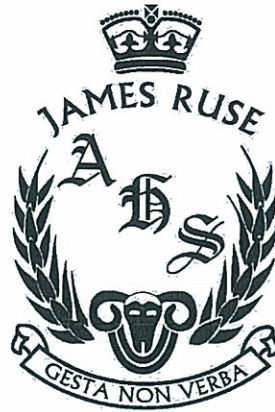


Student Number:	
Class:	



TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2017

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black pen.
- Board-approved calculators & templates may be used.
- A Reference Sheet is provided.
- In Questions 11-16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: 10 marks

- Attempt Questions 1-10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 90 Marks

- Attempt Question 11-16.
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate, *stapled* bundles, clearly labeled as Question 11, Question 12, etc.

Each question must show your Candidate Number.

QUESTION ONE

Which of the following describes the conic section given by

$$4x^2 + 7y^2 + 32x - 56y + 148 = 0?$$

- (A) Ellipse with centre $(-4, 4)$ and foci at $(4 \pm \sqrt{3}, -4)$.
- (B) Hyperbola with centre $(-4, 4)$ and foci at $(4, -4 \pm \sqrt{3})$.
- (C) Ellipse with centre $(-4, 4)$ and foci at $(-4 \pm \sqrt{3}, 4)$.
- (D) Hyperbola with centre $(-4, 4)$ and foci at $(-4, 4 \pm \sqrt{3})$.

QUESTION TWO

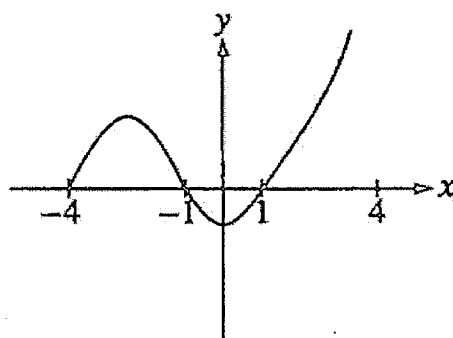
If $\sum_{n=0}^{\infty} \cos^{2n} \theta = 5$, what is the value of $\cos 2\theta$?

- (A) $\frac{4}{5}$
- (B) $\frac{3}{5}$
- (C) $\frac{2}{5}$
- (D) $\frac{1}{5}$

Examination continues on next page ...

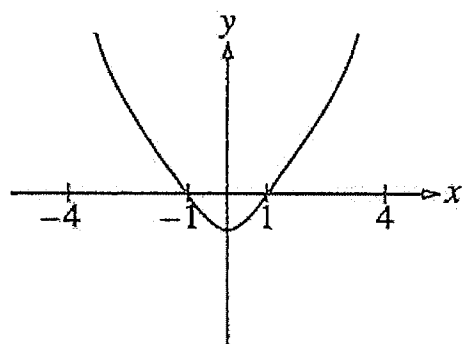
7

QUESTION THREE

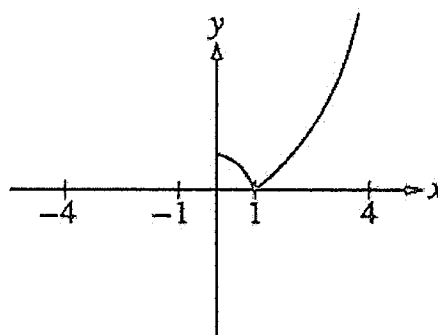


The graph of $y = f(x)$ is shown above. Which of the following is a possible graph of $y = f(|x|)$?

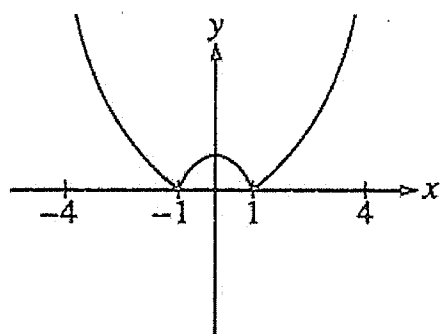
(A)



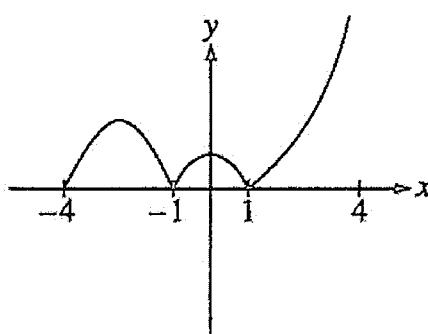
(B)



(C)



(D)



Examination continues on next page ...

QUESTION FOUR

Without evaluating directly, which of the following integrals is positive?

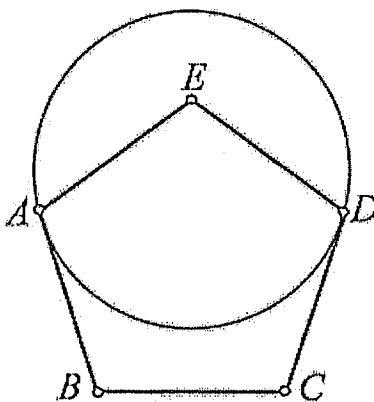
(A) $\int_{-1}^1 \frac{\sin^{-1} x}{1+x^2} dx$

(B) $\int_{-1}^1 \frac{\cos^{-1} x}{1+x^2} dx$

(C) $\int_{-1}^1 \frac{\tan^{-1} x}{\cos x} dx$

(D) $\int_{-1}^1 (x^2 - 1)e^{-x^2} dx$

QUESTION FIVE



The diagram above shows a regular pentagon $ABCDE$, and a circle of unit radius that is tangent to $\overline{DC}$ at D and to $\overline{AB}$ at A . What is the length of the arc AD that is contained within the pentagon?

(A) $\frac{2\pi}{5}$

(B) $\frac{3\pi}{5}$

(C) $\frac{4\pi}{5}$

(D) $\frac{6\pi}{5}$

Examination continues on next page ...

QUESTION SIX

Let $f(x) = ax^4 - bx^2 + x + 5$ with $f(-3) = 2$. What is the value of $f(3)$?

- (A) 8
- (B) 1
- (C) -2
- (D) -5

QUESTION SEVEN

If the complex number z satisfies $z + |z| = 2(1 + 4i)$, which of the following is $|z|^2$?

- (A) 68
- (B) 100
- (C) 208
- (D) 289

QUESTION EIGHT

For $xy = e^{xy}$, what is $\frac{dy}{dx}$ where y is an implicit function of x ?

(A) $\frac{x}{y}$

(B) $-\frac{x}{y}$

(C) $\frac{y}{x}$

(D) $-\frac{y}{x}$

QUESTION NINE

A funfair game has the following setup: a player may toss two balls into any of k chutes, and if both balls are returned via a single chute, the player wins. What is the probability that both balls enter via different chutes, but are returned via the same chute, where that chute is neither of the chutes by which any ball entered?

(A) $\frac{k^2 - 3k + 2}{k^3}$

(B) $\frac{k^3 - 4k^2 + 5k - 2}{k^3}$

(C) $\frac{k^2 - k}{k^2 - 5k + 6}$

(D) $\frac{k^2 - 2k + 2}{k^4}$

Examination continues on next page ...

QUESTION TEN

Suppose f and g are differentiable functions such that $g(x) > 0$ for all x and $f(0) = 1$.

If $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x)$, which of the following is $f(x)$?

(A) 0

(B) 1

(C) e^x

(D) $g(x)$

QUESTION ELEVEN (15 Marks) Use a SEPARATE writing booklet.

Marks

(a) Define $I_n = \int_0^1 x^n e^x dx$.

(i) Show that $I_n = e - nI_{n-1}$.

2

(ii) Hence evaluate I_2 .

2

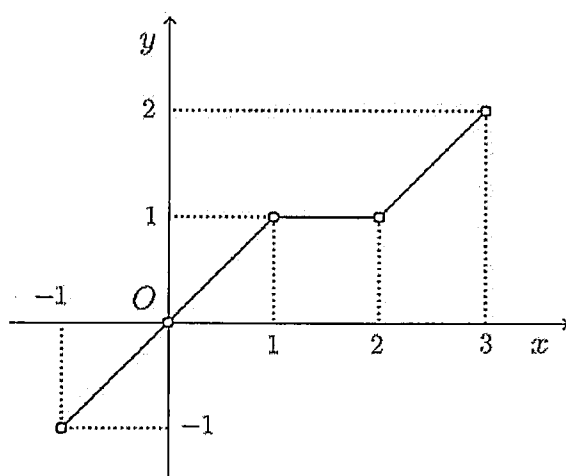
(b) (i) Show that $\int_0^a f(a-x) dx = \int_0^a f(x) dx$.

2

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$, for n a positive integer.

3

(c)



The diagram above shows a piecewise function $f(x)$ comprised of linear segments for all real x over $[-1, 3]$. Sketch, over the real numbers, the graphs of:

(i) $y = \log_e f(x)$

2

(ii) $y = \frac{1}{f(x)}$

2

(iii) $y^2 = f(x)$

2

Examination continues on next page ...

QUESTION TWELVE (15 Marks) Use a SEPARATE writing booklet.

Marks

(a) Solve $z^2 + iz + 1 = 0$.

2

(b) Sketch the region in the Argand plane where

2

$$1 \leq |z - 2i| \leq 2$$

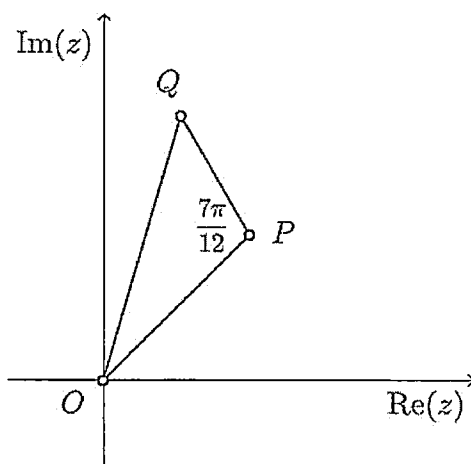
and

$$-\frac{\pi}{4} \leq \arg(z - 2i) \leq \frac{\pi}{4}$$

hold simultaneously.

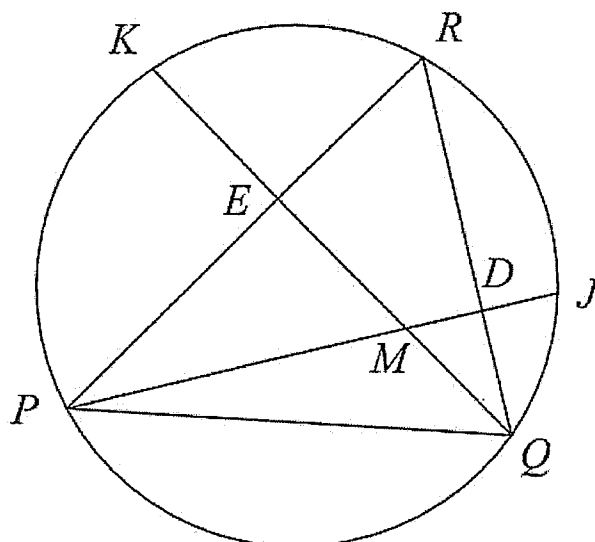
(c)

3



The diagram above shows triangle OPQ in the Argand plane with the point P being represented by the complex number $\frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}}$. If $\angle OPQ = \frac{7\pi}{12}$ and $|PQ| = 2$, find, in Cartesian form, the complex number representing the point Q .

(d)



In the diagram above, C is the circumcircle of triangle PQR . The altitude PD is produced to meet C at J , and the altitude QE is produced to meet C at K . Also, let the altitudes intersect at the point M .

- (i) Copy the diagram and state why quadrilaterals $PQDE$ and $REMD$ are cyclic. 2
 - (ii) Prove that PR bisects $\angle KRM$. 2
 - (iii) Hence show that $KR = JR$. 2
- (e) In how many ways can seven identical cats be put into three identical pens so that all of the pens are occupied? You must state reasoning. 2

QUESTION THIRTEEN (15 Marks) Use a SEPARATE writing booklet.

Marks

- (a) If $P(x) = x^4 - 8x^3 + 18x^2 - 27$, show that $P(x)$ has a multiple zero and determine its multiplicity. 2

- (b) Let $P = (a \sec \theta, b \tan \theta)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

- (i) Show that the equation of the tangent to the hyperbola at the point P is given by 2

$$bx \sec \theta - ay \tan \theta = ab.$$

- (ii) Show that the tangent intersects the asymptotes of the hyperbola at the points Q and R where 3

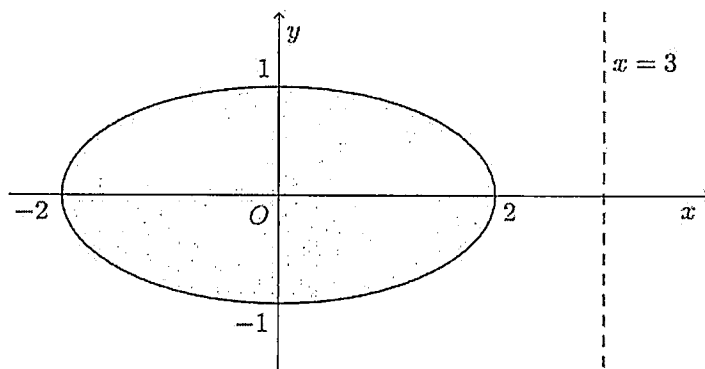
$$Q = (a(\sec \theta + \tan \theta), b(\sec \theta + \tan \theta))$$

and

$$R = (a(\sec \theta - \tan \theta), -b(\sec \theta - \tan \theta)).$$

- (iii) Hence, for O the origin, show that $|OQ| \cdot |OR|$ is a constant. 2

(c)



The area bounded by the ellipse $x^2 + 4y^2 = 4$ is rotated about the line $x = 3$ to form an oblate torus.

- (i) Using the method of cylindrical shells, show that the volume V of the torus is given by the integral 3

$$V = 2\pi \int_{-2}^2 (3 - x)\sqrt{4 - x^2} dx$$

- (ii) Hence show that $V = 12\pi^2$. 3

Examination continues on next page ...

QUESTION FOURTEEN (15 Marks) Use a SEPARATE writing booklet.

Marks

(a) Define $P(x) = x^3 - 2x^2 + 3x + 2$ and let α, β, γ be the roots of $P(x) = 0$.

(i) By considering $\alpha^2 + \beta^2 + \gamma^2$, explain why $P(x) = 0$ has only one real solution. 1

(ii) Find the monic polynomial having roots $\alpha\beta, \alpha\gamma, \beta\gamma$. 3

(b) An object of mass m kg is launched vertically upward into a resistive medium. It reaches a maximum height H metres and then descends vertically downward. In both the upward and downward motions, the object experiences a resistive force of magnitude mkv^2 where v m/s is the velocity of the object and $k > 0$ is a constant. The only other force acting on the object is that due to gravity, which exerts an acceleration of magnitude g m/s².

(i) If the object is launched with an initial velocity of u m/s, show that the maximum height reached is 3

$$H = \frac{1}{2k} \log_e \left(1 + \frac{k}{g} u^2 \right)$$

(ii) After reaching its maximum height, the object descends.

(α) Explain why the terminal velocity of the object is $\sqrt{\frac{g}{k}}$. 1

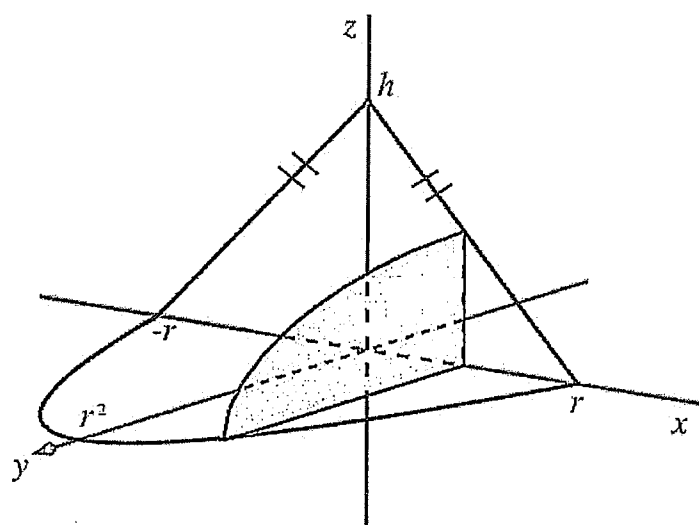
(β) Show that the falling time t is given by 3

$$t = \frac{w}{2g} \log_e \left(\frac{w + v}{w - v} \right)$$

where w is the terminal velocity.

(c)

4



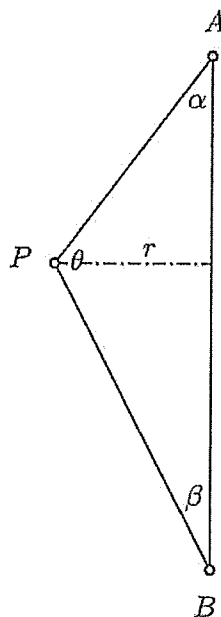
A solid is formed with a parabolic base having equation $y = r^2 - x^2$. A typical vertical cross-section, taken perpendicular to the x -axis, is that of a quarter of an ellipse with one semi-axis bounded by the parabola and the other semi-axis bounded by the side of an isosceles triangle having height h and base length $2r$. Find the volume of the solid given that the area of an ellipse is πab , where a and b are the lengths of the semi-major and semi-minor axes.

Examination continues on next page . . .

QUESTION FIFTEEN (15 Marks) Use a SEPARATE writing booklet.

Marks

(a)



The diagram above shows an object of mass m kg fixed at a point P between two light, taut strings AP and BP . The length AB corresponds to a fixed rod in line with the vertical. The mass rotates in a horizontal circle at uniform speed. The construction forms a triangle with $\angle PAB = \alpha$, $\angle PBA = \beta$ and $\angle APB = \theta$. Let the distance from P to AB be r metres, and let the tensions supplied by strings AP and BP be T and U respectively. Also, let ω rad/s be the angular speed and g m/s² the magnitude of the acceleration due to gravity.

(i) Show that the tensions supplied by the strings are:

3

$$T = \frac{m(r\omega^2 \cos \beta + g \sin \beta)}{\sin \theta}$$

and

$$U = \frac{m(r\omega^2 \cos \alpha - g \sin \alpha)}{\sin \theta}$$

(ii) For the case where $\theta = \frac{\pi}{2}$, show that the system must have

3

$$\frac{\omega^2}{g} \geq \frac{AB}{AP^2}$$

for both strings to remain taut.

Examination continues on next page ...

(b) Define $P(z) = z^5 - 1$ where z is complex.

(i) Use de Moivre's theorem to find the roots of $P(z) = 0$.

1

(ii) Express $P(z)$:

(α) as a product of real linear and irreducible factors;

2

(β) in the form $P(z) = (z - 1)Q(z)$ where $Q(z)$ is a polynomial in z .

1

(iii) Hence show that

2

$$(1 + \cos(\frac{2\pi}{5}))(1 + \cos(\frac{4\pi}{5})) = \frac{1}{4}$$

(c) (i) Given

1

$$\frac{1}{3}(x + y + z) \geq (xyz)^{1/3}$$

for all x, y, z non-negative real numbers, show that

$$xy + yz + zx \geq 3(xyz)^{2/3}$$

(ii) Hence show that

2

$$xyz \leq (a - 1)^3$$

if $1 + (x + y + z) + (xy + yz + zx) + xyz = a^3$ where $a \geq 2$ is a real number.

QUESTION SIXTEEN (15 Marks) Use a SEPARATE writing booklet.

Marks

(a) Define the integral I_n by

$$I_n = \int_0^\pi f_n(x) \sin x \, dx$$

where n is a non-negative integer and where $f_n(x)$ is an n -times differentiable function on the real numbers such that $f_n^{(k)}(0) = f_n^{(k)}(\pi) = 0$ for all integers $k = 0, 1, 2, \dots, n$ where $f_n^{(k)}(x)$ is the k^{th} derivative of $f_n(x)$.

(i) Using integration by parts, show that $I_n = - \int_0^\pi f_n^{(2)}(x) \sin x \, dx$. 2

(ii) If $f_n(x) = \frac{(\pi x - x^2)^n}{n!}$, show that 3

$$I_n = (4n - 2)I_{n-1} - \pi^2 I_{n-2}$$

(iii) Use strong induction to prove that for all integers $n \geq 2$, the recursion formula in (ii) generates polynomials in π of the form, 3

$$I_n = \sum_{k=0}^n c_k \pi^k$$

having all integer coefficients c_k and with degree at most n . You may assume that $I_0 = 2$ and $I_1 = 4$, and that each iteration generates a polynomial in π that may have differing coefficients to the last. Hence, if I_n is as given above, then I_{n+1} would be of the form $I_{n+1} = \sum_{k=0}^{n+1} d_k \pi^k$ where c_k and d_k are not necessarily equal for $k = 0, 1, 2, \dots, n$.

(b) This section will make use of the results determined in part (a).

Assume that $\pi = \frac{a}{b}$ where a and b are both positive integers, and construct a sequence of numbers z_n where $z_n = b^n I_n$ for n a non-negative integer.

(i) By considering I_n in its polynomial form, explain why z_n must be an integer. 1

(ii) By considering I_n in its integral form, explain why z_n must be positive. 2

Examination continues on next page ...

- (iii) It can be easily shown that the integrand of I_n obtains its maximum at $\frac{\pi}{2}$. 2

Given that if $f(x) \leq g(x)$ over an interval $[\alpha, \beta]$, then $\int_{\alpha}^{\beta} f(x) dx \leq \int_{\alpha}^{\beta} g(x) dx$, show that

$$0 < z_n \leq \left(\frac{b\pi^2}{4}\right)^n \frac{\pi}{n!}$$

- (iv) Hence given $\lim_{n \rightarrow \infty} \frac{c^n}{n!} = 0$ for c a constant, show that π is irrational. 2

END OF EXAMINATION

MCQ

- | | |
|----|---|
| 1 | C |
| 2 | B |
| 3 | A |
| 4 | B |
| 5 | C |
| 6 | A |
| 7 | D |
| 8 | D |
| 9 | A |
| 10 | B |

MATHEMATICS X2 2017 trial : Question 11

Suggested Solutions

Marks

Marker's Comments

$$\int_0^a f(a-x) dx = \int_0^a f(u) du$$

$$= \int_0^a f(u) du$$

$$= \int_0^a f(x) dx \text{ (dummy variable)}$$

(ii) let $I = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$

$$= \int_0^{\pi/2} \frac{\cos^n(\frac{\pi}{2} - x)}{\sin^n(\frac{\pi}{2} - x) + \cos^n(\frac{\pi}{2} - x)} dx$$

$$= \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x} dx$$

$$2I = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx + \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$$

$$= \int_0^{\pi/2} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} dx$$

$$= \int_0^{\pi/2} 1 dx$$

$$= \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}$$

1 fairly well done.

1

students equated integrals instead of adding them

1

A number of students failed to divide by 2

Suggested Solutions

Marks

Marker's Comments

$$I_n = \int_0^1 x^n e^x dx$$

$$(i) \quad u = x^n \quad dv = e^x dx$$

$$du = nx^{n-1} dx \quad v = e^x$$

$$I_n = [x^n e^x]_0^1 - n \int_0^1 e^x x^{n-1} dx$$

$$= (e - 0) - n I_{n-1}$$

$$= \underline{e - n I_{n-1}}$$

$$(ii) \quad I_0 = \int_0^1 x^0 e^x dx$$

$$= \int_0^1 e^x$$

$$= e - 1$$

$$I_1 = e - I_0$$

$$= e - e + 1$$

$$= 1$$

$$I_2 = e - n I_1$$

$$= e - 2(1)$$

$$= e - 2$$

$$b) (c) \int_0^a f(a-x) dx$$

let $u = a - x$ then $x = 0 \rightarrow u = a$
 $x = a \rightarrow u = 0$
 $du = -dx$

1

1

1

1

1

poor algebraic skills cost students easy marks.

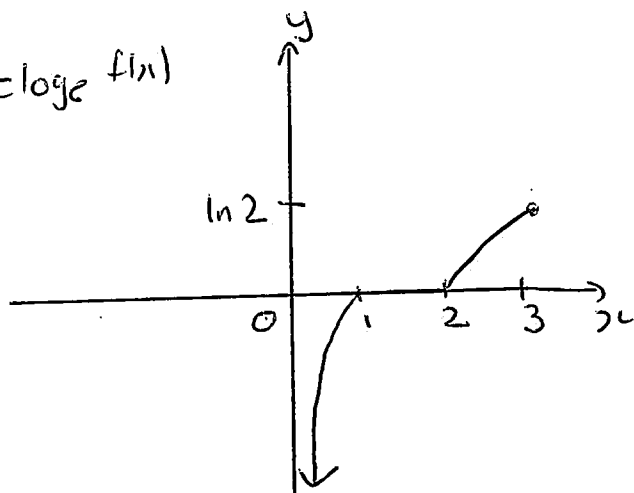
some students left n in instead of substituting.

Suggested Solutions

Marks

Marker's Comments

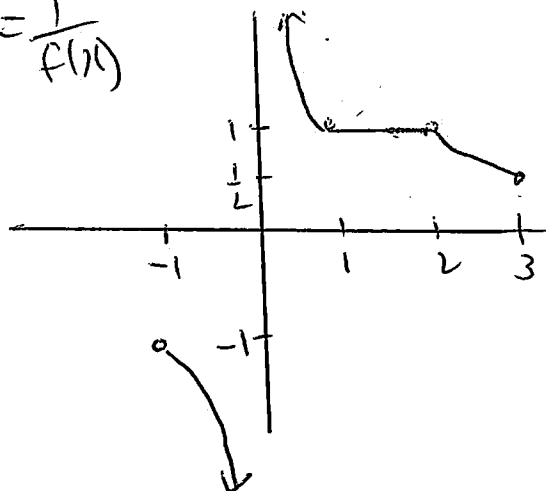
(i) $y = \log_e f(x)$



2

1 for two correct parts
2 for correct with $\ln 2$ and correct concavity from 2 to 3.

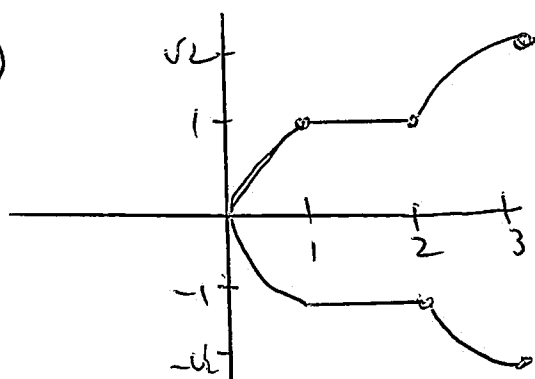
(ii) $y = \frac{1}{f(x)}$



2

1 for two correct parts
2 for correct with labelled points correct concavity

(iii) $y^2 = f(x)$



2

As above
~~no~~ students
same
failed to do reflection
concavity from 2 to 3 was
1 - correct.

Suggested Solutions

Marks

Marker's Comments

a) $z^2 + iz + 1 = 0$

$$z = \frac{-i \pm \sqrt{i^2 - 4}}{2}$$

$$= \frac{-i \pm \sqrt{-5}}{2}$$

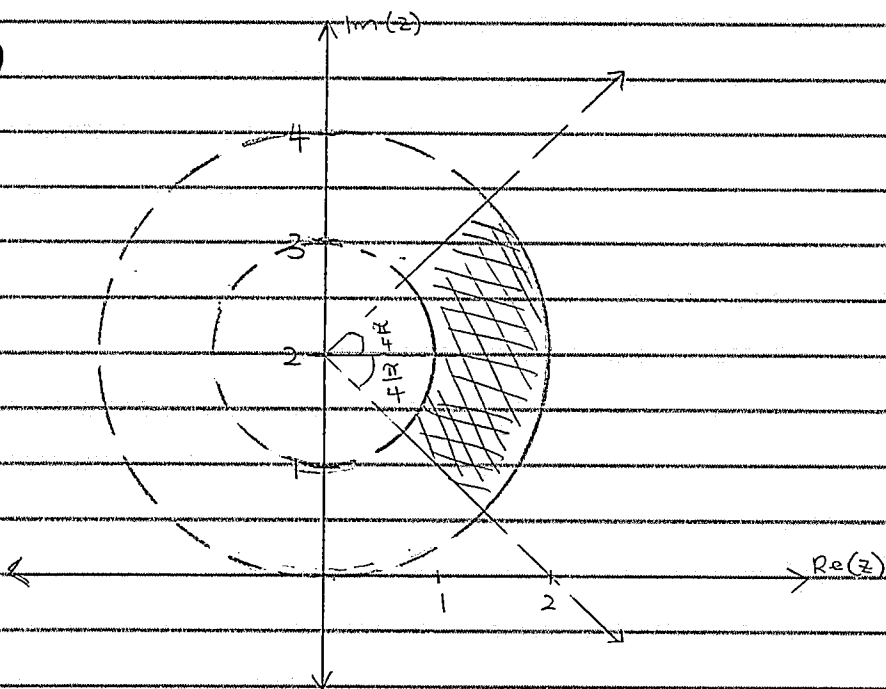
$$= \frac{-i \pm i\sqrt{5}}{2}$$

$$\therefore z = \frac{-i(1+\sqrt{5})}{2} \text{ or } \frac{-i(1-\sqrt{5})}{2}$$

1

1

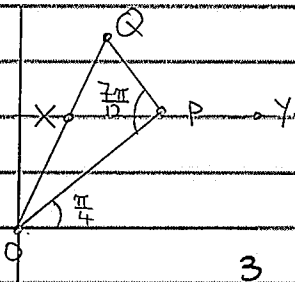
b)



1 correct region

1 correct circles & lines with correct labels & scale

c)



$$\arg \vec{OP} = \tan^{-1} \left(\frac{3/\sqrt{2}}{3/\sqrt{2}} \right)$$

$$= \frac{\pi}{4}$$

Suggested Solutions

Marks

Marker's Comments

$$\begin{aligned} \therefore \angle XPO &= \frac{\pi}{4} \text{ (alternate angles, parallel lines)} \\ \therefore \angle QPX &= \frac{7\pi}{12} - \frac{\pi}{4} \\ &= \frac{\pi}{3} \end{aligned}$$

$$\therefore \angle QPY = \frac{2\pi}{3} \text{ (angle sum of } \angle XPY \text{ is } \pi)$$

$$\begin{aligned} \therefore \vec{PQ} &= 2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \\ &= 2 \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2} \right) \\ &= -1 + i\sqrt{3} \end{aligned}$$

$$\begin{aligned} \therefore \vec{OQ} &= \vec{OP} + \vec{PQ} \\ &= \frac{3}{\sqrt{2}} + \frac{i3}{\sqrt{2}} + (-1 + i\sqrt{3}) \\ &= \left(\frac{3}{\sqrt{2}} - 1 \right) + i \left(\frac{3}{\sqrt{2}} + \sqrt{3} \right) \\ &= \frac{3\sqrt{2} - 2}{2} + \frac{3\sqrt{2} + 2\sqrt{3}}{2} i \end{aligned}$$

OR

$$\begin{aligned} |\vec{OP}| &= 3 \\ \arg \vec{OP} &= \frac{\pi}{4} \\ \therefore \vec{OP} &= 3 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\ \vec{PQ} &\text{ is } \frac{2}{3} \vec{PO} \text{ rotated clockwise by } \frac{7\pi}{12} \end{aligned}$$

$$\begin{aligned} \therefore \vec{PQ} &= \frac{2}{3} \times \left(-3 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right) \\ &\quad \times \left(\cos \left(-\frac{7\pi}{12} \right) + i \sin \left(-\frac{7\pi}{12} \right) \right) \end{aligned}$$

$$\begin{aligned} &= -2 \left(\cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \right) \\ &= -1 + i\sqrt{3} \end{aligned}$$

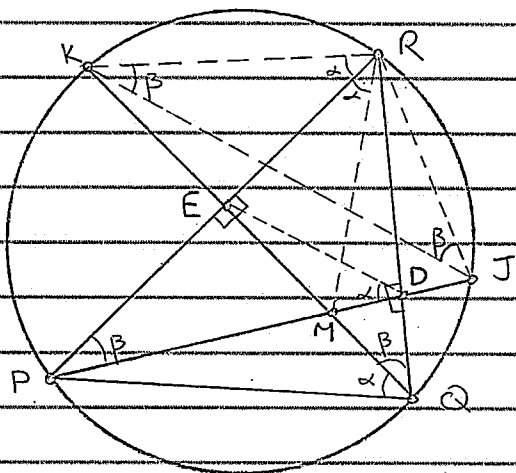
$$\begin{aligned} \therefore \vec{OQ} &= \vec{PQ} + \vec{OP} \\ &= \text{as above} \end{aligned}$$

Suggested Solutions

Marks

Marker's Comments

d.)



i) $\angle PEM = \angle MER = 90^\circ$ (EQ is altitude of $\triangle PRQ$)

$\angle RDM = \angle QDM = 90^\circ$ (PD is altitude of $\triangle PQR$)

$\therefore \angle PEQ = \angle PDQ = 90^\circ$

$\therefore PQDE$ is cyclic (PQ subtends equal angles on the same side at E and D)

$\angle REM + \angle RDM = 180^\circ$

$\therefore REMD$ is cyclic (opposite angles are supplementary)

ii) let $\angle KRP = \alpha$

$\angle KRP = \angle KQP = \alpha$ (angles at the circumference standing on the same arc)

Similarly in cyclic quad PQDE,

$\angle PQE = \angle PDE = \alpha$

Similarly in cyclic quad REMD,

$\angle MDE = \angle MRE = \alpha$

$\therefore \angle KRP = \angle MRE = \alpha$

$\therefore PR$ bisects $\angle KRM$

MATHEMATICS Extension 2: Question.....

Suggested Solutions	Marks	Marker's Comments
iii) let $\angle RKJ = \beta$ $\angle RKJ = \angle RPJ = \beta$ (angles at the circumference standing on the same arc) similarly in cyclic quad PQDE, $\angle RPJ = \angle EQD = \beta$ similarly in cyclic quad KRQP, $\angle EQD = \angle KJR = \beta$ $\therefore \angle RKJ = \angle RJK = \beta$ $\therefore KR = JR$ (equal sides are opposite equal angles in $\triangle KRJ$)	1	Another common proof: $\angle RPJ = \angle KQR$ $\therefore \text{arc } KR = \text{arc } RJ$ (equal angles at the circumference subtend equal <u>arcs</u>) <u>NOT CHORD</u> $\therefore KR = RJ$ (equal arcs subtend equal chords)
e) since they are all identical then the only possible ways are: 1, 1, 5 1, 2, 4 1, 3, 3 2, 2, 3 $\therefore$ 4 ways	2	

Suggested Solutions	Marks	Marker's Comments
a) $P(x) = x^4 - 8x^3 + 18x^2 - 27$ $P'(x) = 4x^3 - 24x^2 + 36x$ $P''(x) = 12x^2 - 48x + 36$ $P'''(x) = 24x - 48$ $P'''(x) = 0 \rightarrow x = 2$ $P''(x) = 0 \rightarrow 12x^2 - 48x + 36 = 0$ $x^2 - 4x + 3 = 0$ $(x-3)(x-1) = 0$ <u>$x = 3$</u> or <u>1</u> check $P'(3) = 4(3)^3 - 24(3)^2 + 36(3)$ $= 108 - 216 + 108$ $= 0$ $P(3) = (3)^4 - 8(3)^3 + 18(3)^2 - 27$ $= 81 - 216 + 162 - 27$ $= 0$ $\therefore x = 3$ is the multiple root with multiplicity <u>3</u>	Any time you get $x = 3$ as a solution to $P(x)$, $P'(x)$ or $P''(x)$. ① ①	

Suggested Solutions

Marks

Marker's Comments

$$b) i) x = a \sec \theta, y = b \tan \theta$$

$$\frac{dx}{d\theta} = a(\sec \theta \tan \theta) \quad \frac{dy}{d\theta} = b \sec^2 \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= b \sec^2 \theta \times \frac{1}{a \sec \theta \tan \theta}$$

$$= \frac{b \sec \theta}{a \tan \theta}$$

①

$\therefore$ Egn of tangent:

$$(y - b \tan \theta) = \frac{b \sec \theta}{a \tan \theta} (x - a \sec \theta)$$

$$ay \tan \theta - ab \tan^2 \theta = bx \sec \theta - ab \sec^2 \theta$$

$$bx \sec \theta - ay \tan \theta = ab \sec^2 \theta - ab \tan^2 \theta$$

$$bx \sec \theta - ay \tan \theta = ab (\sec^2 \theta - \tan^2 \theta) \quad \text{--- ①}$$

$$bx \sec \theta - ay \tan \theta = ab$$

Suggested Solutions	Marks	Marker's Comments
ii) Asymptotes at $y = \pm \frac{bx}{a}$ - Egn 1 Sub Egn 1 into tangent: $\therefore bx \sec \theta - a \left(\pm \frac{b}{a} x \right) \tan \theta = ab$ $bx \sec \theta \mp bx \tan \theta = ab$ $x \sec \theta \mp x \tan \theta = a$ $x = \frac{a}{\sec \theta \mp \tan \theta} \text{ --- (1)}$ $= \frac{a(\sec \theta \pm \tan \theta)}{(\sec \theta \mp \tan \theta)(\sec \theta \pm \tan \theta)}$ $= a(\sec \theta \pm \tan \theta)$ When $x = a(\sec \theta + \tan \theta)$ $y = \frac{b}{a} \times a(\sec \theta + \tan \theta)$ $= b(\sec \theta + \tan \theta)$ $\therefore Q [a(\sec \theta + \tan \theta), b(\sec \theta + \tan \theta)] \text{ --- (1)}$ When $x = a(\sec \theta - \tan \theta)$ $y = \frac{-b}{a} (a \sec \theta - \tan \theta)$ $= -b(\sec \theta - \tan \theta)$ $\therefore R [a(\sec \theta - \tan \theta), -b(\sec \theta - \tan \theta)] \text{ --- (1)}$		

Suggested Solutions

Marks

Marker's Comments

$$\begin{aligned} \text{iii) } |OQ| &= \sqrt{[a(\sec\theta + \tan\theta) - 0]^2 + [b(\sec\theta + \tan\theta) - 0]^2} \\ &= \sqrt{(\sec\theta + \tan\theta)^2 (a^2 + b^2)} \\ &= \sqrt{a^2 + b^2} |\sec\theta + \tan\theta| \end{aligned}$$

①

$$\begin{aligned} |OR| &= \sqrt{[a(\sec\theta - \tan\theta) - 0]^2 + [b(\sec\theta - \tan\theta) - 0]^2} \\ &= \sqrt{(\sec\theta - \tan\theta)^2 (a^2 + b^2)} \\ &= \sqrt{a^2 + b^2} |\sec\theta - \tan\theta| \end{aligned}$$

$$\begin{aligned} \therefore |OQ| + |OR| &= (a^2 + b^2) |\sec^2\theta - \tan^2\theta| \\ &= a^2 + b^2 \end{aligned}$$

①

which is a constant

Note: If you found $|OQ|^2 + |OR|^2$ without giving any explanation/conclusion you miss the last mark.

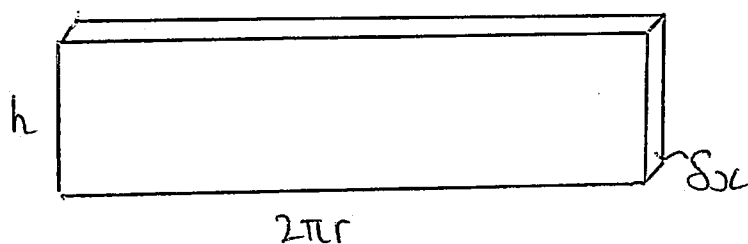
Suggested Solutions

Marks

Marker's Comments

c)

$$\begin{aligned} \text{i) } r &= 3-x \\ h &= 2|y| \end{aligned}$$



$$\begin{aligned} \therefore V_{\text{slice}} &= 2\pi r h \delta x \\ &= 2\pi (3-x)(2|y|) \delta x \end{aligned}$$

$$\text{But } x^2 + 4y^2 = 4$$

$$4y^2 = 4 - x^2$$

$$y^2 = \frac{4-x^2}{4}$$

$$y = \frac{\pm \sqrt{4-x^2}}{2}$$

$$2|y| = \sqrt{4-x^2}$$

$$\therefore V_{\text{slice}} = 2\pi (3-x) \sqrt{4-x^2} \delta x$$

$$\therefore V_{\text{solid}} = \lim_{\delta x \rightarrow 0} \sum_{x=-2}^2 2\pi (3-x) \sqrt{4-x^2} \delta x$$

$$= 2\pi \int_{-2}^2 (3-x) \sqrt{4-x^2} dx$$

Suggested Solutions	Marks	Marker's Comments
ii) $V = 2\pi \int_{-2}^2 (3-x) \sqrt{4-x^2} dx$ $= 2\pi \int_{-2}^2 3\sqrt{4-x^2} dx - 2\pi \int_{-2}^2 x\sqrt{4-x^2} dx$ ——— ① $= 6\pi \times \underbrace{\frac{\pi(2)^2}{2}}_1 - 2\pi \int_{-2}^2 x\sqrt{4-x^2} dx$ Area of semi-circle with radius 2 ——— ① $= 12\pi^2 - 2\pi \underbrace{\int_{-2}^2 x\sqrt{4-x^2} dx}_1$ ——— ① $= 12\pi^2 - 0$ Odd function with symmetrical limits $= 12\pi^2$		

Suggested Solutions

Marks

Marker's Comments

(a) $P(x) = x^3 - 2x^2 + 3x + 2$

(i)
$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma) \\ &= 2^2 - 2(3) \\ &= -2 \end{aligned}$$

If α, β, γ are all real, then $\alpha^2 + \beta^2 + \gamma^2 \geq 0$ but this is not true so at least one root must be complex.

The coefficients of $P(x)$ are all rational so the complex roots must occur in conjugate pairs, so there are 2 complex roots and 1 real root.

needed the full explanation to get the mark.

①

(ii) roots are $\alpha\beta, \beta\gamma, \gamma\alpha$

$$\alpha\beta\gamma = -d/a = -2/1 = -2$$

$$\therefore \alpha\beta = -\frac{2}{\gamma}, \beta\gamma = -\frac{2}{\alpha}, \gamma\alpha = -\frac{2}{\beta}$$

Sub in $y = -\frac{2}{x}$ ie $x = -\frac{2}{y}$

$$P(-\frac{2}{y}) = (-\frac{2}{y})^3 - 2(-\frac{2}{y})^2 + 3(-\frac{2}{y}) + 2$$

$$\therefore 0 = -\frac{8}{y^3} - \frac{8}{y^2} - \frac{6}{y} + 2$$

$$0 = -8 - 8y - 6y^2 + 2y^3$$

$$\therefore 0 = -4 - 4y - 3y^2 + y^3$$

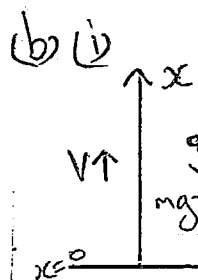
$$\therefore P(x) = x^3 - 3x^2 - 4x - 4$$
 ① (dummy variable)

*some students lost the final mark because their polynomial wasn't monic

Suggested Solutions

Marks

Marker's Comments



$$m\ddot{x} = -mg - mkv^2 \quad (\text{Newton's 2nd Law})$$

$$\ddot{x} = -g - kv^2$$

$$v \frac{dv}{dx} = -(g + kv^2)$$

$$\int_u^0 \frac{v dv}{g + kv^2} = \int_0^H -dx$$

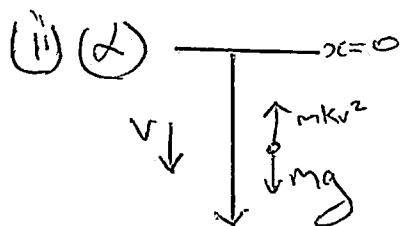
$$-\frac{1}{2k} [\ln(g + kv^2)]_u^0 = [x]_0^H$$

$$H - 0 = -\frac{1}{2k} [\ln g - \ln(g + ku^2)]$$

$$\therefore H = -\frac{1}{2k} \ln\left(\frac{g}{g + ku^2}\right)$$

$$= \frac{1}{2k} \ln\left(\frac{g + ku^2}{g}\right)$$

$$= \frac{1}{2k} \ln\left(1 + \frac{k}{g}u^2\right)$$



$$m\ddot{x} = mg - mkv^2$$

at terminal velocity $\ddot{x} = 0$

$$\therefore 0 = mg - mkv^2$$

$$mg = mkv^2$$

$$\frac{g}{k} = v^2$$

$$v_T = \sqrt{\frac{g}{k}}$$

(β) we need $t = f(v)$ so use $\ddot{x} = \frac{dv}{dt}$

$$\therefore \frac{dv}{dt} = g - kv^2$$

$$\int_0^v \frac{dv}{g - kv^2} = \int_0^t dt$$

①

①

①

①

①

very badly done
and terrible
setting out !!!

If students started
with $v \frac{dv}{dx}$ or
 $\frac{d(\frac{1}{2}v^2)}{dx}$ then they

scored zero mks.

Suggested Solutions

Marks

Marker's Comments

now $\frac{1}{g - Kv^2} = \frac{A}{\sqrt{g} + \sqrt{K}v} + \frac{B}{\sqrt{g} - \sqrt{K}v}$

$1 = A\sqrt{g} + B\sqrt{g} \dots \textcircled{1}$

$1 = A(\sqrt{g} - \sqrt{K}v) + B(\sqrt{g} + \sqrt{K}v) \dots \textcircled{2}$

from $\textcircled{1}$ $1 - B\sqrt{g} = A\sqrt{g}$

$\frac{1}{\sqrt{g}} - B = A$

sub into $\textcircled{2}$

$1 = (\frac{1}{\sqrt{g}} - B)(\sqrt{g} - \sqrt{K}v) + B(\sqrt{g} + \sqrt{K}v)$

$1 = 1 - \frac{\sqrt{K}}{\sqrt{g}} - B\sqrt{g} + B\sqrt{K}v + B\sqrt{g} + B\sqrt{K}v$

$0 = -\frac{\sqrt{K}}{\sqrt{g}} + 2B\sqrt{K}$

$B = \frac{1}{2\sqrt{g}}$

$A = \frac{1}{\sqrt{g}} - \frac{1}{2\sqrt{g}} = \frac{1}{2\sqrt{g}}$

$\therefore \int_0^v \frac{dv}{g - Kv^2} = \frac{1}{2\sqrt{g}} \int_0^v \left(\frac{1}{\sqrt{g} - \sqrt{K}v} + \frac{1}{\sqrt{g} + \sqrt{K}v} \right) dv$

$\therefore \frac{1}{2\sqrt{g}} \int_0^v \left(\frac{1}{\sqrt{g} - \sqrt{K}v} + \frac{1}{\sqrt{g} + \sqrt{K}v} \right) dv = \int_0^t dt$

$\frac{1}{2\sqrt{g}\sqrt{K}} \left[-\ln(\sqrt{g} - \sqrt{K}v) + \ln(\sqrt{g} + \sqrt{K}v) \right]_0^v = t - 0$

$\therefore t = \frac{1}{2\sqrt{g}\sqrt{K}} \left[\ln\left(\frac{\sqrt{g} + \sqrt{K}v}{\sqrt{g} - \sqrt{K}v}\right) \right]_0^v$

$t = \frac{1}{2\sqrt{g}\sqrt{K}} \left[\ln\left(\frac{\sqrt{g} + \sqrt{K}v}{\sqrt{g} - \sqrt{K}v}\right) - \ln\left(\frac{\sqrt{g}}{\sqrt{g}}\right) \right]$

$t = \frac{1}{2} \times \frac{\sqrt{g}}{\sqrt{K}} \cdot \frac{1}{g} \times \ln\left(\frac{\sqrt{g} + \sqrt{K}v}{\sqrt{g} - \sqrt{K}v}\right)$

$= \frac{w}{2g} \ln\left(\frac{\frac{\sqrt{g}}{\sqrt{K}} + v}{\frac{\sqrt{g}}{\sqrt{K}} - v}\right)$

$= \frac{w}{2g} \ln\left(\frac{w+v}{w-v}\right)$

$\textcircled{1}$

* students who lost their constants lost this mark

* If you fudged the answer then $\textcircled{1}$ you lost another mark

$$(B) \quad \frac{dv}{dt} = g - kv^2$$

$$= k \left(\frac{g}{k} - v^2 \right)$$

$$= k(w^2 - v^2)$$

4/5

where $w = \sqrt{\frac{g}{k}}$

$$\int_0^v \frac{dv}{w^2 - v^2} = \int_0^t k dt$$

(1)

$$w^2 = \frac{g}{k}$$

$$k = \frac{g}{w^2}$$

$$\sqrt{k} = \frac{\sqrt{g}}{w}$$

let $\frac{1}{w^2 - v^2} = \frac{A}{w - v} + \frac{B}{w + v}$

$$A(w + v) + B(w - v) \equiv 1$$

$$v = w \Rightarrow A = \frac{1}{2w}$$

$$v = -w \Rightarrow B = +\frac{1}{2w}$$

$$\frac{1}{2w} \int_0^v \frac{dv}{w - v} - \frac{1}{2w} \int_0^v \frac{dv}{w + v} = \int_0^t k dt$$

(1)

$$-\frac{1}{2w} \left[\ln(w - v) \right]_0^v + \frac{1}{2w} \left[\ln(w + v) \right]_0^v = kt$$

$$\frac{1}{2w} \left\{ -\ln(w - v) + \ln w + \ln(w + v) - \ln w \right\} = kt$$

$$\frac{1}{2w} \ln \left(\frac{w + v}{w - v} \right) = kt$$

(1)

$$t = \frac{1}{2wk} \ln \left(\frac{w + v}{w - v} \right)$$

$$= \frac{w}{2g} \ln \left(\frac{w + v}{w - v} \right)$$

$$\left(\frac{1}{2wk} = \frac{\frac{1}{\sqrt{k}}}{2\sqrt{g}k} \right)$$

$$= \frac{1}{2g} \frac{\sqrt{g}}{\sqrt{k}}$$

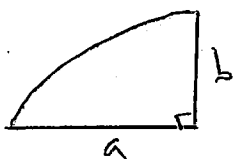
$$= \frac{w}{2g}$$

Suggested Solutions

Marks

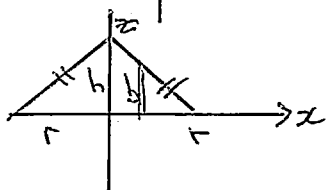
Marker's Comments

(c)



$$V_{\text{slice}} = \frac{1}{4} \pi a b \Delta x$$

$$a = r^2 - x^2 \dots \textcircled{1}$$



$$z = -mx + h \quad (m \text{ is gradient})$$

$$\text{when } z=0, x=r$$

$$0 = -mr + h$$

$$mr = h \quad m = h/r$$

$$\therefore z = -\frac{h}{r}x + h$$

$$\text{when } z=b$$

$$\therefore b = -\frac{h}{r}x + h \dots \textcircled{2}$$

 sub into V_{slice}

$$\therefore V_{\text{slice}} = \frac{\pi}{4} (r^2 - x^2) \left(-\frac{h}{r}x + h\right) \Delta x$$

①

$$= \frac{h\pi}{4} \left(-\frac{x}{r} + 1\right) (r^2 - x^2) \Delta x$$

$$= \frac{h\pi}{4} \left(r^2 - x^2 - xr + \frac{x^3}{r}\right) \Delta x$$

$$V_{\text{solid}} = 2 \times \lim_{\Delta x \rightarrow 0} \sum_{x=0}^r \frac{\pi h}{4} \left(r^2 - x^2 - xr + \frac{x^3}{r}\right) \Delta x$$

* The solid is symmetrical, so total V is twice volume of one half

$$V = \frac{\pi h}{2} \int_0^r \left(r^2 - x^2 - xr + \frac{x^3}{r}\right) dx$$

$$= \frac{\pi h}{2} \left[r^2 x - \frac{1}{3} x^3 - \frac{1}{2} x^2 r + \frac{1}{4r} x^4 \right]_0^r$$

①

$$= \frac{\pi h}{2r} \left[r^4 - \frac{1}{3} r^4 - \frac{1}{2} r^4 + \frac{1}{4} r^4 - 0 \right]$$

$$= \frac{\pi h}{2r} \times \frac{5}{12} r^4 = \frac{5hr^3\pi}{24}$$

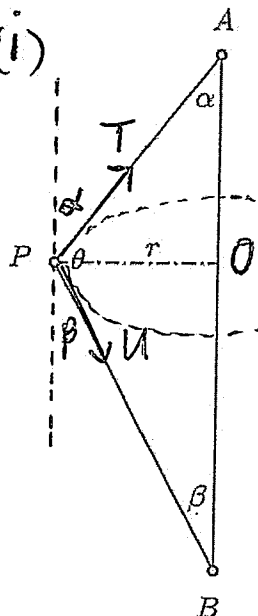
①

Suggested Solutions

Marks

Marker's Comments

a) (i)



Horizontally : $\sum F_x = ma$

$$T \sin \alpha + U \sin \beta = m \omega^2 r \quad \text{--- (1)}$$

Vertically : $\sum F_y = 0$

$$T \cos \alpha - U \cos \beta - mg = 0 \quad \text{--- (2)}$$

① $\times \cos \alpha$ - ② $\times \sin \alpha$:

$$U (\sin \beta \cos \alpha + \cos \beta \sin \alpha) = m \omega^2 r \cos \alpha - mg \sin \alpha$$

$$\therefore U = \frac{m (\omega^2 r \cos \alpha - g \sin \alpha)}{\sin (\alpha + \beta)}$$

$$= \frac{m (\omega^2 r \cos \alpha - g \sin \alpha)}{\sin (\pi - \theta)}$$

$$= \frac{m (\omega^2 r \cos \alpha - g \sin \alpha)}{\sin \theta}$$

Similarly ① $\times \cos \beta$ + ② $\times \sin \beta$: yields

$$T = \frac{m (\omega^2 r \cos \beta + g \sin \beta)}{\sin \theta}$$

Correct
Algebraic
manipulation

Substituting
① into ②
was the
other method:

Suggested Solutions

Marks

Marker's Comments

(ii) $T > 0$ and $u \geq 0$ must hold

$$\Rightarrow \frac{m(w^2 r \cos \alpha - g \sin \alpha)}{\sin \theta} \geq 0$$

But $\theta = \pi/2 \Rightarrow \sin \theta = 1$

$$\therefore w^2 r \cos \alpha - g \sin \alpha \geq 0$$

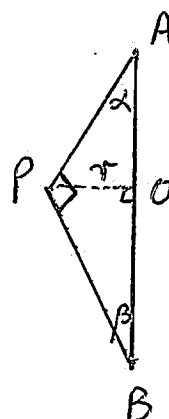
$$\Rightarrow \frac{w^2}{g} \geq \frac{\sin \alpha}{r \cos \alpha}$$

But $\sin \alpha = \frac{r}{AP}$

and $\cos \alpha = \frac{AP}{AB}$

$$\therefore \frac{w^2}{g} \geq \frac{r/AP}{r AP/AB}$$

$$= \frac{AB}{AP^2}$$



1

For all this!

1

or $\frac{\tan \alpha}{r}$

1

For correctly expressing $\sin \alpha$ and $\cos \alpha$

b) (i) $P(z) = 0 \Rightarrow z^5 = 1$

$\Rightarrow \cos 5\theta + i \sin 5\theta = 1$ --- de Moivre's Thm.

So $\theta = \frac{2\pi k}{5}$

If $k = 0, 1, 2, 3, 4$

$z_1 = 1$

$z_2 = \text{cis}(2\pi/5)$

$z_3 = \text{cis}(4\pi/5)$

$z_4 = \text{cis}(6\pi/5)$

$z_5 = \text{cis}(8\pi/5)$

Note: Conjugate Pairs

z_2 and z_5

z_3 and z_4

Check: $\bar{z}_2 = \cos 2\pi/5 - i \sin 2\pi/5$

$z_5 = \cos(-2\pi/5) + i \sin(-2\pi/5)$
 $= \cos 2\pi/5 - i \sin 2\pi/5$
 $= \bar{z}_2$

If $k = 0, \pm 1, \pm 2$

$z_1 = 1$

$z_2 = \text{cis}(2\pi/5)$

$z_3 = \text{cis}(-2\pi/5)$

$z_4 = \text{cis}(4\pi/5)$

$z_5 = \text{cis}(-4\pi/5)$

z_2 and z_3

z_4 and z_5

For all this!

Note: ($k = 0, \pm 1, \pm 2$)

$z_2 + z_3 = 2 \cos \frac{2\pi}{5}$

$z_2 \times z_3 = 1$

If z_2 and z_3 are conjugate pairs

b ii) α)

$$P(z) = (z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)$$

$$= (z-1)(z-\text{cis } \frac{2\pi}{5})(z-\text{cis } (-\frac{2\pi}{5}))(z-\text{cis } (\frac{4\pi}{5}))(z-\text{cis } (-\frac{4\pi}{5}))$$

$$= (z-1)(z^2 - 2\cos \frac{2\pi}{5}z + 1)(z^2 - 2\cos \frac{4\pi}{5}z + 1)$$

irreducible factors

β)

$$P(z) = (z-1)(z^4 + z^3 + z^2 + z + 1)$$

$$\uparrow$$

$$Q(z) \equiv \text{polynomial in } z$$

For combining conjugate pairs into irreducible factors i.e

$$(z-w)(z-\bar{w})$$

$$= z^2 - 2\text{Re}(w)z + 1$$

$$w \cdot \bar{w} = 1$$

OR

just multiplying out the factors

b iii) Hence $\Rightarrow$ use earlier results

Using (α) and (β) : Method I:

$$(z-1)(z^4+z^3+z^2+z+1) = (z-1)(z^2-2z\cos\frac{2\pi}{5}+1)(z^2-2z\cos\frac{4\pi}{5}+1)$$

Choose $z = -1$:

$$(-2)(1-1+1-1+1) = (-2)(1+2\cos\frac{2\pi}{5}+1)(1+2\cos\frac{4\pi}{5}+1)$$

$$1 = (2+2\cos\frac{2\pi}{5})(2+2\cos\frac{4\pi}{5})$$

$$= 4(1+\cos\frac{2\pi}{5})(1+\cos\frac{4\pi}{5})$$

$$\Rightarrow (1+\cos\frac{2\pi}{5})(1+\cos\frac{4\pi}{5}) = \frac{1}{4}$$

Method II: (Sum and Product of Roots)

$$z^5 - 1 = 0$$

$$z^5 + 0z^4 + 0z^3 + 0z^2 + 0z - 1 = 0$$

$$\Rightarrow a=1, b=c=d=e=0, f=-1$$

Suggested Solutions

Marks

Marker's Comments

$$\underline{\text{Sum of Roots (one at a time)}} = \frac{-b}{a} = 0$$

$$\Rightarrow \text{cis } \frac{2\pi}{5} + \text{cis } \left(-\frac{2\pi}{5}\right) + \text{cis } \left(\frac{4\pi}{5}\right) + \text{cis } \left(-\frac{4\pi}{5}\right) + 1 = 0$$

$$\Rightarrow \cos \frac{2\pi}{5} + \cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} + \cos \frac{4\pi}{5} = -1$$

$$\Rightarrow \boxed{\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}} \quad \text{--- (1)}$$

$$\underline{\text{Sum of Roots (taken 2 at a time)}} = \frac{c}{a} = 0$$

$$\text{Let } \alpha = \frac{2\pi}{5}, \quad \beta = \frac{4\pi}{5}$$

$$\begin{aligned} \Rightarrow & \text{cis } \alpha \text{ cis } (-\alpha) + \text{cis } \alpha \text{ cis } \beta + \text{cis } \alpha \text{ cis } (-\beta) + \text{cis } (-\alpha) \text{ cis } (-\beta) \\ & + \\ & \text{cis } (-\alpha) \text{ cis } \beta + \text{cis } \beta \text{ cis } (-\beta) + \text{cis } \alpha \cdot 1 + \text{cis } (-\alpha) \cdot 1 \\ & + \text{cis } \beta \cdot 1 + \text{cis } (-\beta) \cdot 1 = 0 \end{aligned}$$

$$\text{Now } \text{cis } (-\alpha) \cdot \text{cis } (\alpha) = 1$$

$$\text{cis } (-\beta) \cdot \text{cis } (\beta) = 1$$

$$\text{and } \text{cis } \alpha + \text{cis } (-\alpha) = 2 \cos \alpha$$

$$\text{cis } \beta + \text{cis } (-\beta) = 2 \cos \beta$$

$$\Rightarrow \text{cis } \alpha + \text{cis } (-\alpha) + \text{cis } \beta + \text{cis } (-\beta) = 2(\cos \alpha + \cos \beta) = -1$$

$$\Rightarrow \cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$$

Hence

$$1 + \operatorname{cis} \beta (\operatorname{cis} \alpha + \operatorname{cis} (-\alpha)) + \operatorname{cis} (-\beta) (\operatorname{cis} \alpha + \operatorname{cis} (-\alpha)) + 1 - 1 = 0$$

$$\Rightarrow [\operatorname{cis} \alpha + \operatorname{cis} (-\alpha)] [\operatorname{cis} \beta + \operatorname{cis} (-\beta)] = -1$$

$$\text{or } \cos \alpha \cdot \cos \beta = -\frac{1}{4}$$

$$\text{i.e. } \boxed{\cos \frac{2\pi}{5} \cdot \cos \frac{4\pi}{5} = -\frac{1}{4}} \quad \text{--- (2)}$$

$$\therefore (1 + \cos \frac{2\pi}{5})(1 + \cos \frac{4\pi}{5})$$

$$= 1 + \cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} + \cos \frac{2\pi}{5} \cdot \cos \frac{4\pi}{5}$$

$$= 1 + (-\frac{1}{2}) + (-\frac{1}{4}) \quad \text{using (1) \& (2) :}$$

$$= \frac{1}{4} \quad \text{as required}$$

Note: Product of all 5 roots
will not be useful!
Why?

JRAHS TRIALS	MATHEMATICS Extension 2: Question...15	2017
Suggested Solutions	Marks	Marker's Comments
Method III: Put $z = i$ into $(z^4 + z^3 + z^2 + z + 1) = (z^2 - 2z \cos \frac{2\pi}{5} + 1)(z^2 - 2z \cos \frac{4\pi}{5} + 1)$ i.e. $(1 - i - 1 + i + 1) = (-1 - 2i \cos \frac{2\pi}{5} + 1)(-1 - 2i \cos \frac{4\pi}{5} + 1)$ $1 = -4 \cos \frac{2\pi}{5} \cdot \cos \frac{4\pi}{5}$ giving $\cos \frac{2\pi}{5} \cdot \cos \frac{4\pi}{5} = -\frac{1}{4}$ ----- similar to sum of roots taken 2 at a time Now use sum of roots taken once which gives $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$ So it is then easy to show $(1 + \cos \frac{2\pi}{5})(1 + \cos \frac{4\pi}{5}) = \frac{1}{4} !$		

JRAHS TRIALS MATHEMATICS Extension 2: Question.....15			2017
Suggested Solutions		Marks	Marker's Comments
(c) (i) $\frac{1}{3}(x+y+z) \geq (xyz)^{1/3} \dots$ given $x, y, z \in \mathbb{R}^+$ $\Rightarrow x+y+z \geq 3(xyz)^{1/3}$ consider groups $x \equiv xy$ $y \equiv yz$ $z \equiv xz$ $\therefore xy + yz + xz \geq 3(xy \cdot yz \cdot xz)^{1/3}$ $= 3(x^2 \cdot y^2 \cdot z^2)^{1/3}$ $= 3(xyz)^{2/3}$ Hence $xy + yz + xz \geq 3(xyz)^{2/3}$			

Suggested Solutions

Marks

Marker's Comments

C(11)

$$a^3 = (x+y+z) + (xy+yz+xz) + xyz + 1$$

$$\geq 3(xyz)^{1/3} + 3(xyz)^{2/3} + xyz + 1 \quad \text{using (i)}$$

$$= 1 + 3(xyz)^{1/3} + 3(xyz)^{2/3} + xyz$$

$$= [1 + (xyz)^{1/3}]^3$$

1st mark

2nd mark

$$\Rightarrow a \geq 1 + (xyz)^{1/3}$$

$$\text{or } (xyz)^{1/3} \leq a-1$$

$$\Rightarrow xyz \leq (a-1)^3$$

Method II: Consider $xyz - (a-1)^3$

Method III: By Contradiction

i.e Assume $xyz > (a-1)^3$?

Suggested Solutions

Marks

Marker's Comments

$$\begin{aligned}
 (c) \quad I_n &= \int_0^{\pi} f_n(x) \sin x \, dx \\
 &= \left[f_n(x) (-\cos x) \right]_0^{\pi} + \int_0^{\pi} f_n'(x) \cos x \, dx \\
 &= f_n(\pi) \times 1 + 1 \times f_n(0) + \int_0^{\pi} f_n'(x) \cos x \, dx \\
 &\quad \text{Now } f_n(\pi) = f_n(0) = 0 \\
 &= \int_0^{\pi} f_n'(x) \cos x \, dx \\
 &= \left[f_n'(x) \sin x \right]_0^{\pi} - \int_0^{\pi} f_n''(x) \sin x \, dx \\
 &= 0 \\
 &\quad \text{Since } f_n'(\pi) = f_n'(0) = 0 \\
 \therefore I_n &= - \int_0^{\pi} f_n''(x) \sin x \, dx
 \end{aligned}$$

$$\begin{aligned}
 (d) \quad f_n(x) &= \frac{(\pi x - x^2)^n}{n!} \\
 f_n'(x) &= \frac{(\pi x - x^2)^{n-1}}{(n-1)!} (\pi - 2x) \\
 f_n''(x) &= \frac{\cancel{\pi x - x^2} (n-1) (\pi x - x^2)^{n-2} (\pi - 2x)^2}{(n-1)!} \\
 &\quad - \frac{2 (\pi x - x^2)^{n-1}}{(n-1)!}
 \end{aligned}$$

had to clearly indicate why $\left[f_n(x) (-\cos x) \right]_0^{\pi}$ disappeared.

A significant number of students lost the $(\pi - 2x)$ term.

Suggested Solutions

Marks

Marker's Comments

$$f_n''(x) = \frac{\pi x - x^2}{(n-2)!} (\pi^2 - 4\pi x + 4x^2) - 2 \frac{(\pi x - x^2)^{n-1}}{(n-1)!}$$

$$= \frac{\pi^2 (\pi x - x^2)^{n-2}}{(n-2)!} - \frac{4(\pi x - x^2)(\pi x - x^2)^{n-2}}{(n-2)!} - 2 \frac{(\pi x - x^2)^{n-1}}{(n-1)!}$$

$$= \frac{\pi^2 (\pi x - x^2)^{n-2}}{(n-2)!} - \frac{4(\pi x - x^2)^{n-1}}{(n-2)! (n-1)} - 2 \frac{(\pi x - x^2)^{n-1}}{(n-1)!}$$

$$= \pi^2 I_{n-2}(x) - 4(n-1)I_{n-1}(x) - 2I_{n-1}(x)$$

$$= \pi^2 I_{n-2}(x) - (4n-2)I_{n-1}(x)$$

$$\text{so } I_n = - \int_0^\pi f_n''(x) \sin x \, dx$$

$$= - \int_0^\pi \pi^2 I_{n-2}(x) \sin x - (4n-2)I_{n-1}(x) \sin x \, dx$$

$$= \underline{(4n-2)I_{n-1} - \pi^2 I_{n-2}}$$

Suggested Solutions

Marks

Marker's Comments

(iii) Base case $I_0 = 2$ $I_1 = 4$ are both polynomials in π

$$I_2 = (4 \times 2 - 2)I_1 - \pi^2 I_0$$

$$= 24 - 2\pi^2 \text{ which is a polynomial in } \pi.$$

Assume true for $n=k$, $n=k-1$

① $I_n = \sum_{k=0}^n c_k \pi^k$ $I_{k-1} = \sum_{k=0}^{k-1} d_k \pi^k$

$$I_{k+1} = (4n-2)I_k - \pi^2 I_{k-1}$$

$$= (4n-2) \sum_{k=0}^n c_k \pi^k - \pi^2 \sum_{k=0}^{n-1} d_k \pi^k$$

$$= \sum_{k=0}^n c_k (4n-2) \pi^k - \sum_{k=0}^{n-1} d_k \pi^{k+2}$$

All coefficients in summation are

② integers $c_k, d_k \in \mathbb{Z}$ $(4n-2) \in \mathbb{Z}$

Highest possible power on π is

$$k-1+2 = n+1$$

So I_{k+1} is a polynomial in π having all integer coefficients and $\deg \leq n+1$.

Some ~~many~~ students tried to find

I_0 and I_1 , but it was given,

as students got $24 - 2\pi^2$ told it explain its significance

had to assume true for two cases

incorrect limit on sigma a common error

needed ① and ② for third mark.

Suggested Solutions

Marks

Marker's Comments

$$(b) z_n = b^n I_n \quad \text{and} \quad \pi = \frac{a}{b} \in \mathbb{Q}^+$$

$$(c) \text{ If } I_n = \sum_{k=0}^n c_k \pi^k \text{ is a}$$

polynomial in π over $\mathbb{Z}$ then
for $\pi = \frac{a}{b}$

$$b^n I_n = \sum_{k=0}^n c_k b^n \frac{a^k}{b^k}$$

$$= \sum_{k=0}^n c_k a^k b^{n-k}$$

Every term is the sum of an
integer $c_k, a^k, b^{n-k} \in \mathbb{Z}$

$b^{n-k} \in \mathbb{Z}$ since $n \geq k$.

$\therefore z_n$ is an integer

$$(a) z_n = b^n I_n = b^n \int_0^{\pi} \left(\frac{\pi x - x^2}{n!} \right)^n \sin x dx.$$

needed to
state $n \geq k$

MATHEMATICS Extension 2 : Question 16

Suggested Solutions

Marks

Marker's Comments

Now

$$\pi x - x^2 > 0 \text{ for } 0 < x < \pi$$

$$\sin x > 0 \text{ for } 0 < x < \pi$$

1

$$\therefore \frac{\pi x - x^2}{n!} \sin x > 0$$

$$\text{so } \int_0^\pi \frac{\pi x - x^2}{n!} \sin x \, dx \text{ will be positive.}$$

1

$$(iii) f_n(x) = \frac{(\pi x - x^2)^n}{n!} \sin x$$

$$\leq \frac{\left(\pi \times \frac{\pi}{2} - \left(\frac{\pi}{2} \right)^2 \right)^n \sin \frac{\pi}{2}}{n!}$$

$$\therefore \int_0^\pi \frac{\pi x - x^2}{n!} \sin x \, dx \leq \int_0^\pi \left(\frac{\pi^2}{4} \right)^n \frac{1}{n!} \, dx$$

$$= \left[\frac{\left(\frac{\pi^2}{4} \right)^n}{n!} x \right]_0^\pi$$

$$= \frac{\left(\frac{\pi^2}{4} \right)^n}{n!} \times \pi$$

1

Suggested Solutions

Marks

Marker's Comments

$$z_n = \frac{\left(\frac{b\pi^2}{4}\right)^n}{n!} \pi$$

$$\therefore 0 < z_n < \frac{\left(\frac{b\pi^2}{4}\right)^n}{n!} \pi$$

(iv) Now $\frac{\left(\frac{b\pi^2}{4}\right)^n}{n!}$ is of form

$\frac{e_n^N}{n!}$ from given information

$$\therefore \lim_{n \rightarrow \infty} \frac{\left(\frac{b\pi^2}{4}\right)^n}{n!} = 0$$

Now there exists a value for $n=N$ such that $\frac{\left(\frac{b\pi^2}{4}\right)^N}{N!} \pi < \frac{1}{2}$

so we have

$$0 < z_n < \frac{\left(\frac{b\pi^2}{4}\right)^N}{N!} < \frac{1}{2}$$

But z_n is always an integer so contradiction as it can't be an integer. So original assumption π is ~~irrational~~ is incorrect