Student ID:

Maths Class: ______________________

2025	YEAR 12 TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Student ID and Maths Class at the top of this page

Total marks: 100

Section I – 10 marks (pages 3–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–14)

- Attempt Questions 11–15
- Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the distance of the point $(2, 3, 7)$ from the x-z plane?

A. 2
B. 3
C. 7
D. 9

2 If a, b, c and d are consecutive integers, then which of the following statements may be false?

A. $a + b + c + d$ is divisible by 2.
B. $a + b + c + d$ is divisible by 4.
C. $a \times b \times c \times d$ is divisible by 2.
D. $a \times b \times c \times d$ is divisible by 4.

3 The acceleration, a ms^{-2}, of a particle that starts from rest and moves in a straight line is described by $a = v + 1$, where v ms^{-1} is its velocity after t seconds.

The velocity of the particle after $\ln(e + 1)$ seconds is

A. e
B. $e + 1$
C. $e^2 + 1$
D. $\ln[(1 + e)] + 1$

4 A skydiver jumps out of a plane and experiences a resistive force of $m(v+v^2)$. Let the acceleration due to gravity be $10\ \text{ms}^{-2}$.

What is the terminal velocity of the skydiver?

A. $\dfrac{\sqrt{39}+1}{2}\ \text{ms}^{-1}$

B. $\dfrac{\sqrt{39}-1}{2}\ \text{ms}^{-1}$

C. $\dfrac{\sqrt{41}+1}{2}\ \text{ms}^{-1}$

D. $\dfrac{\sqrt{41}-1}{2}\ \text{ms}^{-1}$

5 The area bound by the curve $y = \cos^{-1} x$ and the coordinate axes is rotated about the y-axis. The volume of the solid formed is given by:

A. $\pi \displaystyle\int_0^1 (\cos^{-1} x)^2\, dx$

B. $\pi \displaystyle\int_0^1 \cos y\sqrt{1-\sin^2 y}\, dy$

C. $\pi \displaystyle\int_0^{\pi/2} \sqrt{1-u^2}\, du$, where $u = \sin y$

D. $\pi \displaystyle\int_0^1 \sqrt{1-u^2}\, du$, where $u = \sin y$

6 The sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = 1$ lies entirely within the sphere $x^2 + y^2 + z^2 = 9$. What is the maximum value of $a^2 + b^2 + c^2$?

A. 8

B. 4

C. 3

D. 2

7 Consider the following propositions:

$$P: \quad \forall x > 0, \exists n \in \mathbb{Z} \text{ such that } x > \frac{1}{n}$$
$$Q: \quad \exists n \in \mathbb{Z} \text{ such that } \forall x > 0, x > \frac{1}{n}$$

Which of the following is **false**?

A. $Q \implies \neg P$

B. $\neg P \implies Q$

C. $\neg Q \implies P$

D. $P \implies Q$

8 For what values of a and b, with $a < b$, does the integral

$$\int_a^b \left(5x - 6 - x^2\right) e^{|x|} \, dx$$

attain its maximum value?

A. $a = -1, b = 6$

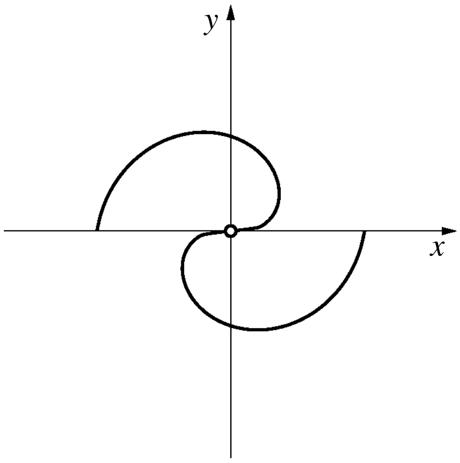
B. $a = -2, b = 3$

C. $a = 1, b = 6$

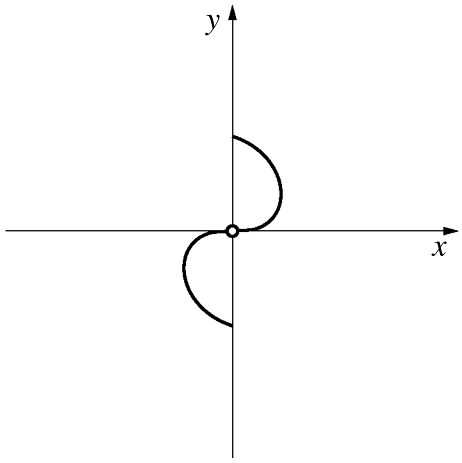
D. $a = 2, b = 3$

9 Which of the following represents the curve $\text{Arg}(z^2) = |z|$, where $\text{Arg}(z)$ refers to the pricinipal argument of z?

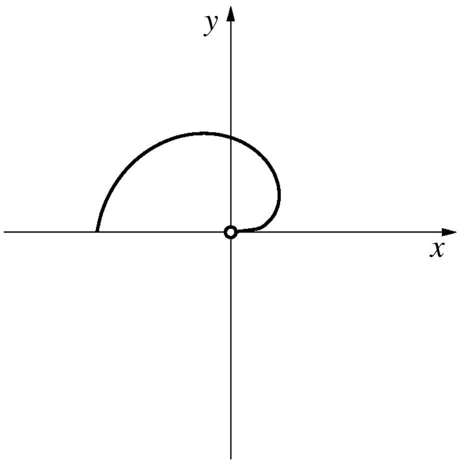
A.



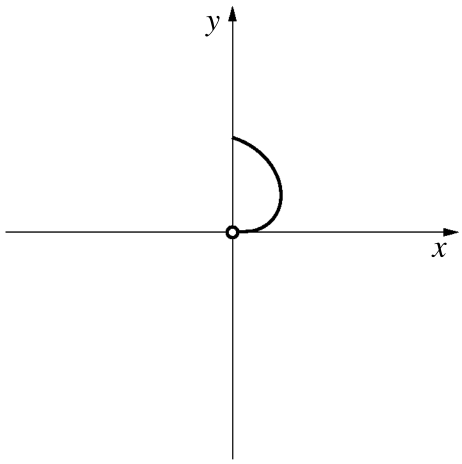
B.



C.



D.



10 Let $\underset{\sim}{u} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, and suppose $\underset{\sim}{v}$ is a vector in three dimensions.

Let $\underset{\sim}{p}$ be the vector projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$, and let $\underset{\sim}{n}$ be the vector component of $\underset{\sim}{u}$ that is perpendicular to $\underset{\sim}{v}$, that is, $\underset{\sim}{u} = \underset{\sim}{p} + \underset{\sim}{n}$.

If $|\underset{\sim}{p}| > |\underset{\sim}{n}|$, which of the following could be $\underset{\sim}{v}$?

A. $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

B. $3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

C. $\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$

D. $3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$

Section II

90 marks
Attempt Questions 11–15
Allow about 2 hours 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (19 marks) Use the Question 11 Writing Booklet

(a) Let $u = 2 + i$ and $v = -1 - i$.

(i) Find $\text{Im}(uv)$. **1**

(ii) Evaluate $|u - v|^2$. **1**

(iii) Express $\dfrac{u}{v}$ in the form $a + ib$ where a and b are real numbers. **2**

(b) (i) Find the square roots of $2i$, giving the answers in the form $x + iy$, where x and y are real numbers. **2**

(ii) Hence, solve $2z^2 + 2\sqrt{2}z + 1 - i = 0$. **3**

(c) Let a, b be positive, real numbers.

(i) Show that $\dfrac{a+b}{2} \geq \sqrt{ab}$. **1**

(ii) Hence, show that for $c \geq 0$, **2**

$$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c).$$

(d) By making an appropriate substitution, or otherwise, evaluate $\displaystyle\int_0^1 \frac{e^{3x}}{1 + e^x}\,dx$. **3**

Question 11 continues on page 8

Question 11 (continued)

(e) Let $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Find the area of the triangle formed by $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{u} - \underset{\sim}{v}$, given that the area of a triangle can be calculated using the formula **2**

$$A = \sqrt{(|\underset{\sim}{u}||\underset{\sim}{v}|)^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v})^2}.$$

You do NOT need to prove this formula.

(f) A particle moves along a straight line. When the particle is x m from a fixed point O, its velocity, v ms^{-1}, is given by **2**

$$v = \frac{3x+2}{2x-1},$$

where $x \geq 1$.

Find the acceleration of the particle in ms^{-2}, when $x = 2$.

End of Question 11

Question 12 (19 marks) Use the Question 12 Writing Booklet

(a) Prove for all integers n, that n^2 is odd if and only if n is odd. **3**

(b) Evaluate $\displaystyle\int_0^{\pi/3} \sin x \ln(\cos x)\ dx$. **3**

(c) Use mathematical induction to prove that, for all integers $n > 0$, **3**

$$\frac{d^n}{dx^n}(xe^{2x}) = 2^{n-1}(2x+n)e^{2x}.$$

(d) The vertical displacement, $y(t)$ centimetres, of the seat of a toy horse from the mean position after t seconds is measured as the toy horse moves up and down a pole during a ride on a merry-go-round.

The vertical motion of the toy horse obeys the equation $\dfrac{d^2y}{dt^2} = -\left(\dfrac{\pi}{2}\right)^2 y$.

The vertical motion of each toy horse spans 50 centimetres, and the seat of the toy horse is at the mean position at the start of the ride.

(i) Determine the vertical displacement function, $y(t)$. **3**

(ii) Determine the maximum vertical speed of the toy horse, correct to the nearest centimetre per second. **1**

(e) An object of mass 70 kg, initially at rest, is pulled along a horizontal surface by a constant horizontal force of 140N. It experiences a resistance proportional to its speed, v.

When the speed is 10 ms^{-1}, the acceleration is 1 ms^{-2}.

Let x represent the displacement in metres from the initial position of the object.

(i) Show that the equation of motion is $\ddot{x} = 2 - \dfrac{1}{10}v$. **2**

(ii) Find an expression for x as a function of v. **3**

(iii) Show that the object's speed cannot exceed 20 ms^{-1}. **1**

End of Question 12

Question 13 (16 marks) Use the Question 13 Writing Booklet

(a) (i) Find constants A and B such that **2**

$$\frac{1}{u^2+u-1} = \frac{A}{u+\frac{1+\sqrt{5}}{2}} + \frac{B}{u+\frac{1-\sqrt{5}}{2}}.$$

(ii) Hence, find $\displaystyle\int \frac{\cos x}{\sin x - \cos^2 x}\,dx$. **2**

(b) (i) Prove that if a and b have no common factors other than 1, then a^2 and b also have no common factors other than 1. **2**

(ii) Let x be a positive rational number different from 1. Using part (i), or otherwise, prove that $x + \dfrac{1}{x}$ can never be an integer. **2**

(c) (i) Prove that $\cos\theta = \dfrac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$. **2**

(ii) Hence, prove that for all real α and β, **3**

$$\cos 2(\alpha+\beta) - \cos 2\alpha - \cos 2\beta + 1 = -4\sin\alpha\sin\beta\cos(\alpha+\beta).$$

(d) Suppose that the non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ satisfy **3**

$$\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a} = \lambda\underset{\sim}{b} \qquad \text{and} \qquad \text{proj}_{\underset{\sim}{a}}\,\underset{\sim}{b} = (1-\lambda)\underset{\sim}{a}$$

where λ is a real constant. Prove that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not parallel.

End of Question 13

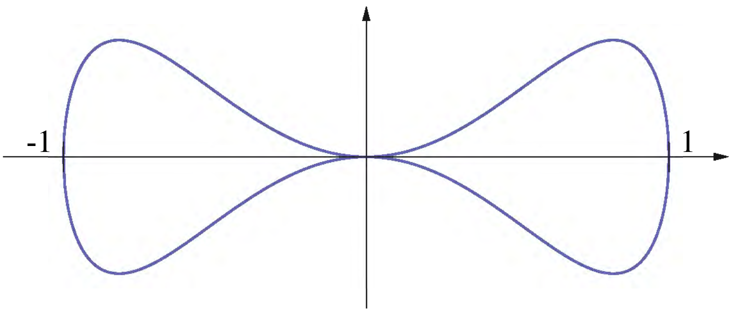
Question 14 (19 marks) Use the Question 14 Writing Booklet

(a) Let $I_n = \displaystyle\int_0^1 x^n \sqrt{1 - x^2}\, dx$.

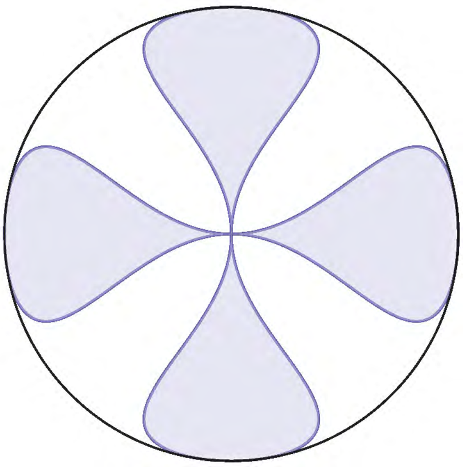
(i) Show that, for $n \geq 2$, **4**

$$I_n = \frac{n-1}{n+2} I_{n-2}.$$

The graph of the relation $y^2 = x^4(1 - x^2)$ is shown below.



A new gaming company logo has been designed using the curves $y^2 = x^4(1 - x^2)$ and $x^2 = y^4(1 - y^2)$ inscribed inside the unit circle.



(ii) Find the ratio of the shaded area to the unshaded area in the company logo. **2**

Question 14 continues on page 12

(b) The lines given by **3**

$$\mathbf{r_1} = \begin{bmatrix}1\\1\\1\end{bmatrix} + \lambda\begin{bmatrix}2\\-1\\1\end{bmatrix} \text{ where } \lambda \in \mathbb{R}$$

and

$$\mathbf{r_2} = \begin{bmatrix}3\\4\\5\end{bmatrix} + \mu\begin{bmatrix}3\\s\\t\end{bmatrix} \text{ where } \mu \in \mathbb{R}$$

meet at a right angle.

Find the values of λ and μ where the lines meet.

(c) z is a complex number such that $\arg\left(1 - \dfrac{1}{z}\right) = \dfrac{\pi}{3}$.

(i) Sketch the path that z traces out, showing all important information. **3**

(ii) Hence, show that z satisfies **2**

$$|z|^2 + |z-1|^2 = |z^2 - z| + 1.$$

(d) A 300 kg space probe is launched vertically upwards from the surface of Saturn's moon, Titan, at a speed of v_0 ms^{-1}. The acceleration due to gravity on Titan is 1.4 ms^{-2}. The space probe experiences a resistive force from the dense atmosphere of magnitude $150v$ Newtons, where v ms^{-1} is the speed of the space probe. Eventually, the space probe falls back onto Titan's surface.
A sensor on the space probe calculates that the sum of the initial speed and the speed at which it hits Titan's surface is 50 ms^{-1}.

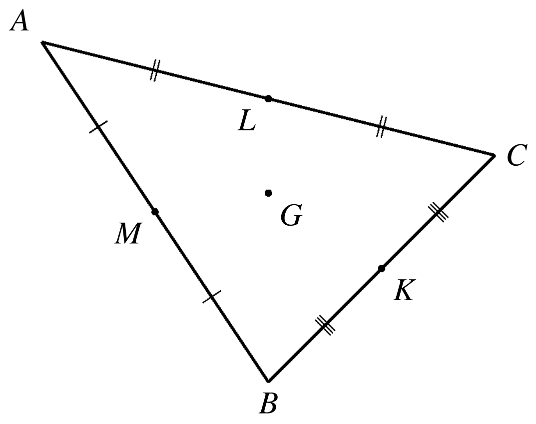
(i) Find v_0, correct to one decimal place. **3**

(ii) Determine the total flight time, correct to one decimal place. **2**

End of Question 15

Question 15 (17 marks) Use the Question 15 Writing Booklet

(a) In $\triangle ABC$ shown below, K is the midpoint of BC, L is the midpoint of CA and M is the midpoint of AB. Let the points A, B, C have position vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ respectively. Let G be the point with position vector $\underset{\sim}{g} = \frac{1}{3}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$.



You are given the triangle inequality, that is

$$|\underset{\sim}{u} + \underset{\sim}{v}| \leq |\underset{\sim}{u}| + |\underset{\sim}{v}| \qquad \text{(Do NOT prove this).}$$

(i) Show that $2|\overrightarrow{AK}| \leq |\overrightarrow{AB}| + |\overrightarrow{AC}|$. **1**

(ii) Hence, show that $|\overrightarrow{AK}| + |\overrightarrow{BL}| + |\overrightarrow{CM}| \leq |\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CA}|$. **1**

(iii) By expressing $\overrightarrow{OK}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$, show that points A, G, K are collinear. **3**

(iv) Deduce that AK, BL, CM are concurrent at G. **1**

(v) Show that the ratio $|\overrightarrow{AG}| : |\overrightarrow{GK}| = 2 : 1$. **1**

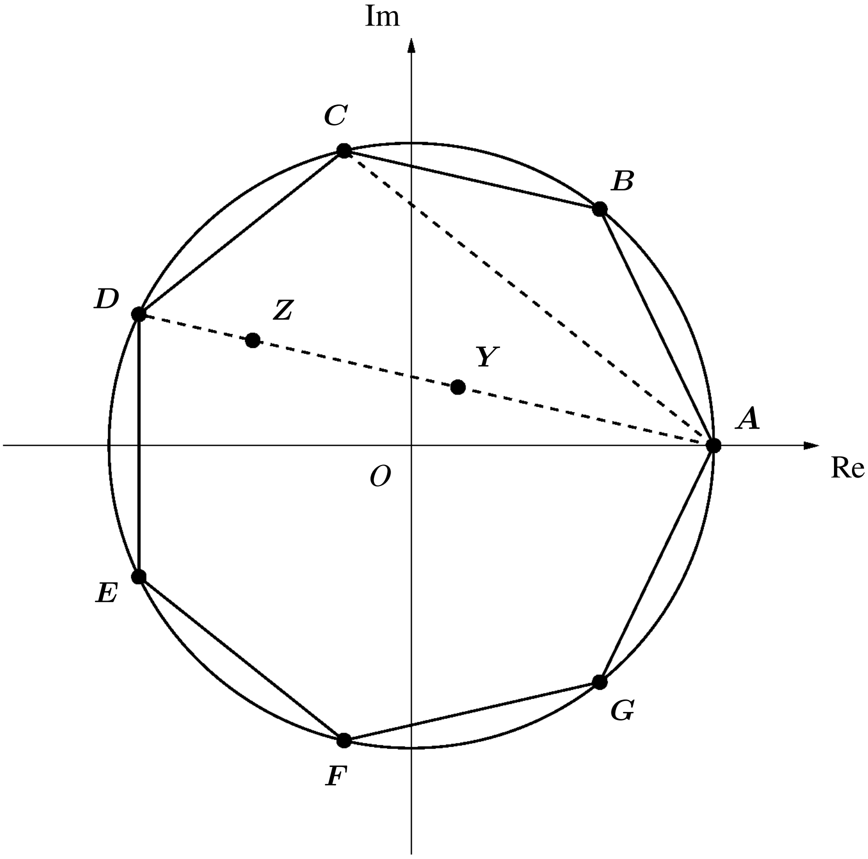
(vi) Hence, prove that **3**

$$|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CA}| \leq \frac{4}{3}\left(|\overrightarrow{AK}| + |\overrightarrow{BL}| + |\overrightarrow{CM}|\right).$$

Question 15 continues on page 14

Question 15 (continued)

(b) The points A, B, C, D, E, F and G, corresponding to the complex roots of $z^7 = 1$, form the vertices of a regular heptagon in the Argand Diagram, as shown in the diagram below.



(i) By considering appropriate arguments, show that $\angle CAB = \dfrac{\pi}{7}$ and $\angle DAB = \dfrac{2\pi}{7}$. **2**

Let $\omega = \cos\dfrac{\pi}{7} + i\sin\dfrac{\pi}{7}$.

(ii) The points Y and Z are chosen on the line segment AD, such that **2**

$$AY = AB \quad \text{and} \quad AZ = AC.$$

Find the complex numbers representing points Y and Z in terms of ω.

(iii) Hence, or otherwise, prove that **3**

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}.$$

End of paper

Multiple Choice

1	2	3	4	5	6	7	8	9	10
B	B	A	D	D	B	A	D	B	B

Question 11

a)

i.

$$\begin{aligned} uv &= (2+i)(-1-i) \\ &= 2(-1-i) + i(-1-i) \\ &= -2-2i-i-i^2 \\ &= -1-3i \end{aligned}$$

$Im(uv) = -3$ 1 mark (must get all prior working out before getting the mark)

Special Note: Way too many students wrote $-3i$ as the answer for our comfort.

ii.

$$\begin{aligned} u-v &= (2+i)-(-1-i) \\ &= 3+2i \end{aligned}$$

$$\begin{aligned} |u-v|^2 &= 3^2+2^2 \\ &= 13 \end{aligned}$$ 1 mark (must get all prior working out before getting the mark)

Special Note: Most common mistake was people forgetting to square the modulus at the end.

iii.

$$\frac{u}{v} = \frac{2+i}{-1-i}$$

$$= \frac{2+i}{-1-i} \times \frac{-1+i}{-1+i}$$ 1 mark for knowing to multiply by conjugate

$$= \frac{-2+2i-i-1}{(-1)^2-i^2}$$

$$= \frac{-3+i}{2}$$

$$= -\frac{3}{2} + \frac{1}{2}i$$ 1 mark for answer

Special Note: The question asks for $x+iy$ form! $\frac{1}{2}(-3+i)$ is not quite that, consider yourselves VERY lucky not to have lost a mark. Don't make the same mistake in the HSC!

b)

i.

Let $z = x + iy$ where $z^2 = 2i$ and $x, y \in \mathbb{R}$

$(x^2 - y^2) + 2xyi = 2i$

$x^2 - y^2 = 0 \Rightarrow x = \pm y$ Equation 1

$2xy = 2 \Rightarrow xy = 1$ Equation 2 1 mark for setting up the simultaneous equations

Sub Equation 1 into Equation 2:

$\pm y^2 = 1$

$y = \pm 1$ (Since $y \in \mathbb{R}$)

When $y = 1, x = 1$

When $y = -1, x = -1$

$\therefore z = 1 + i$ or $-1 - i$ 1 mark

Done well generally, with the exception of a few arithmetic errors.

ii.

$$z = \frac{-2\sqrt{2} \pm \sqrt{8 - 4(2)(1 - i)}}{4}$$ 1 mark for applying quadratic formula correctly

$$= \frac{-2\sqrt{2} \pm \sqrt{8i}}{4}$$

$$= \frac{-2\sqrt{2} \pm 2\sqrt{2i}}{4}$$ 1 mark for simplifying discriminant appropriate to use part i.

$$= \frac{-\sqrt{2} \pm \sqrt{2i}}{2}$$

$$= \frac{-\sqrt{2} \pm (1 + i)}{2}$$

$$= \frac{-\sqrt{2} + 1}{2} + \frac{1}{2}i \text{ or } \frac{-\sqrt{2} - 1}{2} - \frac{1}{2}i$$ 1 mark for answer

Special Note: You DO NOT get the mark after just substituting $\pm(1 + i)$ into $\sqrt{2i}$, MUST write your answer into are recognised form of a complex number.

c)

i.

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$

$$a - 2\sqrt{ab} + b \geq 0$$

$$a + b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$ 1 mark

ii.

From i:

$$a^2b^2 + b^2c^2 \geq 2\sqrt{a^2b^4c^2}$$

$$= 2ab^2c \quad (a, b, c \geq 0) \text{ Equation 1}$$ 1 mark for using part i. appropriately.

$$c^2a^2 + b^2c^2 \geq 2\sqrt{a^2b^2c^4}$$

$$= 2abc^2 \quad (a, b, c \geq 0) \text{Equation 2}$$

$$a^2b^2 + c^2a^2 \geq 2\sqrt{a^4b^2c^2}$$

$$= 2a^2bc \quad (a, b, c \geq 0) \text{Equation 3}$$

Equation 1 + Equation 2 + Equation 3:

$$2(a^2b^2 + b^2c^2 + c^2a^2) \geq 2(a^2bc + ab^2c + abc^2)$$

$$a^2b^2 + b^2c^2 + c^2a^2 \geq a^2bc + ab^2c + abc^2$$

$$= abc(a + b + c)$$ 1 mark completing proof without skipping steps

d)

$$I = \int_2^{1+e} \frac{(u-1)^3}{u} \times \frac{du}{u-1}$$ 1 mark

$$= \int_2^{1+e} \frac{(u-1)^2}{u} du$$

$$= \int_2^{1+e} \left(\frac{u^2 - 2u + 1}{u}\right) du$$

$$= \int_2^{1+e} \left(u - 2 + \frac{1}{u}\right) du$$

$$= \left[\frac{u^2}{2} - 2u + \ln|u|\right]_2^{1+e}$$ 1 mark

$$= \frac{(1+e)^2}{2} - 2(1+e) + \ln(1+e) - (2 - 4 + \ln 2)$$

$$= \frac{1}{2} + e + \frac{e^2}{2} - 2 - 2e + \ln(1+e) - 2 + 4 - \ln 2$$

$$= \frac{1}{2} - e + \frac{e^2}{2} + \ln\left(\frac{1+e}{2}\right)$$ 1 mark

Note: Final mark is only awarded if you have attempted to, and successfully clean up your expression by adding/subtracting your like terms.

Let $u = 1 + e^x$

$$\frac{du}{dx} = e^x$$

$$dx = \frac{du}{e^x} = \frac{du}{u-1}$$

When $x = 1, u = 1 + e$

When $x = 0, u = 2$

Common mistake: Changing the limits incorrectly, or forgetting to change it altogether.

e)

$$\left|\underset{\sim}{u}\right|^2 = 1^2 + 2^2 + 3^2 = 14$$

$$\left|\underset{\sim}{v}\right|^2 = 2^2 + (-1)^2 + 1^2 = 6$$

$$\left(\underset{\sim}{u} \cdot \underset{\sim}{v}\right)^2 = \left((1 \times 2) + (2 \times -1) + (3 \times 1)\right)^2 = 9$$

$$\therefore A = \frac{1}{2}\sqrt{(14 \times 6) - 9}$$

$$= \frac{\sqrt{75}}{2}$$

$$= \frac{5\sqrt{3}}{2} \text{ unit}^2$$

1 mark awarded for either working out the two moduli correctly, or $\underset{\sim}{u} \cdot \underset{\sim}{v}$.

Full marks for correct answer and working

Very well done generally, the most common mistake came from simple arithmetic errors.

f)

$$v = \frac{3x+2}{2x-1} \Rightarrow v^2 = \left(\frac{3x+2}{2x-1}\right)^2$$

$$\frac{1}{2}v^2 = \frac{1}{2}\left(\frac{3x+2}{2x-1}\right)^2$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2} \times 2\left(\frac{3x+2}{2x-1}\right) \times \left(\frac{3(2x-1)-2(3x+2)}{(2x-1)^2}\right)$$

$$= \left(\frac{3x+2}{2x-1}\right) \times \frac{-7}{(2x-1)^2}$$

$$= \frac{-21x-14}{(2x-1)^2}$$

$$a = \frac{-21x-14}{(2x-1)^3}$$

When $x = 2$

$$a = \frac{-42-14}{27} = -\frac{56}{27}\text{ms}^{-2}$$

or

$$v = \frac{3x+2}{2x-1}$$

$$\frac{dv}{dx} = \frac{3(2x-1)-2(3x+2)}{(2x-1)^2}$$

$$= \frac{-7}{(2x-1)^2}$$

$$a = v\frac{dv}{dx}$$

$$= \left(\frac{3x+2}{2x-1}\right) \times \frac{-7}{(2x-1)^2}$$

$$= \frac{-21x-14}{(2x-1)^2}$$

When $x = 2$

$$a = \frac{-42-14}{27} = -\frac{56}{27}\text{ms}^{-2}$$

1 mark for showing the knowledging for squaring velocity, multiply by $\frac{1}{2}$, then diff WRT x.

1 mark for final answer

Note: It was VERY worrying to see students unable to demonstrate the relationship between $v(x)$ and $a(x)$. Too many students thought that $a(x) = v'(x)$. Other errors also included students accidentally integrated instead of differentiated.

Student Number ____________

Question 12.

a. prove for all integers n, that n^2 is odd iff n is odd

n is odd $\Rightarrow n^2$ is odd ①

n^2 is odd $\Rightarrow$ n is odd ②

① n is odd, $\exists m \in \mathbb{Z} / n = 2m+1$

$n^2 = (2m+1)^2 = 4m^2 + 4m + 1$

$= 2(2m^2 + 2m) + 1$ ✓

$= 2p + 1$, where $p = 2m^2 + 2m \in \mathbb{Z}$

$\therefore n^2$ is odd

② Method 1: The contrapositive of n^2 is odd $\Rightarrow n$ is odd is: n is even $\Rightarrow n^2$ is even ✓

n is even, $\exists m \in \mathbb{Z} / n = 2m$

$n^2 = 4m^2 = 2(2m^2) = 2q$, $q = 2m^2 \in \mathbb{Z}$ ✓

$\therefore n^2$ is even

Method 2: n^2 is odd, $\exists m \in \mathbb{Z} / n^2 = 2m+1$

then by contradiction, if n is not odd (n is even)

$\exists p \in \mathbb{Z} / n = 2p$ ✓

$n^2 = (2p)^2 = 4p^2$ but $n^2 = 2m+1$

$\therefore 4p^2 = 2m+1$ contradiction ✓

$\therefore n$ is odd

Method 3: n^2 is odd, $\exists m \in \mathbb{Z} / n^2 = 2m+1$

$n^2 - 1 = 2m$

$(n-1)(n+1) = 2m$ ✓

$n-1$ and $n+1$ must be both even because their product is even and their differences is even ✓

Student Number ______

$\therefore$ n is odd

Comments:

* iff $\equiv$ two cases to be considered.
* n is odd $\Rightarrow$ n^2 is odd (majority of students did it correctly)
* n^2 is odd $\Rightarrow$ n is odd

3 Methods: ① prove the contrapositive
② prove by contradiction
③ direct proof

- If you want to prove the contrapositive, you need to state clearly the contrapositive statement then prove it
- proving by contraction, Assume that n is even while n^2 is odd
[Consider that n^2 is even and n is odd to prove by contradiction is wrong]

* For the direct proof method

$$(n-1)(n+1) = 2m$$

need to explain that n-1 and n+1 must be odd or even, but since their products is even $\therefore$ n-1 and n+1 are both even

* Major mistake is to design n^2 as a square of an odd number. If n^2 is odd, $\exists m \in \mathbb{Z} / n^2 = 2m+1$
m is an arbitary integer.
* using square root / using fraction

Student Number ______________________

(b) Evaluate $\int_0^{\pi/3} \sin x \ln(\cos x)\,dx$

Method 1: let $u = \ln\cos x$ and $\sin x\,dx = dv$

$du = \frac{-\sin x}{\cos x}dx \qquad v = -\cos x$

$$\int_0^{\pi/3} \sin x \ln\cos x\,dx = \Big[-\cos x \ln\cos x\Big]_0^{\pi/3} - \int_0^{\pi/3} (-\cos x)\left(\frac{-\sin x}{\cos x}dx\right) ✓$$

$$= -\frac{1}{2}\ln\frac{1}{2} - \int_0^{\pi/3} \sin x\,dx ✓$$

$$= -\frac{1}{2}\ln\frac{1}{2} + \Big[\cos x\Big]_0^{\pi/3}$$

$$= \frac{1}{2}\ln 2 + \frac{1}{2} - 1 ✓$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}$$

Method 2: let $u = \cos x \therefore du = -\sin x\,dx$

For $x = 0,\ u = 1$ and $x = \pi/3,\ u = \frac{1}{2}$

$$\int_0^{\pi/3} \sin x \ln\cos x\,dx = -\int_1^{1/2} \ln u\,du ✓$$

$$= \int_{1/2}^{1} \ln x\,dx \qquad (x \text{ dummy variable})$$

let $u = \ln x$ and $dx = dv$

$du = \frac{1}{x}dx$ and $v = x$

$$\int_{\frac{1}{2}}^{1} \ln x\,dx = \Big[x\ln x - x\Big]_{1/2}^{1} = 0 - 1 - \left(\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}\ln 2 - \frac{1}{2}$$ ✓ ✓

Student Number ______________

Method 3: let $p = \ln \cos x \therefore dp = \frac{-\sin x}{\cos x} dx$

$\cos x = e^{p}$

$$\int_0^{\pi/3} \sin x \ln \cos x \, dx = -\int_0^{\pi/3} \cos x \ln \cos x \times \frac{-\sin x}{\cos x} dx \checkmark$$

For $x = 0$, $p = 0$, $x = \pi/3$, $p = -\ln 2$

$$\int_0^{\pi/3} \sin x \ln \cos x \, dx = -\int_0^{-\ln 2} p e^{p} \, dp$$

let $p = u$ and $e^{p} dp = dv$

$du = dp \qquad v = e^{p} \checkmark$

$$-\int_0^{-\ln 2} p e^{p} \, dp = -\left[\left[p e^{p} \right]_0^{-\ln 2} - \int_0^{-\ln 2} e^{p} \, dp \right]$$

$$= -\left[p e^{p} - e^{p} \right]_0^{-\ln 2} \checkmark$$

$$= \ln 2 \, e^{-\ln 2} + e^{-\ln 2} + 0 - 1$$

$$= \frac{1}{2} \ln 2 + \frac{1}{2} - 1 = \frac{1}{2} \ln 2 - \frac{1}{2}$$

Comments:

- no dx in the integral
- Careless errors

* use the formula sheet for basic issue such the derivation of $\sin x$ or so.

Student Number ____________

(c) Use mathematical induction to prove that for all $n > 0$, n is an integer

$$\frac{d^n}{dx^n}(xe^{2x}) = 2^{n-1}(2x+n)e^{2x}$$

For $n=1$,

$$LHS = \frac{d}{dx}(xe^{2x}) = e^{2x} + x(2e^{2x}) = (2x+1)e^{2x}$$

$$RHS = 2^{1-1}(2x+1)e^{2x} = (2x+1)e^{2x}$$ ✓

$LHS = RHS$ ∴ it is true for $n=1$

Assume it is true for $n=k$, $k \in \mathbb{Z}^+$

$$\frac{d^k}{dx^k}(xe^{2x}) = 2^{k-1}(2x+k)e^{2x} \quad (*)$$

prove it true for $n=k+1$

$$\frac{d^{k+1}}{dx^{k+1}}(xe^{2x}) = 2^k(2x+k+1)e^{2x} \ ?$$

$$\frac{d^{k+1}}{dx^{k+1}}(xe^{2x}) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(xe^{2x})\right)$$ ✓

$$= \frac{d}{dx}\left(2^{k-1}(2x+k)e^{2x}\right) \quad \text{using } (*)$$

$$= 2^{k-1}\left[2e^{2x} + (2x+k)(2e^{2x})\right]$$

$$= 2^k\left[e^{2x} + (2x+k)e^{2x}\right]$$ ✓

$$= 2^k(2x+k+1)e^{2x}$$ ∴ it is true for $n=k+1$

By mathematical induction it is true for all $n>0$.

Student Number

Comments: In general very good

- For the base case, try to be very clear.
  (Find the LHS and RHS and show they are equal)
- Clearly indicate where you are using the assumption.

* $\frac{d^{k+1}}{dx^{k+1}} f(x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(f(x))\right)$
* Wrong use or bad notation
  $\frac{d^{k+1}}{dx^{k+1}} f(x) = \frac{d^k}{dx^k} f(x) \frac{d}{dx}$ (very bad)
* need to show all working, especially when you are using the product rule for differentiation.
* Omitting by mathematical induction
  (need to state clearly that this proof is granted by the principle of mathematical induction).

Student Number ______________

(d) $\frac{d^2y}{dt^2} = -\left(\frac{\pi}{2}\right)^2 y$

(i) $y = a \sin bt$, where $b = \frac{\pi}{2}$

Span of motion is 50cm $\therefore$ $a = 25$

$y = 25 \sin \frac{\pi}{2} t$

(ii) $v(t) = 25 \frac{\pi}{2} \cos \frac{\pi}{2} t$

Maximum speed $= \frac{25\pi}{2} \simeq 39$ cm/s

comments:

* lots of student did not recognise that the problem is SHM
* lots of student integrate the differential equation to derive the SHM solution
* student failed to find the amplitude correctly
* some student they sub the period = 4 for b instead of $\frac{\pi}{2}$.

Student Number ______________________

(e) (i) $F = ma$

$70\ddot{x} = 140 - kV$

$\ddot{x} = 2 - \frac{k}{70}V$

for $V = 10$ m/s , $\ddot{x} = 1$

$1 = 2 - \frac{k}{70} \times 1 \quad \therefore k = 7$

$\therefore \ddot{x} = 2 - \frac{1}{10}V$

(ii) $\frac{V\,dV}{dx} = \frac{20 - V}{10}$

$\frac{V\,dV}{20 - V} = \frac{1}{10}\,dx \qquad V \neq 20$

$\frac{V - 20 + 20}{20 - V}\,dV = \frac{1}{10}\,dx$

$\left(-1 + \frac{20}{20 - V}\right)dV = \frac{1}{10}\,dx$

$-V - 20\ln|20 - V| + C = \frac{1}{10}x$

For $x = 0$, $V = 0 \quad \therefore C = 20\ln 20$

$x = -10V + 200\ln 20 - 200\ln|20 - V|$

(iii) Terminal velocity when acceleration is zero

$\therefore V_T = 20$ m/s.

Student Number

Comments: Majority did well

* Some careless errors of finding k in part (i)
* in part (ii) majority of student did it correctly
  * minor careless errors
  * Major problem that some student did the following:

$$\int \frac{V\,dV}{20-V} = \ln|20-V|$$

which is wrong and makes the problem much easier (terminating error).

(iii) student wrote essay.

All you need here to say that the terminal velocity occurs when $\ddot{x}=0$, hence $V=20\,m/s$.

Q13

a i, $\frac{1}{u^2+u-1} = \frac{A}{u-\frac{1+\sqrt{5}}{2}} + \frac{B}{u-\frac{1-\sqrt{5}}{2}}$

$1 \equiv A\left(u-\frac{1-\sqrt{5}}{2}\right) + B\left(u-\frac{1+\sqrt{5}}{2}\right)$

let $u = \frac{1+\sqrt{5}}{2} \rightarrow 1 \equiv A\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$

$A = 1/\sqrt{5}$

1 mark for each constant

$u = \frac{1-\sqrt{5}}{2} \rightarrow 1 \equiv B\left(\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right)$

$B = -1/\sqrt{5}$

ii, $\int \frac{\cos x}{\sin x - \cos^2 x}\,dx$

$= \int \frac{\cos x}{\sin x + \sin^2 x - 1}\,dx$ let $u = \sin x$

$= \int \frac{du}{u^2+u-1}$ — 1 making valid sub

$= \int\left(\frac{1/\sqrt{5}}{u-\frac{1-\sqrt{5}}{2}} - \frac{1/\sqrt{5}}{u-\frac{1+\sqrt{5}}{2}}\right)du$

$= \frac{1}{\sqrt{5}} \ln\left|\frac{u-\frac{1-\sqrt{5}}{2}}{u-\frac{1+\sqrt{5}}{2}}\right| + C$

$= \frac{1}{\sqrt{5}} \ln\left|\frac{2\sin x - 1+\sqrt{5}}{2\sin x - 1 - \sqrt{5}}\right| + C$ — 1 Final

bi, For contradiction, suppose $\gcd(a,b)=1$ and $\gcd(a^2,b)=k>1$

Then, there is some prime $p \le k$ such that

$p \mid a^2$ and $p \mid b$ — 1 Set up

$\Rightarrow p \mid a$ and $p \mid b$, by FTA since p prime

But this is a contradiction since $p>1$ and $\gcd(a,b)=1$.

$\therefore$ If $\gcd(a,b)=1$, then $\gcd(a^2,b)=1$ — 1 Finalised proof

ALTERNATE SOLN:

Let $a = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ and $b = q_1^{m_1} q_2^{m_2} \cdots q_l^{m_l}$ be the prime factorisations of a and b. Since $\gcd(a,b) = 1$, then

$$p_i \neq q_j \quad \forall i, j$$ — 1 Set up

Now $a^2 = p_1^{2n_1} p_2^{2n_2} \cdots p_k^{2n_k}$. There are no new prime factors, and thus no common prime factors with b!

$\therefore \gcd(a^2, b) = 1$. — 1 Final reasoning

Feedback:

- Very very poorly done by the cohort.
- Successful attempts used the alternate solution.
- Common mistake was saying that if $\gcd(a^2,b) > 1$, then $\frac{a^2}{b} = k \in \mathbb{Z}$.
  - $\Rightarrow$ This is false! e.g. $a^2 = 36$, $b = 24$
- Other unsuccessful attempts resulted from an incorrect negation when trying proof by contradiction or attempting a contrapositive which become too difficult for most to handle.

ii, let $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$, $p \neq q$ and p, q coprime.

Suppose $x + \frac{1}{x} \in \mathbb{Z}$

$$x + \frac{1}{x} = k, \qquad k \in \mathbb{Z}$$

$$\frac{p}{q} + \frac{q}{p} = k$$

$$p^2 + q^2 = kpq$$ — 1 equivalent valid progress.

$$\frac{p^2}{q} = kp - q$$

since $q \neq p$ and p, q coprime, then LHS is not an integer by part (i), which RHS is an integer

Contradiction

$\therefore x + \frac{1}{x} \notin \mathbb{Z}$ — 1 Finalised proof

Feedback:

- Also poorly done by the cohort.
- Most common error was arguing with $x \in \mathbb{Z}$. This was an instant "fatal error" and zero marks is awarded. The question states x is rational.
- Few candidates offered other valid proofs.

c i, We have $e^{i\theta} = \cos\theta + i\sin\theta$ ①

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$e^{-i\theta} = \cos\theta - i\sin\theta$ ② since sine odd and cosine even

$\frac{①+②}{2}$: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ ①

$\frac{①-②}{2i}$: $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ ①

ii, RHS $= -4\sin\alpha\sin\beta\cos(\alpha+\beta)$

$= -4\left(\frac{e^{i\alpha}-e^{-i\alpha}}{2i}\right)\left(\frac{e^{i\beta}-e^{-i\beta}}{2i}\right)\left(\frac{e^{i(\alpha+\beta)}+e^{-i(\alpha+\beta)}}{2}\right)$ ① using (i)

$= \frac{1}{2}\left(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)} + e^{-i(\alpha+\beta)}\right)\left(e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}\right)$

$= \frac{1}{2}\left(e^{i2(\alpha+\beta)} + 1 - e^{2i\alpha} - e^{-2i\beta} - e^{2i\beta} - e^{-2i\alpha} + 1 + e^{-2i(\alpha+\beta)}\right)$ ① expanding out

$= \frac{1}{2}\left(e^{i2(\alpha+\beta)} + e^{-2i(\alpha+\beta)}\right) - \frac{1}{2}\left(e^{2i\alpha}+e^{-2i\alpha}\right) - \frac{1}{2}\left(e^{2i\beta}+e^{-2i\beta}\right) + 1$

$= \cos 2(\alpha+\beta) - \cos 2\alpha - \cos 2\beta + 1$

$=$ LHS ① Final

Feedback:

- Those who started w/ LHS struggled more than those who started w/ RHS.
- Candidates need to write clearer and more coherently!

d, Suppose, for contradiction, that $\underset{\sim}{a} \parallel \underset{\sim}{b}$.

i.e. $\underset{\sim}{a} = \mu \underset{\sim}{b}$, $\mu \neq 0$. — ① correct negation

Then, $\text{proj}_{\underset{\sim}{b}}(\underset{\sim}{a}) = \text{proj}_{\underset{\sim}{b}}(\mu \underset{\sim}{b})$

$\Rightarrow \lambda \underset{\sim}{b} = \mu \underset{\sim}{b}$

$\Rightarrow \lambda = \mu$, since $\underset{\sim}{b} \neq \underset{\sim}{0}$ — ①

Now, $\underset{\sim}{a} = \text{proj}_{\underset{\sim}{a}}(\underset{\sim}{a})$

$= \text{proj}_{\underset{\sim}{a}}(\mu \underset{\sim}{b})$

$= \mu \, \text{proj}_{\underset{\sim}{a}}(\underset{\sim}{b})$

$= \mu (1-\lambda) \underset{\sim}{a}$

$\underset{\sim}{a} = \lambda(1-\lambda)\underset{\sim}{a}$

$\Rightarrow 1 = \lambda(1-\lambda)$ since $\underset{\sim}{a} \neq \underset{\sim}{0}$

$\Rightarrow \lambda^2 - \lambda + 1 = 0$

$\Delta = -3 < 0$

$\therefore \lambda \notin \mathbb{R}$

contradiction, so $\underset{\sim}{a} \nparallel \underset{\sim}{b}$. — ① Finalised proof.

ALTERNATE SOLN:

We have $\text{proj}_{\underset{\sim}{b}}(\underset{\sim}{a}) = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \underset{\sim}{b}$

$\lambda \underset{\sim}{b} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \underset{\sim}{b}$

$\Rightarrow \lambda = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}$

Similarly, $\text{proj}_{\underset{\sim}{a}}(\underset{\sim}{b}) = \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{a}|^2} \underset{\sim}{a}$

$(1-\lambda)\underset{\sim}{a} = \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{a}|^2} \underset{\sim}{a}$

$\Rightarrow 1-\lambda = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|^2}$

$$\therefore \quad 1 = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|^2} + \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|^2}$$

$$\Rightarrow \frac{|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2}{|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2} = \underset{\sim}{a}\cdot\underset{\sim}{b} \quad \text{— } 1$$

Now, if $\underset{\sim}{a} \parallel \underset{\sim}{b}$, then $\underset{\sim}{a}\cdot\underset{\sim}{b} = \pm|\underset{\sim}{a}||\underset{\sim}{b}|$.

$$\therefore \frac{|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2}{|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2} = \pm|\underset{\sim}{a}||\underset{\sim}{b}|$$

$$\pm|\underset{\sim}{a}||\underset{\sim}{b}| = |\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2 \quad \text{— } 1$$ Significant progress

Clearly the negative case is not possible as RHS > 0.

For the positive case, note that by AM-GM

$$|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2 \geq 2|\underset{\sim}{a}||\underset{\sim}{b}|$$

$$> |\underset{\sim}{a}||\underset{\sim}{b}| \quad , \text{since } |\underset{\sim}{a}|, |\underset{\sim}{b}| \neq 0$$

Thus, not possible for $\underset{\sim}{a} \parallel \underset{\sim}{b}$. — 1 Finalised proof.

Feedback:

- Poorly done by cohort
- Most candidates used second approach, but couldn't correctly justify beyond

$$\frac{|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2}{|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2} = \underset{\sim}{a}\cdot\underset{\sim}{b}$$

- Terrifying mistakes included

$$\Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{b}=0 \Rightarrow \underset{\sim}{a}\parallel\underset{\sim}{b}$$

$$\Rightarrow \underset{\sim}{a}\parallel\underset{\sim}{b} \Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{b}=1$$

PLEASE REVISE!!!

Suggested Solutions

Question 14

a, i,

Given $I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$

$v = \frac{-1}{3}(1-x^2)^{3/2}$ → 1m

$v = \frac{-1}{2}\cdot\frac{2}{3}(1-x^2)^{3/2}$

Using integration by parts

$u = x^{n-1}$

$u' = (n-1)x^{n-2}$

$v' = \frac{-1}{2}\cdot 2x\sqrt{1-x^2}$

$v' = x\sqrt{1-x^2}$

$\int u\, dv = uv - \int v\, du$

$= x^{n-1}\left[\frac{-1}{3}(1-x^2)^{3/2}\right]_0^1 - \int_0^1 \frac{-1}{3}(1-x^2)^{3/2}\cdot(n-1)x^{n-2}\, dx$

$= \left[-\frac{x^{n-1}}{3}(1-x^2)^{3/2}\right]_0^1 + \frac{1}{3}(n-1)\int_0^1 (1-x^2)^{3/2}\cdot x^{n-2}\, dx$

$= 0 + \frac{1}{3}(n-1)\int_0^1 x^{n-2}\cdot(1-x^2)(1-x^2)^{\frac{1}{2}}\, dx$ → 1m

Marker's Feedback

Page _____

Suggested Solutions

Question

$$I_n = \frac{1}{3}(n-1)\left[\int_0^1 x^{n-2}(1-x^2)^{1/2}\,dx - \int_0^1 x^n(1-x^2)^{1/2}\,dx\right] \to 1m$$

$$3I_n = (n-1)\int_0^1 x^{n-2}(1-x^2)^{1/2}\,dx - (n-1)\int_0^1 x^n(1-x^2)^{1/2}\,dx$$

$$3I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$(n+2)I_n = (n-1)I_{n-2}$$

$$I_n = \frac{(n-1)}{n+2}I_{n-2} \longrightarrow 1m$$

Marker's Feedback

Most students have shown good understanding

Some students chose wrong u & v', hence lost marks.

Alternate sol: let $x = \sin\theta$, method also works

a students were given full marks if the approach was correct.

Suggested Solutions

Question

14aii, Shaded area : Unshaded area

$$y^2 = x^4(1-x^2)$$

$$y = x^2(1-x^2)^{1/2}$$

$$\text{Shaded area} = 8\int_0^1 x^2\sqrt{1-x^2}\,dx$$

$$= 8I_2 \qquad \left(I_n = \frac{n-1}{n+2} I_{n-2}\right)$$

$$= 8\left(\frac{2-1}{2+2}\right)I_0 \longrightarrow 1m$$

$$= 2\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 2(\text{Total area}) \times \frac{1}{4}$$ (Note: Integral is only representing Quarter of a circle)

Shaded · Area = $\frac{1}{2}$(Shaded Area + unshaded area)

$\frac{1}{2}$ Shaded · A = $\frac{1}{2}$ unshaded · A => 1 : 1 ⟶ 1m

Marker's Feedback

Poorly done.

Most students went wrong with the logic of the 2nd part of the Question.

Most common answers were 1:2 or 1:3 -

Suggested Solutions

Question 14b

$$r_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix} ; \quad r_2 = \begin{pmatrix}3\\4\\5\end{pmatrix} + \mu\begin{pmatrix}3\\s\\t\end{pmatrix}$$

$$\begin{pmatrix}2\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}3\\s\\t\end{pmatrix} = 0 \quad (\because r_1 \,\&\, r_2 \text{ meet at right angle})$$

$$6 - s + t = 0$$
$$t = s - 6 \quad \text{— 1m}$$

Given the lines intersect

$$1 + 2\lambda = 3 + 3\mu \Rightarrow 2\lambda = 2 + 3\mu \rightarrow \text{eq 1}$$
$$1 - \lambda = 4 + s\mu \Rightarrow -\lambda = 3 + s\mu \rightarrow \text{eq 2}$$
$$1 + \lambda = 5 + t\mu \Rightarrow \lambda = 4 + t\mu \rightarrow \text{eq 3}$$

eq 3 − eq 2

$$2\lambda = 1 + (t - s)\mu$$
$$2\lambda = 1 - 6\mu \quad (\text{but } 2\lambda = 2 + 3\mu \rightarrow \text{eq } 1)$$
$$2 + 3\mu = 1 - 6\mu \Rightarrow \mu = -\frac{1}{9}$$
$$2\lambda = 2 - \frac{1}{3} = 5/3 \Rightarrow \lambda = \frac{5}{6}$$

2m

Marker's Feedback

→ Mostly done well apart from few calculation Errors.

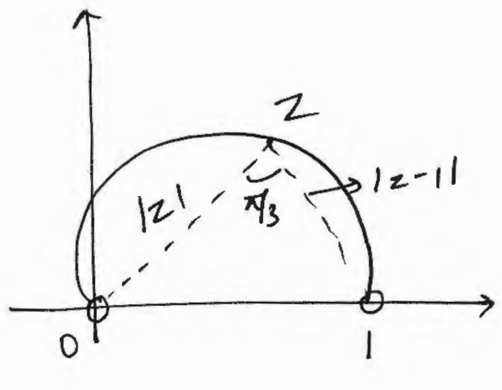
Page ______

Suggested Solutions

Question 14 C (i)

$$\arg\left(1-\frac{1}{z}\right) = \frac{\pi}{3}$$

$$\arg\left(\frac{z-1}{z}\right) = \frac{\pi}{3}$$



3m

14c (ii) using cosine rule

$$1^2 = |z|^2 + |z-1|^2 - 2|(z)|\,|z-1| \times \frac{1}{2}$$

$$|z|^2 + |z-1|^2 = 1 + |z^2 - z|$$

$\longrightarrow$ 2m

Marker's Feedback

Poorly done!
Students encouraged to revise drawing Argand diagrams.

Most students lost marks in 14C(ii) because the Question asked to use information from (i) to do (ii) (It is a HENCE question)

Page ______

Suggested Solutions

Question 14 d

i, upwards $\uparrow 300\ddot{y}$, $\downarrow g \ \downarrow 150v$

$$300\ddot{y} = -300g - 150v$$

$$\ddot{y} = -g - \frac{1}{2}v$$

$$v\frac{dv}{dy} = \frac{-2g - v}{2}$$

$$v\frac{dv}{dy} = \frac{-1}{2}(2g+v)$$

$$\int_{v_0}^{0} \frac{v}{v+2g}\,dv = \frac{-1}{2}\int_{0}^{H} dy$$

$$\int_{v_0}^{0} \frac{v+2g-2g}{v+2g}\,dv = \frac{-1}{2}\left[y\right]_{0}^{H}$$

$$\int_{v_0}^{0} 1 - \int_{v_0}^{0} \frac{2g}{v+2g}\,dv = \frac{-1}{2}H$$

$$\left[v - 2g\ln|v+2g|\right]_{v_0}^{0} = \frac{-1}{2}H$$

$$\left[-v_0 - \left[2g\ln 2g - 2g\ln(v_0+2g)\right]\right] = \frac{-1}{2}H$$

$$-v_0 - 2g\ln 2g + 2g\ln(v_0+2g) = \frac{-1}{2}H$$

$$H = 2v_0 + 4g\ln 2g - 4g\ln(v_0+2g)$$

$$H = 2v_0 + 4g\ln\left(\frac{2g}{v_0+2g}\right) \rightarrow \text{eq 1}$$

Marker's Feedback

Suggested Solutions

Question 14 d i, continued

equating 1 & 2

$$2g\ln\left(\frac{2g}{2g-V_1}\right)-V_1 = 2g\ln\left(\frac{2g}{2g+V_0}\right)+V_0$$

$$2g\ln\left(\frac{2g+V_0}{2g-V_1}\right) = V_0+V_1$$

$$2g\ln\left(\frac{2g+V_0}{2g-V_1}\right) = 50$$

$$\ln\left(\frac{2g+V_0}{2g-V_1}\right) = \frac{25}{g}$$

$$2g+V_0 = (2g-V_1)e^{25/g}$$ $(V_1+V_0=50;\ V_1=50-V_0)$

$$2g+V_0 = \left(2g-(50-V_0)\right)e^{25/g}$$

$$2g+V_0 = \left(2g-50+V_0\right)e^{25/g}$$

$$2g+V_0 = 2ge^{25/g}-50e^{25/g}+V_0e^{25/g}$$

$$V_0\left(1-e^{25/g}\right) = (2g-50)e^{25/g}-2g$$

$$V_0 = \frac{(2g-50)e^{25/g}-2g}{\left(1-e^{25/g}\right)}$$

$V_0 \simeq 47.2$ m/s

1m

Marker's Feedback

Very few students have shown full correct working out.

Students have trouble forming motion equations.

All students are encouraged to redo this question in your own time.

Suggested Solutions

Question 14 d, i, continued

<u>Downwards</u>:

$$\uparrow 150v \qquad \downarrow 300g$$

$$300\ddot{y} = 300g - 150v$$

$$\ddot{y} = g - \frac{1}{2}v$$

$$\ddot{y} = \frac{2g - v}{2}$$

$$v\frac{dv}{dy} = \frac{1}{2}(2g - v)$$

$$\int_0^{V_1} \frac{v}{2g - v}\,dv = \frac{1}{2}\int_0^H dy$$

$$-\int_0^{V_1} \frac{-v + 2g - 2g}{2g - v}\,dv = \frac{1}{2}\Big[y\Big]_0^H$$

$$\int_0^{V_1} 1 \cdot dv - \int_0^{V_1} \frac{2g}{2g - v}\,dv = \frac{1}{2}H$$

$$-\Big[v + 2g\ln(2g - v)\Big]_0^{V_1} = \frac{1}{2}H$$

$$-(V_1 + 2g\ln(2g - V_1) - 2g\ln 2g) = \frac{1}{2}H$$

$$-V_1 - 2g\ln(2g - V_1) + 2g\ln 2g = \frac{1}{2}H$$

$$\frac{1}{2}H = 2g\ln\left(\frac{2g}{2g - V_1}\right) - V_1$$

$\hookrightarrow$ eq 2

$\hookrightarrow$ 1m

Marker's Feedback

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Suggested Solutions

Question 14 d ii,

↑ upward

$$\ddot{y} = \frac{-1}{2}(2g + v)$$

$$\int_{v_0}^{0} \frac{1}{2g+v}\,dv = \frac{-1}{2}\int_0^{T_1} dt$$

$$\Big[\ln(2g+v)\Big]_{v_0}^{0} = \frac{-1}{2}T_1$$

$$T_1 = -2(\ln 2g - \ln(2g+v_0))$$

$$T_1 = 2\ln\left(\frac{2g+v_0}{2g}\right)$$

→ 1m

↓ downward

$$\ddot{y} = \frac{1}{2}(2g - v)$$

$$\int_0^{v_1} \frac{1}{2g-v}\,dv = \frac{1}{2}\int_0^{T_2} dt$$

$$-\int_0^{v_1} \frac{-1}{2g-v}\,dv = \frac{1}{2}\int_0^{T_2} dt$$

$$-\Big[\ln(2g-v)\Big]_0^{v_1} = \frac{1}{2}\Big[t\Big]_0^{T_2}$$

$$-\Big[\ln(2g-v_1) - \ln 2g\Big] = \frac{1}{2}T_2$$

$$(\ln 2g - \ln(2g-v_1)) = \frac{1}{2}T_2$$

$$T_2 = 2\ln\left(\frac{2g}{2g-v_1}\right)$$

$$T_1 + T_2 = 2\ln\left(\frac{2g+v_0}{2g}\right) + 2\ln\left(\frac{2g}{2g-v_1}\right) = 2\ln\left(\frac{2g+v_0}{2g-v_1}\right)$$ (from i)

$$= 2\left(\frac{25}{g}\right)$$

1m

← $= \frac{50}{g} \simeq 35.7$ seconds

Marker's Feedback

→ Students who have shown understanding of part d i, have shown good understanding of d ii, otherwise poorly done!

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15)

a) i)

$$2|\overrightarrow{AK}| = |\overrightarrow{AK} + \overrightarrow{AK}|$$
$$= |\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BK} + \overrightarrow{CK}|$$
$$= |\overrightarrow{AB} + \overrightarrow{AC}| \leq |\overrightarrow{AB}| + |\overrightarrow{AC}|$$

OR $\overrightarrow{AK} = \overrightarrow{AC} + \overrightarrow{CK}$

$$= \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB}$$
$$= \overrightarrow{AC} + \frac{1}{2}(\overrightarrow{CA} + \overrightarrow{AB})$$
$$= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$$

$$\therefore 2|\overrightarrow{AK}| = |\overrightarrow{AB} + \overrightarrow{AC}|$$
$$2|\overrightarrow{AK}| \leq |\overrightarrow{AB}| + |\overrightarrow{AC}| \quad \text{— } ①$$

ii) Similar to (i)

$$2|\overrightarrow{CM}| \leq |\overrightarrow{CA}| + |\overrightarrow{CB}| \quad \text{— } ②$$
$$2|\overrightarrow{BL}| \leq |\overrightarrow{BA}| + |\overrightarrow{BC}| \quad \text{— } ③$$

①+②+③

$$2(|\overrightarrow{AK}| + |\overrightarrow{CM}| + |\overrightarrow{BL}|) \leq 2(|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CA}|)$$
$$\therefore |\overrightarrow{AK}| + |\overrightarrow{CM}| + |\overrightarrow{BL}| \leq |\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CA}|$$

iii) $\overrightarrow{OK} = \overrightarrow{OB} + \overrightarrow{BK}$

$$= \underset{\sim}{b} + \frac{1}{2}\overrightarrow{BC} = \underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b})$$
$$= \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c})$$

$\vec{AG} = \vec{AO} + \vec{OG} = -\underset{\sim}{a} + \frac{1}{3}(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})$

$= \frac{1}{3}(\underset{\sim}{b}+\underset{\sim}{c}-2\underset{\sim}{a})$

$\vec{AK} = \frac{1}{2}(\underset{\sim}{b}+\underset{\sim}{c}) - \underset{\sim}{a}$

$= \frac{1}{2}(\underset{\sim}{b}+\underset{\sim}{c}-2\underset{\sim}{a}) = \frac{3}{2}\vec{AG}$ —(*)

$\therefore$ A, G and K are colinear

iv) similarly B, G, L are colinear

and M, G, C are colinear

$\therefore$ AK, BL and MC intersect at G

$\therefore$ they are concurrent.

v) From (*) $|\vec{AK}| : |\vec{AG}| = 3 : 2$

$\therefore |\vec{AG}| : |\vec{GK}| = 2 : 1$

vi) $|\vec{AB}| + |\vec{BC}| + |\vec{CA}|$

$= |\vec{AG} + \vec{GB}| + |\vec{BG} + \vec{GC}| + |\vec{CG} + \vec{GA}|$

$\leq 2(|\vec{AG}| + |\vec{BG}| + |\vec{CG}|)$

$= 2(\frac{2}{3}|\vec{AK}| + \frac{2}{3}|\vec{BL}| + \frac{2}{3}|\vec{CM}|)$

$= \frac{4}{3}(|\vec{AK}| + |\vec{BL}| + |\vec{CM}|)$

b) $\text{Cis}\,7\theta = \text{Cis}\,0$

$$7\theta = 2k\pi$$

$$\theta = \frac{2k\pi}{7} \quad ; \; k = 0, \pm 1, \pm 2, \pm 3$$

Let z_1, z_2 and z_3 represent the points A, B and C respectively

a) $$\angle CAB = \arg \frac{z_3 - z_1}{z_2 - z_1} = \arg \frac{e^{i\left(\frac{4\pi}{7}\right)} - 1}{e^{i\left(\frac{2\pi}{7}\right)} - 1}$$

$$= \arg\left(e^{i\left(\frac{2\pi}{7}\right)} + 1\right)$$

$$= \arg\left[e^{i(\pi/7)}\left(e^{i\frac{\pi}{7}} + e^{-i\frac{\pi}{7}}\right)\right]$$

Max 1 mark if not considered the arguments

$$= \arg\left(2\cos\frac{\pi}{7}\, e^{i\frac{\pi}{7}}\right)$$

$$= \frac{\pi}{7} \text{ since } 2\cos\frac{\pi}{7} > 0$$

$$\angle DAB = \arg\left(\frac{e^{i\left(\frac{6\pi}{7}\right)} - 1}{e^{i\left(\frac{2\pi}{7}\right)} - 1}\right)$$

$$= \arg\left(e^{i\left(\frac{4\pi}{7}\right)} + e^{i\left(\frac{2\pi}{7}\right)} + 1\right)$$

$$= \arg\left[e^{i\frac{2\pi}{7}}\left(e^{i\frac{2\pi}{7}} + 1 + e^{-i\frac{2\pi}{7}}\right)\right]$$

$$= \arg\left[\left(2\cos\frac{2\pi}{7} + 1\right) e^{i\frac{2\pi}{7}}\right]$$

$$= \frac{2\pi}{7} \text{ since } 2\cos\frac{2\pi}{7} + 1 > 0$$

ii) Let the complex numbers α, β represent the points Y and Z respectively

Let z_1, z_2 and z_3 represents the points A, B and C respectively.

$$z_1 - \alpha = (z_1 - z_2)\, e^{i\left(\frac{2\pi}{7}\right)}$$

$$1 - \alpha = \left(1 - e^{i\left(\frac{2\pi}{7}\right)}\right) e^{i\left(\frac{2\pi}{7}\right)}$$

$$\alpha = 1 - (1 - w^2)w^2$$

$$= w^4 - w^2 + 1$$

$$1 - \beta = \left(1 - e^{i\left(\frac{4\pi}{7}\right)}\right) e^{i\left(\frac{\pi}{7}\right)}$$

$$\beta = 1 - (1 - w^4)w$$

$$= w^5 - w + 1$$

iii)

$$AB = AZ = |1 - \beta| = \left|1 - e^{i\left(\frac{4\pi}{7}\right)}\right|$$

$$= \left|e^{i\frac{2\pi}{7}}\left(e^{-i\frac{2\pi}{7}} - e^{i\frac{2\pi}{7}}\right)\right| = 2\sin\frac{2\pi}{7}$$

2 mark if any two correct

$$AD = \left|1 - e^{i\left(\frac{6\pi}{7}\right)}\right| = \left|e^{i\frac{3\pi}{7}}\left(e^{-i\frac{3\pi}{7}} - e^{i\frac{3\pi}{7}}\right)\right|$$

$$= 2\sin\frac{3\pi}{7}$$

$$AB = \left|1 - e^{i\frac{2\pi}{7}}\right| = \left|e^{i\pi/7}\left(e^{-i\frac{\pi}{7}} - e^{i\pi/7}\right)\right|$$

$$= 2\sin\pi/7$$

$$R.H.S = \frac{1}{AC} + \frac{1}{AD}$$

$$= \frac{1}{2\sin\frac{2\pi}{7}} + \frac{1}{2\sin\frac{3\pi}{7}}$$

$$\frac{\sin\frac{3\pi}{7} + \sin\frac{2\pi}{7}}{2\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}$$

$$= \frac{2\sin\left(\frac{5\pi}{14}\right)\cos\left(\frac{\pi}{14}\right)}{2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{7}\right)}$$

$$= \frac{2\sin\left(\frac{5\pi}{14}\right)}{4\sin\frac{\pi}{7}\sin\left(\frac{\pi}{2}-\frac{\pi}{7}\right)}$$

$$= \frac{1}{2\sin\frac{\pi}{7}} = \frac{1}{AB} = L.H.S$$

Alternate method

$AB = AY$ and $AC = AZ$; $\omega^7 = -1$

$$\frac{1}{\vec{AZ}} + \frac{1}{\vec{AD}} = \frac{1}{\omega^5 - \omega^2} + \frac{1}{\omega^6 - 1}$$

$$= \frac{\omega^2}{-1-\omega^3} + \frac{\omega}{-1-\omega}$$

$$= -\left[\frac{\omega^2 + \omega(1-\omega+\omega^2)}{1+\omega^3}\right]$$

$$= -\frac{(w^3+w)}{(1+w^3)} \times \frac{(w^4-w^2)}{(w^4-w^2)}$$

$$= -\frac{(-1-w^5+w^5-w^3)}{(1+w^3)(w^4-w^2)}$$

$$= \frac{1}{w^4-w^2} = \frac{1}{\overrightarrow{AY}}$$

$$\frac{1}{\overrightarrow{AZ}} + \frac{1}{\overrightarrow{AD}} = \frac{1}{\overrightarrow{AY}}$$

Since A, Y, Z and D are colinear in that order

$$\frac{1}{|\overrightarrow{AZ}| \arg \overrightarrow{AD}} + \frac{1}{|\overrightarrow{AD}| \arg \overrightarrow{AD}} = \frac{1}{|\overrightarrow{AY}| \arg \overrightarrow{AD}}$$

$$\frac{1}{AC} + \frac{1}{AD} = \frac{1}{AB}$$

(a)

(i) It is recommended to first show that $2\,\overrightarrow{AK} = (\overrightarrow{AB} + \overrightarrow{AC})$ and then apply the triangle inequality

(ii) & (iii) These parts are well done. However, it is important to mention the common point:

since $\overrightarrow{AK} \parallel \overrightarrow{AG}$ and $\boldsymbol{A}$ **is a common point**, it follows that A, G and K are colinear points

(iv) Avoid repeating identical proofs multiple times. Simply writing "similarly" is sufficient.

(v) Note that $\overrightarrow{AK} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c} - 2\underset{\sim}{a})$ and $\overrightarrow{AK} \neq \frac{1}{2}\sqrt{1^2 + 1^2 + (-2)^2}$ because $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are **NOT** component vectors

(v) Many students had difficulty manipulating the vector effectively to achieve the required result.

(b) This part was generally done poorly, possibly due to time pressure at the end of the exam.

(i) Since this is a "show" question, it is essential to present all relevant steps clearly.
A common mistake was writing $\overrightarrow{OB} = \omega$. It is in fact $\overrightarrow{OB} = \omega^2$.

(ii) Some failed to find $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ in terms of ω .

(iii) A few students attempted this part.