) is sitting on the edge of a smooth horizontal circular plate, 3 m in diameter, which is spinning at a constant rate about a vertical axis through its centre C at 20 revolutions per minute. The boy holds onto two separate ropes which are attached to the ends of the diameter (AB) of the plate.

The ropes are such that $AP = 1.8 \text{ m}$ and $BP = 2.4 \text{ m}$. (Given $\angle APB = 90^\circ$ and $\angle PAB = \theta^\circ$)

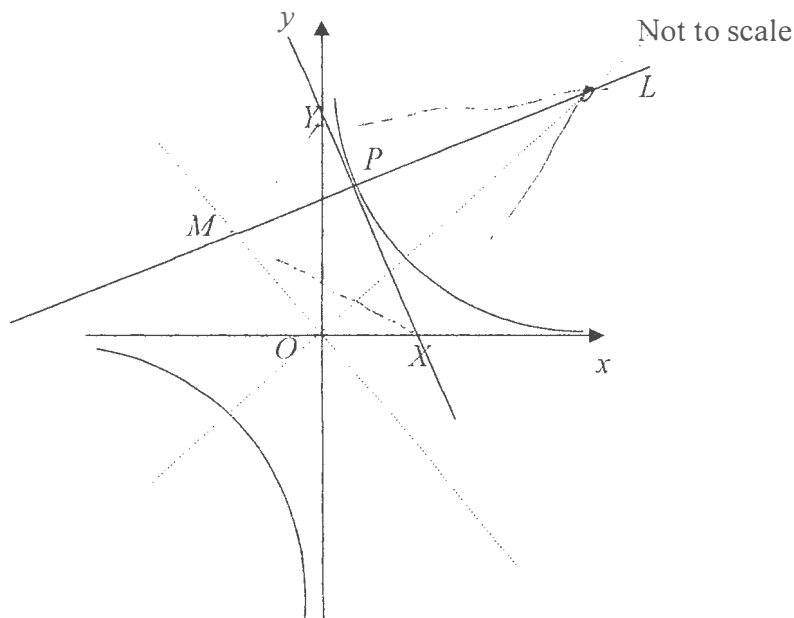


- (i) Write motion equations in the tangential and radial directions for the boy. 3
 (ii) Calculate the tension in the shorter rope. 2
- (c) (i) Show that the equation of the tangent to the rectangular hyperbola $xy = c^2$ at the point $P\left(cp, \frac{c}{p}\right)$ is $x + p^2y = 2cp$. 2
 (ii) The tangent at P meets the x -axis at X and the y -axis at Y and the normal at P meets the lines $y = x$ and $y = -x$ at L and M respectively, as shown in the diagram below.

Prove that $LYMX$ is a rhombus provided that $p \neq 1$.

4

You may assume that the equation of the normal at P is $px - \frac{1}{p}y = c\left(p^2 - \frac{1}{p^2}\right)$.



Question 14 (15 marks) Start a new page

(a) (i) Show that $\int_{-a}^a \frac{x^4}{1+e^x} dx = \int_{-a}^a \frac{x^4 e^x}{1+e^x} dx$ 2

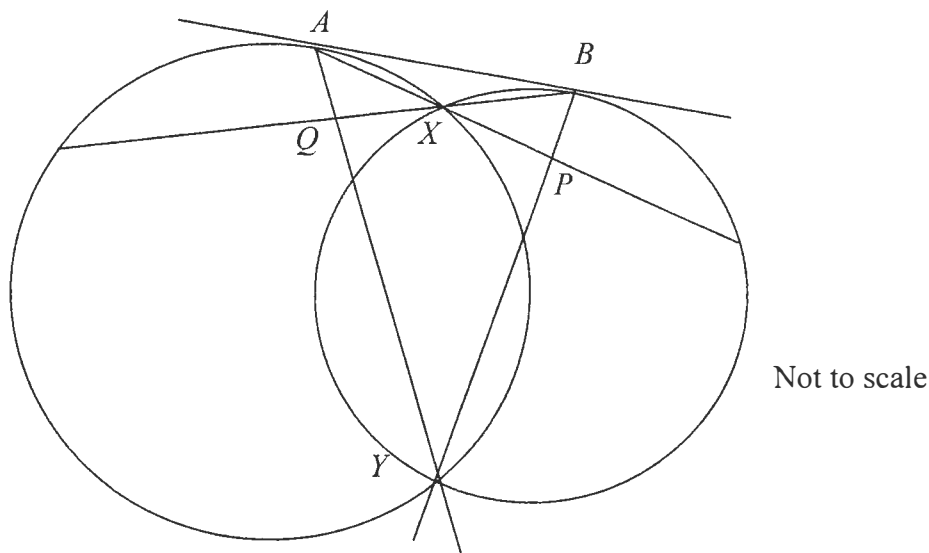
(ii) Hence, or otherwise, evaluate $\int_{-2}^2 \frac{x^4}{e^x+1} dx$ 2

(b) The equation $z^2 + (a + ib)z + m + in = 0$ has one real root, where a, b, m and $n \in \mathbb{R}$.

Show that $n^2 - abn + mb^2 = 0$. 3

(c) Two boys each throw two dice. Each person adds the two numbers uppermost on his dice. What is the probability that they gain an equal score? 2

(d) Two circles intersect at X and Y and AB is a common tangent. The lines AX and BY meet at P . The lines AY and BX meet at Q as shown in the diagram below.



(i) Copy the diagram and show that $XPYQ$ is a cyclic quadrilateral, giving reasons. 2

(ii) Prove that $QP \parallel AB$. 1

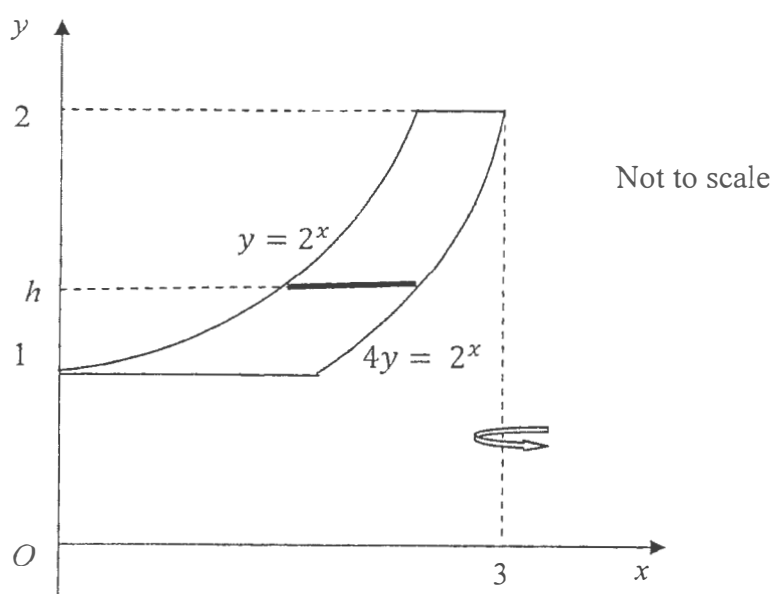
(ii) Hence, or otherwise, prove that XY bisects PQ . 3

Question 15 (15 marks) Start a new page

- (a) The equation $x^3 - x - 1 = 0$ has roots α , β and γ . 3

Find an equation whose roots are $\frac{1+\alpha}{1-\alpha}$, $\frac{1+\beta}{1-\beta}$ and $\frac{1+\gamma}{1-\gamma}$ in the form $Ax^3 + Bx^2 + Cx + D = 0$.

- (b) A model of part of the roof of a new art gallery is generated by rotating the curves $y = 2^x$ and $4y = 2^x$ from $1 \leq y \leq 2$ around the line $x = 3$ as shown in the diagram below.



- (i) Show that the area (A) of a cross-sectional slice of the roof taken at $y = h$, for $1 \leq h \leq 2$, is given by the formula $A = 4\pi[2 - \log_2 h]$. 3
- (ii) Calculate the volume of the material required to build the model of this part of the roof. 3
- (c) (i) Given $z^n = (z + 1)^n$.
- Show that:
$$Z = \frac{1}{\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) - 1} \quad \text{for } k = 1, 2, \dots$$
 2
- (ii) Hence, or otherwise, find the roots of $z^n = (z + 1)^n$ in Cartesian form. 3
- (iii) Find the Cartesian equation of the line on which the roots lie. 1

Question 16 (15 marks) Start a new page

(a) Given $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}} dx$, $n = 0, 1, 2, 3, \dots$

(i) Show that $(2n + 1)I_n = 2\sqrt{2} - 2nI_{n-1}$ $n = 1, 2, 3, \dots$ 4

(ii) Evaluate $\int_0^1 \frac{x^3}{\sqrt{x+1}} dx$ 3

- (b) A projectile of mass m kg is launched from the origin O at an angle θ° above the horizontal where the air resistance has a magnitude mkv N, v m/s is the speed of the particle and k is a positive constant.

Initially the horizontal and vertical components of its velocity are U_0 m/s² and V_0 m/s² respectively.

- (i) Show that the vertical displacement (y m), whilst the projectile is rising, at time t seconds after the launch is given by: 4

$$y = \left(\frac{g}{k^2} + \frac{V_0}{k} \right) (1 - e^{-kt}) - \frac{gt}{k}$$

where g m/s² is the acceleration due to gravity.

- (ii) Find the Cartesian equation of the trajectory of the projectile before it reaches its maximum height. 4

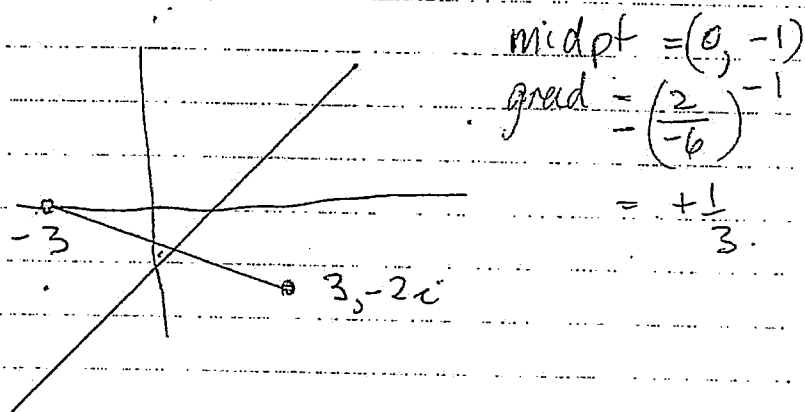
END OF EXAMINATION

multiple choice

1. $|z - 3 + 2i| = |z + 3|$

$|z - (3 - 2i)| = |z - (-3)|$

line is perp bisector



2. Acceleration is in direction of motion $\therefore$ not stop

3. $\int e^{2x} \sqrt{e^x - 1} dx$

$\int e^x \sqrt{e^x - 1} e^x dx$

$\int (u+1) u^{\frac{1}{2}} du$

$\int (u^{3/2} + u^{1/2}) du$

$u = e^x - 1$

$\frac{du}{dx} = e^x$

$du = e^x dx$

D

4

$$x^2 + 6x + 1 = 0$$

$$x = \frac{-6 \pm \sqrt{36 - 4}}{2}$$

$$= -3 \pm 2\sqrt{2}$$

$$\therefore (x + 3 - 2\sqrt{2})(x + 3 + 2\sqrt{2}) = 0$$

A

5

$$f(x) = e^{\tan x} \quad -\pi/2 < x < \pi/2$$

$$\tan x \in \mathbb{R}$$

$$0 < f(x) \quad y > 0$$

C

6

$$y^2 - xy + 9 = 0$$

$$2y \frac{dy}{dx} - x \frac{dy}{dx} - y = 0$$

$$\frac{dy}{dx} [2y - x] = y$$

$$\frac{dy}{dx} = \frac{y}{2y - x}$$

vertical when $x = 2y$

$$y^2 - 2y^2 + 9 = 0$$

$$-y^2 + 9 = 0 \quad \therefore y = \pm 3$$

B

7

20 nos

3 dividers in spaces

19

$${}^{19}C_3 = 969$$

A

8

$$x = 3t \quad y = e^t$$

$$\frac{dx}{dt} = 3$$

$$\frac{dy}{dt} = e^t$$

$$\frac{d^2x}{dt^2} = 0$$

$$\frac{d^2y}{dt^2} = e^t$$

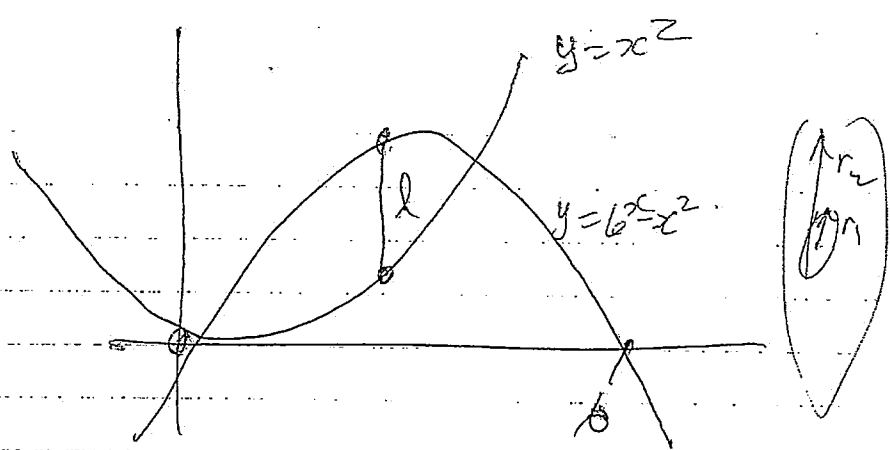
No x acceleration any

y acceleration

$\therefore$ constant in direction

C

9



$$\begin{aligned}
 A &= \pi (r_2^2 - r_1^2) \\
 &= \pi ((6x - x^2)^2 - (x^2)^2) \\
 &= \pi (6x - x^2 - x^2)(6x - x^2 + x^2)
 \end{aligned}$$

$$\begin{aligned}
 &= \pi (6x - 2x^2) 6x \\
 &= \pi (36x^2 - 12x^3)
 \end{aligned}$$

$$\begin{aligned}
 \frac{dA}{dx} &= \pi (72x - 36x^2) = 0 \\
 72x &= 36x^2 \quad x \neq 0 \\
 x &= 2
 \end{aligned}$$

D

10

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{\pi/4}^{\pi/4+h} \frac{\sin x}{x} dx$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} [F(\frac{\pi}{4}+h) - F(\frac{\pi}{4})]$$

where $F'(x) = \frac{\sin x}{x}$

→ ALMA

D

$$\begin{aligned}
 &= \frac{\sin \pi/4}{\pi/4} = \frac{\frac{1}{\sqrt{2}}}{\pi/4} \\
 &= \frac{4}{\pi\sqrt{2}} \\
 &= \frac{2\sqrt{2}}{\pi}
 \end{aligned}$$

Q11 Ext 2 Y12 TRIAL 2016

$$a) \frac{z}{w} = \frac{1+i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{5+i}{13}$$

$$ii) \bar{z} = 1-i = \sqrt{2} \cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})$$

$$b) (1+2w+3w^2)(1+3w+2w^2)$$

$$= 1 + 3w + 2w^2 + 2w + 6w^2 + 9w^3 + 3w^2 + 9w^4 + 6w^5$$

$$= 14 + 11(w + w^2)$$

$$= 14 + 11(-1)$$

$$= 3$$

$$c) (a-b)^2 > 0 \quad (a \neq b)$$

$$a^2 + b^2 > 2ab$$

$$ii) (1-a)(1-b)(1-c)$$

$$= (b+c)(a+c)(a+b) > (2\sqrt{bc})(2\sqrt{ac})(2\sqrt{ab})$$

$$> 8\sqrt{a^2b^2c^2}$$

$$> 8abc$$

$$d) \frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$$

$$\int_0^1 \frac{dx}{(x+2)(x+3)} = \int_0^1 \frac{1}{x+2} - \frac{1}{x+3} dx = \ln\left(\frac{x+2}{x+3}\right) \Big|_0^1$$

$$= \ln \frac{9}{8}$$

$$e) A = \left(0, \frac{2}{\sin \theta}\right) \quad B = \left(\frac{3}{\cos \theta}, 0\right)$$

$$L^2 = \frac{4}{(\sin \theta)^2} + \frac{9}{(\cos \theta)^2} \Rightarrow 2L \frac{dL}{d\theta} = \frac{8 \cos \theta}{\sin^3 \theta} + \frac{18 \sin \theta}{\cos^3 \theta}$$

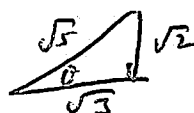
$$SP \text{ when } \frac{dL}{d\theta} = 0 \quad 8 \cos^4 \theta = 18 \sin^4 \theta$$

$$\Rightarrow \tan^4 \theta = \frac{4}{9} \Rightarrow \tan \theta = \sqrt{\frac{2}{3}}$$

Check max/min

$$\sin \theta = \sqrt{\frac{3}{5}}, \cos \theta = \sqrt{\frac{2}{5}}$$

$$\therefore L^2 = \frac{9}{\frac{3}{5}} + \frac{4}{\frac{2}{5}} = 25 \Rightarrow \min L = 5 \quad (L > 0)$$



1m

1m

1m

1m

1m

1m

1m

1m

1m

1m

1m

1m

Do Not accept

$$\sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) \text{ not}$$

non principal argument

Do Not get any marks

just expanding

$$w^3 = 1 \Rightarrow 1 + w + w^2 = 0$$

2 1/2 mark given when

simplified and very close

to final answer = 3

like one step away

Some students wrote $\geq$

only 1m

Alternatively $(b+c)(a+c)(b+a)$

$$= a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2) + 2abc$$

$$> 2ab + 2ab + 2ab + 2abc$$

$$> 8abc$$

1m for A & B

$$1m \text{ for SP } \tan \theta = \sqrt{\frac{2}{3}}$$

1m for check max/min

1m for correct $L = 5$

To get the 3 1/2 x 4

mark, $0 < \theta < \frac{\pi}{2}$ and

not a random guess

MATHEMATICS: Question.....12
Extension 2

Extension 2		Suggested Solutions	Marks	Marker's Comments
(a)(i)	$x^2 + \left(\frac{s}{2}\right)^2 = s^2$ (Pythagoras) $\frac{s^2}{4} - \frac{s^2}{4} = x^2$ $\frac{3s^2}{4} = x^2$ $s^2 = \frac{4x^2}{3}$ $s = \frac{2x}{\sqrt{3}}$ $OA = \frac{\sqrt{3}b}{2}$ (Pythagoras) using similar triangles: $\frac{d}{b/2} = \frac{x}{\sqrt{3}b/2}$ $d = \frac{2x}{\sqrt{3}}$ $\therefore \text{side length} = 2d = \frac{2x}{\sqrt{3}}$	1	side = $\frac{2x}{\sqrt{3}}$	
(ii)	Area of square = $\left(\frac{2x}{\sqrt{3}}\right)^2 = \frac{4x^2}{3}$ volume of slice = $\Delta V = \frac{4x^2}{3} \Delta x$ volume of solid = $\lim_{\Delta x \rightarrow 0} \sum_{i=1}^n \frac{4x_i^2}{3} \Delta x$ $= \frac{4}{3} \int_0^{\sqrt{3}b/2} x^2 dx$ $= \frac{4}{3} \left[\frac{x^3}{3} \right]_0^{\sqrt{3}b/2}$ $= \frac{4}{9} \left[\left(\frac{\sqrt{3}b}{2}\right)^3 - 0 \right]$ $= \frac{4}{9} \times \frac{3\sqrt{3}b^3}{8}$ $= \frac{\sqrt{3}b^3}{6} \text{ units}^3$	1	TC the lim $\Delta x \rightarrow 0$ statement is missing then students lost one mark 1 $\frac{4b^3}{9} \text{ or } \frac{4b^3}{9} = 2\text{mks}$ <u>maximum</u> 1	

MATHEMATICS: Question 12
Extension 2

Suggested Solutions	Marks	Marker's Comments								
(b) $y = \frac{2}{3}x^{5/3} - 3x^{2/3}$ $\frac{dy}{dx} = x^{2/3} - 2x^{-1/3}$ poss. stat. points when $\frac{dy}{dx} = 0$ ie $0 = x^{2/3} - 2x^{-1/3}$ $0 = x^{-1/3}(x - 2)$ $x^{-1/3} \neq 0 \quad \therefore x = 2$ only $\frac{d^2y}{dx^2} = \frac{2}{3}x^{-1/3} + \frac{2}{3}x^{-4/3}$ when $x = 2$, $\frac{d^2y}{dx^2} = 0.52913 + 0.2645668$ $= 0.793696$ $> 0 \quad \therefore$ concave up $\therefore$ rel. min at $x = 2$ (ii) when $x < 0$, what happens to $\frac{d^2y}{dx^2}$? $\frac{d^2y}{dx^2} = \frac{2}{3}x^{-1/3} + \frac{2}{3}x^{-4/3}$ $= \frac{2}{3}x^{-1/3}(1 + x^{-1})$ poss pts of inflexion when $\frac{d^2y}{dx^2} = 0$ ie $0 = \frac{2}{3}x^{-1/3}(1 + \frac{1}{x})$ $x^{-1/3} \neq 0, \quad 1 + \frac{1}{x} = 0$ $x \neq 0, \quad \frac{1}{x} = -1$ $x = -1$ <table border="1"><tr><td>x</td><td>-1.1</td><td>-1</td><td>-0.9</td></tr><tr><td>$\frac{d^2y}{dx^2}$</td><td>-0.059</td><td>0</td><td>0.0767</td></tr></table> $\therefore$ change in concavity at $x = -1$ $\therefore$ pt of inflexion at $(-1, -3^{3/5})$ $\therefore$ concave up for $-1 < x < 0$ $\therefore$ concave down for $x < -1$	x	-1.1	-1	-0.9	$\frac{d^2y}{dx^2}$	-0.059	0	0.0767	① ① ① ① ①	Students who went on to "prove" that $x=0$ was a horizontal pt. of inflexion or a maximum - lost <u>one mark</u> A lot of students just quoted that $\frac{d^2y}{dx^2} < 0$ without substituting in values $\rightarrow$ max. <u>1 mark</u>
x	-1.1	-1	-0.9							
$\frac{d^2y}{dx^2}$	-0.059	0	0.0767							

MATHEMATICS: Question 12...
Extension 2

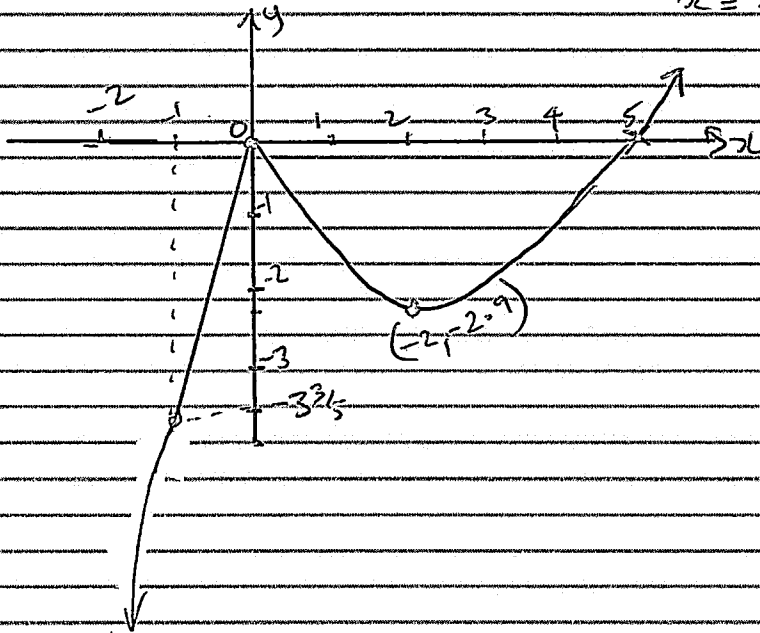
Suggested Solutions

Marks

Marker's Comments

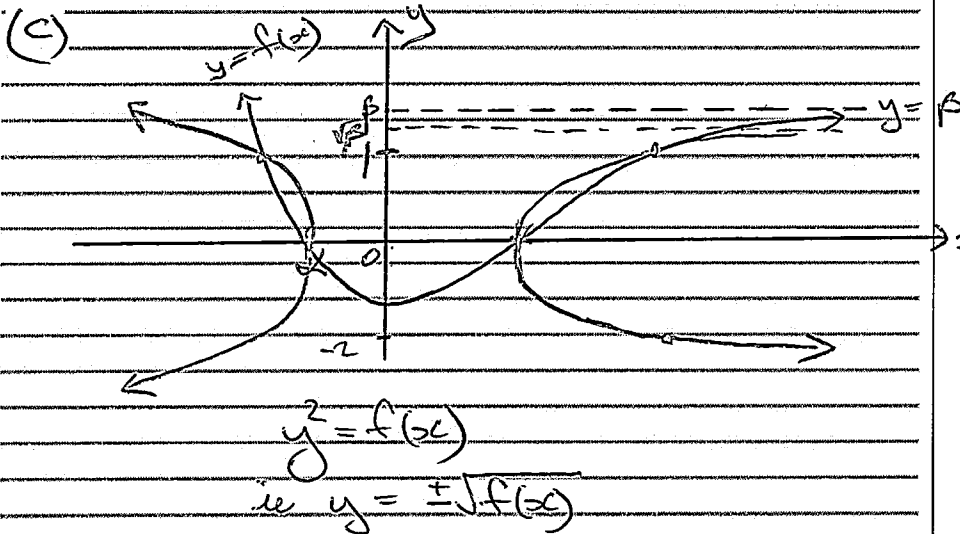
(iii) $y = \frac{3}{5}x^{5/3} - 3x^{2/3}$
 $= x^{2/3}(\frac{3}{5}x - 3)$

$y=0$ when $x=0$ or $\frac{3}{5}x=3$
 $x=5$



- 1 for x intercepts
 1 cusp at $x=0$
 1 for concavity change at $x=-1$ and minimum point

* If the graph did NOT represent what the students found in (i) and (ii) they lost a mark.



- * no graph $2 \leq x < 1$
 * $y = f(x)$ and $y^2 = f(x)$ intersect at $y=1$ } ①
 * vertical tangents at $x=2$ and $x=1$ } ①
 * horizontal asymptote is lower than $y=2$ but higher than $y=1$ } ①
 * symmetrical about $y=0$

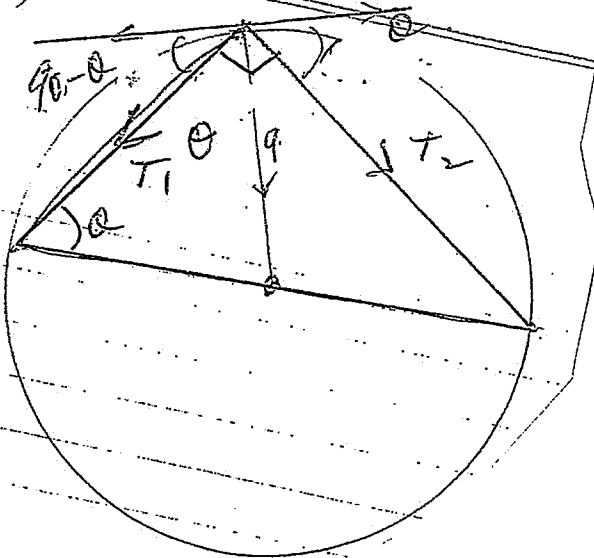
* If they didn't draw the original graph, they lost one mark.

* A lot of students forgot to change the asymptote to $y = \sqrt{2}$ or they put the asymptote $y = \sqrt{2}$ in the wrong place.

MATHEMATICS Extension 2 : Question 13...

Suggested Solutions	Marks Awarded	Marker's Comments
a) If $u_0 = 0$ $u_n = \sqrt{u_{n-1} + 2}$ $n = 1, 2, 3, \dots$ Test $u_0 = 0$ $u_0 = 2 \cos \frac{\pi}{2}$ $= 0$ $\therefore$ true for $n = 0$ Assume true for $k = 0, 1, 2, 3, \dots$ $u_k = 2 \cos \left(\frac{\pi}{2^{k+1}} \right)$ Prove true for $n = k+1$ $u_{k+1} = 2 \cos \left(\frac{\pi}{2^{k+2}} \right)$ Now $u_{k+1} = \sqrt{u_k + 2}$ $= \sqrt{2 \cos \left(\frac{\pi}{2^{k+1}} \right) + 2}$ $= \sqrt{2 \left[2 \cos^2 \frac{\pi}{2^{k+2}} - 1 \right] + 2}$ Since $\cos 2\theta = 1 - 2 \cos^2 \theta$ so $\cos \theta = 1 - 2 \cos^2 \frac{\theta}{2}$ $= \sqrt{4 \cos^2 \frac{\pi}{2^{k+2}} - 2 + 2}$ $= 2 \cos \frac{\pi}{2^{k+2}}$ $= \text{RHS} \quad 0 < \frac{\pi}{2^{k+2}} < \frac{\pi}{2}$ $\therefore$ by process of mathematical induction true for $n \geq 0$ $n \in \mathbb{Z}$	1 1 1 1	Did not need to test for 1 but needed to test for 0 Needed some explanation or clear setting out. (do not use pmt)

b) (i)



(ii)

Let tensions be T_1 and T_2
Tangential (no acceleration)

$$T_2 \cos \theta - T_1 \sin \theta = 0$$

Radial

$$T_2 \sin \theta + T_1 \cos \theta = m \omega^2 r$$

$$m = 50 \quad \omega = \frac{2\pi}{T}$$

$$= 2\pi \times \frac{20}{60}$$

$$= \frac{2\pi}{3} \text{ rad/s}$$

$$(ii) \quad \cos \theta = \frac{1.8}{3} = 0.6 \quad \sin \theta = \frac{2.4}{3} = 0.8$$

Tangential

$$0.6 T_2 = 0.8 T_1 \therefore T_2 = \frac{4}{3} T_1$$

Radial

$$0.8 \times \frac{4}{3} T_1 + 0.6 T_1 = \frac{100 \pi^2}{3}$$

$$\therefore \frac{5}{3} T_1 = \frac{100 \pi^2}{3}$$

$$T_1 = 20 \pi^2 \text{ Newtons}$$

Marks Awarded

Marker's Comments

1

1

1

1

1

did not get both marks if not clear what was radial and tangential.
Students did not clearly indicate which tension went with which string.

Equations needed to make some sense to get carry forward errors.

Suggested Solutions

Marks

Marker's Comments

c) (i) $xy = c^2$

$$y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

when $x = cp$

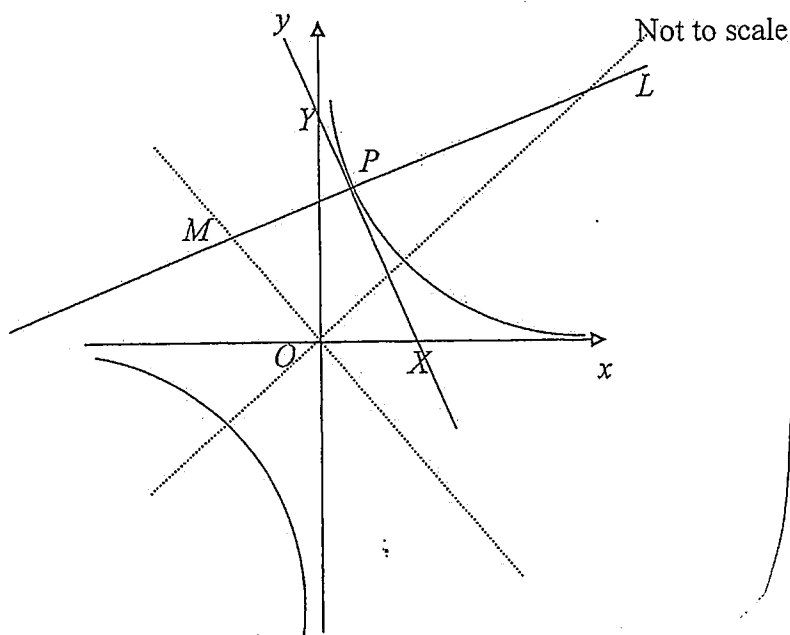
$$\frac{dy}{dx} = -\frac{1}{p^2}$$

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$

$$p^2y - cp = -x + cp$$

$$x + p^2y = 2cp$$

(ii)



For rhombus diagonals bisect at right angles.

coords $X = (2cp, 0)$ midpoint

$Y = (0, \frac{2c}{p}) = P(cp, \frac{c}{p})$

1

well done

1

MATHEMATICS Extension 2: Question 13

Suggested Solutions

Marks

Marker's Comments

Coordinates L $y = x$

Normal

$$px - \frac{1}{p}y = c \left(p^2 - \frac{1}{p^2} \right)$$

$$x \left(p - \frac{1}{p} \right) = c \left(p - \frac{1}{p} \right) \left(p + \frac{1}{p} \right)$$

$$x = c \left(p + \frac{1}{p} \right) y = c \left(p + \frac{1}{p} \right)$$

$$L \left(c \left(p + \frac{1}{p} \right), c \left(p + \frac{1}{p} \right) \right)$$

For M $y = -x$

$$M : x \left(p + \frac{1}{p} \right) = c \left(p^2 - \frac{1}{p^2} \right)$$

$$x = c \left(p - \frac{1}{p} \right) y = -c \left(p - \frac{1}{p} \right)$$

$$M \left(c \left(p - \frac{1}{p} \right), c \left(p - \frac{1}{p} \right) \right)$$

$$L \left(c \left(p + \frac{1}{p} \right), c \left(p + \frac{1}{p} \right) \right)$$

$$\text{Midpoint} \left(cp, \frac{c}{p} \right)$$

which is P

$\therefore$ diagonals are normal
and tangent $\therefore$ perpendicular.

$\therefore$ Rhombus since diagonals
bisect at right angles

Students didn't
simplify and
ended up
with messy
expressions

Students didn't
know sufficient
conditions for
rhombus

could also
prove
parallelogram
with adjacent
sides equal
using gradients
and distance
or all 4 sides
equal

a:) Let $u = -x$ $du = -dx$

$$\int_{-a}^a \frac{x^4 dx}{1+e^x} = \int_{-a}^a \frac{(-u)^4}{1+e^{-u}} (-du) = \int_{-a}^a \frac{u^4}{1+e^{-u}} du = \int_{-a}^a \frac{e^u u^4}{1+e^u} du$$

$$\therefore \int_{-a}^a \frac{x^4 dx}{1+e^x} = \int_{-a}^a \frac{e^x x^4}{1+e^x} dx \quad (\text{change of variable})$$

ii) Let $I = \int_{-2}^2 \frac{x^4}{1+e^x} dx = \int_{-2}^2 \frac{e^x x^4}{1+e^x} dx$

$$2I = \int_{-2}^2 \frac{(1+e^x)x^4}{1+e^x} dx = \int_{-2}^2 x^4 dx = \left[\frac{x^5}{5} \right]_{-2}^2$$

$$2I = \frac{1}{5} [32 - (-32)] = \frac{64}{5}$$

$$\therefore I = \frac{32}{5} \quad \text{Im}$$

b) Let x be the real root, then

$$x^2 + x(a+ib) + m+in = 0 + 0i$$

$$\therefore \text{Real part} \quad x^2 + ax + m = 0 \quad (1) \quad \text{Im}$$

Imaginary part

$$bx + n = 0 \Rightarrow x = -\frac{n}{b} \quad \text{Im}$$

Sub $x = -\frac{n}{b}$ into (1) $\left(-\frac{n}{b}\right)^2 + a\left[-\frac{n}{b}\right] + m = 0 \quad \text{Im}$

$$n^2 - anb + mb^2 = 0$$

Alternatively let α is the real root and β be the complex root

$$\therefore \begin{matrix} \text{sum roots} & \alpha + \beta = -(a+ib) \\ \text{product roots} & \alpha\beta = m+in \end{matrix} \quad \text{Im}$$

$$\beta = (-\alpha - a) - ib \quad \alpha\beta = \alpha[-\alpha - a] - (iab)$$

$$\alpha\beta = m+in$$

P1

Alternatively

$$\text{Let } f(x) = \frac{(1-e^x)x^4}{1+e^x}$$

$$-f(-x) = \frac{(-x)^4(e^{-x}-1)}{1+e^{-x}} \cdot \frac{e^x}{e^x}$$

$$-f(-x) = \frac{(1-e^x)x^4}{1+e^x} = f(x)$$

$\therefore f(x)$ is odd Im

$$\therefore \int_{-a}^a f(x) dx = 0$$

$$\therefore \int_{-a}^a \frac{x^4 dx}{1+e^x} = \int_{-a}^a \frac{e^x x^4}{1+e^x} dx \quad \text{Im}$$

Im (a) for equating real part / imaginary part

Im for final proof

Im for sum & product of roots

Note A lot of students think only 1 real root

$$\therefore \Delta = 0$$

This only apply to real polynomial.

Om

$$\therefore -a[a+b] = m \quad \text{and} \quad h = -a b$$

$\underbrace{\hspace{10em}}_{|m} \quad \underbrace{\hspace{10em}}_{(\text{ie } a = -\frac{h}{b})}$

Solving simultaneously

$$-\frac{h}{b} \left[-\frac{h}{b} + a \right] = m$$

$$\frac{-h^2}{b^2} + \frac{a h}{b} = m \quad |m$$

$$h^2 - a b h + m b^2 = 0$$

c)

f	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$P(X=2) = \frac{1}{36}$$

$$P(X=3) = \frac{2}{36}$$

$$P(X=4) = \frac{3}{36}$$

$$P(X=5) = \frac{4}{36}$$

$$P(X=6) = \frac{5}{36}$$

$$P(X=7) = \frac{6}{36}$$

$$P(X=8) = \frac{5}{36}$$

$$P(X=9) = \frac{4}{36}$$

$$P(X=10) = \frac{3}{36}$$

$$P(X=11) = \frac{2}{36}$$

$$P(X=12) = \frac{1}{36}$$

$$P(\text{equal scores}) = \frac{1}{36} [1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2]$$

|m

$$= \frac{73}{648} \quad |m$$

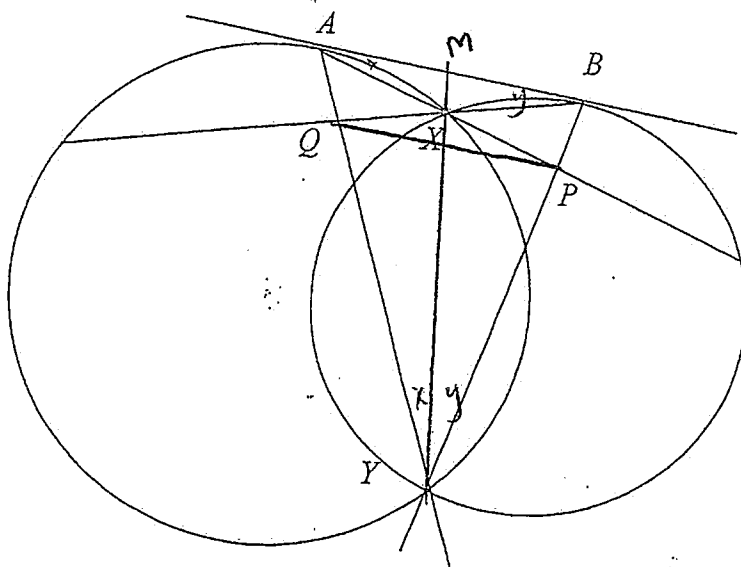
1 m for
Equating real &
imaginary pts

1 m final line

Getting the right
formula

final answer

Well done overall



- (i) Join XY Let $\angle BAX = x$, $\angle ABX = y$
 $\angle BXP = x+y$ (exterior angle of $\triangle ABX$)
 $\angle BAX = \angle AYX = x$ (angle between tangent and chord at point of contact equals angle in alternate segment) 1m
 Similarly
 $\angle ABX = \angle BYX = y$

$$\therefore \angle BXP = \angle QYP = x+y$$

$\therefore XPYQ$ is a cyclic quadrilateral
 (exterior angle equals opposite interior angle) 1m

- (ii) Join PQ $\angle XPQ = \angle QYX = x$ (angle at circumference standing on same arc as $XPYQ$ is cyclic)

$$\therefore \angle BAP = \angle XPQ = x$$

$\therefore AB \parallel QP$ (alternate angles equal.) 1m

- (iii) Extend YX to meet AB at M , & P at C

$$AM^2 = MX \cdot MY \quad \text{(: product of intercepts on secants equal to tangents squared)}$$

$$\text{Similarly } BM^2 = MX \cdot MY$$

$$\therefore AM^2 = BM^2$$

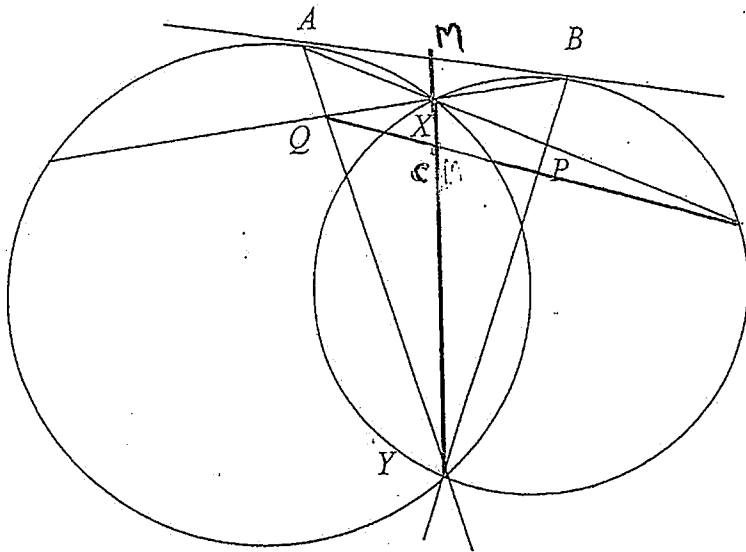
$$\therefore AM = BM \quad (AM > 0, BM > 0) \quad \text{1m}$$

Some students do not know the exact theorem.

must mention alternate angles ^{names of} $\wedge$

Most students do not realize this theorem

& give up.



In $\triangle AMY$, $\triangle QCY$

$AB \parallel QP$ (proved)

$\angle AMY = \angle QCY$ (corresponding angles on parallel lines)

$\angle AYM$ is common

$\triangle AMY \parallel \triangle QCY$ (equiangular)

$\therefore \frac{AM}{QC} = \frac{MY}{CY}$ (ratio of corresponding sides of similar triangles) I m

Similarly $\frac{BM}{PC} = \frac{MY}{CY}$

$$\therefore \frac{AM}{QC} = \frac{BM}{PC}$$

I m

As $AM = BM \therefore QC = PC$

$\therefore XY$ bisects PQ .

Students must show the ratio of corresponding sides of similar triangles to get I m not stopping at equiangular.

Note Students did not prove $AM = BM$ max I m

2016 Trial 22 MATHEMATICS: Question 15....

Extension 2

Suggested Solutions

Marks

Marker's Comments

(a) Let $m = \frac{1+\alpha}{1-\alpha}$

$$m(1-\alpha) = 1+\alpha$$

$$m-1 = \alpha(1+m)$$

$$\frac{m-1}{m+1} = \alpha$$

$$\alpha \text{ is a root} \Rightarrow \alpha^3 - \alpha - 1 = 0$$

$$\therefore \left(\frac{m-1}{m+1}\right)^3 - \left(\frac{m-1}{m+1}\right) - 1 = 0$$

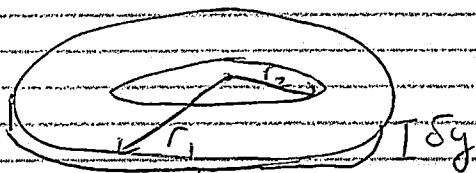
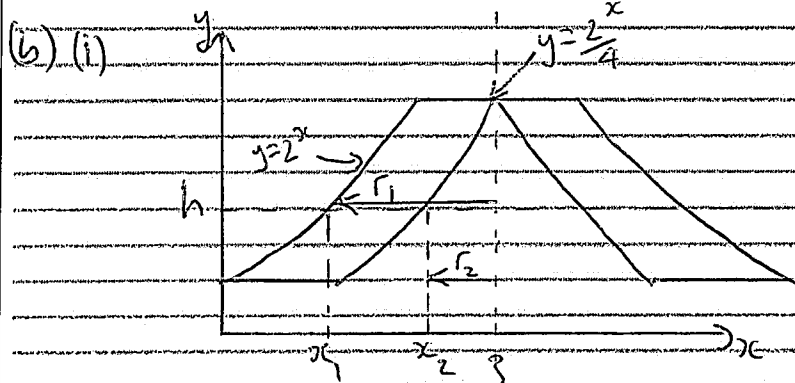
$$(m-1)^3 - (m-1)(m+1)^2 - (m+1)^3 = 0$$

$$m^3 - 3m^2 + 3m - 1 + m^2 + m^3 - 1 - m - m^3 - 3m^2 - 3m - 1 = 0$$

$$-m^3 - 7m^2 + m - 1 = 0$$

Equation in x is

$$x^3 + 7x^2 - x + 1 = 0$$



$$r_1 = 3 - x_1$$

$$r_2 = 3 - x_2$$

A diagram with dimension was necessary to earn the mark

MATHEMATICS: Question.....
Extension 2

Suggested Solutions

Marks

Marker's Comments

Note

$$y = 2^x \Rightarrow h = 2^{x_1}$$

$$\log_2 h = x_1 \Rightarrow r_1 = 3 - \log_2 h$$

$$4y = 2^x \Rightarrow 4h = 2^{x_2}$$

$$\log_2 4h = x_2 \Rightarrow r_2 = (\log_2 4 + \log_2 h) + 3 \\ = 1 + \log_2 h$$

$$A = \pi(r_1^2 - r_2^2)$$

$$= \pi((3 - \log_2 h)^2 - (1 + \log_2 h)^2)$$

$$= \pi(9 - 6\log_2 h + (\log_2 h)^2 - 1 - 2\log_2 h - (\log_2 h)^2)$$

$$= \pi(8 - 4\log_2 h)$$

$$= 4\pi(2 - \log_2 h) \text{ as required}$$

$$(ii) \Delta V = A \Delta y$$

$$V \doteq \lim_{\Delta y \rightarrow 0} \sum_{i=1}^2 4\pi(2 - \log_2 y) \Delta y$$

$$= 4\pi \int_1^2 2 - \log_2 y \, dy$$

$$= 4\pi \left(\int_1^2 2 \, dy - \int_1^2 1 \times \log_2 y \, dy \right)$$

$$= 4\pi \left([2y]_1^2 - [y \log_2 y]_1^2 + \int_1^2 \frac{y \times \frac{1}{y}}{\ln 2} \, dy \right)$$

Many algebraic methods were possible, including noting

$$A = \pi(r_1 - r_2)(r_1 + r_2)$$

Faulty algebra or misused log laws

resulted in a carried error IF subsequent logic was sound.

MATHEMATICS: Question.....
Extension 2

Suggested Solutions	Marks	Marker's Comments
$= 4\pi \left(\ln 2 - 2 \log_2 2 + 0 + \int_1^2 \frac{1}{\ln 2} dy \right)$ $= 4\pi \left(0 + \left[\frac{y}{\ln 2} \right]_1^2 \right)$ $= \frac{4\pi}{\ln 2}$	1	If student made an error in integration by parts, a carried error was awarded
$(1) \quad z^n = (z+1)^n$ $\frac{(z+1)^n}{z^n} = 1$ $= \cos\left(\frac{2k\pi}{n}\right) \quad k \neq 0, k=1, 2, 3, \dots$	1	only if the process showed progress towards the end goal and did not simplify the computation
$\frac{(z+1)}{z} = \cos\left(\frac{2k\pi}{n}\right) \quad \text{By De Moivre's theorem}$ $z+1 = z \cos\left(\frac{2k\pi}{n}\right)$ $1 = z \left(\cos\left(\frac{2k\pi}{n}\right) - 1 \right)$ $z = \frac{1}{\cos\left(\frac{2k\pi}{n}\right) - 1} \quad k \neq 0, k=1, 2, 3, \dots$	1	Dubbing De Moivre's theorem was necessary to earn the second mark. Simply substituting the result into the given expressions is circular reasoning and earned <u>no marks</u>

MATHEMATICS: Question.....
Extension 2

Suggested Solutions

Marks

Marker's Comments

$$(i) z = \frac{\cos \frac{2\pi k}{n} - 1 + i \sin \frac{2\pi k}{n}}{1 - 2\sin^2 \frac{\pi k}{n} - 1 + 2i \sin \frac{\pi k}{n} \cos \frac{\pi k}{n}}$$

1

A clear

$$= \frac{2i \sin \frac{\pi k}{n} (\cos \frac{\pi k}{n} + i \sin \frac{\pi k}{n})}{2i \sin \frac{\pi k}{n} (\cos \frac{\pi k}{n} + i \sin \frac{\pi k}{n})} \times \frac{\cos \frac{\pi k}{n}}{\cos \frac{\pi k}{n}}$$

1

computational
path needed to
be shown for
marks to be

$$= \frac{1}{2} \left(\frac{-1 \cos \frac{\pi k}{n}}{\sin \frac{\pi k}{n}} - \frac{i \sin \frac{\pi k}{n}}{i \sin \frac{\pi k}{n}} \right)$$

1

awarded.

$$= -\frac{1}{2} \left(1 - i \cot \frac{\pi k}{n} \right)$$

Possible alternatives include:

$$z = \frac{\cos \left(\frac{2\pi k}{n} \right) - 1 + i \sin \left(\frac{2\pi k}{n} \right)}{2(1 - \cos \left(\frac{2\pi k}{n} \right))}$$

$$= \frac{-1 - i \sin \left(\frac{2\pi k}{n} \right)}{2(1 - \cos \left(\frac{2\pi k}{n} \right))}$$

(ii) $z = x + iy$ where x is constant

z lies on the line $x = -\frac{1}{2}$

1

16. a) i) $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}} dx$, $n \in \mathbb{Z}$, $n \geq 0$

$$= \left[2(x+1)^{\frac{1}{2}} \times x^n \right]_0^1 - n \int_0^1 x^{n-1} \times 2\sqrt{x+1} dx \quad \text{--- (1)}$$

① $\text{---} = 2\sqrt{2} - 2n \int_0^1 x^{n-1} \sqrt{x+1} dx$

$$= 2\sqrt{2} - 2n \int_0^1 x^{n-1} \times \frac{(x+1)}{\sqrt{x+1}} dx$$

$$= 2\sqrt{2} - 2n \int_0^1 \frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}} dx \quad \text{--- (1)}$$

$$= 2\sqrt{2} - 2n I_n - 2n I_{n-1}$$

$$\therefore (2n+1) I_n = 2\sqrt{2} - 2n I_{n-1} \quad \text{--- (1)}$$

ii) $\int_0^1 \frac{x^3}{\sqrt{x+1}} dx = I_3$

$$I_3 = \frac{2\sqrt{2} - 6I_2}{7} \quad \text{--- (1)}$$

$$= \frac{2\sqrt{2}}{7} - \frac{6}{7} \left[\frac{2\sqrt{2}}{5} - \frac{4I_1}{5} \right]$$

$$= \frac{2\sqrt{2}}{7} - \frac{12\sqrt{2}}{35} + \frac{24}{35} \left[\frac{2\sqrt{2}}{3} - \frac{2I_0}{3} \right]$$

$$= \frac{-2\sqrt{2}}{35} + \frac{48\sqrt{2}}{105} - \frac{48}{105} I_0$$

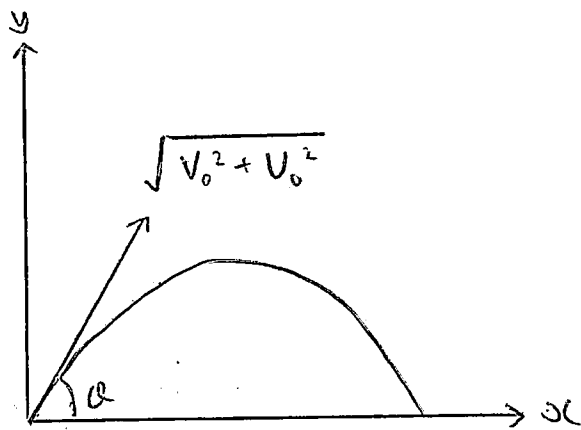
$$= \frac{42\sqrt{2}}{105} - \frac{48}{105} \int_0^1 \frac{1}{\sqrt{x+1}} dx$$

$$= \frac{42\sqrt{2}}{105} - \frac{48}{105} \left[2\sqrt{x+1} \right]_0^1 \quad \text{--- (1)}$$

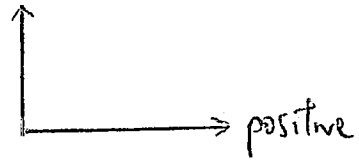
$$= \frac{42\sqrt{2}}{105} - \frac{48}{105} (2\sqrt{2} - 2)$$

$$= \frac{96}{105} - \frac{54\sqrt{2}}{105} = \frac{32}{105} - \frac{18\sqrt{2}}{105} \quad \text{--- (1)}$$

b)



positive



vertically - $m \ddot{y} = -mg - mkv$ ————— (1)

$$\frac{dv}{dt} = -g - kv$$

$$\frac{dt}{dv} = \frac{-1}{g+kv}$$

$$\int_0^t dt = - \int_{v_0}^v \frac{1}{g+kv} dv$$

$$t = -\frac{1}{k} [\ln(g+kv)]_{v_0}^v$$

$$= -\frac{1}{k} \ln \frac{g+kv}{g+kv_0} \text{ ————— (1)}$$

$$\therefore -kt = \ln \left(\frac{g+kv}{g+kv_0} \right)$$

$$\frac{g+kv}{g+kv_0} = e^{-kt}$$

$$g+kv = (g+kv_0)e^{-kt}$$

$$kv = (g+kv_0)e^{-kt} - g$$

$$v = \frac{(g+kv_0)e^{-kt}}{k} - \frac{g}{k}$$

1st Approach

$$\frac{dy}{dt} = \frac{(g+kv_0)e^{-kt} - g}{k}$$

$$\int_0^y dy = \int_0^t \frac{(g+kv_0)e^{-kt} - g}{k} \text{ ————— (1)}$$

$$y = \left[\frac{(g+kv_0)e^{-kt}}{k^2} - \frac{gt}{k} \right]_0^t$$

$$= \frac{(g+kv_0)}{k^2} - \frac{(g+kv_0)e^{-kt}}{k^2} - \frac{gt}{k} = \left(\frac{g}{k^2} + \frac{v_0}{k} \right) (1 - e^{-kt}) - \frac{gt}{k} \quad (1)$$

Alternatively:

$$v \frac{dv}{dy} = -g - kv$$

$$\frac{dv}{dy} = \frac{-g - kv}{v}$$

$$\frac{dy}{dv} = \frac{-v}{g + kv}$$

$$\int_0^y dy = - \int_{v_0}^v \frac{v}{g + kv} dv$$

$$y = -\frac{1}{k} \int_{v_0}^v \left(1 - \frac{g}{g + kv}\right) dv$$

$$= -\frac{1}{k} \left[v - \frac{g}{k} \ln(g + kv) \right]_{v_0}^v$$

$$= -\frac{1}{k} \left[v - \frac{g}{k} \ln(g + kv) - v_0 + \frac{g}{k} \ln(g + kv_0) \right]$$

$$= -\frac{1}{k} \left[v - v_0 + \frac{g}{k} \ln\left(\frac{g + kv_0}{g + kv}\right) \right] \quad \text{--- (1)}$$

Sub $v = \frac{(g + kv_0)e^{-kt} - g}{k}$ into y.

$$y = -\frac{1}{k} \left[\frac{(g + kv_0)e^{-kt} - g}{k} - v_0 + \frac{g}{k} \ln\left(\frac{g + kv_0}{g + (g + kv_0)e^{-kt} - g}\right) \right]$$

$$= -\frac{1}{k} \left[\frac{(g + kv_0)e^{-kt}}{k} - \frac{g}{k} - v_0 + \frac{g}{k} \ln\left(\frac{1}{e^{-kt}}\right) \right]$$

$$= -\frac{1}{k} \left[\frac{(g + kv_0)e^{-kt}}{k} - \frac{g}{k} - v_0 + gt \right]$$

$$= \frac{v_0}{k} - \frac{(g + kv_0)e^{-kt}}{k^2} + \frac{g}{k^2} - \frac{gt}{k}$$

$$= \frac{v_0}{k} (1 - e^{-kt}) + \frac{g}{k^2} (1 - e^{-kt}) - \frac{gt}{k}$$

$$= \left(\frac{g}{k^2} + \frac{v_0}{k}\right) (1 - e^{-kt}) - \frac{gt}{k} \quad \text{--- (1)}$$

$$\left[\frac{v}{g + kv} = \frac{g + kv - g}{g + kv} \times \frac{1}{k} \right. \\ \left. = \frac{1}{k} \left(1 - \frac{g}{g + kv}\right) \right]$$

ii) Horizontally:

$$m\ddot{x} = -kx$$

$$\ddot{x} = -kx$$

$$\frac{du}{dt} = -ku$$

$$\int_0^t dt = -\frac{1}{k} \int_{u_0}^u \frac{1}{u} du$$

$$t = \frac{1}{k} \ln\left(\frac{u_0}{u}\right) \quad \text{--- (1)}$$

$$e^{kt} = \frac{u_0}{u}$$

$$u = u_0 e^{-kt}$$

$$\frac{dx}{dt} = u_0 e^{-kt}$$

$$x = u_0 \int_0^t e^{-kt} dt$$

$$= \frac{u_0}{k} (e^{-kt} - 1) \quad \text{--- (1)}$$

$$\frac{-kx}{u_0} = e^{-kt} - 1$$

$$e^{kt} = \frac{-kx}{u_0} + 1$$

$$e^{kt} = \frac{u_0 - kx}{u_0}$$

$$t = \frac{1}{k} \ln\left(\frac{u_0 - kx}{u_0}\right) \quad \text{--- (1)}$$

Sub into y from i)

$$y = \left(\frac{g}{k^2} + \frac{v_0}{k}\right) \left(1 - e^{\ln\left(\frac{u_0 - kx}{u_0}\right)}\right) - \frac{g}{k^2} \ln\left(\frac{u_0}{u_0 - kx}\right)$$

$$= \left(\frac{g}{k^2} + \frac{v_0}{k}\right) \left(1 - \frac{u_0 - kx}{u_0}\right) - \frac{g}{k^2} \ln\left(\frac{u_0}{u_0 - kx}\right)$$

$$= \left(\frac{g}{k^2} + \frac{v_0}{k}\right) \left(\frac{kx}{u_0}\right) - \frac{g}{k^2} \ln\left(\frac{u_0}{u_0 - kx}\right) \quad \text{--- (1)}$$