NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2025 Year 12 Course Assessment Task 4 (Trial Examination)
Friday, 1 August 2025

General instructions

- Working time – 3 hours. (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 17)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12MXX.1 – Ms Ham ◯ 12MXX.2 – Mr Ho

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	%
MARKS	/10	/17	/15	/14	/14	/16	/14	/100

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 17).

Glossary

- $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ – set of all natural numbers
- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – set of all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers
- $\mathbb{C}$ – set of all complex numbers

Questions **Marks**

1. Let $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ be the position vectors of points P, Q, R respectively. It is given that **1**

$$\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 1 \\ 8 \end{pmatrix} \quad , \quad \overrightarrow{QR} = \begin{pmatrix} \lambda + 2 \\ 2 \\ \mu - 5 \end{pmatrix}$$

What are the values of λ and μ if P, Q and R are collinear?

(A) $\lambda = -4$ and $\mu = 9$

(B) $\lambda = -6$ and $\mu = 11$

(C) $\lambda = -6$ and $\mu = 13$

(D) $\lambda = -10$ and $\mu = 21$

2. Consider the two lines with vector equations **1**

$$\underset{\sim}{r_1} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \underset{\sim}{r_2} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$$

Which of the following is the exact value of the angle between the two lines?

(A) $\cos^{-1}\left(\dfrac{7}{\sqrt{174}}\right)$

(B) $\cos^{-1}\left(\dfrac{-13}{\sqrt{174}}\right)$

(C) $\cos^{-1}\left(\dfrac{7}{\sqrt{294}}\right)$

(D) $\cos^{-1}\left(\dfrac{-13}{\sqrt{294}}\right)$

3. Consider $z^6 = 1$. Which of the following are the roots of the equation? **1**

(A) $e^{i\frac{k\pi}{3}}$ for $k = \pm 1, \pm 2, \pm 3$

(B) $e^{i\frac{k\pi}{6}}$ for $k = \pm 1, \pm 2, \pm 3$

(C) $e^{i\frac{k\pi}{3}}$ for $k = 0, 1, 2, \cdots, 5$

(D) $e^{i\frac{k\pi}{6}}$ for $k = 0, 1, 2, \cdots, 5$

4. Consider the statement: **1**

$$\forall x \in \mathbb{R}, \quad \exists y \in \mathbb{R} \quad \text{such that} \quad x^y > 1$$

Which of the following is the negation of the statement above?

(A) $\forall x \in \mathbb{R}, \quad \exists x \in \mathbb{R}, \quad x^y \leq 1$

(B) $\exists x \in \mathbb{R}, \quad \forall y \in \mathbb{R}, \quad x^y \leq 1$

(C) $\forall x, y \in \mathbb{R}, \quad x^y \leq 1$

(D) $\exists x, y \in \mathbb{R}, \quad x^y \leq 1$

5. A particle moving along a horizontal line has acceleration **1**

$$a = \frac{12 - 4x}{x^3}$$

where x is the displacement of the particle. The particle starts from rest at $x = 6$.

Which of the following describes the velocity of the particle?

(A) $v^2 = -\dfrac{6}{x^2} + \dfrac{4}{x}$

(B) $v^2 = -\dfrac{12}{x^2} + \dfrac{8}{x}$

(C) $v^2 = -\dfrac{6}{x^2} + \dfrac{4}{x} - \dfrac{1}{2}$

(D) $v^2 = -\dfrac{12}{x^2} + \dfrac{8}{x} - 1$

6. Let ω be an non-real cube root of unity. **1**

Which of the following is equivalent to $\left(1 + 2\omega - \omega^2 + \omega^3\right)^5$?

(A) -243ω

(B) $-243\omega^2$

(C) 243ω

(D) $243\omega^2$

Examination continues overleaf...

7. Let z be a complex number such that $\text{Im}(z) \neq 0$. **1**

Which of the following expressions is equivalent to $\dfrac{8\overline{z}z}{(\overline{z}-z)^2}$?

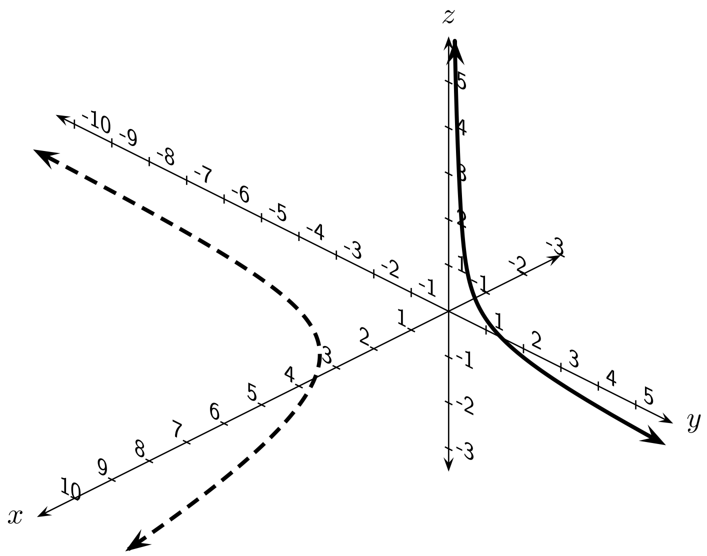
(A) $2 + 2\left(\dfrac{\text{Re}(z)}{\text{Im}(z)}\right)^2$

(B) $2 + 2\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)^2$

(C) $-2 - 2\left(\dfrac{\text{Re}(z)}{\text{Im}(z)}\right)^2$

(D) $-2 - 2\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)^2$

8. Which position vector best describes the curve in diagram? **1**



(A) $\underset{\sim}{r} = \begin{pmatrix} t \\ -\frac{1}{t} \\ e^{-t} \end{pmatrix}$

(B) $\underset{\sim}{r} = \begin{pmatrix} t \\ \frac{1}{t} \\ e^{t} \end{pmatrix}$

(C) $\underset{\sim}{r} = \begin{pmatrix} e^{-t} \\ -\frac{1}{t} \\ t \end{pmatrix}$

(D) $\underset{\sim}{r} = \begin{pmatrix} e^{t} \\ -\frac{1}{t} \\ -t \end{pmatrix}$

9. Which of the following integrals is equivalent to **1**

$$\int_0^{\frac{\pi}{2}} \frac{x}{1+\sin x+\cos x}\,dx$$

(A) $\dfrac{\pi}{2}\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x-\cos x}\,dx$

(B) $\dfrac{\pi}{4}\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x-\cos x}\,dx$

(C) $\dfrac{\pi}{2}\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x+\cos x}\,dx$

(D) $\dfrac{\pi}{4}\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x+\cos x}\,dx$

10. Consider the statement: **1**

Period 0 is cancelled if and only if there is an excursion.

Which of the following statements is FALSE?

(A) The negation is an 'or' statement.

(B) The negation is 'Period 0 being cancelled does not imply that there is an excursion'.

(C) A counterexample is 'There is no excursion and Period 0 is cancelled'.

(D) A counterexample is 'Period 0 is not cancelled and there is an excursion'.

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (17 marks) **Marks**

(a) Find $\dfrac{-3+i}{1-2i}$, showing all working out. **2**

(b) Given that $w = 2 - i\left(2\sqrt{3}\right)$ and $\text{Arg}(zw) = \dfrac{\pi}{3}$, find the exact value of $\text{Arg}(z)$. **2**

(c) The complex numbers $z_1 = 2 - 3i$, z_2 and z_3 are roots of the polynomial **2**

$$p(z) = z^3 - 8z^2 + 29z - 52$$

Find z_2 and z_3.

(d) Find the square roots of $21 - 20i$. **2**

(e) Prove that $\dfrac{3}{2}k^2 - 6k - \dfrac{63}{2}$ is divisible by 12 for all positive odd integer k. **2**

(f) i. By decomposing $\dfrac{9x}{(x^2+2)(x+1)}$ into partial fractions, show that **2**

$$\frac{9x}{(x^2+2)(x+1)} = \frac{3x+6}{x^2+2} - \frac{3}{x+1}$$

ii. Hence find $\displaystyle\int \frac{9x}{(x^2+2)(x+1)}\,dx$. **2**

(g) Using an appropriate substitution, find $\displaystyle\int \frac{2}{1+\sqrt{2x}}\,dx$. **3**

Question 12 (15 marks) **Marks**

(a) Consider the statement:

For all irrational numbers p, the number $\left(\frac{1}{3} - 5p\right)$ is irrational.

i. State the negation of the statement above. **1**

ii. Hence prove the given statement by contradiction. **2**

(b) Rosa's office chair oscillates along a straight path with an amplitude of 3 cm.

The acceleration $\ddot{x}$ cm/s^2 of the chair is

$$\ddot{x} = 20 - 4x$$

where x is the displacement of the chair from her desk.

i. Briefly explain why the chair is undergoing simple harmonic motion. **1**

ii. By integrating $\ddot{x}$ with respect to x, show that the velocity of the chair is given by **2**

$$v^2 = -4\Big((x-5)^2 - 9\Big)$$

iii. Find the fastest speed that the chair reaches. **1**

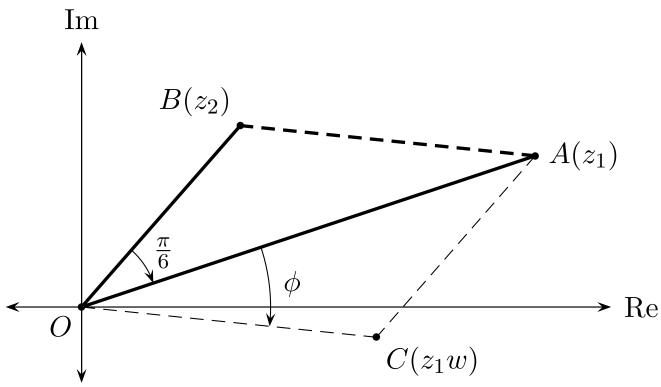
Given that the chair starts at the position closest to the desk:

iv. Determine a possible equation for the displacement of the chair. **1**

v. Determine whether the chair is moving towards or away from the desk after $\frac{37\pi}{24}$ seconds. Justify your answer. **2**

Examination continues overleaf...

(c) The complex numbers z_1 and z_2 are represented by the points A and B respectively. In $\triangle OAB$, $OA = 2OB$ and $\angle AOB = \frac{\pi}{6}$ as shown in the Argand diagram.



It is given that $z_1 = r\left(\cos\theta + i\sin\theta\right)$, where $r \in \mathbb{R}$ and $0 < \theta < \dfrac{\pi}{2}$.

i. Express z_2 in exponential form, in terms of r and θ. **2**

Let C be the point such that $ABOC$ is a parallelogram and $\angle AOC = \phi$, where $0 < \phi < \dfrac{\pi}{2}$. Consider the complex number w for which $z_1 w$ represents C.

ii. By finding w, calculate ϕ correct to two decimal places. **3**

Question 13 (14 marks) **Marks**

(a) Consider the region defined by

$$|z - 2 + 2i| \leq 4 \quad \text{and} \quad -\frac{\pi}{3} \leq \text{Arg}(z - 1 + 3i) \leq \frac{\pi}{4}$$

i. On an Argand diagram, clearly sketch the region above. **3**

Let w represent any complex number that lies in the given region.

ii. Find the possible range of values for $\text{Arg}(w)$, giving your answer correct to two decimal places. **3**

(b) Prove that $|a - b| + |c - b| + |c - d| \geq a - d$. **2**

(c) By considering the contrapositive, prove that for all $a, b \in \mathbb{Z}^+$ and where $n \in \mathbb{Z}$, **2**

If $a + b \geq 2n + 1$, then $a \geq n + 1$ or $b \geq n + 1$.

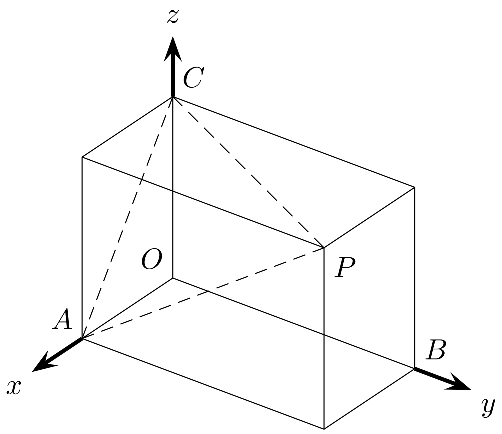
(d) Use mathematical induction to prove that, for all integers $n \geq 3$, **4**

$$(3n)! < (n!)^3 \times 27^n$$

Examination continues overleaf...

Question 14 (14 marks) **Marks**

(a) Consider the rectangular prism in the diagram. **3**



Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be the non-zero position vectors of the points A, B and C.

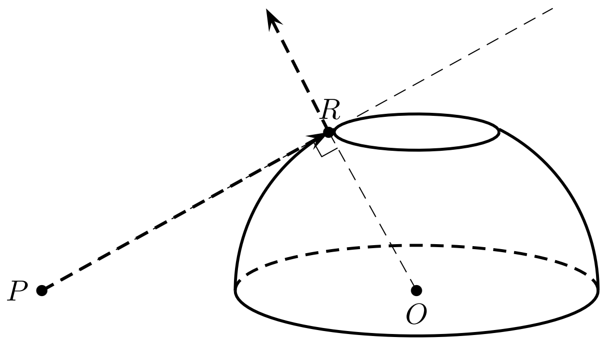
Using vector methods, prove that triangle ACP **cannot** be a right-angled triangle.

(b) Find the exact value of **4**

$$\int_{3\sqrt{2}}^{6} \frac{dx}{x^3\sqrt{x^2-9}}$$

(c) A rock band is performing at a small arena which is shaped as a hemi-sphere and has a diameter of 26 metres. Let the centre of the arena be the origin O.

A mirror is place at point $R(a, b, c)$ on the circular opening of the arena's roof, which is h metres above the ground. As a part of the lighting effects, a light will be shone from point $P(-46, 5, 0)$ outside the arena, to the mirror at R.



i. Given that $PR \perp OR$, prove that $46a - 5b + 169 = 0$. **2**

ii. If the mirror at R is directly above a seat in the arena located at $(k, -3, 0)$, find the coordinates of R. **2**

The mirror is angled so that the light is reflected upwards in the direction of $\overrightarrow{OR}$. Fireworks are shot along the path

$$\underset{\sim}{f} = \begin{pmatrix} -14 \\ -10 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \qquad , \qquad \mu \in \mathbb{R}$$

iii. Determine whether the trajectory of the fireworks will cross through the reflected ray of light. **3**

Examination continues overleaf...

Question 15 (16 marks) **Marks**

(a) Let a, b and c be positive real numbers.

i. Prove that $a^2 + b^2 + c^2 \geq ab + ac + bc$. **2**

ii. Hence or otherwise, prove that $a^3 + b^3 + c^3 \geq 3abc$. **1**

iii. Hence prove that $ac(a+c) + ab(a+b) + bc(b+c) \geq 6abc$. **2**

(b) Consider the following series:

$$C = 1 + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta$$
$$S = \sin\theta + \sin 2\theta + \cdots + \sin(n-1)\theta$$

Given that $z = \cos\theta + i\sin\theta$:

i. Show that $C + iS = \dfrac{1 - z^n}{1 - z}$ where $z \neq 1$. **2**

ii. Hence show that **2**

$$\sin\theta + \sin 2\theta + \cdots + \sin(n-1)\theta = \frac{(\sin\theta - \sin n\theta) + \sin\left[(n-1)\theta\right]}{2(1-\cos\theta)}$$

(c) Let $I_{3n+2} = \displaystyle\int_0^1 x^{3n+2} e^{x^3}\, dx$, where $n \in \mathbb{Z}$ and $n \geq 0$.

i. Prove that $I_{3n+2} = \dfrac{e}{3} - nI_{3n-1}$. **2**

ii. Hence find I_8 as an exact value. **2**

iii. Using part (i), prove that **3**

$$3\int_0^1 (1 + x^3)x^{3n-1}e^{x^3}\, dx \leq e$$

Question 16 (14 marks) **Marks**

(a) i. By considering the a possible graph of $y = f(x)$, prove that: **3**

If f is a continuous function where $f'(x) > 0$, $f''(x) < 0$ and $f(0) = 0$, then

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy \geq ab$$

where f^{-1} is the inverse function of f, $a \geq 0$ and $b \geq 0$.

ii. Hence prove that **4**

$$\tan^{-1}\left(\frac{1}{2}\right) \geq \frac{\pi}{6} + \ln\left(\frac{15}{16}\right)$$

(b) A object with a mass of 2 kg is at a height of 90 metres above the ground. It is dropped from rest and falls vertically downwards through a gaseous medium which exerts a resistance to the motion equal to $\dfrac{v^2}{45}$ newtons, where v ms^{-1} is the velocity of the particle.

After falling 25 metres, the object exits the gaseous medium and continues to fall vertically downwards, subject to air resistance equivalent to $\dfrac{v}{10}$ newtons.

The acceleration due to gravity is 10 ms^{-2}.

i. Show that object exits the gaseous medium approximately 2.34 seconds after it was dropped. **4**

At the same time that the first object was dropped, a identical second object is launched from the ground directly below. The second object is launched vertically upwards at a speed of 40 ms^{-1} such that it will collide with the first object.

ii. Determine in which medium would the two object collide. Justify your answer with the appropriate calculations. **3**

End of paper.

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Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

NESA STUDENT #: ....

Class (please ✔)

◯ 12MXX.1 – Ms Ham ◯ 12MXX.2 – Mr Ho

Directions for multiple choice answers

- Read each question and its suggested answers.
- Select the alternative Ⓐ, Ⓑ, Ⓒ, or Ⓓ that best answers the question.
- Mark only one circle per question. There is only *one* correct choice per question.
- Fill in the response circle completely, using blue or black pen, e.g.

  Ⓐ Ⓑ ● Ⓓ

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

  Ⓐ Ⓑ ⊗ ●

- If you continue to change your mind, write the word `correct` and clearly indicate your final choice with an arrow as shown below:

  Ⓐ Ⓑ ⊗ ⊗ correct

1 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**6 –**	Ⓐ	Ⓑ	Ⓒ	Ⓓ
2 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**7 –**	Ⓐ	Ⓑ	Ⓒ	Ⓓ
3 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**8 –**	Ⓐ	Ⓑ	Ⓒ	Ⓓ
4 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**9 –**	Ⓐ	Ⓑ	Ⓒ	Ⓓ
5 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ	**10 –**	Ⓐ	Ⓑ	Ⓒ	Ⓓ

Sample Band E4 Responses

1. (D) **2.** (B) **3.** (C) **4.** (B) **5.** (D)
6. (A) **7.** (C) **8.** (D) **9.** (D) **10.** (B)

Note: Working out is not marked for MCQ, though solutions are provided as a learning opportunity on mathematical writing.

Question 1

If P, Q and R are collinear, $2\overrightarrow{PQ} = \overrightarrow{QR}$:

$$\begin{aligned} -8 &= \lambda + 2 \\ 16 &= \mu - 5 \end{aligned}$$

$$\therefore \quad \lambda = -10 \quad \text{and} \quad \mu = 21$$

Correct option: (D)

Question 2

Using the direction vectors of lines $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ respectively:

$$\cos\theta = \frac{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}}{\sqrt{(-1)^2 + 2^2 + 1^2}\sqrt{3^2 + (-4)^2 + (-2)^2}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-13}{\sqrt{174}}\right)$$

Correct option: (B)

Question 3

- Six unique sixth roots of unity, evenly spaced apart by $\dfrac{\pi}{3}$ $\quad \rightarrow \quad$ not (B) or (D)
- $e^{i0} = 0$ is a root of $z^6 = 1$
- $e^{-i\pi} = e^{i\pi} = -1$ for $k = -3$ and $k = 3$ $\quad \rightarrow \quad$ not (A)

Correct option: (C)

Question 4

Negating each component:

$$\begin{array}{ccc} \forall x \in \mathbb{R} & \to & \exists x \in \mathbb{R} \\ \exists y \in \mathbb{R} & \to & \forall y \in \mathbb{R} \\ x^y > 1 & \to & x^y \leq 1 \end{array}$$

Negation: $\exists x \in \mathbb{R}, \quad \forall y \in \mathbb{R}$ such that $x^y \leq 1$

Correct option: (B)

Question 5

Testing option (B) and (D) and splitting the fraction:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{d}{dx}\left(-\frac{6}{x^2} + \frac{4}{x} + C\right) \\ &= \frac{12}{x^3} - \frac{4}{x^2} \\ a &= \frac{12-4x}{x^3} \end{aligned}$$

$$\begin{aligned} \frac{1}{2}v^2 &= -\frac{6}{x^2} + \frac{4}{x} + C \\ v^2 &= -\frac{12}{x^2} + \frac{8}{x} + C \end{aligned}$$

When $x = 6$, $v = 0$:

$$\begin{aligned} &v^2 = -\frac{12}{6^2} + \frac{8}{6} + C \\ &0 = 1 + C \\ &C = 0 \end{aligned}$$

$$\therefore \quad v^2 = -\frac{12}{6^2} + \frac{8}{6} - 1$$

Correct option: (D)

Question 6

Given $\omega^3 = 1$:

$$\begin{aligned}(\omega - 1)(\omega^2 + \omega + 1) &= 0 \\ \omega^2 + \omega + 1 &= 0 && \text{(given } \mathrm{Im}(\omega) \neq 0\text{)} \\ \omega + 1 &= -\omega^2\end{aligned}$$

$$\begin{aligned}\left(1 + 2\omega - \omega^2 + \omega^3\right)^5 &= \left(1 + 2\omega - \omega^2 + 1\right)^5 \\ &= \left(-2\omega^2 - \omega^2\right)^5 \\ &= -243\omega^{10} \\ &= -243\Big(\omega \cdot \left(\omega^3\right)^2\Big) \\ &= -243\omega\end{aligned}$$

Correct option: (A)

Question 7

$$\begin{aligned}\frac{8\overline{z}z}{(\overline{z} - z)^2} &= \frac{8|z|^2}{[-2\mathrm{Im}(z)i]^2} \\ &= \frac{8\left([\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2\right)}{-4\,[\mathrm{Im}(z)]^2} \\ &= -2\left(\frac{[\mathrm{Re}(z)]^2}{[\mathrm{Im}(z)]^2} + 1\right) \\ &= -2 - 2\left(\frac{\mathrm{Re}(z)}{\mathrm{Im}(z)}\right)^2\end{aligned}$$

Correct option: (C)

Question 8

In the given graph, as $z \to \infty$, $x \to 0^+$ and $y \to 0^+$. By assuming $z \to \infty$ in the given options:

- Option (A): $t \to -\infty$, but then $x \to -\infty$.
- Option (B): $t \to \infty$, but then $x \to \infty$.
- Option (C): $t \to \infty$, but then $y \to 0^-$.
- Option (D): $t \to -\infty$, and $x \to 0^+$ and $y \to 0^+$.

Correct option: (D)

Question 9

Using the property $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{x}{1+\sin x+\cos x}\,dx &= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{1+\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{1+\cos x+\sin x}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}}{1+\sin x+\cos x}\,dx - \int_0^{\frac{\pi}{2}} \frac{x}{1+\sin x+\cos x}\,dx
\end{aligned}$$

Correct option: (D)

Question 10

Let:

- p be 'Period 0 is cancelled'
- q be 'There is an excursion'

Rewriting the statement: $(p \Rightarrow q) \wedge (q \Rightarrow p)$

Negation of the statement: $(p \wedge \ q) \vee (q \wedge \ p)$

Correct option: (B)

Question 11 ( )

(a) (2 marks)

✓ [1] for an equivalent fraction using the conjugate of the denominator

✓ [1] for the correct answer

$$\begin{aligned}\frac{-3+i}{1-2i} &= \frac{(-3+i)(1+2i)}{(1-2i)(1+2i)} \quad \checkmark\\ &= \frac{-5-5i}{1+4}\\ &= -1-i \quad \checkmark\end{aligned}$$

(b) (2 marks)

✓ [1] for stating $\text{Arg}(z_2) + \text{Arg}(z_2) = \text{Arg}(z_2 z_3)$

✓ [1] for the correct answer

$$\begin{aligned}\text{Arg}(z_2) + \text{Arg}(z_3) &= \text{Arg}(z_2 z_3) \quad \checkmark\\ \pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) + \text{Arg}(z_3) &= \frac{\pi}{3}\\ \frac{2\pi}{3} + \text{Arg}(z_3) &= \frac{\pi}{3}\\ \therefore \quad \text{Arg}(z_3) &= -\frac{\pi}{3} \quad \checkmark\end{aligned}$$

(c) (2 marks)

✓ [1] for one correct root

✓ [1] for all correct roots with sufficient working out

$2+3i$ is also a root since $p(z)$ real coefficients. ✓

By sum of roots:

$$\begin{aligned}(2-3i)+(2+3i)+z_3 &= -(-8)\\ z_3 &= 4 \quad \checkmark\end{aligned}$$

$\therefore$ Roots z_2 and z_3 are $2-3i$ and $z_3 = 4$.

(d) (2 marks)

- ✓ [1] for correctly equating the real and imaginary components
- ✓ [1] for all correct answers

Let $(x+iy)^2 = 21 - 20i$.

$$x^2 - y^2 + (2xy)i = 21 - 20i$$

Equating real and imaginary components:

$$x^2 - y^2 = 21$$

$$\begin{aligned} 2xy &= -20 \\ xy &= -10 \qquad \checkmark \end{aligned}$$

By inspection: $x = \pm 5$ and $y = \mp 2$

$\therefore$ The square roots are $5 - 2i$ and $-5 + 2i$. ✓

(e) (2 marks)

- ✓ [1] for correctly substituting an expression for k as an odd integer
- ✓ [1] for correctly proving the result with sufficient justifications

Let $k = 2a + 1$, where $a \in \mathbb{Z}$.

$$\begin{aligned} \frac{3}{2}k^2 - 6k - \frac{63}{2} &= \frac{3}{2}\Big((2a+1)^2 - 4(2a+1) - 21\Big) \qquad \checkmark \\ &= \frac{3}{2}\Big(4a^2 - 4a - 24\Big) \\ &= 6(a^2 - a - 6) \\ &= 6\Big(a(a-1) - 6\Big) \\ &= 6(2m - 6) \qquad \text{where } m \in \mathbb{Z} \\ &\qquad \text{(since one of the consecutive integers } a-1 \text{ or } a \text{ must be negative)} \\ &= 12(m - 3) \qquad \text{where } m - 3 \in \mathbb{Z} \end{aligned}$$

$\therefore$ $\dfrac{3}{2}k^2 - 6k - \dfrac{63}{2}$ is divisible by 12 for any positive odd integer k. ✓

(f) i. (2 marks)

✓ [1] for significant progress

✓ [1] for correctly showing the required result

$$\frac{9x}{(x^2+2)(x+1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x+1}$$
$$9x = (Ax+B)(x+1) + C(x^2+2)$$

Let $x=-1$: $\quad -9 = 3C \quad \to \quad C = -3$ ✓

Let $x=0$: $\quad 0 = B - 6 \quad \to \quad B = 6$

Let $x=1$: $\quad 9 = 2(A+6) - 9 \quad \to \quad A = 3$

$$\therefore \quad \frac{9x}{(x^2+2)(x+1)} = \frac{3x+6}{x^2+2} - \frac{3}{x+1}$$

ii. (2 marks)

✓ [2] for the correct integral

$$\begin{aligned}\int \frac{9x}{(x^2+2)(x+1)}\,dx &= \int \frac{3x+6}{x^2+2} - \frac{3}{x+1}\,dx\\ &= \int \frac{3x}{x^2+2} + \frac{6}{x^2+2} - \frac{3}{x+1}\,dx\\ &= \frac{3}{2}\ln|x^2+2| + \frac{6}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - 3\ln|x+1| + C \qquad ✓✓\end{aligned}$$

(g) (3 marks)

✓ [1] for identifying a correct substitution

✓ [1] for correctly substituting the x variable for the u variable

✓ [1] for the correct answer

Let $u = 1 + \sqrt{2x}$. ✓

$$\frac{du}{dx} = \frac{1}{2}\cdot\frac{2}{\sqrt{2x}}$$
$$\sqrt{2x}\,du = dx$$

$$\begin{aligned}\int \frac{2}{1+\sqrt{2x}}\,dx &= 2\int \frac{1}{u}\cdot\sqrt{2x}\,du\\ &= 2\int \frac{u-1}{u}\,du\\ &= 2\int 1 - \frac{1}{u}\,du \qquad ✓\\ &= 2u - 2\ln|u| + C\\ &= 2 + 2\sqrt{2x} - 2\ln\left(1+\sqrt{2x}\right) + C \qquad ✓\end{aligned}$$

Question 12 ( )

(a) i. (1 mark)

✓ [1] for the correct negation

There exists an irrational number p such that $\dfrac{1}{3} - 5p$ is rational. ✓

ii. (2 marks)

✓ [1] for correctly expressing p as a rational number

✓ [2] for correctly explaining the contradiction and thus proving by contradiction.

Assume the negation from part (i) is true, i.e. $p \notin \mathbb{Q}$ and $\dfrac{1}{3} - 5p \in \mathbb{Q}$.

Let $a, b \in \mathbb{Z}$, $b \neq 0$ and HCF of a and b be is 1, such that

$$\begin{aligned} \frac{1}{3} - 5p &= \frac{a}{b} \\ -5p &= \frac{3a - b}{3b} \\ p &= \frac{b - 3a}{15b} \qquad ✓ \end{aligned}$$

So p is always rational since $b - 3a \in \mathbb{Z}$ and $15b \in \mathbb{Z}$, given that $a, b \in \mathbb{Z}$ (and $b \neq 0$).

But $p \notin \mathbb{Q}$ from assumption.

Hence by contradiction, the given statement is true. ✓

(b) i. (1 mark)

✓ [1] for correctly showing the acceleration can be expressed as $\ddot{x} = -n^2(x-c)$

$$\begin{aligned}\ddot{x} &= 20 - 4x \\ &= -2^2(x-5) \quad \checkmark\end{aligned}$$

$\therefore$ The chair in SHM as its acceleration can be expressed in the form $\ddot{x} = -n^2(x-c)$ with a centre of motion $c = 5$ and frequency $n = 2$.

ii. (2 marks)

✓ [1] for the correct indefinite integral of $\frac{1}{2}v^2$ in terms of x

✓ [2] for the correct final answer

At extremeties $x = 5 \pm 3$, $v = 0$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= 20 - 4x \\ \int_0^v d\left(\frac{1}{2}v^2\right) &= \int_2^x 20 - 4x\, dx \\ \left[\frac{1}{2}v^2\right]_0^v &= \left[20x - 2x^2\right]_2^x \quad \checkmark \\ \frac{1}{2}v^2 &= 20x - 2x^2 - 32 \\ v^2 &= 40x - 4x^2 - 64 \\ &= -4(x^2 - 10x + 16) \\ &= -4(x^2 - 10x + 25 - 9) \\ \therefore \quad v^2 &= -4\big((x-5)^2 - 9\big) \quad \checkmark\end{aligned}$$

iii. (1 mark)

✓ [1] for the correct answer

Fastest speed occurs at $x = 5$.

$$\begin{aligned}v^2 &= 36 \qquad \text{(from ii.)} \\ |v| &= 6\end{aligned}$$

$\therefore$ The fastest speed reached is 6 metres.

iv. (1 mark)

✓ [1] for a correct displacement equation

Given $a = 3$, $n = 2$, $c = 5$ and $x(0) = 2$.

$$x = -3\cos(2t) + 5 \quad \checkmark$$

v. (2 marks)

✓ [1] for calculating the velocity at $t = \frac{37\pi}{24}$, or describing the motion using $T = \pi$

✓ [1] for the correct conclusion from relevant working out

$$\dot{x} = 6\sin(2t)$$

When $t = \dfrac{37\pi}{24}$,

$$\dot{x} = -1.55\ldots \qquad ✓$$

$\therefore$ The particle is moving towards the desk. ✓

Alternate method:

The motion has a period of π seconds, and $\dfrac{3\pi}{2} < \dfrac{37\pi}{24} < 2\pi$.

The chair starts at the minimum displacement $x = 2$.

After exactly $\dfrac{3\pi}{2}$ seconds, it will be at the maximum displacement $x = 8$. ✓

Hence the chair will be moving towards the desk at $t = \dfrac{37\pi}{24}$ seconds. ✓

(c) i. (2 marks)

✓ [1] for multiplying to a correct modulus OR argument

✓ [1] for the correct answer in exponential form

$$z_1 = re^{i\theta}$$

$$z_2 = re^{i\theta} \times \frac{1}{2}e^{i\frac{\pi}{6}} \qquad ✓$$

$$\therefore \quad z_2 = \frac{1}{2}re^{i\left(\theta + \frac{\pi}{6}\right)} \qquad ✓$$

ii. (3 marks)

✓ [1] for a correct expression for $z_1 w$

✓ [1] for finding w

✓ [1] for the correct answer

Opposite sides of a parallelogram are equal and parallel:

$$\overrightarrow{BA} = \overrightarrow{OC}$$

$$\begin{aligned}
z_1 w &= z_1 - z_2 \quad ✓ \\
&= re^{i\theta} - \frac{1}{2}re^{i\left(\theta+\frac{\pi}{6}\right)} \\
re^{i\theta} w &= re^{i\theta}\left(1 - \frac{1}{2}e^{i\frac{\pi}{6}}\right) \\
w &= 1 - \frac{1}{2}e^{i\frac{\pi}{6}} \quad ✓ \\
&= 1 - \frac{1}{2}\cos\frac{\pi}{6} - \frac{1}{2}i\sin\frac{\pi}{6} \\
&= \left(1 - \frac{\sqrt{3}}{4}\right) - \frac{1}{4}i \\
\arg(w) &= -0.42 \text{ radians} \\
\therefore \quad \phi &= 0.42 \text{ radians} \quad ✓
\end{aligned}$$

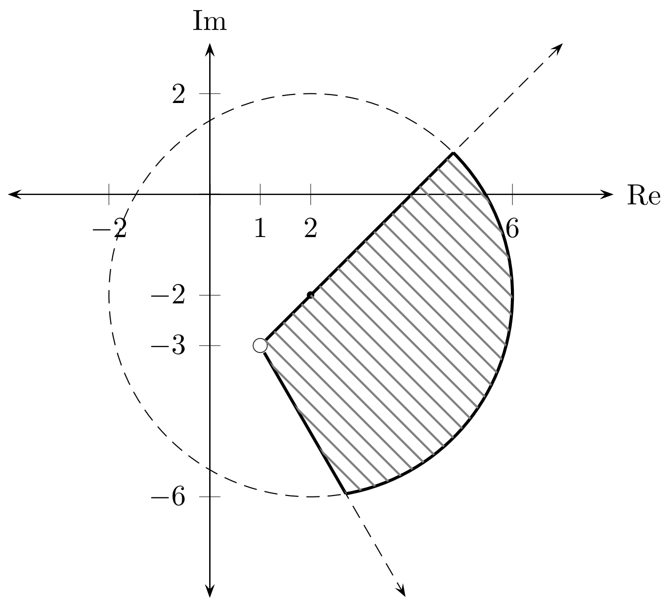
Question 13 ( )

(a) i. (3 marks)

✓ [1] for the correct circle representing $|z - 2 + 2i| \leq 4$

✓ [1] for the correct rays representing $-\frac{\pi}{3} \leq \operatorname{Arg}(z - 1 + 3i) \leq \frac{\pi}{4}$ with $(1 - 3i)$ excluded

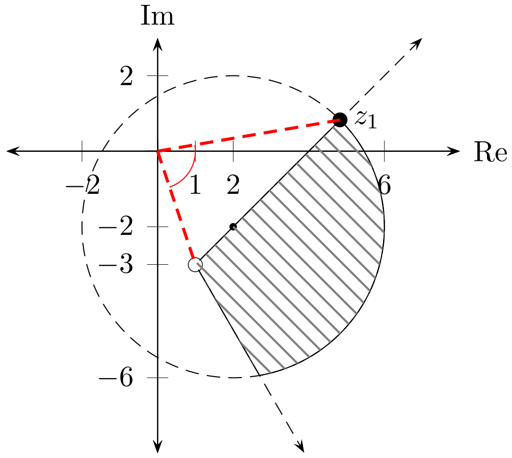
✓ [1] for the correct shaded region



ii. (3 marks)

- ✓ [1] for identifying the values of w for which $\text{Arg}(w)$ is a minimum and maximum
- ✓ [1] for the correct minimum or maximum value of $\text{Arg}(w)$
- ✓ [1] for the correct range of values for $\text{Arg}(w)$ with an exclusive lower limit

Let z_1 be a complex number such that $|z_1 - 2 + 2i| = 4$ and $\text{Arg}(z_1 - 1 + 3i) = \dfrac{\pi}{4}$.



For z_1:

$$(x-2)^2 + (y+2)^2 = 16 \tag{1}$$
$$y = x - 4 \tag{2}$$

Substitute (2) into (1):

$$\begin{aligned}(x-2)^2 + (x-2)^2 &= 16\\ (x-2)^2 &= 8\\ x &= 2 + 2\sqrt{2} \qquad \text{(since } \text{Re}(z_1) > 0)\end{aligned}$$

Substitute $x = 2 + 2\sqrt{2}$ into (3):

$$y = -2 + 2\sqrt{2}$$

$$\begin{aligned}z_1 &= (2 + 2\sqrt{2}) + i(-2 + 2\sqrt{2})\\ \text{Arg}(z_1) &= 0.17 \quad \text{radians} \qquad ✓\end{aligned}$$

From the Argand diagram:

$$\begin{aligned}&\text{Arg}(1 - 3i) < \text{Arg}(w) \le \text{Arg}(z_1) \qquad ✓\\ \therefore\quad &-1.25 < \text{Arg}(w) \le 0.17 \qquad ✓\end{aligned}$$

(b) (2 marks)

✓ [1] for correctly applying the triangle inequality once

✓ [1] for correctly proving the required result with justification for $|a-d| \geq a-d$

$$\textbf{RTP:} \qquad |a-b|+|c-b|+|c-d| \geq a-d$$

$$\begin{aligned}
|a-b|+|c-b|+|c-d| &= \Big(|a-b|+|b-c|\Big)+|c-d| \\
&\geq |a-b+b-c|+|c-d| && \text{(triangle inequality)} \\
&= |a-c|+|c-d| \\
&\geq |a-d| && \text{(triangle inequality)}
\end{aligned}$$

$|a-d| \geq a-d$ since:

- if $a \geq d$, then $|a-d| = a-d$
- if $a < d$, then $|a-d| > a-d$

$$\therefore \quad |a-b|+|c-b|+|c-d| \geq a-d$$

(c) (2 marks)

✓ [1] for the correct contrapositive

✓ [1] for correctly proving the required result with sufficient conclusions

Contrapositive: If $a < n+1$ and $b < n+1$, then $a+b < 2n+1$

for all a, $b \in \mathbb{Z}$ and where $n \in \mathbb{Z}$

Since $a, b \in \mathbb{Z}$, $a \leq n$ and $b \leq n$.

$$\begin{aligned}
a+b &\leq 2n \\
&< 2n+1
\end{aligned}$$

$\therefore$ The contrapositive is true.

Hence for all $a, b \in \mathbb{Z}$ and where $n \in \mathbb{Z}$, if $a+b \geq 2n+1$, then $a \geq n+1$ or $b \geq n+1$.

(d) (3 marks)

✓ [1] for finding r_{AM}, or equivalent merit.

✓ [2] for substantial progress in setting up a system of equations by equating $\underset{\sim}{r}_{AM}$ with $\underset{\sim}{r}_{BT}$.

✓ [3] for fully correct solution and justification.

Let $P(n)$ be the proposition.

Base case: For $n = 3$,

$$\begin{aligned}\text{LHS} &= 9! \\ &= 362\,880\end{aligned}$$

$$\begin{aligned}\text{RHS} &= (3!)^3 \times 27^3 \\ &= 4\,251\,528\end{aligned}$$

$\therefore$ LHS < RHS, and $P(3)$ is true.

Inductive step: Assume $P(k)$ is true.

$$(3k)! < (k!)^3 \times 27^k$$

Consider $n = k + 1$:

RTP: $(3k+3)! < ((k+1)!)^3 \times 27^{k+1}$

$$\begin{aligned}\text{LHS} &= (3k+3)(3k+2)(3k+1)\cdot(3k)! \\ &< (3k+3)(3k+2)(3k+1)\left((k!)^3 \times 27^k\right) && \text{(from assumption)} \\ &= 3^3(k+1)\left(k+\frac{2}{3}\right)\left(k+\frac{1}{3}\right)\cdot(k!)^3 \times 27^k \\ &= (k+1)\left(k+\frac{2}{3}\right)\left(k+\frac{1}{3}\right)\cdot(k!)^3 \times 27^{k+1} \\ &< (k+1)^3(k!)^3 \times 27^{k+1} && (\text{since } k \geq 3) \\ &= ((k+1)!)^3 \times 27^{k+1}\end{aligned}$$

$\therefore$ LHS < RHS, and $P(k+1)$ is true if $P(k)$ is true.

Hence by mathematical induction, $P(n)$ is true for $n \geq 3$.

Question 14 ( )

(a) (3 marks)

- ✓ [1] for correctly showing that one pair of sides cannot be perpendicular OR significant progress in showing no interiors can be right-angles
- ✓ [1] for correctly showing that two pairs of sides cannot be perpendicular
- ✓ [1] for correctly proving the required result

Let $\overrightarrow{OA} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$ where $a, b, c \in \mathbb{Z}^+$.

$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{AP} &= \begin{pmatrix} -a \\ 0 \\ c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} \\ &= c^2 \end{aligned}$$

$\therefore$ AC cannot be perpendicular to AP since c is non-zero. ✓

$$\begin{aligned} \overrightarrow{CA} \cdot \overrightarrow{CP} &= \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \\ &= a^2 \end{aligned}$$

$\therefore$ CA cannot be perpendicular to CP since a is non-zero. ✓

$$\begin{aligned} \overrightarrow{PA} \cdot \overrightarrow{PC} &= \begin{pmatrix} 0 \\ -b \\ -c \end{pmatrix} \cdot \begin{pmatrix} -a \\ -b \\ 0 \end{pmatrix} \\ &= b^2 \end{aligned}$$

$\therefore$ PA cannot be perpendicular to PC since b is non-zero.

Hence triangle ACP cannot be a right-angled triangle. ✓

Alternate method:

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

Since coordinate axes are perpendicular to each other,

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$$

$$\begin{aligned}\overrightarrow{AC} \cdot \overrightarrow{AP} &= (\underset{\sim}{c} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} + \underset{\sim}{c}) \\ &= \underset{\sim}{b} \cdot \underset{\sim}{c} + |\underset{\sim}{c}|^2 - \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{c} \\ &= |\underset{\sim}{c}|^2\end{aligned}$$

$\therefore$ AC cannot be perpendicular to AP since $|\underset{\sim}{c}|$ is non-zero. ✓

$$\begin{aligned}\overrightarrow{CA} \cdot \overrightarrow{CP} &= (\underset{\sim}{a} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) \\ &= |\underset{\sim}{a}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{c} \\ &= |\underset{\sim}{a}|^2\end{aligned}$$

$\therefore$ CA cannot be perpendicular to CP since $|\underset{\sim}{a}|$ is non-zero. ✓

$$\begin{aligned}\overrightarrow{PA} \cdot \overrightarrow{PC} &= (-\underset{\sim}{b} - \underset{\sim}{c}) \cdot (-\underset{\sim}{a} - \underset{\sim}{b}) \\ &= \underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{b}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c} \\ &= |\underset{\sim}{b}|^2\end{aligned}$$

$\therefore$ PA cannot be perpendicular to PC since $|\underset{\sim}{b}|$ is non-zero.

Hence triangle ACP cannot be a right-angled triangle. ✓

(b) (4 marks)

- ✓ [1] for selecting a correct substitution with significant progress
- ✓ [1] for correctly simplifying the expression after substitution
- ✓ [1] for the correct integral from the double angle formula of $\cos^2\theta$
- ✓ [1] for the correct final answer

Let $x = 3\sec\theta$.

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{-3(-\sin\theta)}{\cos^2\theta}\\ dx &= 3\sec\theta\tan\theta\,d\theta \qquad \checkmark\end{aligned}$$

$$\begin{aligned} x = 6 \qquad &\rightarrow \qquad \theta = \frac{\pi}{3}\\ x = 3\sqrt{2} \qquad &\rightarrow \qquad \theta = \frac{\pi}{4}\end{aligned}$$

$$\begin{aligned}\int_{3\sqrt{2}}^{6} \frac{dx}{x^3\sqrt{x^2-9}} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3\sec\theta\tan\theta}{27\sec^3\theta\sqrt{9\sec^2\theta-9}}\,d\theta\\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan\theta}{9\sec^2\theta(3\tan\theta)}\,d\theta\\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{27\sec^2\theta}\,d\theta \qquad \checkmark\\
&= \frac{1}{27}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\,d\theta\\
&= \frac{1}{27}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2}\left(\cos 2\theta + 1\right)\,d\theta\\
&= \frac{1}{54}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \qquad \checkmark\\
&= \frac{1}{54}\left[\left(\frac{\sqrt{3}}{4}+\frac{\pi}{3}\right) - \left(\frac{1}{2}+\frac{\pi}{4}\right)\right]\\
&= \frac{1}{54}\left(\frac{\sqrt{3}-2}{4}+\frac{\pi}{12}\right) \qquad \checkmark\\
&= \frac{\pi+3\sqrt{3}-6}{648}\end{aligned}$$

(c) i. (2 marks)

✓ [1] for correctly simplifying $\overrightarrow{PR} \cdot \overrightarrow{OR}$

✓ [1] for correctly showing the required result by considering the sphere

$$\begin{aligned}\overrightarrow{PR} &= \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} -46 \\ 5 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a+46 \\ b-5 \\ c \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} \cdot \overrightarrow{OR} &= 0 && \text{(given } PR \perp OR\text{)} \\ \begin{pmatrix} a+46 \\ b-5 \\ c \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= 0 \\ a^2 + 46a + b^2 - 5b + c^2 &= 0 \quad \checkmark && (1)\end{aligned}$$

Since R lies on the sphere:

$$\begin{aligned}x^2 + y^2 + z^2 &= 13^2 \\ a^2 + b^2 + c^2 &= 169 && (2)\end{aligned}$$

Substitute (2) into (1):

$$\therefore \quad 46a - 5b + 169 = 0 \quad \checkmark$$

ii. (2 marks)

✓ [1] for the correct x-coordinate for R

✓ [1] for the correct coordinates for R

Let $b = -3$. From (i):

$$\begin{aligned}46a + 15 + 169 &= 0 \\ a &= -4 \quad \checkmark\end{aligned}$$

From (2):

$$\begin{aligned}(-4)^2 + (-3)^2 + c^2 &= 169 \\ c &= 12\end{aligned}$$

$\therefore \quad R(-4, -3, 12) \quad \checkmark$

iii. (3 marks)

✓ [1] for correctly equating the vector equations of the two lines

✓ [1] for a correct set of values for λ and μ from the system of equations

✓ [1] for the correct conclusion with relevant working

Let $\underset{\sim}{r}$ be the equation of line OR :

$$\underset{\sim}{r} = \lambda \begin{pmatrix} -4 \\ -3 \\ 12 \end{pmatrix} \qquad \text{where } \lambda \in \mathbb{R}$$

Let $\underset{\sim}{f} = \underset{\sim}{r}$:

$$\begin{pmatrix} -14 \\ -10 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \lambda \begin{pmatrix} -4 \\ -3 \\ 12 \end{pmatrix} \qquad \checkmark$$

Equating the components:

$$-14 + 2\mu = -4\lambda \tag{3}$$
$$-10 + \mu = -3\lambda \tag{4}$$
$$5\mu = 12\lambda \tag{5}$$

$(3) - 2 \times (4)$:

$$6 = 2\lambda$$
$$\lambda = 3$$

Substitute $\lambda = 3$ into (4):

$$\mu = 1 \qquad \checkmark$$

Substitute $\lambda = 3$ and $\mu = 1$ into (5):

$$\text{LHS} = 5$$
$$\text{RHS} = 36$$
$$\text{LHS} \neq \text{RHS}$$

$\therefore$ No values of λ and μ such that $\underset{\sim}{f} = \underset{\sim}{r}$.

Hence the trajectory of the fireworks will **NOT** cross through the reflected ray of light. ✓

Question 15 ( )

(a) i. (2 marks)

✓ [1] for correctly showing $a^2 + b^2 \geq 2ab$ or equivalent

✓ [1] for correctly showing the required result

$$(a-b)^2 \geq 0$$
$$a^2 - 2ab + b^2 \geq 0$$
$$a^2 + b^2 \geq 2ab \quad ✓ \tag{1}$$

Similarly:

$$a^2 + c^2 \geq 2ac \tag{2}$$
$$b^2 + c^2 \geq 2bc \tag{3}$$

$(1) + (2) + (3)$:

$$2(a^2 + b^2 + c^2) \geq 2(ab + ac + bc)$$
$$\therefore \quad a^2 + b^2 + c^2 \geq ab + ac + bc \quad ✓ \tag{4}$$

ii. (3 marks)

✓ [1] for correctly showing the required result

Multiply (4) from (i), by $(a + b + c)$:

$$(a + b + c)(a^2 + b^2 + c^2) \geq (a + b + c)(ab + ac + bc)$$

$$a^3+b^3+c^3+ab^2+ac^2+a^2b+bc^2+a^2c+b^2c \geq a^2b+a^2c+ab^2+b^2c+ac^2+bc^2+abc+abc+abc$$

$$\therefore \quad a^3 + b^3 + c^3 \geq 3abc \quad ✓$$

iii. (2 marks)

✓ [1] for correctly multiplying two applications of part (ii)

✓ [1] for correctly showing the required result using (ii)

$$a^3 + b^3 + c^3 \geq 3abc \tag{5}$$

Let $a \to \dfrac{1}{a}$, $b \to \dfrac{1}{b}$ and $c \to \dfrac{1}{c}$:

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{3}{abc} \tag{6}$$

Multiply (1) and (2):

$$\left(a^3 + b^3 + c^3\right)\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) \geq 9 \quad ✓$$

$$1 + 1 + 1 + \frac{b^3 + c^3}{a^3} + \frac{a^3 + c^3}{b^3} + \frac{a^3 + b^3}{c^3} \geq 9$$

$$b^3c^3(b^3 + c^3) + a^3c^3(a^3 + c^3) + a^3b^3(a^3 + b^3) \geq 6a^3b^3c^3$$

Let $a \to \sqrt[3]{a}$, $b \to \sqrt[3]{b}$ and $c \to \sqrt[3]{c}$:

$$\therefore \quad bc(b + c) + ac(a + c) + ab(a + b) \geq 6abc \quad ✓$$

(b) i. (2 marks)

✓ [1] for correctly applying De Moivre's Theorem

✓ [1] for correctly showing the required result

$$\begin{aligned} C + iS &= \big(1 + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta\big) + i\big(1 + \sin 2\theta + \cdots + \sin(n-1)\theta\big) \\ &= 1 + (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + \cdots \big(\cos(n-1)\theta + i\sin(n-1)\theta\big) \\ &= 1 + (\cos\theta + i\sin\theta) + (\cos\theta + i\sin\theta)^2 \cdots + (\cos\theta + i\sin\theta)^{n-1} \qquad \checkmark \\ &= 1 + z + z^2 + \cdots + z^{n-1} \\ &= \frac{(1-z)(1 + z + z^2 + \cdots + z^{n-1})}{1-z} \end{aligned}$$

$$\therefore \quad C + iS = \frac{1-z^n}{1-z} \qquad \checkmark$$

ii. (2 marks)

✓ [1] for correctly rationalising $C + iS$

✓ [1] for correctly equating the imaginary components to show the required result

$$\begin{aligned} C + iS &= \frac{1-z^n}{1-z} \\ &= \frac{1 - (\cos\theta + i\sin\theta)^n}{1 - (\cos\theta + i\sin\theta)} \\ &= \frac{1 - (\cos n\theta + i\sin n\theta)}{(1-\cos\theta) - i\sin\theta} \\ &= \frac{\big((1-\cos n\theta) - i\sin n\theta\big)\big((1-\cos\theta) + i\sin\theta\big)}{\big((1-\cos\theta) - i\sin\theta\big)\big((1-\cos\theta) + i\sin\theta\big)} \\ &= \frac{(1-\cos n\theta)(1-\cos\theta) + \sin n\theta\sin\theta + i\big(\sin\theta(1-\cos n\theta) - \sin n\theta(1-\cos\theta)\big)}{(1-\cos\theta)^2 + \sin^2\theta} \qquad \checkmark \end{aligned}$$

Equate imaginary components:

$$\begin{aligned} S &= \frac{\sin\theta - \cos n\theta\sin\theta - \sin n\theta + \sin n\theta\cos\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} \\ &= \frac{(\sin\theta - \sin n\theta) + \sin(n\theta - \theta)}{2 - 2\cos\theta} \end{aligned}$$

$$\therefore \quad 1 + \sin 2\theta + \cdots + \sin(n-1)\theta = \frac{(\sin\theta - \sin n\theta) + \sin\big[(n-1)\theta\big]}{2(1-\cos\theta)} \qquad \checkmark$$

(c) i. (2 marks)

✓ [1] for correctly applying integration by parts

✓ [1] for correctly proving the required result

$$\begin{aligned} \text{Let} \quad u &= x^{3n} \quad \to \quad u' = 3nx^{3n-1} \\ v' &= x^2 e^{x^3} \quad \to \quad v = \frac{1}{3}e^{x^3} \end{aligned}$$

$$\begin{aligned} I_{3n+2} &= \int_0^1 x^{3n+2} e^{x^3}\, dx \\ &= \left[\frac{1}{3}x^{3n}e^{x^3}\right]_0^1 - \int_0^1 nx^{3n-1}e^{x^3}\, dx \qquad \checkmark \\ &= \frac{1}{3}(e-0) - nI_{3n-1} \\ \therefore \quad I_{3n+2} &= \frac{e}{3} - nI_{3n-1} \qquad \checkmark \end{aligned}$$

ii. (2 marks)

✓ [1] for correctly applying the reduction formula from part (i)

✓ [1] for the correct final answer

$$\begin{aligned} I_8 &= \frac{e}{3} - 2I_5 \qquad \checkmark \\ &= \frac{e}{3} - 2\left(\frac{e}{3} - I_2\right) \\ &= -\frac{e}{3} + 2\int_0^1 x^2 e^{x^3}\, dx \\ &= -\frac{e}{3} + 2\left[\frac{1}{3}e^{x^3}\right]_0^1 \\ &= -\frac{e}{3} + \frac{2}{3}(e-1) \\ \therefore I_8 &= \frac{e-2}{3} \qquad \checkmark \end{aligned}$$

iii. (3 marks)

- ✓ [1] for correctly simplifying the giving integral
- ✓ [1] for correctly showing that $I_{3n-1} \geq 0$
- ✓ [1] for correctly proving the required result with reference to $(n-1)$

$$\begin{aligned} 3\int_0^1 (1+x^3)x^{3n-1}e^{x^3}\,dx &= 3\left(\int_0^1 x^{3n-1}e^{x^3}\,dx + \int_0^1 x^{3n+2}e^{x^3}\,dx\right) \\ &= 3(I_{3n-1} + I_{3n+2}) \\ &= 3\left(I_{3n-1} + \frac{e}{3} - nI_{3n-1}\right) && \text{(from i.)} \\ &= e - 3(n-1)I_{3n-1} \quad ✓ \end{aligned}$$

For $0 \leq x \leq 1$:

$$\begin{gathered} x^{3n-1} \geq 0 \qquad \text{(since } n \geq 1\text{)} \\ x^3 \geq 0 \quad \text{so} \quad e^{x^3} > 0 \qquad \text{(since } f(0) = 0\text{)} \\ x^{3n-1}e^{x^3} \geq 0 \end{gathered}$$

$$\begin{gathered} \therefore \int_0^1 x^{3n-1}e^{x^3}\,dx \geq 0 \\ I_{3n-1} \geq 0 \quad ✓ \\ 3(n-1)I_{3n-1} \geq 0 \qquad \text{(since } n \geq 1\text{)} \end{gathered}$$

$$\text{Hence} \quad 3\int_0^1 (1+x^3)x^{3n-1}e^{x^3} \leq e \quad ✓$$

Question 16 ( )

(a) i. (3 marks)

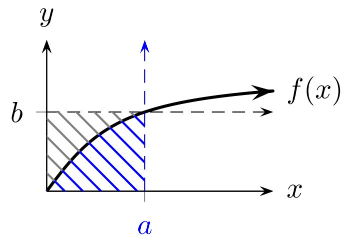
- ✓ [1] for correctly explaining one case with a graph
- ✓ [1] for correctly explaining two cases with graph(s)
- ✓ [1] for correctly proving the required result by explaining all cases

$\displaystyle\int_0^b f^{-1}(y)\,dy$ is the area bounded by $y = f(x)$, the y-axis, $y = 0$ and $y = b$:

$$f(x) = y \qquad \to \qquad x = f^{-1}(y)$$

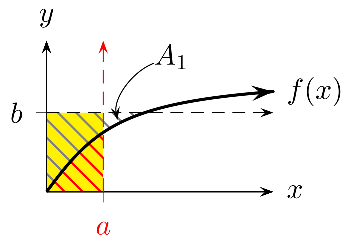
Consider the graph of a continuous function that is increasing and concave down:

Case 1: If $f(a) = b$,



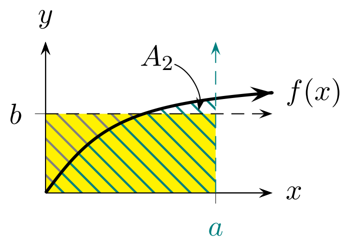
$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy = ab \qquad ✓$$

Case 2: If $f(a) < b$,



$$\begin{aligned}\int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy &= ab + A_1 \\ &> ab \qquad ✓\end{aligned}$$

Case 3: If $f(a) > b$,



$$\begin{aligned}\int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy &= ab + A_2 \\ &> ab\end{aligned}$$

Hence if f is a continuous function where $f'(x) > 0$, $f''(x) < 0$ and $f(0) = 0$, then

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy \geq ab$$

where f^{-1} is the inverse function of f, $a \geq 0$ and $b \geq 0$. ✓

Trivial case: If $a = 0$ or $b = 0$, $ab = 0$.

$$\int_0^a f(x)\,dx \geq 0 \quad \text{or} \quad \int_0^b f^{-1}(y)\,dy \geq 0$$

$$\therefore \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy \geq ab$$

ii. (4 marks)

- ✓ [1] for correctly identifying $f(x) = \tan^{-1} x$ as a suitable function
- ✓ [1] for the correct integration by parts
- ✓ [1] for a correct inequality for $\tan^{-1} a$
- ✓ [1] for correctly proving the required result

Let $f(x) = \tan^{-1} x$, which satisfies the given properties in the sufficient condition.

Note: For $\displaystyle\int_0^a \tan^{-1} x \, dx$:

$$\begin{array}{lllll} \text{Let} & u = \tan^{-1} x & \rightarrow & u' = \dfrac{1}{1+x^2} \\ & v' = 1 & \rightarrow & v = x \end{array}$$

Then by the implication from part (i):

$$\begin{aligned} \int_0^a \tan^{-1} x \, dx + \int_0^b \tan y \, dy &\geq ab \qquad \checkmark \\ \left(\Big[x \tan^{-1} x \Big]_0^a - \int_0^a \frac{x}{1+x^2} \, dy \right) + \int_0^b \frac{\sin y}{\cos y} \, dy &\geq ab \qquad \checkmark \\ a \tan^{-1} a - \frac{1}{2} \Big[\ln \left(1+x^2\right) \Big]_0^a - \Big[\ln(\cos y) \Big]_0^b &\geq ab \end{aligned}$$

(Note: $1 + x^2 > 0$ and $\cos y > 0$ for $x \in [0, a]$ and $y \in [0, b]$)

$$a \tan^{-1} a - \frac{1}{2} \ln(1+a^2) - \ln(\cos b) \geq ab \qquad \checkmark$$

$$\begin{aligned} \tan^{-1} a &\geq b + \frac{1}{a} \ln \left(\sqrt{1+a^2} \right) + \frac{1}{a} \ln(\cos b) \\ &= b + \frac{1}{a} \ln \left(\cos b \cdot \sqrt{1+a^2} \right) \end{aligned}$$

Let $a = \dfrac{1}{2}$ and $b = \dfrac{\pi}{6}$:

$$\begin{aligned} \tan^{-1}\left(\frac{1}{2}\right) &\geq \frac{\pi}{6} + 2 \ln \left(\frac{\sqrt{3}}{2} \cdot \sqrt{\frac{5}{4}} \right) \\ &= \frac{\pi}{6} + 2 \ln \left(\frac{\sqrt{15}}{16} \right) \\ \therefore \quad \tan^{-1}\left(\frac{1}{2}\right) &\geq \frac{\pi}{6} + \ln \left(\frac{15}{16} \right) \qquad \checkmark \end{aligned}$$

(b) i. (4 marks)

- ✓ [1] for the correct acceleration equation from the sum of forces
- ✓ [1] for the correct exit speed of the object
- ✓ [1] for the correct integral from $\frac{dv}{dt}$
- ✓ [1] for correctly showing the required result with justifications (resolving $\ln|f(x)|$)

Let Object A be the first object, and downwards be the positive direction of its motion.

$$\begin{aligned}\sum F_A &= mg - \frac{v^2}{45}\\ 2\ddot{x} &= 20 - \frac{v^2}{45}\\ \ddot{x}_A &= \frac{900 - v^2}{90} \qquad \checkmark \qquad (1)\end{aligned}$$

Let $v = V$ when $x = 25$:

$$\begin{aligned}v\frac{dv}{dx} &= \frac{900 - v^2}{90}\\ \int_0^V \frac{v}{900 - v^2}\,dv &= \int_0^{25} \frac{1}{90}\,dx\\ -\frac{1}{2}\Big[\ln\left|900 - v^2\right|\Big]_0^V &= \frac{1}{90}\Big[x\Big]_0^{25}\\ \ln\left(\frac{900 - V^2}{900}\right) &= -\frac{1}{45}(25 - 0) \qquad (\text{since} \quad \ddot{x}_A > 0\ , \quad 900 - v^2 > 0)\\ 900 - V^2 &= 900e^{-\frac{5}{9}}\\ V^2 &= 900\left(1 - e^{-\frac{5}{9}}\right)\\ V &= 19.5862...\ \text{ms}^{-1} \qquad \checkmark \qquad (\text{since } V > 0)\end{aligned}$$

(i.e. V is the velocity at which Object A exits the gaseous medium.)

From (1):

$$\begin{aligned}\frac{dv}{dt} &= \frac{(30 + v)(30 - v)}{90}\\ \int_0^v \frac{1}{(30 + v)(30 - v)}\,dv &= \int_0^t \frac{1}{90}\,dt\\ \int_0^v \frac{\frac{1}{60}}{30 + v} + \frac{\frac{1}{60}}{30 - v}\,dv &= \frac{1}{90}\Big[t\Big]_0^t\\ \Big[\ln|30 + v| - \ln|30 - v|\Big]_0^v &= \frac{2}{3}t \qquad \checkmark\end{aligned}$$

$$(\text{since} \quad \ddot{x} > 0 \quad \text{and} \quad v > 0\ , \quad (30 + v)(30 - v) > 0 \quad \text{so} \quad 30 - v > 0)$$

$$\begin{aligned}\ln\left(\frac{30 + v}{30 - v}\right) - \ln\left(\frac{30}{30}\right) &= \frac{2}{3}t\\ t_A &= \frac{3}{2}\ln\left(\frac{30 + v}{30 - v}\right)\end{aligned}$$

When $v = V \approx 19.5862...$:

$$t = 2.3408...$$

Hence Object A exits the gaseous medium approximately 2.34 seconds after it was dropped. ✓

Alternate method:

Let Object A be the first object, and downwards be the positive direction of its motion.

$$\begin{aligned}\sum F_A &= mg - \frac{v^2}{45} \\ 2\ddot{x} &= 20 - \frac{v^2}{45} \\ \ddot{x}_A &= \frac{900 - v^2}{90} \qquad \checkmark \end{aligned} \tag{1}$$

$$\begin{aligned}\frac{dv}{dt} &= \frac{(30+v)(30-v)}{90} \\ \int_0^v \frac{1}{(30+v)(30-v)}\,dv &= \int_0^t \frac{1}{90}\,dt \\ \int_0^v \frac{\frac{1}{60}}{30+v} + \frac{\frac{1}{60}}{30-v}\,dv &= \frac{1}{90}\Big[t\Big]_0^t \\ \Big[\ln|30+v| - \ln|30-v|\Big]_0^v &= \frac{2}{3}t \qquad \checkmark \\ &\text{(since } \ddot{x} > 0 \text{ and } v > 0\text{, } (30+v)(30-v) > 0 \text{ so } 30 - v > 0) \\ \ln\left(\frac{30+v}{30-v}\right) - \ln\left(\frac{30}{30}\right) &= \frac{2}{3}t \\ \frac{30+v}{30-v} &= e^{\frac{2}{3}t} \\ \left(1 + e^{\frac{2}{3}t}\right)v &= 30\left(e^{\frac{2}{3}t} - 1\right) \\ v &= \frac{30(e^{\frac{2}{3}t} - 1)}{e^{\frac{2}{3}t} + 1}\end{aligned}$$

$$\begin{aligned}\Big[x\Big]_0^x &= \int_0^t \frac{e^{\frac{1}{3}t} - e^{-\frac{1}{3}t}}{e^{\frac{1}{3}t} + e^{-\frac{1}{3}t}}\,dt \\ x &= 90\Big[\ln\left|e^{\frac{1}{3}t} + e^{-\frac{1}{3}t}\right|\Big]_0^t \\ &= 90\Big(\ln\left(e^{\frac{1}{3}t} + e^{-\frac{1}{3}t}\right) - \ln 2\Big) \qquad \checkmark \\ &\text{(since } e^{\frac{1}{3}t} > 0 \text{ and } e^{-\frac{1}{3}t} > 0)\end{aligned}$$

When $t = 2.34$,

$$\begin{aligned}x &= 90\Big(\ln\left(e^{\frac{2.34}{3}} + e^{-\frac{2.34}{3}t}\right) - \ln 2\Big) \\ &= 24.98\ldots \\ &\approx 25\end{aligned}$$

Hence Object A exits the gaseous medium approximately 2.34 seconds after it was dropped. ✓

ii. (3 marks)

- ✓ [1] for the correct height of Object B when Object A exits the gas
- ✓ [1] for correctly showing that Object B is moving upwards when Object A exits the gas
- ✓ [1] for the correct conclusion with sufficient justifications

Let Object B be the second object, and upwards be the positive direction of its motion.

$$
\begin{aligned}
\sum F_B &= -mg - \frac{v}{10} \\
2\ddot{x} &= -20 - \frac{v}{10} \\
\ddot{x_B} &= \frac{-(200+v)}{20} \\
\frac{dv}{dt} &= -\frac{1}{20}(200+v) \\
\int_{40}^{v} \frac{1}{200+v}\,dv &= \int_0^t -\frac{1}{20}\,dt \\
\Big[\ln|200+v|\Big]_{40}^{v} &= -\frac{1}{20}\Big[t\Big]_0^t \\
\ln\left(\frac{200+v}{240}\right) &= -\frac{1}{20}t \qquad\qquad (\text{since} \quad \ddot{x}_B < 0\ , \quad 200+v>0) \\
200+v &= 240e^{-\frac{1}{20}t} \\
v_B &= 240e^{-\frac{1}{20}t} - 200 \qquad\qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx}{dt} &= 240e^{-\frac{1}{20}t} \\
\Big[x\Big]_0^x &= \Big[-4800e^{-\frac{1}{20}t} - 200t\Big]_0^t \\
x &= -4800e^{-\frac{1}{20}t} - 200t + 4800 \\
x_B &= 4800\left(1 - e^{-\frac{1}{20}t}\right) - 200t
\end{aligned}
$$

From part (i), when $t = 2.3408...$:

$$
\begin{aligned}
x_B &= 62.00... \qquad ✓ \\
v_B &= 13.49... > 0
\end{aligned}
$$

i.e. when Object A exits the gaseous medium, Object B is 62 m above the ground and still moving upwards (has yet to reach its peak or fall back down). ✓

The gaseous medium has a depth of 25 m, below which is 65 m of air before the ground. Thus Object B cannot enter the gaseous medium before Object A exits it.

Hence the two objects will collide in the air. ✓