NORTH SYDNEY BOYS HIGH SCHOOL

2025 HSC ASSESSMENT TASK 3

Mathematics Extension 2

General Instructions	Class Teacher:
• Reading time – 10 minutes • Working time – 3 hours • For Section I, shade the correct box on the sheet provided • For Section II, write in the booklets provided • Each new question is to be started in a new answer booklet • Write using blue or black pen • Board approved calculators may be used • All necessary working should be shown in every question • A reference sheet is provided	○ Mrs Moss ○ Ms Lee ○ Mr Berry ○ Dr Vranešević

Student Number

(To be used by the exam markers only.)

Question	1-10	11	12	13	14	15	16	Total
Mark	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{100}$

Section I

10 Marks
Attempt Question $\mathbf{1-10}$
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Question $1-10$

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1 The displacement of a particle is given by

$$x = \frac{e^{2t} - e^{-2t}}{2}$$

The acceleration of the particle is

A. Twice the displacement in the same direction

B. Twice the displacement in the opposite direction

C. Four times the displacement in the same direction

D. Four times the displacement in the opposite direction

2 Which of the following is the negation of the statement

If I know how to negate this statement, then I will select the correct answer

A. If I don't know how to negate this statement, then I will not select the correct answer

B. I know how to negate this statement and I will not select the correct answer

C. If I do not select the correct answer, then I do not know how to negate this statement

D. I do not know how to negate this statement and I will select the correct answer

3 For the complex number $z = a + ib$, which of the following is the correct expression for

$$\sum_{k=0}^{2025} i^k z$$

A. $(a+b) + (a+b)i$

B. $(a-b) + (a+b)i$

C. $(a+b) + (a-b)i$

D. $(a-b) + (b-a)i$

4 Using the substitution $u = 1 + e^x$,

$$\int_0^{\ln 2} \frac{1}{1+e^x}\,dx$$

can be expressed as

A. $\displaystyle\int_1^{1+e^2} \left(\frac{1}{u-1} + \frac{1}{u}\right) du$

B. $\displaystyle\int_2^{3} \left(\frac{1}{u-1} - \frac{1}{u}\right) du$

C. $\displaystyle\int_2^{3} \left(\frac{1}{u} - \frac{1}{u-1}\right) du$

D. $\displaystyle\int_1^{1+e^2} \left(\frac{1}{u} - \frac{1}{u-1}\right) du$

5 For the vectors $\underset{\sim}{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}$. Which of the following is the expression for $|\text{proj}_{\underset{\sim}{q}}\underset{\sim}{p}|$?

A. $a^2 + b^2 + c^2$

B. $|ab + bc + ca|$

C. $\dfrac{a^2 + b^2 + c^2}{|ab + bc + ca|}$

D. $\dfrac{|ab + bc + ca|}{\sqrt{a^2 + b^2 + c^2}}$

6 Let a and b be real numbers with $a \neq b$. Let $z = x + iy$ be a complex number such that $|z - a|^2 - |z - b|^2 = 1$. Which of the following best describes the locus of z?

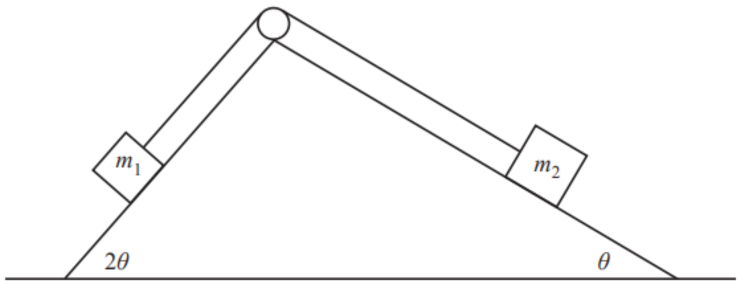
A. A circle with centre $(0,0)$

B. A circle with centre (a, b)

C. A ray

D. A vertical line

7 Two particles with masses m_1 kilograms and m_2 kilograms are connected by a taught light string that passes over a smooth pulley. The particles sit on a smooth inclined plane as shown in the diagram below.



If the system is in equilibrium, then $\dfrac{m_1}{m_2}$ is equal to

A. $\dfrac{\sec\theta}{2}$

B. $\dfrac{\sin 2\theta}{2}$

C. $\dfrac{1}{2}$

D. 2

8 How many ordered pairs of positive real numbers (a, b) exist that satisfy the equation

$$(1+2a)(2+2b)(2a+b) = 32ab$$

A. 0

B. 1

C. 2

D. Infinitely many

9 Points P and Q have position vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ respectively.

$$\frac{1}{2}\sqrt{\left|\underset{\sim}{p}\right|^2\left|\underset{\sim}{q}\right|^2-(\underset{\sim}{p}\cdot\underset{\sim}{q})^2}$$

is an expression for

A. The perimeter of ΔOPQ

B. The area of ΔOPQ

C. The magnitude of the position vector of the midpoint of PQ

D. The cosine of $\angle POQ$

10 For $|1 + z + z^2| = 4$

The maximum magnitude of the imaginary component of z can be represented as $\frac{\sqrt{m}}{n}$

What is $m + n$?

A. 13

B. 17

C. 21

D. 25

Section II

90 marks
Attempt Questions $11 - 16$
Allow about 2 hours and 45 minutes for the section

Answer each question in a separate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include all relevant mathematical reasoning and/or calculations

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Question 11 (15 marks) Start a new writing booklet

(a) (i) What is the contrapositive to the statement **1**

$$\textit{For any integers } a \textit{ and } b,\ a + b \geq 15 \textit{ implies that } a \geq 8 \textit{ or } b \geq 8.$$

(ii) Hence prove the statement from (i). **1**

(b) Find **2**

$$\int x \sin x \, dx$$

(c) (i) Show that $x = 2i$ is a solution to $x^4 - 2x^3 + 8x^2 - 8x + 16 = 0$ **1**

(ii) Hence express $P(x) = x^4 - 2x^3 + 8x^2 - 8x + 16$ as a product of two quadratic factors. **2**

(iii) Express $P(x) = x^4 - 2x^3 + 8x^2 - 8x + 16$ as the product of four linear factors. **2**

(d) Points A, B and C are respectively defined by position vectors $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = p\underset{\sim}{i} - 2\underset{\sim}{j}$, where p is a real constant. **3**

Given that $\angle ABC = \dfrac{\pi}{3}$, find the value(s) of p.

(e) Evaluate **3**

$$\int_0^{\frac{\pi}{3}} (\sec x \tan x)^3 \, dx$$

Question 12 (15 marks) Start a new writing booklet

(a) Giant's Deep is a large extra-terrestrial planet covered mostly by water. Large watery tornados regularly burst from the surface projecting any debris, or unlucky explorer, high into the air.

The displacement, from origin O, of a small statue caught in a particular tornado is given by the vector

$$\underset{\sim}{r} = \left(50 + 25\cos\frac{\pi t}{30}\right)\underset{\sim}{i} + \left(50 + 25\sin\frac{\pi t}{30}\right)\underset{\sim}{j} + \frac{2t}{5}\underset{\sim}{k}$$

Where $\underset{\sim}{i}$ is a unit vector East, $\underset{\sim}{j}$ is a unit vector North and $\underset{\sim}{k}$ is a unit vector vertically up. Displacement components are measured in metres and time is measured in seconds.

(i) Find the time, in seconds, for the statue to reach a height of 60 m. **1**

(ii) Find, to the nearest degree, the angle of elevation from O of the statue when it is at a height of 60 m. **2**

(iii) After how many seconds will the statue first be directly above its original point on the surface of the planet? **1**

(iv) Show that the velocity of the statue is perpendicular to its acceleration. **3**

(v) Find the speed, in metres per second, that the statue is travelling. Answer correct to two decimal places. **2**

(b) (i) Let ω be a non-real root of the equation $z^3 + 8 = 0$ **2**

Show that $\omega^2 = 2\omega - 4$

(ii) Hence evaluate $(2\omega - 4)^6$ **1**

(c) (i) Prove that for $a, b, c > 0$ that **2**

$$\frac{a^2}{b^3c} - \frac{a}{b^2} \geq \frac{c}{b} - \frac{c^2}{a}$$

(ii) Under what conditions does equality occur? **1**

Question 13 (15 marks) Start a new writing booklet

(a) (i) Use partial fractions to find: **3**

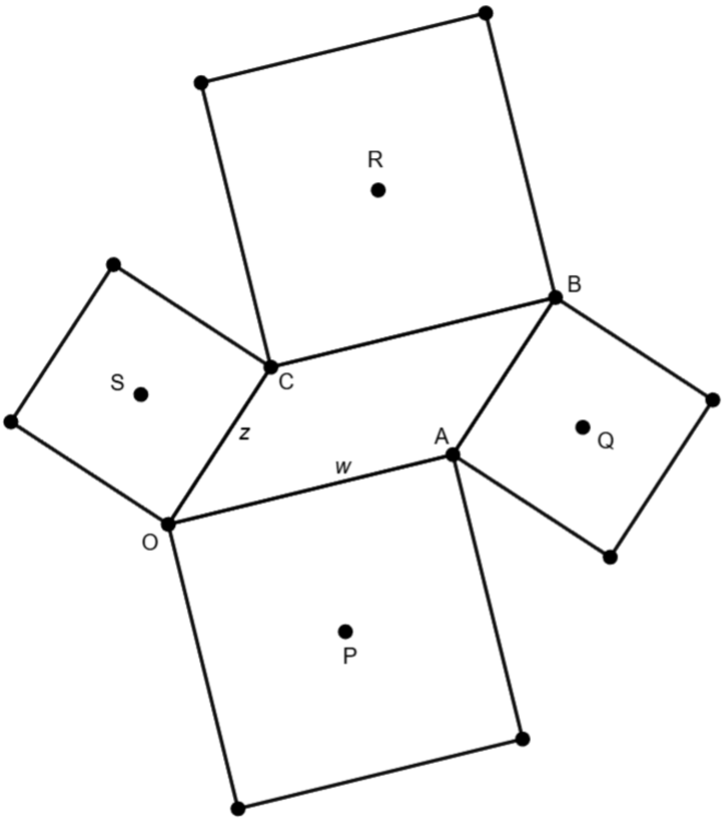
$$\int \frac{dx}{x^2(x-1)}$$

(ii) Use the substitution $t = \tan\frac{\theta}{2}$ to find: **3**

$$\int \operatorname{cosec}(\theta) d\theta$$

(b) $OABC$ is a parallelogram drawn on the Argand plane such that point O is the origin, point A is represented by the complex number w and point C is represented by the complex number z.

Squares are constructed off each of the four sides of parallelogram $OABC$ with centres P, Q, R and S as shown below.



(i) Show that $\overrightarrow{PR}$ is given by the complex number $z + iw$ **2**

(ii) Find a similar expression in terms of w, z and i for $\overrightarrow{QS}$ **2**

(iii) Hence, or otherwise, show that $PQRS$ is a square. **2**

(c) Find the points of intersection between the sphere: **3**

$$x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$$

and the line:

$$\underset{\sim}{r} = \begin{pmatrix} 7 \\ -1 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

Question 14 (15 marks) Start a new writing booklet

(a) A particle moves with acceleration $a = 2x^3 + 2x$ metres per second.

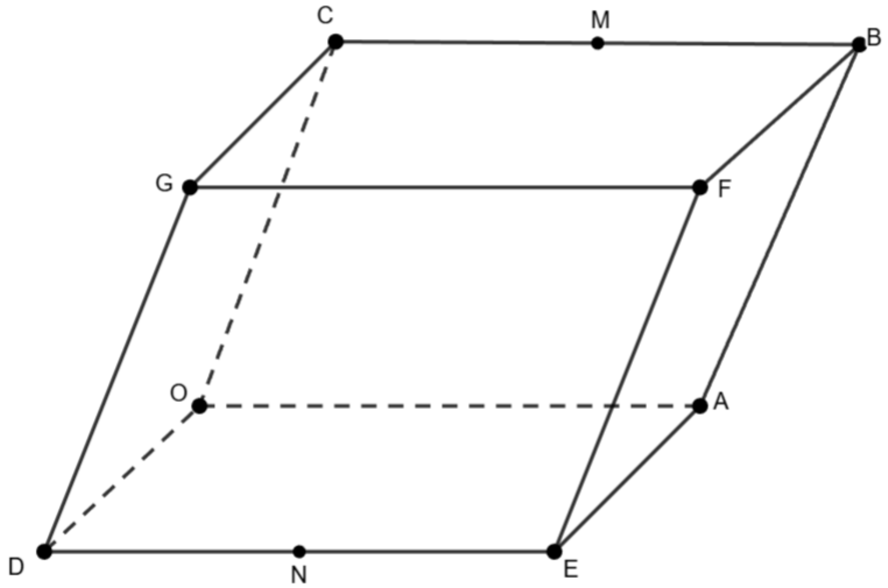
It starts at the origin with an initial velocity of 1 metre per second. Find

(i) Its velocity in terms of displacement. **2**

(ii) Its displacement in terms of time. **2**

(b) A parallelepiped is a three-dimensional figure formed by six parallelograms. Note that the opposite faces of a parallelepiped are identical.

Consider a parallelepiped $OABCDEFG$ as shown.



(i) Prove that the diagonals OF and CE bisect each other **3**

(ii) Let M be the midpoint of CB and N be the midpoint of DE. **1**
Prove that the midpoint of MN is the point where diagonals OF ande CE intersect.

Question 14 continues on page 10

(c) (i) Prove that for $a, b \in \mathbb{R}^+$: **1**

$$\frac{a+b}{2} \geq \sqrt{ab}$$

(ii) Hence or otherwise prove that for $a, b \in \mathbb{R}^+$: **1**

$$\sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$$

(iii) Prove that for $a, b \in \mathbb{R}^+$: **2**

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

(iv) Using any of the above identities, or otherwise, prove that for $x, y, z \in \mathbb{R}^+$: **3**

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \leq \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

End of Question 14

Question 15 (15 Marks) Start a new writing booklet

(a) By considering the function $y = \sqrt{x}$ from $x = 0$ to $x = n + 1$, use $n + 1$ applications of the trapezoidal rule to show that: **3**

$$\sum_{k=1}^{n} \sqrt{k} < \frac{(4n+1)\sqrt{n+1}}{6}$$

(b) (i) Let **3**

$$I_n = \int_0^1 \sqrt{(1-x^2)^n}\,dx$$

Show that for integers $n \geq 2$:

$$I_n = \frac{n}{n+1} I_{n-2}$$

(ii) Hence or otherwise evaluate **2**

$$\int_0^1 \sqrt{(1-x^2)^5}\,dx$$

Question 15 continues on page 12

(c) Gabbro is an astronaut who is about to embark upon a voyage to explore the extra-terrestrial planet Giant's Deep.

Gabbro's rocket ship, of mass m kilograms, is initially stationary on the surface of his home planet of Timber Hearth.

The engines in Gabbro's ship deliver a constant force of F newtons.

Gabbro engages the ship's engines until he reaches a height of h metres above Timber Hearth's surface.

Up until the ship reaches height h metres it experiences a constant gravity of g metres per second squared and an air resistance of 50% of the square of the ship's velocity.

(i) Show that when Gabbro has reached a height of h metres he is travelling at a speed of $\sqrt{2(F-mg)\left(1-e^{-\frac{h}{m}}\right)}$ metres per second. **3**

Once he reaches a height of h metres, Gabbro notices that the atmospheric conditions have changed.

As the air is much thinner, the air resistance is now negligible and as the ship is further from the planet, the force due to gravity is no longer constant but is now inversely proportional to the square of the distance of the ship from the centre of Timber Hearth.

At this height Gabbro turns off the engines and relies upon the current velocity of the ship to maintain its continued journey into the outer wilds of space.

(ii) Given that the gravity is g metres per second squared at the surface of Timber Hearth, show that once the ship as reached a height of h the gravity now has a magnitude of $\frac{gR^2}{(R+x)^2}$ where R is the radius of Timber Hearth and x is the height of the ship above the surface of the planet. **1**

(iii) The ship's designer, Hornfels, must work out how powerful the ship's engines need to be to ensure that Gabbro will be able to escape Timber Hearth's atmosphere. **3**

They begin by weighing the ship while it is still on the surface of the planet (weight is measured in newtons and is calculated by multiplying mass by gravity) and then consulting with Gabbro who tells them of their plan to switch off the engines at height h.

Show that the force of the engines must exceed the weight of the ship by

$$\frac{gR^2}{(h+R)\left(1-e^{-\frac{h}{m}}\right)} \text{ newtons}$$

End of Question 15

Question 16 (15 Marks) Start a new writing booklet

(a) Evaluate the integral **3**

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$$

(You may quote the identity $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ without proof at any point in your working)

(b) z is a complex number such that $|z| = 1$ and $z^{n+1} - z^n - 1 = 0$. **3**

Prove that $n + 2$ is a multiple of 6.

Question 16 continues of page 14

Question 16 (continued)

(c) Coprime integers are integers which share no common factors other than 1.

(i) If a and b are coprime and $ax = by$, show that a is a factor of y, for integers a, b, x & y. **1**

If a and b are coprime you may assume without proof that a^m and b^n are also coprime for positive integers m and n.

(ii) $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$ is a polynomial of degree n with integer coefficients. **2**

Let the rational roots of $P(x)$ be expressed in the form $\frac{p}{q}$ such that p and q are coprime integers.

Prove that q is a factor of a_n.

(iii) By letting $\cos\theta = x$ we can write the following expansions of $\cos(n\theta)$ in terms of x. **3**

$\cos(0\theta) = 1$
$\cos(1\theta) = x$
$\cos(2\theta) = 2x^2 - 1$
$\cos(3\theta) = 4x^3 - 3x$
$\cos(4\theta) = 8x^4 - 8x^2 + 1$
...

By considering the first few terms in the expansion of $(\cos\theta + i\sin\theta)^n$ or otherwise, show that the leading coefficient of the polynomial expansion of $\cos(n\theta)$ is 2^{n-1}.

(Note: you may use the abbreviation $c = \cos\theta$ and $s = \sin\theta$ if you wish for ***this question only*** to make your working more concise).

(iv) Given that n is an odd integer greater than or equal to 3 and that **3**

$$\cos^{-1}\left(\frac{1}{\sqrt{n}}\right) = k\pi$$

Prove that k is irrational.

End of paper

Multiple Choice

Q1./ $x = \dfrac{e^{2t} - e^{-2t}}{2}$

$\dot{x} = \dfrac{2e^{2t} + 2e^{-2t}}{2}$

$\ddot{x} = \dfrac{4e^{2t} - 4e^{-2t}}{2}$

$= 4\left(\dfrac{e^{2t} - e^{-2t}}{2}\right)$

$= 4x$

$\therefore$ C

Q2./ B

Q3./ $\sum_{k=0}^{2025} i^k z = i^0 z + iz + \sum_{k=2}^{2025} i^k z$

$= a + ib + i(a+ib) + 0$ (2024 terms cancels as its a multiple of 4)

$= a + ib + ai - b$

$= a - b + i(a+b)$

$\therefore$ B

4./ $u = 1 + e^x$ $\qquad x = \ln 2$ $\qquad x = 0$

$\frac{du}{dx} = e^x$ $\qquad u = 1 + e^{\ln 2}$ $\qquad u = 1 + e^0$

$\qquad = 3$ $\qquad = 2$

$\therefore dx = \frac{du}{e^x}$

$$\int_0^{\ln} \frac{1}{1+e^x}\,dx = \int_2^3 \frac{1}{u} \times \frac{du}{e^x} \qquad \frac{A}{u} + \frac{B}{u-1}$$

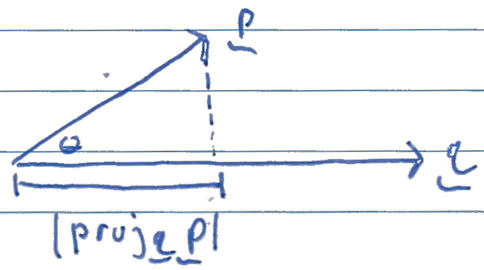
$$= \int_2^3 \frac{1}{u(u-1)}\,du \qquad = \frac{Au - A + Bu}{u(u-1)}$$

$$\therefore A = -1,\ B = 1$$

$$= \int_2^3 \left(\frac{1}{u-1} - \frac{1}{u}\right) du$$

$\therefore$ B

5./ $\underset{\sim}{p} \cdot \underset{\sim}{q} = |p||q| \cos\theta$



$$|p|\cos\theta = \frac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{|q|}$$

$$= \frac{ab + bc + ca}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore |\text{proj}_{\underset{\sim}{q}} \underset{\sim}{p}| = \frac{|ab + bc + ca|}{\sqrt{a^2 + b^2 + c^2}}$$

$\therefore$ D

6./ $|z-a|^2 - |z-b|^2 = 1$

$|x+iy-a|^2 - |x+iy-b|^2 = 1$

$|(x-a)+iy|^2 - |(x-b)+iy|^2 = 1$

$\therefore (x-a)^2 + y^2 - [(x-b)^2 + y^2] = 1$

$x^2 - 2ax + a^2 + y^2 - [x^2 - 2bx + b^2 + y^2] = 1$

$\cancel{x^2} - 2ax + a^2 + \cancel{y^2} - \cancel{x^2} + 2bx - b^2 - \cancel{y^2} = 1$

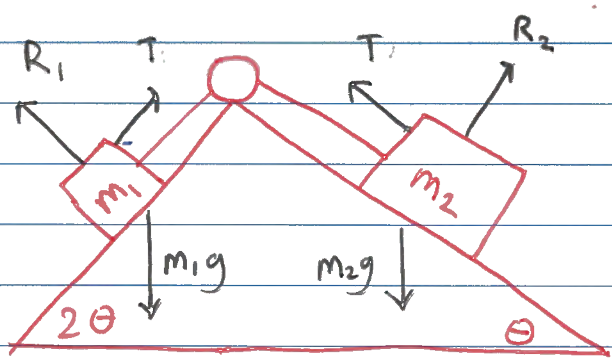
$\therefore 2bx - 2ax = 1 + b^2 - a^2$

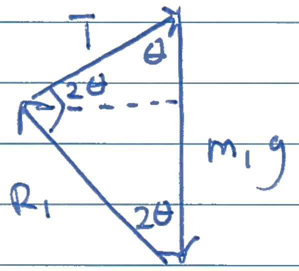
$x(2b - 2a) = 1 + b^2 - a^2$

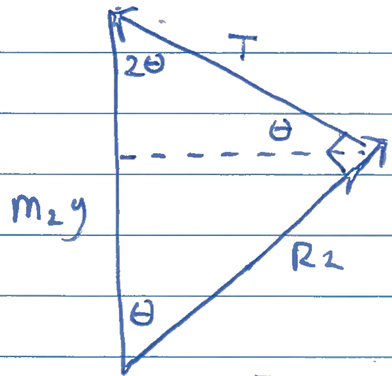
$\therefore x = \dfrac{1 + b^2 - a^2}{2(b-a)}$

$\therefore$ D

7./







$\sin 2\theta = \dfrac{T_1}{m_1 g}$

$T = m_1 g \sin 2\theta \ldots (1)$

$\sin\theta = \dfrac{T}{m_2 g}$

$\therefore T = m_2 g \sin\theta \ldots (2)$

$(1) = (2)$

$m_1 \cancel{g} \sin 2\theta = m_2 \cancel{g} \sin\theta$

$\therefore \dfrac{m_1}{m_2} = \dfrac{\sin\theta}{\sin 2\theta}$

$= \dfrac{\cancel{\sin\theta}}{2\cos\theta \cancel{\sin\theta}}$

$= \dfrac{1}{2\cos\theta}$

$= \dfrac{\sec\theta}{2}$

$\therefore$ A

8./ $1+2a \geqslant 2\sqrt{2a}$ ... ①

$2+2b \geqslant 2\sqrt{(2)(2b)}$ ... ②

$2a+b \geqslant 2\sqrt{2ab}$ ... ③

① × ② × ③

$$(1+2a)(2+2b)(2a+b) \geqslant 8\sqrt{2^4a^2b^2}$$
$$= 32ab$$

for equality

$1 = 2a$

$a = 1/2$

$2a = b$

$\therefore b = 1$

$\therefore (\frac{1}{2}, 1)$ is the only pair

$\therefore$ B

9./ $\frac{1}{2}\sqrt{|\underset{\sim}{p}|^2|\underset{\sim}{q}|^2 - (\underset{\sim}{p}.\underset{\sim}{q})^2} = \frac{1}{2}\sqrt{|\underset{\sim}{p}|^2|\underset{\sim}{q}|^2 - (|\underset{\sim}{p}||\underset{\sim}{q}|\cos\theta)^2}$

$$= \frac{1}{2}\sqrt{|\underset{\sim}{p}|^2|\underset{\sim}{q}|^2 - |\underset{\sim}{p}|^2|\underset{\sim}{q}|^2\cos^2\theta}$$

$$= \frac{1}{2}\sqrt{|\underset{\sim}{p}|^2|\underset{\sim}{q}|^2(1-\cos^2\theta)}$$

$$= \frac{1}{2}\sqrt{|\underset{\sim}{p}|^2|\underset{\sim}{q}|^2\sin^2\theta}$$

$$= \frac{1}{2}|\underset{\sim}{p}||\underset{\sim}{q}||\sin\theta|$$

$\therefore$ B

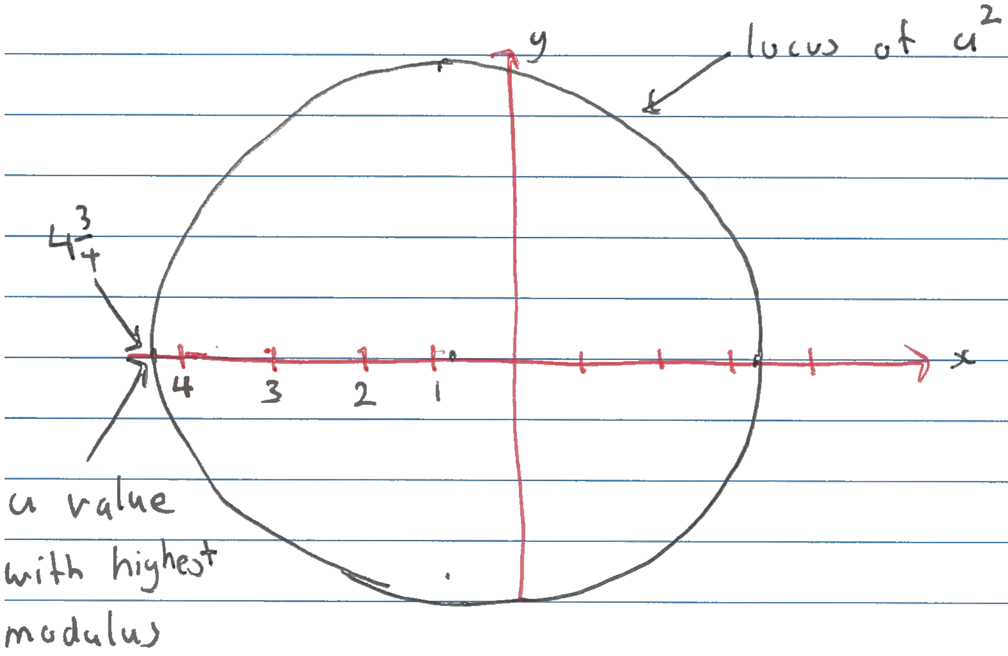
$10.$ / $|1 + z + z^2| = 4$

$\therefore |z^2 + z + \frac{1}{4} + \frac{3}{4}| = 4$

$|(z + \frac{1}{2})^2 + \frac{3}{4}| = 4$

let $u = z + \frac{1}{2}$ (note z & u have the same imaginary value)

$|u^2 + \frac{3}{4}| = 4$



$\therefore |u^2| = \frac{19}{4}$

as $|u^2| = |u|^2$

$|u| = \sqrt{\frac{19}{4}}$

$= \frac{\sqrt{19}}{2}$

$19 + 2 = 21$

$\therefore$ C

Question 11

(a) (i) For any integers a and b,
$a < 8$ and $b < 8$ implies that $a+b < 15$

(ii) For integers a & b
$a < 8 \Rightarrow a \leq 7$ & $b < 8 \Rightarrow b \leq 7$

$$\begin{aligned} \therefore a + b &\leq 7 + 7 \\ &\leq 14 \\ &< 15 \end{aligned}$$

$\therefore$ statement true as contrapositive is true

(b) $\int x \sin x \, dx = x(-\cos x) - \int (1)(-\cos x) \, dx$

$= -x\cos x + \sin x + C$

$$\boxed{\begin{array}{ll} u = x & v' = \sin x \\ u' = 1 & v = -\cos x \end{array}}$$

(c) (i) $(2i)^4 - 2(2i)^3 + 8(2i)^2 - 8(2i) + 16$
$= 16 + 16i - 32 - 16i + 16$
$= 0$

(ii) $x - 2i$ is a factor so $x + 2i$ is a factor by conjugate root theorem

$\therefore x^2 + 4$ is a factor

$$
\begin{array}{r|l}
 & x^2 - 2x + 4 \\
\hline
x^2 + 4 & x^4 - 2x^3 + 8x^2 - 8x + 16 \\
 & \underline{x^2 + 0x^3 + 4x^2} \\
 & \quad -2x^3 + 4x^2 - 8x \\
 & \quad \underline{-2x^3 + 0x^2 - 8x} \\
 & \qquad\qquad 4x^2 + 0x + 16 \\
 & \qquad\qquad \underline{4x^2 + 0x + 16} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

$\therefore (x^2+4)(x^2-2x+4)$

(iii) solving $x^2 - 2x + 4 = 0$

$$x = \frac{2 \pm \sqrt{(-2)^2 - (4)(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-16}}{2}$$

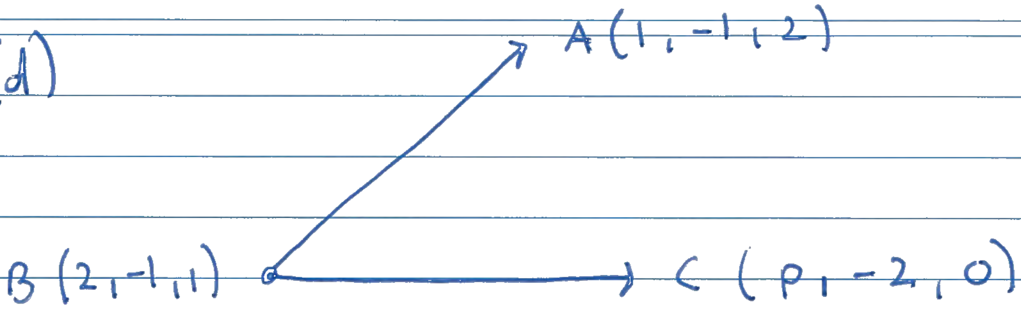
$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}\,i}{2}$$

$$= 1 \pm \sqrt{3}\,i$$

$\therefore (x-2i)(x+2i)(x-1-\sqrt{3}\,i)(x-1+\sqrt{3}\,i)$

(d)



$\overrightarrow{BA} = (-1, 0, 1) \qquad BC = (p-2, -1, -1)$

$|\overrightarrow{BA}| = \sqrt{(-1)^2 + (1)^2} \qquad |\overrightarrow{BC}| = \sqrt{(p-2)^2 + (-1)^2 + (-1)^2}$

$= \sqrt{2} \qquad = \sqrt{p^2 - 4p + 4 + 1 + 1}$

$= \sqrt{p^2 - 4p + 6}$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-1)(p-2) + 0(-1) + (1)(-1)$

$= -p + 2 - 1$

$= 1 - p$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}| \cos \pi/3$

$1 - p = \sqrt{2} \times \sqrt{p^2 - 4p + 6} \times \frac{1}{2}$

$1 - 2p + p^2 = 2(p^2 - 4p + 6) \times \frac{1}{4}$

$2 - \cancel{4p} + 2p^2 = p^2 - \cancel{4p} + 6$

$\left(\begin{array}{l} \text{as RHS} \geqslant 0 \\ \text{LHS} \geqslant 0 \\ 1 - p \geqslant 0 \\ \therefore p \leqslant 1 \end{array}\right)$

$\therefore p^2 - 4 = 0$

$p^2 = 4$

$p = \pm 2$

$\therefore p = -2 \quad (\text{as } p \leqslant 1)$

(e) $\int_0^{\pi/3} (\sec x \tan x)^3 \, dx$

$= \int_0^{\frac{\pi}{3}} \sec x \tan x \sec^2 x \tan^2 x \, dx$

$= \int_0^{\pi/3} \sec x \tan x \sec^2 x (\sec^2 x - 1) \, dx$

$= \int_0^{\pi/3} (\sec x \tan x \sec^4 x - \sec x \tan x \sec^2 x) \, dx$

$= \left[\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} \right]_0^{\pi/3}$

$= \frac{1}{5} (\sec \pi/3)^5 - \frac{1}{3} (\sec \frac{\pi}{3})^3 - (\frac{1}{5} (\sec 0)^5 - \frac{1}{3} (\sec 0)^3)$

$= \frac{2^5}{5} - \frac{2^3}{3} - \frac{1^5}{5} + \frac{1^3}{3}$

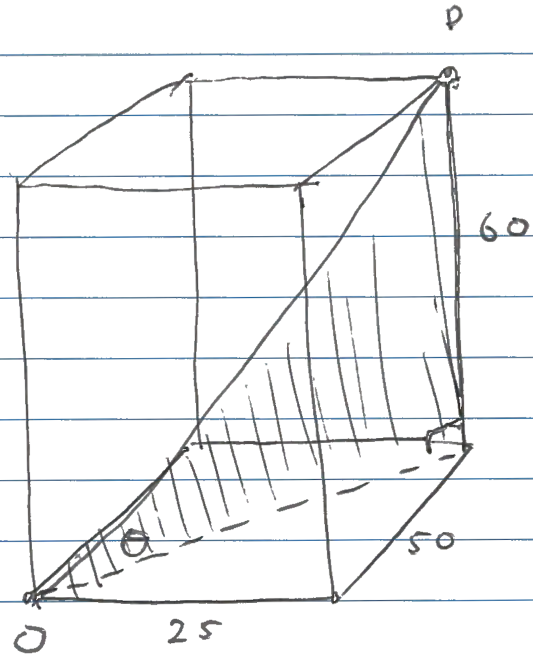
$= \frac{58}{15}$

Question 12

(a)(i) $\frac{2t}{5} = 60$

$2t = 300$

$t = 150$ seconds



(ii) $x = 50 + 25\cos 5\pi$

$= 25$

$y = 50 + 25\sin 5\pi$

$= 50$

$z = 60$

$\theta = \tan^{-1}\left(\frac{60}{\sqrt{25^2 + 50^2}}\right)$

$= 47°$ (nearest degree)

(iii) 1 period

$T = \frac{2\pi}{\pi/30}$

$= 2 \times 30$

$= 60$ seconds

(iv) $\underset{\sim}{v} = -\frac{25\pi}{30}\sin\left(\frac{\pi t}{30}\right)\underset{\sim}{i} + \frac{25\pi}{30}\cos\frac{\pi t}{30}\underset{\sim}{j} + \frac{2}{5}\underset{\sim}{k}$

$= -\frac{5\pi}{6}\sin\left(\frac{\pi t}{30}\right)\underset{\sim}{i} + \frac{5\pi}{6}\cos\left(\frac{\pi t}{30}\right)\underset{\sim}{j} + \frac{2}{5}\underset{\sim}{k}$

$$\mathbf{a} = -\frac{5\pi^2}{180}\cos\left(\frac{\pi t}{30}\right)\mathbf{i} - \frac{5\pi^2}{180}\sin\left(\frac{\pi t}{30}\right)\mathbf{j}$$

$$= -\frac{\pi^2}{36}\cos\left(\frac{\pi t}{30}\right)\mathbf{i} - \frac{\pi^2}{36}\sin\left(\frac{\pi t}{30}\right)\mathbf{j}$$

$$\mathbf{a}\cdot\mathbf{v} = \left(\frac{-\pi^2}{36}\cos\left(\frac{\pi t}{30}\right)\right)\left(\frac{-5\pi}{6}\sin\left(\frac{\pi t}{30}\right)\right) + \left(\frac{-\pi^2}{36}\sin\left(\frac{\pi t}{30}\right)\right)\left(\frac{5\pi}{6}\cos\left(\frac{\pi t}{30}\right)\right)$$

$$+ \left(\frac{2}{5}\right)(0)$$

$$= \frac{5\pi^3}{216}\cos\left(\frac{\pi t}{30}\right)\sin\left(\frac{\pi t}{30}\right) - \frac{5\pi^3}{216}\cos\left(\frac{\pi t}{30}\right)\sin\left(\frac{\pi t}{30}\right)$$

$$= 0$$

$\therefore \mathbf{a} \perp \mathbf{v}$

(v) let $\theta = \frac{\pi t}{30}$

$$|\mathbf{v}|^2 = \left(-\frac{5\pi}{6}\sin\theta\right)^2 + \left(\frac{5\pi}{6}\cos\theta\right)^2 + \left(\frac{2}{5}\right)^2$$

$$= \frac{25\pi^2}{36}\sin^2\theta + \frac{25\pi^2}{36}\cos^2\theta + \frac{4}{25}$$

$$= \frac{25\pi^2}{36}\left(\sin^2\theta + \cos^2\theta\right) + \frac{4}{25}$$

$$= \frac{25\pi^2}{36} + \frac{4}{25}$$

$$= 7.01\ m/s$$

$\therefore$ speed $= 2.65\ m/s$

(b)(i) $z^3 + 8 = 0$

$\therefore (z+2)(z^2 - 2z + 4) = 0$

as w is a solution

$(w+2)(w^2 - 2w + 4) = 0$

$w \neq -2$ (as w is non-real)

$\therefore w^2 - 2w + 4 = 0$

$\therefore w^2 = 2w - 4$

(ii) $(2w-4)^6 = (w^2)^6$

$= w^{12}$

$= (w^3)^4$

$= (-8)^4$

$= 4096$

(c) $\text{LHS} - \text{RHS} = \dfrac{a^2}{b^3c} - \dfrac{a}{b^2} - \dfrac{c}{b} + \dfrac{c^2}{a}$

$= \dfrac{a^3}{ab^3c} - \dfrac{a^2bc}{ab^3c} - \dfrac{ab^2c^2}{ab^3c} + \dfrac{b^3c^3}{ab^3c}$

$= \dfrac{a^3 - a^2bc - ab^2c^2 + b^3c^3}{ab^3c}$

$= \dfrac{a^2(a-bc) - b^2c^2(a-bc)}{ab^3c}$

$= \dfrac{(a-bc)(a^2-b^2c^2)}{ab^3c}$

$= \dfrac{(a-bc)(a-bc)(a+bc)}{ab^3c}$

$$= \frac{(a-bc)^2(a+bc)}{ab^3c}$$

$$\geq 0 \qquad \text{for } a, b, c > 0$$

as $LHS - RHS \geq 0$

$LHS \geq RHS$

(ii) when $a = bc$

Question 13

(a) (i) $$\frac{1}{x^2(x-1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2}$$

$$= \frac{Ax^2 + Bx(x-1) + C(x-1)}{x^2(x-1)}$$

$$= \frac{Ax^2 + Bx^2 - Bx + Cx - C}{x^2(x-1)}$$

$$= \frac{(A+B)x^2 + (C-B)x - C}{x^2(x-1)}$$

$$\therefore C = -1, \quad B = -1, \quad A = 1$$

$$\int \frac{1}{x^2(x-1)}\,dx = \int\left(\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}\right)dx$$

$$= \ln|x-1| - \ln|x| + \frac{1}{x} + C$$

(ii) $$\int \operatorname{cosec}\theta \, d\theta$$

$$= \int \frac{\cancel{1+t^2}}{\cancel{2}t} \times \frac{\cancel{2}}{\cancel{1+t^2}}\,dt$$

$$= \int \frac{1}{t}\,dt$$

$$= \ln|t| + c$$

$$= \ln\left|\tan\frac{\theta}{2}\right| + c$$

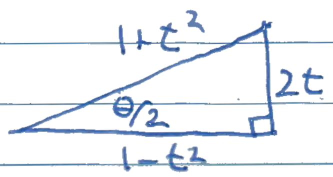
$$t = \tan\left(\frac{\theta}{2}\right)$$

$$\frac{\theta}{2} = \tan^{-1} t$$

$$\theta = 2\tan^{-1} t$$

$$\frac{d\theta}{dt} = \frac{2}{1+t^2}$$

$$d\theta = \frac{2\,dt}{1+t^2}$$



(b)(i) Let D, E, F & G be the midpoints of OA, AB, BC & CO respectively.

$$\vec{OP} = \vec{OD} + \vec{DP}$$
$$= \tfrac{1}{2}w - \tfrac{1}{2}iw$$

$$\vec{OR} = \vec{OC} + \vec{CF}$$
$$= z + \tfrac{1}{2}w + \tfrac{1}{2}iw$$

$$\vec{PR} = \vec{OR} - \vec{OP}$$
$$= z + \tfrac{1}{2}w + \tfrac{1}{2}iw - \left(\tfrac{1}{2}w - \tfrac{1}{2}iw\right)$$
$$= z + \cancel{\tfrac{1}{2}w} + \tfrac{1}{2}iw - \cancel{\tfrac{1}{2}w} + \tfrac{1}{2}iw$$
$$= z + iw$$

(ii) $$\vec{QS} = \vec{OS} - \vec{OQ}$$
$$= (\vec{OG} + \vec{GS}) - (\vec{OA} + \vec{AE} + \vec{EQ})$$
$$= \tfrac{1}{2}z + \tfrac{1}{2}iz - \left(w + \tfrac{1}{2}z - \tfrac{1}{2}iz\right) \quad \ldots \quad (1)$$
$$= \cancel{\tfrac{1}{2}z} + \tfrac{1}{2}iz - w - \cancel{\tfrac{1}{2}z} + \tfrac{1}{2}iz$$
$$= iz - w$$

(iii) $$i\vec{PR} = iz - w$$
$$= \vec{QS}$$

$\therefore$ diagonals are perpendicular and equal in length

$$\text{midpoint of } \vec{PR} = \tfrac{1}{2}\left(\vec{OP} + \vec{OR}\right)$$
$$= \tfrac{1}{2}\left(\tfrac{1}{2}w - \cancel{\tfrac{1}{2}iw} + z + \tfrac{1}{2}w + \cancel{\tfrac{1}{2}iw}\right)$$
$$= \tfrac{1}{2}(w + z)$$

midpoint of $\overrightarrow{QS} = \frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OS})$

$= \frac{1}{2}(\underbrace{w + \frac{1}{2}z - \cancel{\frac{1}{2}iz}}_{OQ} + \underbrace{\frac{1}{2}z + \cancel{\frac{1}{2}iz}}_{OS})$ from ①

$= \frac{1}{2}(w + z)$

$\therefore$ diagonals bisect

$\therefore$ square as equal, perpendicular diagonals bisect

(c) $x^2 + y^2 + z^2 - 4x + 2y - 6z = -5$

$x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 - 6z + 9 = -5 + 4 + 1 + 9$

$(x-2)^2 + (y+1)^2 + (z-3)^2 = 9$

$\therefore (7 + 3\lambda - 2)^2 + (-1 - \lambda + 1)^2 + (9 + 4\lambda - 3)^2 = 9$

$(5 + 3\lambda)^2 + (-\lambda)^2 + (6 + 4\lambda)^2 = 9$

$25 + 30\lambda + 9\lambda^2 + \lambda^2 + 36 + 48\lambda + 16\lambda^2 = 9$

$\therefore 26\lambda^2 + 78\lambda + 52 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 2)(\lambda + 1) = 0$

$\therefore \lambda = -2$ or $\lambda = -1$

$\therefore (7-3, -1+1, 9-4)$ and $(7-6, -1+2, 9-8)$

$(4, 0, 5)$ and $(1, 1, 1)$

Question 14

(a) (i) $a = 2x^3 + 2x$

$= 2x(x^2+1)$

$\therefore v\frac{dv}{dx} = 2x(x^2+1)$

$v\,dv = 2x(x^2+1)\,dx$

$\int_1^v v\,dv = \int_0^x 2x(x^2+1)\,dx$

$\left[\cancel{\tfrac{1}{2}}v^2\right]_1^v = \left[\cancel{\tfrac{1}{2}}(x^2+1)^2\right]_0^x$

$v^2 - 1^2 = (x^2+1)^2 - (0+1)^2$

$v^2 \cancel{-1} = (x^2+1)^2 \cancel{-1}$

$\therefore v = (x^2+1)$ (as $v_0 > 0$ and $v \neq 0 \ \forall x$)

(ii) $\frac{dx}{dt} = x^2+1$

$\frac{dt}{dx} = \frac{1}{x^2+1}$

$dt = \frac{dx}{1+x^2}$

$\int_0^t dt = \int_0^x \frac{dx}{1+x^2}$

$[t]_0^t = [\tan^{-1}(x)]_0^x$

$t - 0 = \tan^{-1}(x) - \cancel{\tan^{-1}(0)}$

$\therefore x = \tan(t)$

(b)(i) Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{c}$ & $\overrightarrow{OD} = \underset{\sim}{d}$

Let the midpoint of $\overrightarrow{OF}$ be X and the midpoint of $\overrightarrow{CE}$ be Y.

$$\begin{aligned}\overrightarrow{OX} &= \tfrac{1}{2}\overrightarrow{OF}\\ &= \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BF})\\ &= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c} + \underset{\sim}{d}) \qquad (\text{as } \overrightarrow{AB} = \overrightarrow{OC} = \underset{\sim}{c}\\ &\qquad\qquad\qquad\qquad\quad \& \ \overrightarrow{BF} = \overrightarrow{OD} = \underset{\sim}{d})\end{aligned}$$

$$\begin{aligned}\overrightarrow{OY} &= \overrightarrow{OC} + \overrightarrow{CY}\\ &= \overrightarrow{OC} + \tfrac{1}{2}\overrightarrow{CE}\\ &= \overrightarrow{OC} + \tfrac{1}{2}(\overrightarrow{OE} - \overrightarrow{OC})\\ &= \overrightarrow{OC} + \tfrac{1}{2}\overrightarrow{OE} - \tfrac{1}{2}\overrightarrow{OC}\\ &= \tfrac{1}{2}(\overrightarrow{OC} + \overrightarrow{OE})\\ &= \tfrac{1}{2}(\overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{DE})\\ &= \tfrac{1}{2}(\underset{\sim}{c} + \underset{\sim}{d} + \underset{\sim}{a}) \qquad (\text{as } \overrightarrow{DE} = \overrightarrow{OA} = \underset{\sim}{a})\\ &= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c} + \underset{\sim}{d})\\ &= \overrightarrow{OX}\end{aligned}$$

$\therefore$ X & Y coincide so OF & CE BISECT

(ii) Let Z be the midpoint of $\overrightarrow{MN}$

$$\begin{aligned}\overrightarrow{OZ} &= \tfrac{1}{2}(\overrightarrow{OM} + \overrightarrow{ON})\\ &= \tfrac{1}{2}(\overrightarrow{OC} + \overrightarrow{CM} + \overrightarrow{OD} + \overrightarrow{DN})\\ &= \tfrac{1}{2}(\overrightarrow{OC} + \tfrac{1}{2}\overrightarrow{CB} + \overrightarrow{OD} + \tfrac{1}{2}\overrightarrow{DE})\\ &= \tfrac{1}{2}(\underset{\sim}{c} + \tfrac{1}{2}\underset{\sim}{a} + \underset{\sim}{d} + \tfrac{1}{2}\underset{\sim}{a}) \qquad (\text{as } \overrightarrow{CB} = \overrightarrow{DE} = \overrightarrow{OA} = \underset{\sim}{a})\\ &= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c} + \underset{\sim}{d})\end{aligned}$$

$\therefore$ Z coincides with x & y so
X & Y intersect at Z

(c)(i) $(\sqrt{a}-\sqrt{b})^2 \geq 0$

$\therefore a - 2\sqrt{ab} + b \geq 0$

$a + b \geq 2\sqrt{ab}$

$\therefore \frac{a+b}{2} \geq \sqrt{ab}$

(ii) let $a = \frac{1}{a}$ & $b = \frac{1}{b}$ & sub into (i)

$$\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \sqrt{\frac{1}{a}\times\frac{1}{b}}$$

$$\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \frac{1}{\sqrt{ab}}$$

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab}$$ (reciprocating reverses inequality)

$$\therefore \sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$$

(iii) $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab$

$\therefore 2a^2 + 2b^2 \geq a^2 + 2ab + b^2$ (add a^2+b^2 to both sides)

$2(a^2+b^2) \geq (a+b)^2$

$\therefore \frac{a^2+b^2}{2} \geq \frac{(a+b)^2}{4}$

$\therefore \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$ (as $\sqrt{x}$ is monotonically increasing and LHS & RHS are positive)

$$\therefore \frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$ (from (i), (ii) & (iii))

(iv) $a = x \quad b = y$

$$\therefore \frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \frac{x+y}{2} \qquad \left(\text{as } \frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \frac{a+b}{2} \text{ from (iii)}\right)$$

$$\frac{2}{\frac{x+y}{xy}} \leq \frac{x+y}{2}$$

$$\frac{2xy}{x+y} \leq \frac{x+y}{2}$$

$$\frac{x+y}{2xy} \geq \frac{2}{x+y} \qquad (\text{reciprucating reverses inequality})$$

$$\frac{1}{2}\left(\frac{x+y}{xy}\right) \geq 2\left(\frac{1}{x+y}\right)$$

$$\frac{1}{2}\left(\frac{x}{xy}+\frac{y}{xy}\right) \geq 2\left(\frac{1}{x+y}\right)$$

$$\frac{1}{2}\left(\frac{1}{y}+\frac{1}{x}\right) \geq 2\left(\frac{1}{x+y}\right)$$

$$\therefore \frac{1}{x+y} \leq \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right) \quad \ldots (1)$$

similarly $\frac{1}{y+z} \leq \frac{1}{4}\left(\frac{1}{y}+\frac{1}{z}\right) \quad \ldots (2)$

$$\frac{1}{z+x} \leq \frac{1}{4}\left(\frac{1}{z}+\frac{1}{x}\right) \quad (3)$$

$(1) + (2) + (3)$

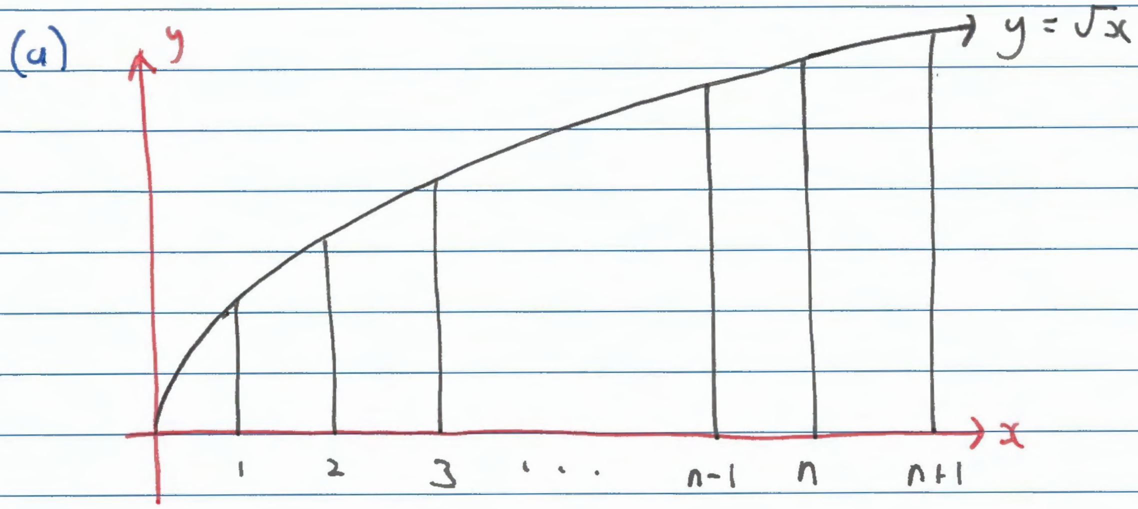
$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \leq \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{y}+\frac{1}{z}+\frac{1}{z}+\frac{1}{x}\right)$$

$$= \frac{1}{4}\left(\frac{2}{x}+\frac{2}{y}+\frac{2}{z}\right)$$

$$= \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Question 15

(a)



As $y=\sqrt{x}$ is concave down the trapezoidal rule will underestimate the area.

$$\therefore \frac{h}{2}\left[y_0+y_n+2(y_1+y_2+\dots+y_{n-1})\right] < \int_a^b f(x)\,dx$$

$$\frac{1}{2}\left[0+\sqrt{n-1}+2(\sqrt{1}+\sqrt{2}+\dots+\sqrt{n})\right] < \int_0^{n+1}\sqrt{x}\,dx$$

$$\frac{1}{2}\sqrt{n-1}+\sqrt{1}+\sqrt{2}+\dots+\sqrt{n} < \left[\frac{2}{3}x^{3/2}\right]_0^{n+1}$$

$$\frac{1}{2}\sqrt{n+1}+\sum_{k=1}^{n}\sqrt{k} < \frac{2}{3}\sqrt{(n+1)^3}-\cancel{\frac{2}{3}\sqrt{0^3}}$$

$$\sum_{k=1}^{n}\sqrt{k} < \frac{2\sqrt{(n+1)^3}}{3}-\frac{\sqrt{n+1}}{2}$$

$$=\frac{4\sqrt{(n+1)^3}-3\sqrt{n+1}}{6}$$

$$=\frac{\sqrt{n+1}\,[4(n+1)-3]}{6}$$

$$=\frac{\sqrt{n+1}\,(4n+4-3)}{6}$$

$$=\frac{\sqrt{n+1}\,(4n+1)}{6}$$

$$=\frac{(4n+1)\sqrt{n+1}}{6}$$

(b)(i) $I_n = \int_0^1 (1-x^2)^{n/2}\,dx$

$$= \int_0^1 (1)(1-x^2)^{n/2}\,dx$$

$$= [uv]_0^1 - \int_0^1 u'v\,dx$$

$u = (1-x^2)^{n/2}$ $\quad v' = 1$

$u' = \left(\frac{n}{2}\right)(-2x)(1-x^2)^{\frac{n}{2}-1}$ $\quad v = x$

$= -nx(1-x^2)^{\frac{n-2}{2}}$

$$= \left[(1-x^2)^{n/2}(x)\right]_0^1 - \int_0^1 -nx(1-x^2)^{\frac{n-2}{2}}(x)\,dx$$

$$= \left[(1-1^2)^{n/2}(1) - (1-0^2)^{n/2}(0)\right] + \int_0^1 nx^2(1-x^2)^{\frac{n-2}{2}}\,dx$$

$$= 0 + n\int_0^1 x^2(1-x^2)^{\frac{n-2}{2}}\,dx$$

$$= n\int_0^1 (x^2-1+1)(1-x^2)^{\frac{n-2}{2}}\,dx$$

$$= n\int_0^1 (x^2-1)(1-x^2)^{\frac{n-2}{2}}\,dx + n\int_0^1 (1-x^2)^{\frac{n-2}{2}}\,dx$$

$$= -n\int_0^1 (1-x^2)(1-x^2)^{\frac{n-2}{2}}\,dx + n\int_0^1 (1-x^2)^{\frac{n-2}{2}}\,dx$$

$$= -n\int_0^1 (1-x^2)^{n/2} + n\int_0^1 (1-x^2)^{\frac{n-2}{2}}$$

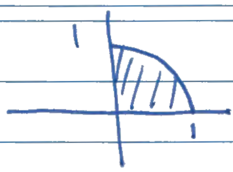
$\therefore I_n = -nI_n + nI_{n-2}$

$I_n + nI_n = nI_{n-2}$

$I_n(1+n) = nI_{n-2}$

$\therefore I_n = \dfrac{n}{n+1} I_{n-2}$

(ii) $I_1 = \int_0^1 \sqrt{1-x^2}\, dx$



$= \pi/4$

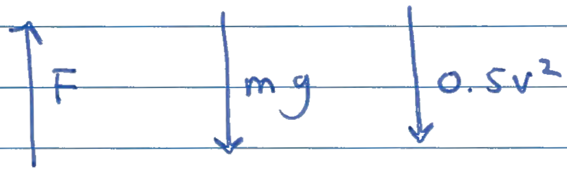
$I_3 = \frac{3}{4} \times \frac{\pi}{4}$

$= \frac{3\pi}{16}$

$I_5 = \frac{5}{\cancel{6}\,2} \times \frac{\cancel{3}\pi}{16}$

$= \frac{5\pi}{32}$

(c) (i) Let up be the positive direction



Force $= ma$

$ma = F - mg - 0.5v^2$

$a = \frac{F - mg - 0.5v^2}{m}$

$v\frac{dv}{dx} = \frac{F - mg - 0.5v^2}{m}$

$\frac{dv}{dx} = \frac{F - mg - 0.5v^2}{mv}$

$$\frac{dx}{dv} = \frac{mv}{F - mg - 0.5v^2}$$

$$dx = \frac{mv}{F - mg - 0.5v^2}\,dv$$

$$\int_0^h dx = \int_0^v \frac{mv}{F - mg - 0.5v^2}\,dv$$

$$= -m\int_0^v \frac{-v}{F - mg - 0.5v^2}\,dv$$

$$\left[x\right]_0^h = -m\left[\ln|F - mg - 0.5v^2|\right]_0^v$$

$$h - 0 = -m\ln\left|\frac{F - mg - 0.5v^2}{F - mg - 0.5(0)^2}\right|$$

$$-\frac{h}{m} = \ln\left|\frac{F - mg - 0.5v^2}{F - mg}\right|$$

$$\frac{F - mg - 0.5v^2}{F - mg} = e^{-h/m}$$

$$1 - \frac{0.5v^2}{F - mg} = e^{-h/m}$$

$$\frac{0.5v^2}{F - mg} = 1 - e^{-h/m}$$

$$0.5v^2 = (F - mg)(1 - e^{-h/m})$$

$$v^2 = 2(F - mg)(1 - e^{-h/m})$$

$$v = \sqrt{2(F - mg)(1 - e^{-h/m})} \qquad (v > 0)$$

(ii) let $\ddot{x}$ be gravity

$$\ddot{x} \propto \frac{1}{(x+R)^2}$$

$$\therefore \ddot{x} = \frac{k}{(x+R)^2}$$

when $x = 0$ $\ddot{x} = g$

$$g = \frac{k}{(0+R)^2}$$

$$k = gR^2$$

$$\therefore \ddot{x} = \frac{gR^2}{(x+R)^2}$$

(iii) <u>at height h</u>

up is positive

gravity is only force

let velocity at $x = h$ be u.

$$ma = -m\ddot{x}$$

$$a = -\ddot{x}$$

$$= -\frac{gR^2}{(x+R)^2}$$

$$v\frac{dv}{dx} = -\frac{gR^2}{(x+R)^2}$$

$$v\,dv = -\frac{gR^2}{(x+R)^2}dx$$

$$\int_u^v v\,dv = \int_h^x \frac{-gR^2}{(x+R)^2}dx$$

$$= gR^2 \int_h^x -\frac{1}{(x+R)^2}dx$$

$$\left[\frac{1}{2}v^2\right]_u^v = gR^2\left[\frac{1}{(x+R)}\right]_h^x$$

$$\frac{1}{2}v^2 - \frac{1}{2}u^2 = gR^2\left(\frac{1}{x+R} - \frac{1}{h+R}\right)$$

$$v^2 - u^2 = 2gR^2\left(\frac{1}{x+R} - \frac{1}{h+R}\right)$$

$$v^2 = 2gR^2\left(\frac{1}{x+R} - \frac{1}{h+R}\right) + u^2$$

$$v = \sqrt{2gR^2\left(\frac{1}{x+R} - \frac{1}{h+R}\right) + u^2}$$

to continue upwards $v^2 > 0$ for all x (as $a < 0$, if $v = 0$ then the ship will return to Timber Hearth)

$$\therefore \text{ as } x \to \infty \quad 2gR^2\left(\frac{1}{x+R} - \frac{1}{h+R}\right) + u^2 > 0$$

$$\therefore 2gR^2\left(0 - \frac{1}{h+R}\right) + u^2 > 0$$

$$\therefore -\frac{2gR^2}{h+R} + u^2 > 0$$

$$\therefore u^2 > \frac{2gR^2}{h+R}$$

$$\therefore \cancel{2}(F - mg)(1 - e^{-h/m}) > \frac{\cancel{2}gR^2}{h+R} \quad \text{(from (i))}$$

$$\therefore F - mg > \frac{gR^2}{(h+R)(1 - e^{-h/m})}$$

Question 16

(a) $$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$$

let $x = \tan u$

$\frac{dx}{du} = \sec^2 u$

$dx = \sec^2 u\, du$

when $x = 1$, $u = \pi/4$

when $x = 0$, $u = 0$

$$= \int_0^{\frac{\pi}{4}} \frac{\ln(\tan u + 1)}{\tan^2 u + 1} \sec^2 u\, du$$

$$= \int_0^{\frac{\pi}{4}} \frac{\ln(\tan u + 1)}{\cancel{\sec^2 u}} \cancel{\sec^2 u}\, du$$

$$= \int_0^{\frac{\pi}{4}} \ln(\tan u + 1)\, du$$

$$\int_0^{\pi/4} \ln(\tan u + 1)\, du = \int_0^{\pi/4} \ln\left[\tan\left(\frac{\pi}{4} - u\right) + 1\right] du$$

$$= \int_0^{\pi/4} \ln\left[\frac{\tan \pi/4 - \tan u}{1 + \tan \pi/4 (u)} + 1\right] du$$

$$= \int_0^{\pi/4} \ln\left[\frac{1 - \tan u}{1 + \tan u} + 1\right] du$$

$$= \int_0^{\pi/4} \ln\left[\frac{1 - \cancel{\tan u} + 1 + \cancel{\tan u}}{1 + \tan u}\right] du$$

$$= \int_0^{\pi/4} \ln\left|\frac{2}{1 + \tan u}\right| du$$

$$= \int_0^{\pi/4} \left[\ln 2 - \ln(1 + \tan u)\right] du$$

$$= \int_0^{\pi/4} \ln 2\, du - \int_0^{\pi/4} \ln(1 + \tan u)\, du$$

$$\therefore\ I_n = \int_0^{\frac{\pi}{4}} \ln 2\, du - I_n$$

$$2 I_n = \left[u \ln 2\right]_0^{\pi/4}$$

$$I_n = \frac{1}{2}\left(\frac{\pi}{4}\ln 2 - 0 \ln 2\right)$$

$$= \frac{\pi}{8}\ln 2$$

(b) $z^{n+1} - z^n - 1 = 0$

$z^{n+1} - z^n = 1$

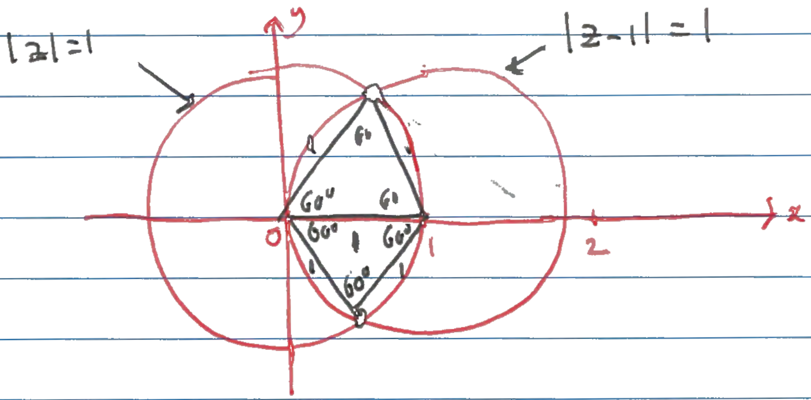
$z^n(z-1) = 1 \quad \ldots \text{①}$

$\therefore |z^n(z-1)| = 1$

$|z^n||z-1| = 1$

$1\,|z-1| = 1$

$|z-1| = 1$



equilateral triangles have angle 60°

$\therefore \operatorname{Arg}(z) = \pm \pi/3$

$\therefore z = e^{\pm i\pi/3}$

$\therefore z^3 = -1$

$z^3 + 1 = 0$

$(z+1)(z^2 - z + 1) = 0$

but $z \neq -1$

so $z^2 - z + 1 = 0$

$z^2 = z - 1$

sub into ①

$z^n \times z^2 = 1$

$z^{n+2} = 1$

but as $z^3 = -1$

$z^6 = 1$

$z^{6k} = 1 \quad k \in \mathbb{Z}^+$

$\therefore n + 2 = 6k$

so $n+2$ is a multiple of 6

(c) (i) $ax = by$

$\therefore \frac{x}{y} = \frac{b}{a}$

as $\frac{b}{a}$ is simplified

$\frac{x}{y} = \frac{kb}{ka}$ $\quad k \in \mathbb{Z}$

$y = ka$

$\therefore$ a is a factor of y

(ii) $P\left(\frac{p}{q}\right) = 0$ $\quad$ ($\frac{p}{q}$ is factor)

$\therefore a_n\left(\frac{p^n}{q^n}\right) + a_{n-1}\left(\frac{p^{n-1}}{q^{n-1}}\right) + \dots + a_1\left(\frac{p}{q}\right) + a_0 = 0 \quad \times q^n$

$a_n p^n + a_{n-1} p^{n-1} q + a_{n-2} p^{n-2} q^2 + \dots + a_1 p q^{n-1} + a_0 q^n = 0$

$\therefore a_n p^n = -a_{n-1} p^{n-1} q - a_{n-2} p^{n-2} q^2 - \dots - a_1 p q^{n-1} - a_0 q^n$

$a_n p^n = q\left(-a_{n-1} p^{n-1} - a_{n-2} p^{n-2} q - \dots \; a_1 p q^{n-2} - a_0 q^{n-1}\right)$

$\therefore$ q is a factor of a_n (from (i) as p & q are coprime)

(iii) $(c+is)^n = \binom{n}{0}c^n + \binom{n}{1}c^{n-1}(is)^1 + \binom{n}{2}c^{n-2}(is)^2 + \binom{n}{3}c^{n-3}(is)^3 + \dots$

$= \binom{n}{0}c^n + i\binom{n}{1}c^{n-1}s - \binom{n}{2}c^{n-2}s^2 - i\binom{n}{3}c^{n-3}s^3 + \binom{n}{4}c^{n-4}s$

equate real parts

$\cos n\theta = \binom{n}{0}c^n - \binom{n}{2}c^{n-2}s^2 + \binom{n}{4}c^{n-4}s^4 - \binom{n}{6}c^{n-6}s^6 + \dots$

$= \binom{n}{0}c^n - \binom{n}{2}c^{n-2}(1-c^2) + \binom{n}{4}c^{n-4}(1-c^2)^2 - \binom{n}{6}c^{n-6}(1-c^2)^3$

we can see the coefficient of c^n is

$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \binom{n}{8} + \dots$

consider $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots$

when $x = 1$ we get

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots \qquad (1)$$

when $x = -1$ we get

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \dots \qquad (2)$$

$(1) + (2)$

$$2^n = 2\binom{n}{0} + 2\binom{n}{2} + 2\binom{n}{4} + \dots \qquad \div 2$$

$$\therefore 2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots$$

which is our coefficient for c^n

(iv) Proof by contradiction

assume k is rational

i.e $k = \frac{p}{q}$ $\quad p, q \in \mathbb{Z}$

$$\therefore \cos^{-1}\left(\frac{1}{\sqrt{n}}\right) = \frac{p\pi}{q}$$

$$\frac{1}{\sqrt{n}} = \cos\left(\frac{p\pi}{q}\right)$$

$$\frac{1}{n} = \cos^2\left(\frac{p\pi}{q}\right)$$

let $\alpha = \frac{p\pi}{q}$

$$\frac{1}{n} = \cos^2\alpha$$

$$\frac{1}{n} = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$$

$\therefore \; 1 + \cos 2\alpha = \frac{2}{n}$

$\cos 2\alpha = \frac{2}{n} - 1$

$= \frac{2-n}{n}$ (note that as n is odd this fraction is fully simplified as 2 is prime)

now as $\alpha = \frac{p\pi}{q}$

$2\alpha = \frac{2p\pi}{q}$

$2q\alpha = 2p\pi$

but as $\cos(2p\pi) = 1 \;\therefore\; \cos(2q\alpha) = 1$

$\therefore \; \cos[q(2\alpha)] - 1 = 0$

expanding as a polynomial in $\cos 2\alpha$ the leading coefficient is 2^{q-1} (from (iii))

$\therefore$ $\cos 2\alpha$ cannot be expressed as $\frac{2-n}{n}$ as n is odd and can't be a factor of 2^{q-1} (from (ii)).

$\therefore$ there is a contradiction so k is irrational