North Sydney Girls High School

2023 **HSC TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 14)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME:________________________ TEACHER:________________________

STUDENT NUMBER:

Question	1-10	11	12	13	14	15	16	Total
Mark	/10	/15	/14	/15	/15	/15	/16	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1 Given $A(1,-2,3)$ and $B(5,0,-1)$, which of the following is a unit vector in the direction of $\overrightarrow{BA}$?

A. $-4\mathbf{i}-2\mathbf{j}+4\mathbf{k}$

B. $4\mathbf{i}+2\mathbf{j}-4\mathbf{k}$

C. $-\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$

D. $\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$

2 Consider the statement: "If it rains then they cancel the game."

If this statement is true, which of the following can be inferred to also be true?

A. If it doesn't rain, they will not cancel the game.

B. If they have not cancelled the game, it is not raining.

C. If they have cancelled the game, it must be raining.

D. If it doesn't rain, they will cancel the game.

3 A particle is in simple harmonic motion and the equation describing its motion is

$$v^2 = 16 + 4x - 2x^2$$

Which of the following gives the maximum displacement S of the particle and the period T of the motion?

A. $S = 3$ $\quad T = \pi$

B. $S = 4$ $\quad T = \pi$

C. $S = 3$ $\quad T = \sqrt{2}\pi$

D. $S = 4$ $\quad T = \sqrt{2}\pi$

4 Given that z and w are complex numbers such that $\overline{z} + i\,\overline{w} = 0$ and $\arg(zw) = \pi$, which of the following is the value of $\arg(z)$?

A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. $\frac{3\pi}{4}$

D. $\frac{5\pi}{4}$

5 The vector equation of straight line l_1 is given by $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

A second line l_2 passes through the point $A(4,2,1)$ with a direction vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

What is the value of the parameter λ at the point of intersection of the two lines?

A. -1

B. $\frac{1}{3}$

C. 1

D. 3

6 A variable force of F Newtons acts on a particle of mass 3 kg so that it moves in a straight line. The velocity of the particle in metres per second is given by $v = 3 - x^2$, where x is the displacement of the particle from a fixed origin.

Which of the following gives the expression for the force F?

A. $F = -2x$

B. $F = -6x$

C. $F = 2x^3 - 6x$

D. $F = 6x^3 - 18x$

7 For all complex numbers z_1 and z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 + 4i| = 5$, what is the minimum value of $|z_1 - z_2|$?

A. 0

B. 2

C. 7

D. 17

8 Consider the statement below.

$\forall x \in \mathbb{R}\left(x < 1 \Rightarrow x^2 < 1\right)$

Which of the following is the negation of this statement?

A. $\exists x \in \mathbb{R}\left(x < 1 \text{ and } x^2 \geq 1\right)$

B. $\forall x \in \mathbb{R}\left(x < 1 \text{ and } x^2 \geq 1\right)$

C. $\exists x \in \mathbb{R}\left(x \geq 1 \text{ and } x^2 \geq 1\right)$

D. $\forall x \in \mathbb{R}\left(x \geq 1 \text{ and } x^2 \geq 1\right)$

9 Given that $-\frac{1}{2} \leq f(x) \leq 3$ and $-3 \leq g(x) \leq 2$ for $x \in [0, 2]$,

which of the following inequalities is satisfied by $I = \int_0^2 \left[f(x) - g(x)\right] dx$?

A. $-\frac{5}{2} \leq I \leq 6$

B. $1 \leq I \leq \frac{5}{2}$

C. $-5 \leq I \leq 12$

D. $2 \leq I \leq 5$

10 Given that $f(x+a)=f(a-x)$ for all values of x, which of the following is NOT necessarily true?

A. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$

B. $\int_0^a f(x-a)dx = \int_0^a f(-x)dx$

C. $\int_0^a f(x+a)dx = \int_0^a f(x)dx$

D. $\int_0^a f(2x)dx = \frac{1}{2}\int_0^a f(x)dx$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Given the complex numbers $z_1 = 2+i$ and $z_2 = 3-2i$, express $\frac{z_1}{z_2}$ in the form $x+iy$, where x and y are real numbers. Show all working. **2**

(b) Consider the polynomial $P(z) = z^3 + 5z^2 + 9z + 5$.

(i) Show that $z+1$ is a factor of $P(z)$. **1**

(ii) Hence solve $P(z) = 0$. **2**

(c) The complex numbers z and w are defined as $z = 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$ and $w = 1 + i\sqrt{3}$.

(i) Write down the value of Arg(z), the principal argument of z. **1**

(ii) Find $\frac{\overline{w}}{z^2}$ in exponential form. Show working. **3**

(iii) Find the set of possible values of n such that $\left(\frac{\overline{w}}{z^2}\right)^n$ is a real number. **2**

(d) Use integration by parts to find $\int \ln x \, dx$. **2**

(e) Prove that for all odd positive integers n, $n^2 - 1$ is divisible by 8. **2**

Question 12 (14 marks) Use a SEPARATE writing booklet

(a) Find $\int \sin^3\theta \cos^3\theta \, d\theta$. **2**

(b) (i) Sketch the locus of z in the Argand plane, where $\arg\left(\dfrac{z-2}{z+i}\right) = 0$. **2**

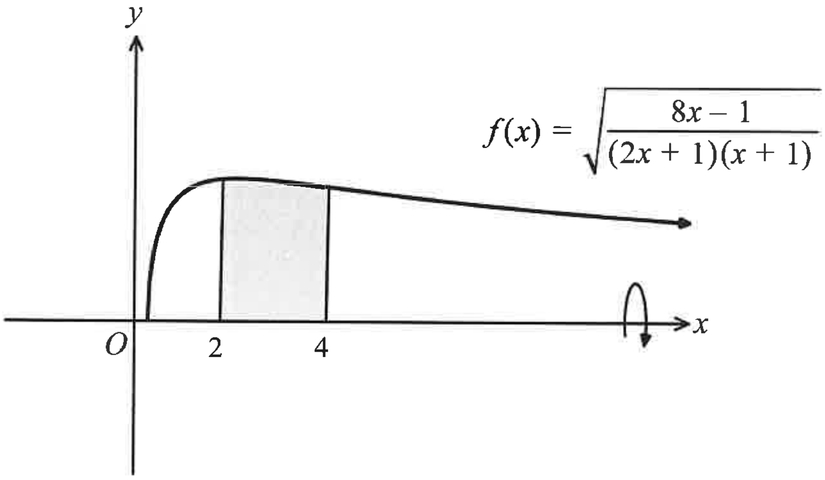
(ii) Find the Cartesian equation of the locus, stating any restrictions on the domain. **1**

(c) An object is bobbing up and down with the waves, executing simple harmonic motion at a frequency of 30 cycles per minute. During the motion, the object rises 35 cm above its mean position. Initially the object is rising through the mean position.

(i) Write down an equation for the displacement of the object about the mean position at any given time. **2**

(ii) Find the first time the object rises through a point 10 cm below the mean position. **2**

(d) The graph of $f(x) = \sqrt{\dfrac{8x-1}{(2x+1)(x+1)}}$ is shown below. **5**



A solid of revolution is formed when the region between the curve and the x-axis between $x = 2$ and $x = 4$ is rotated about the x-axis.

Find the volume of the solid of revolution, giving your answer correct to two decimal places.

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Use the substitution $t = \tan\frac{x}{2}$ to find $\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x}\,dx$. **4**

(b) The two points $A(-1,2,3)$ and $B(0,2,5)$ lie in a three-dimensional coordinate system.

(i) Find the vector equation of the sphere centred at A and passing through B. **2**

(ii) The line l_1 is defined by $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$. Find any intersection points between the sphere and the line. **3**

(c) Consider the statement S below:

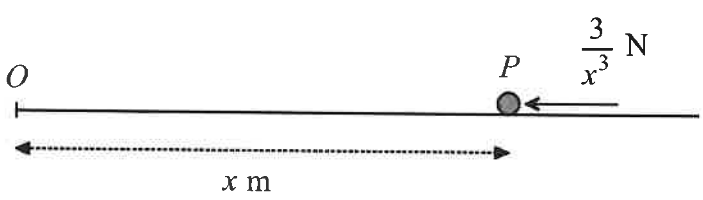
$S : \forall x, y \in \mathbb{R}^+$, If $y^3 - x^2 y \le xy^2 - x^3$ then $y \le x$

(i) State the contrapositive of the statement S. **1**

(ii) Hence, prove the statement by proving the contrapositive. **2**

(d) A particle P of mass 0.6 kg moves in a straight line on a smooth horizontal surface. Initially the particle is at rest 10 m from a fixed point O. **3**

A force of magnitude $\frac{3}{x^3}$ Newtons acts on the particle directed towards O as shown.



Find the velocity of the particle when it is 2.5 m from O.

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) (i) Show that for all positive integers n, $(z^n - e^{i\theta})(z^n + e^{-i\theta}) = z^{2n} - (2i\sin\theta)z^n - 1$. **1**

(ii) Hence solve $z^6 - iz^3 = 1$, giving your answers in exponential form. **3**

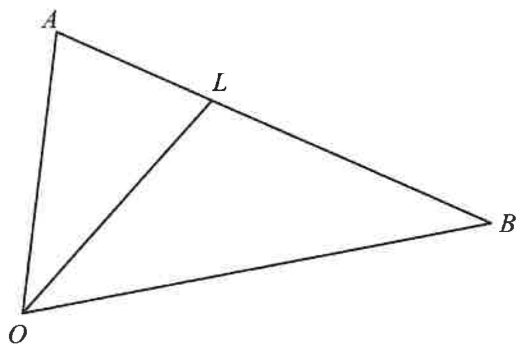
(b) (i) Use a suitable substitution to show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ **1**

(ii) Hence or otherwise, evaluate $\int_0^1 \frac{\ln(x+1)}{\ln((2-x)(1+x))}\,dx$. **2**

(c) Use mathematical induction to prove that for all integer values $a > 1$, **3**

$$\frac{1}{a+1} + \frac{1}{a+2} + \frac{1}{a+3} + \ldots + \frac{1}{2a} > \frac{13}{24}$$

(d) Consider the triangle OAB shown below.
L is a point on AB such that $OA : OB = AL : LB = 1 : k$.



Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$.

(i) Show that $\overrightarrow{OL} = \frac{k}{k+1}\underset{\sim}{a} + \frac{1}{k+1}\underset{\sim}{b}$. **2**

(ii) Use vector methods to establish that OL bisects $\angle AOB$. **3**

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) Let $I_n = \int_0^1 x^n \sqrt{1-x^2}\,dx$, $n \geq 0$. **3**

Show that $I_n = \dfrac{n-1}{n+2} I_{n-2}$ for $n \geq 2$.

(b) (i) Use Mathematical Induction to prove that for all positive integer values of n **3**

$$\sum_{r=1}^{n} \cos(2r\theta) = \frac{\sin[(2n+1)\theta] - \sin\theta}{2\sin\theta}$$

You may use the result $\sin 3x = 3\sin x - 4\sin^3 x$.

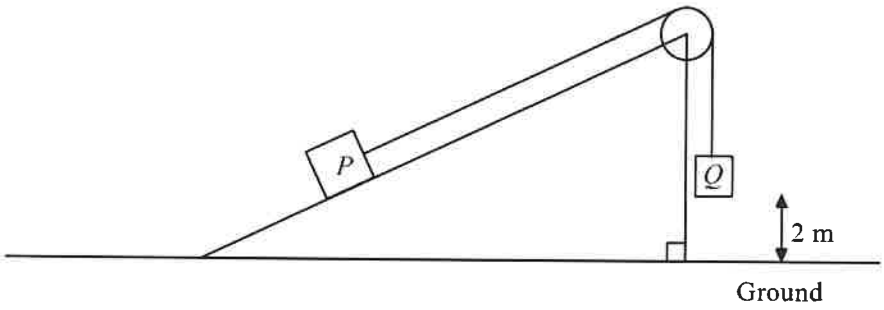
(ii) Use the result in (i) to find an expression for $\sum_{r=1}^{n} \cos^2\left(\dfrac{r\pi}{5}\right)$ and hence find the exact **2**

value of $\sum_{r=1}^{32} \cos^2\left(\dfrac{r\pi}{5}\right)$.

Question 15 continues on Page 12

(c) Two particles P and Q of mass M kg and $2M$ kg respectively are connected by a light inextensible string that passes over a frictionless pulley mounted on top of a smooth ramp of gradient $\frac{3}{4}$.

Initially, Particle P rests on the ramp and Q is suspended vertically from the pulley 2 metres above the ground as shown in the diagram below.



The particles are released from rest so that Q begins to move down and P moves up the ramp.

(i) Show that Q has an acceleration of $\frac{7g}{15}$. **2**

(ii) Find how long it takes for Q to reach the ground. Use $g = 10\text{ m/s}^2$. **2**
Answer in exact form.

(iii) How much further will P travel up the ramp before it comes to rest instantaneously? **3**
You may assume that P comes to rest before reaching the top of the ramp.

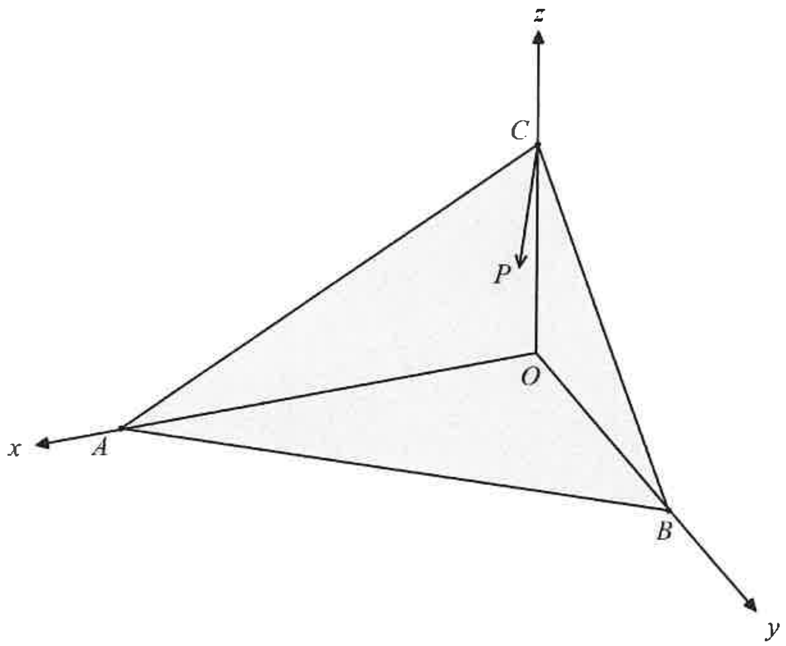
End of Question 15

Question 16 (16 marks) Use a SEPARATE writing booklet

(a) (i) Given ω is a non-real root of $x^3 - 1 = 0$, show that ω is also a root of $x^2 + x + 1 = 0$. **1**

(ii) Use a proof by contradiction to show that $(x+1)^{2n} + x^{2n} + 1$ is not divisible by $(x^2 + x + 1)$ if n is divisible by 3. **3**

(b) A smooth inclined plane is positioned on a three-dimensional coordinate system as shown. $A(6,0,0)$, $B(0,2,0)$ and $C(0,0,3)$ are points on the plane.



A ball of mass m kg is released from C and will roll down the line of shortest distance from C to AB.

(i) $\overrightarrow{CP}$ is a unit vector in the direction of the line of shortest distance from C to AB. **3**
Use vector methods to show that $\overrightarrow{CP} = \frac{1}{\sqrt{35}}\mathbf{i} + \frac{3}{\sqrt{35}}\mathbf{j} - \frac{5}{\sqrt{35}}\mathbf{k}$.

(ii) Show that the net acceleration of the ball down the plane is given by $\ddot{\underset{\sim}{r}} = \frac{10}{7}\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$. **1**

Use acceleration due to gravity as 10 m/s^2.

(iii) Find the location of the ball after it has travelled for 0.5 seconds. **3**

Question 16 continues on Page 14

Question 16 (continued)

(c) It is given that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{r!} + ...$

(i) Use the Binomial Theorem to show that $\left(1 + \frac{1}{n}\right)^n < e$. **2**

(ii) The product $P(n)$ is defined as $P(n) = \frac{3}{2} \times \frac{5}{4} \times \frac{9}{8} \times ... \times \frac{2^n + 1}{2^n}$. **3**

Using the AM-GM inequality $\frac{x_1 + x_2 + x_3 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 ... x_n}$ or otherwise, show that $P(n) < e$ for all n.

End of paper

Multiple Choice

1. $\vec{BA} = \begin{pmatrix}1\\-2\\3\end{pmatrix} - \begin{pmatrix}5\\0\\-1\end{pmatrix} = \begin{pmatrix}-4\\-2\\4\end{pmatrix}$

$|\vec{AB}| = \sqrt{(-4)^2 + 2^2 + 4^2} = \sqrt{36} = 6$

unit vector $= \frac{1}{6}\begin{pmatrix}-4\\-2\\4\end{pmatrix} = -\frac{2}{3}\underset{\sim}{i} - \frac{1}{3}\underset{\sim}{j} + \frac{2}{3}\underset{\sim}{k}$ $\boxed{C}$

2. $P \Rightarrow Q$

P : It rains

Q : They cancel the game

Contrapositive is $\sim Q \Rightarrow \sim P$ $\boxed{B}$

3. $v^2 = 16 + 4x - 2x^2$

$= 2(8 + 2x - x^2)$

$= 2(9 - (x-1)^2)$

$n^2 = 2$

$n = \sqrt{2}$

$T = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

$a^2 = 9$ centre of motion

$a = 3$ $x = 1$

$\therefore x_{max} = 4$ $\boxed{D}$

4. $\bar{z} + i\bar{w} = 0 \Rightarrow \bar{z} = -i\bar{w}$

$\Rightarrow z = iw$

$\arg z = \arg w + \frac{\pi}{2}$ ①

$\arg(zw) = \pi$ $\arg z + \arg w = \pi$ ②

① + ② $2\arg z + \cancel{\arg w} = \cancel{\arg w} + \frac{3\pi}{2}$

$\arg z = \frac{3\pi}{4}$ $\boxed{C}$

5. $l_2 : \begin{pmatrix} 4+\mu \\ 2+\mu \\ 1 \end{pmatrix}$ $l_1 : \begin{pmatrix} 1+2\lambda \\ -\lambda \\ 1 \end{pmatrix}$

① $4+\mu = 1+2\lambda$

② $2+\mu = -\lambda \Rightarrow \mu = -\lambda - 2$ Sub into ①

③ $1 = 1$

$4-\lambda-2 = 1+2\lambda$

$3\lambda = 1$

$\lambda = \frac{1}{3}$

$\boxed{B}$

6. $v = 3-x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}(3-x^2)^2\right)$

$= \cancel{\frac{1}{2}} \times \cancel{2}\,(3-x^2)(-2x)$

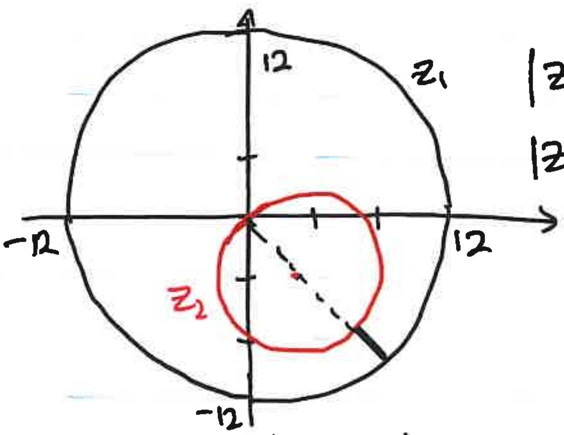
$= -6x + 2x^3$

$\therefore F = ma = 3(-6x+2x^3)$

$= 6x^3 - 18x$

$\boxed{D}$

7.



$|z_1| = 12$ circle centre O, radius 12

$|z_2 - 3 + 4i| = 5$ circle centre $(3,-4)$ radius 5

Min distance along line of centres. $12 - 10 = 2$

$\boxed{B}$

8. Negation of $\forall x \in \mathbb{R}$ is $\exists x \in \mathbb{R}$

Negation of $P \Rightarrow Q$ is P and $\sim Q$

ie. $x < 1$ and $x^2 \geq 1$

$\boxed{A}$

9. $-\frac{1}{2} \leq f(x) \leq 3$ $\qquad -3 \leq g(x) \leq 2$

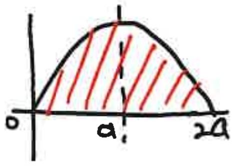
$-2 \leq -g(x) \leq 3$

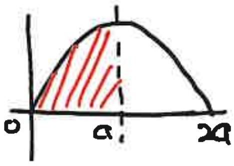
$-\frac{5}{2} \leq f(x) - g(x) \leq 6$

$-5 \leq \int_0^2 f(x) - g(x)\,dx \leq 12$

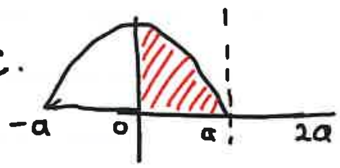
$\boxed{C}$

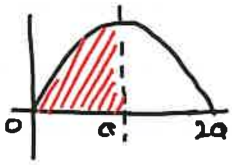
10. $f(a+x) = f(a-x)$. Function is symmetric about $x = a$

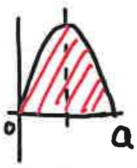
A. 

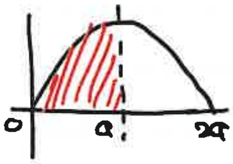
$= 2 \times$ 

B. $\int_0^a f(x-a)\,dx = \int_0^a f(a-x-a)\,dx = \int_0^a f(-x)\,dx$

C. 

$=$ 

D. 

$\neq \frac{1}{2} \times$ 

$\boxed{D}$

These areas are equal

Question 11

(a) $z_1 = 2+i \quad z_2 = 3-2i$

$$\frac{z_1}{z_2} = \frac{2+i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{6+4i+3i+2i^2}{3^2+2^2}$$

$$= \frac{4+7i}{13}$$

$$= \frac{4}{13} + \frac{7}{13}i$$

(b) $P(z) = z^3 + 5z^2 + 9z + 5$

(i) consider $P(-1) = (-1)^3 + 5(-1)^2 + 9(-1) + 5$

$= -1 + 5 - 9 + 5$

$= 0$

By the factor theorem $(z+1)$ is a factor of $P(z)$

(ii) $P(z) = z^3 + 5z^2 + 9z + 5$

$= (z+1)(z^2 + 4z + 5)$ by inspection

$= (z+1)(z^2 + 4z + 4 + 1)$

$= (z+1)((z+2)^2 - i^2)$

$= (z+1)(z+2+i)(z+2-i)$

$\therefore$ roots of $P(z) = 0$ are $z = -1, -2+i, -2-i$

Alternately, solve $z^2 + 4z + 5 = 0$ using quadratic formula or sum and product of roots

Question 11 (continued)

(c) $z = 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$ $\omega = 1 + i\sqrt{3}$

(i) $z = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$ $\therefore \operatorname{Arg}(z) = -\frac{\pi}{3}$

(ii) $\overline{\omega} = 1 - i\sqrt{3}$

$= 2e^{-i\frac{\pi}{3}}$

$z = 4e^{-i\frac{\pi}{3}}$

$z^2 = 16\, e^{-i\frac{2\pi}{3}}$ de moivre's Theorem

$\therefore \dfrac{\overline{\omega}}{z^2} = \dfrac{2e^{-i\frac{\pi}{3}}}{16\, e^{-i\frac{2\pi}{3}}} = \dfrac{1}{8} e^{i\frac{\pi}{3}}$

(iii) $\left(\dfrac{\overline{\omega}}{z^2}\right)^n = \left(\dfrac{1}{8}\right)^n e^{\frac{in\pi}{3}}$ de moivre's theorem

If $\left(\dfrac{\overline{\omega}}{z^2}\right)^n$ is real $\operatorname{Arg}\left(\dfrac{\overline{\omega}}{z^2}\right)^n = k\pi$, $k \in \mathbb{Z}$

so $\dfrac{n\cancel{\pi}}{3} = k\cancel{\pi}$

$n = 3k$, $k \in \mathbb{Z}$

so n is a multiple of 3.

(d) $\int \ln x \, dx = x\ln x - \int \frac{1}{\cancel{x}} \times \cancel{x} \, dx$

$= x\ln x - x + C$

$u = \ln x$	$v' = 1$
$u' = \frac{1}{x}$	$v = x$

Question 11 (continued)

(e) RTP: n^2-1 is divisible by 8, for odd integers n

Let $n = 2k+1$, $k \in \mathbb{Z}$ (n is an odd integer)

$$\begin{aligned} n^2-1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k \cancel{+1} \cancel{-1} \\ &= 4k(k+1) \end{aligned}$$

Now either k is even or if k is odd then $k+1$ is even so $k(k+1)$ is even. Let $k(k+1) = 2m$, $m \in \mathbb{Z}$

Then
$$\begin{aligned} n^2-1 &= 4(2m) \\ &= 8m \end{aligned}$$
which is divisible by 8.

Question 12

(a) $\int \sin^3\theta \cos^3\theta \, d\theta = \int \sin^3\theta \, (1-\sin^2\theta)\cos\theta \, d\theta$

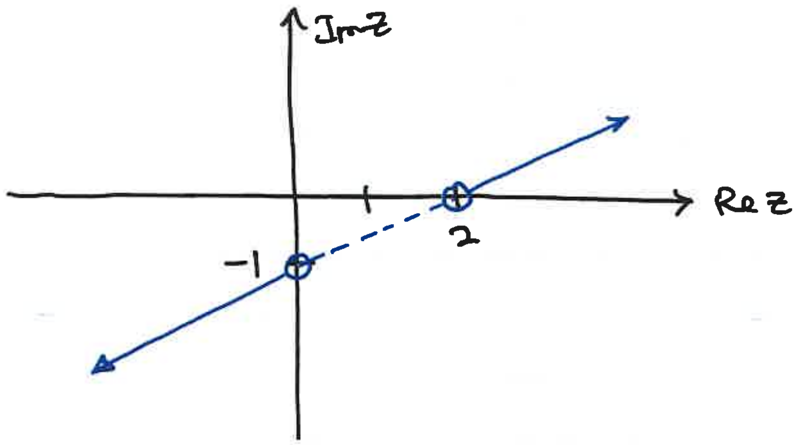
$= \int (\sin^3\theta - \sin^5\theta)\cos\theta \, d\theta$

$= \frac{1}{4}\sin^4\theta - \frac{1}{6}\sin^6\theta + C$

(b) $\arg\left(\frac{z-2}{z+i}\right) = 0 \quad \therefore \arg(z-2) - \arg(z+i) = 0$

$\arg(z-2) = \arg(z+i)$

(i)



(ii) Cartesian Equation is $y = \frac{1}{2}x - 1$

where $x < 0$ or $x > 2$

(c) (i) Amplitude $= 35$

Freq $= 30$ cycles per min

Period $= \frac{1}{30}$ min or 2 seconds

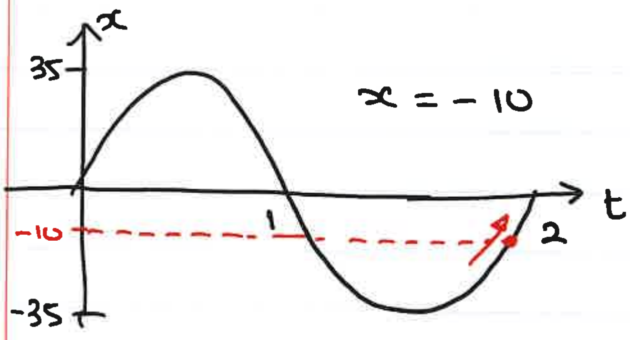
At $t = 0$, $x = 0$

$n = \frac{2\pi}{2} = \pi$

$x = 35 \sin \pi t$

Question 12 (continued)

(c) (ii)



$x = -10$ (10 cm below mean position)

$\therefore \quad 35 \sin \pi t = -10$

$\sin \pi t = \dfrac{-10}{35}$

$\pi t = \pi + 0.2898, \; 2\pi - 0.2898$

$t = 1.0922, \; 1.9078 \quad (4dp)$

Rises through $x = -10$ when $t = 1.91$ s (2dp).

(d) $V = \pi \displaystyle\int_2^4 \frac{8x-1}{(2x+1)(x+1)}\,dx \qquad y^2 = \frac{8x-1}{(2x+1)(x+1)}$

Consider $\dfrac{8x-1}{(2x+1)(x+1)} = \dfrac{A}{2x+1} + \dfrac{B}{x+1}$

$= \dfrac{A(x+1) + B(2x+1)}{(2x+1)(x+1)}$

$8x - 1 = A(x+1) + B(2x+1)$

Sub $x = -1$: $\quad -9 = A(0) + B(-1)$

$\therefore B = 9$

Sub $x = -\frac{1}{2}$: $\quad -5 = A\left(\frac{1}{2}\right) + B(0)$

$A = -10$

$\therefore \dfrac{8x-1}{(2x+1)(x+1)} = \dfrac{-10}{2x+1} + \dfrac{9}{x+1}$

Question 12 (continued)

(d) $\therefore \quad V = \pi \int_2^4 \left(\frac{-10}{2x+1} + \frac{9}{x+1} \right) dx$

$$= \pi \Big[-5 \ln|2x+1| + 9 \ln|x+1| \Big]_2^4$$

$$= \pi \Big[(-5 \ln 9 + 9 \ln 5) - (-5 \ln 5 + 9 \ln 3) \Big]$$

$$\doteqdot 5.21 \text{ units}^3 \quad (2\,dp)$$

Question 13

(a) $t = \tan\frac{x}{2}$

$x = 2\tan^{-1} t$

$dx = \frac{2}{1+t^2}\,dt$

,

x	0	$\frac{\pi}{2}$
t	0	1

, $\cos x = \frac{1-t^2}{1+t^2}$

$$\int_0^{\pi/2} \frac{1}{2-\cos x}\,dx = \int_0^1 \frac{1}{2-\frac{(1-t^2)}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2}{2(1+t^2)-(1-t^2)}\,dt$$

$$= \int_0^1 \frac{2}{2+2t^2-1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{1+3t^2}\,dt$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3}t\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right)$$

$$= \frac{2}{\sqrt{3}} \times \left(\frac{\pi}{3} - 0\right)$$

$$= \frac{2\pi}{3\sqrt{3}}$$

(b) (i) $\vec{AB} = \begin{pmatrix}0\\2\\5\end{pmatrix} - \begin{pmatrix}-1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$|\vec{AB}| = \sqrt{1^2+0^2+2^2} = \sqrt{5}$

Vector eqn of sphere is $\left|\underset{\sim}{s} - \begin{pmatrix}-1\\2\\3\end{pmatrix}\right| = \sqrt{5}$

Question 13 continued

b)(ii) $l_1: \underset{\sim}{r} = \begin{pmatrix}-1\\4\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}-1+\mu\\4-3\mu\\2+2\mu\end{pmatrix}$

Sub $\underset{\sim}{r}$ into equation of sphere

$$\left|\begin{pmatrix}-1+\mu\\4-3\mu\\2+2\mu\end{pmatrix} - \begin{pmatrix}-1\\2\\3\end{pmatrix}\right| = \sqrt{5}$$

$$\left|\begin{matrix}\mu\\2-3\mu\\-1+2\mu\end{matrix}\right| = \sqrt{5}$$

$$\mu^2 + (2-3\mu)^2 + (-1+2\mu)^2 = 5$$

$$\mu^2 + 4 - 12\mu + 9\mu^2 + 1 - 4\mu + 4\mu^2 = 5$$

$$14\mu^2 - 16\mu + 5 = 5$$

$$14\mu^2 - 16\mu = 0$$

$$2\mu(7\mu - 8) = 0$$

$$\mu = 0,\ \frac{8}{7}$$

$\mu = 0$: point is $(-1, 4, 2)$

$\mu = \frac{8}{7}$ point is $\left(\frac{1}{7}, \frac{4}{7}, \frac{30}{7}\right)$

(c) (i) $\forall\ x, y \in \mathbb{R}^+$, If $y > x$ then $y^3 - x^2y > xy^2 - x^3$

(ii) Given $y > x$ prove $y^3 - x^2y > xy^2 - x^3$

ie. $y^3 - x^2y - xy^2 - x^3 > 0$

consider $y^3 - x^2y - xy^2 - x^3 = y(y^2 - x^2) - x(y^2 - x^2)$

$= (y-x)(y^2-x^2)$

$= (y-x)^2(y+x)$

> 0 $\quad$ $y - x > 0$ as $y > x$

$y + x > 0$ as $x, y \in \mathbb{R}^+$

Question 13 (continued)

(d) $m = 0.6$ At $t=0$, $x=10$, $v=0$ $F = -\frac{3}{x^3}$

$F = ma$ $\therefore$ $0.6a = -\frac{3}{x^3}$

$$a = -\frac{3}{0.6x^3}$$

$$= \frac{-5}{x^3}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{5}{x^3}$$

Integrate wrt. x

$$\int_0^v \frac{d}{dx}\left(\frac{1}{2}v^2\right)dx = \int_{10}^{2.5} \frac{-5}{x^3}\,dx$$

$$\left[\frac{1}{2}v^2\right]_0^v = \frac{-5}{-2}\left[\frac{1}{x^2}\right]_{10}^{2.5}$$

$$\frac{1}{\cancel{2}}v^2 = \frac{5}{\cancel{2}}\left(\frac{1}{2.5^2} - \frac{1}{10^2}\right)$$

$$v^2 = \cancel{5}^{1} \times \frac{3}{\cancel{20}\,4}$$

$$v^2 = \frac{3}{4}$$

$$v = -\frac{\sqrt{3}}{2} \quad (v<0 \text{ moving left})$$

Question 14

(a) (i) $(z^n - e^{i\theta})(z^n + e^{-i\theta}) = (z^n)^2 + z^n e^{-i\theta} - z^n e^{i\theta} - (e^{i\theta})(e^{-i\theta})$

$= z^{2n} - z^n(e^{i\theta} - e^{-i\theta}) - e^0$

$= z^{2n} - z^n(2i\sin\theta) - 1$

$= z^{2n} - (2i\sin\theta)z^n - 1$

(ii) $z^6 - iz^3 - 1 = 0$

Using part (i) with $n=3$ and $2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

the equation is equivalent to

$(z^3 - e^{i\frac{\pi}{6}})(z^3 + e^{-i\frac{\pi}{6}}) = 0$

$z^3 = e^{i\frac{\pi}{6}}$

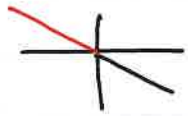
$= e^{i(\frac{\pi}{6} + 2k\pi)}, k \in \mathbb{Z}$

$z = e^{i(\frac{\pi}{18} + \frac{2k\pi}{3})}$

$z = e^{i\frac{\pi}{18}}, e^{i\frac{13\pi}{18}}, e^{-i\frac{11\pi}{18}}$

$k=0 \quad k=1 \quad k=-1$

OR



$z^3 = -e^{-i\frac{\pi}{6}}$

$z^3 = e^{i(\frac{5\pi}{6} + 2k\pi)}, k \in \mathbb{Z}$

$z = e^{i(\frac{5\pi}{18} + \frac{2k\pi}{3})}$

$z = e^{i\frac{5\pi}{18}}, e^{i\frac{17\pi}{18}}, e^{-i\frac{7\pi}{18}}$

$k=0 \quad k=1 \quad k=-1$

(b) (i) Let $u = a - x$

$du = -dx$

x	0	a
u	a	0

$\int_0^a f(a-x)\,dx = \int_a^0 f(u)(-du)$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$ $\quad$ u is a dummy variable

Question 14 (continued)

(b) (ii) $I = \int_0^1 \frac{\ln(x+1)}{\ln((2-x)(1+x))}\,dx = \int_0^1 \frac{\ln(1-x+1)}{\ln(2-(1-x))(1+1-x)}\,dx$ using (i)

$$= \int_0^1 \frac{\ln(2-x)}{\ln[(1+x)(2-x)]}\,dx$$

$$2I = \int_0^1 \frac{\ln(x+1)}{\ln[(2-x)(1+x)]}\,dx + \int_0^1 \frac{\ln(2-x)}{\ln[(2-x)(1+x)]}\,dx$$

$$= \int_0^1 \frac{\ln(x+1)+\ln(2-x)}{\ln(2-x)+\ln(1+x)}\,dx \qquad \ln(ab)=\ln a+\ln b$$

$$= \int_0^1 dx$$

$$= [x]_0^1$$

$$= 1$$

$$\therefore I = \frac{1}{2}$$

Question 14 (continued)

(c) RTP: $\dfrac{1}{a+1}+\dfrac{1}{a+2}+\dots+\dfrac{1}{2a}>\dfrac{13}{24}$

Test $a=2$

LHS $=\dfrac{1}{2+1}+\dfrac{1}{2(2)}=\dfrac{1}{3}+\dfrac{1}{4}=\dfrac{7}{12}=\dfrac{14}{24}$

LHS > RHS $\therefore$ stmt is true for $a=2$

Assume true for $a=k$, $k\in\mathbb{Z}$, $k>2$

ie. $\dfrac{1}{k+1}+\dfrac{1}{k+2}+\dots+\dfrac{1}{2k}>\dfrac{13}{24}$

Prove true for $a=k+1$

RTP: $\dfrac{1}{k+2}+\dfrac{1}{k+3}+\dots+\dfrac{1}{2(k+1)}>\dfrac{13}{24}$

LHS $=\dfrac{1}{k+2}+\dfrac{1}{k+3}+\dots+\dfrac{1}{2k}+\dfrac{1}{2k+1}+\dfrac{1}{2k+2}$

$=\underbrace{\dfrac{1}{k+1}+\dfrac{1}{k+2}+\dfrac{1}{k+3}+\dots+\dfrac{1}{2k}}+\dfrac{1}{2k+1}+\dfrac{1}{2k+2}-\dfrac{1}{k+1}$

$>\dfrac{13}{24}+\dfrac{1}{2k+1}+\dfrac{1}{2(k+1)}-\dfrac{1}{k+1}$ by assumption

$>\dfrac{13}{24}+\left(\dfrac{1}{2k+2}+\dfrac{1}{2(k+1)}-\dfrac{1}{k+1}\right)$

$=\dfrac{13}{24}+\left(\dfrac{1}{2(k+1)}+\dfrac{1}{2(k+1)}-\dfrac{1}{k+1}\right)$

$=\dfrac{13}{24}+\left(\dfrac{\cancel{2}\,1}{\cancel{2}(k+1)}-\dfrac{1}{k+1}\right)$

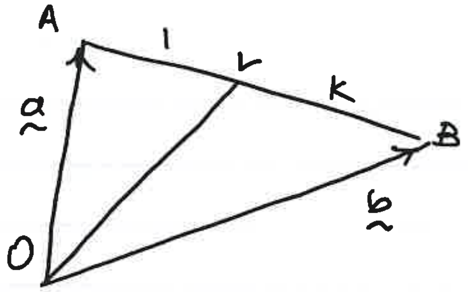
$=\dfrac{13}{24}$

$=$ RHS

$\therefore$ true for $a=k+1$ when true for $a=k$

Hence, the statement is true by Mathematical Induction

Question 14 (continued)



(d)(i) $\dfrac{OA}{OB} = \dfrac{AL}{LB} = \dfrac{1}{k}$

$\overrightarrow{OL} = \overrightarrow{OA} + \overrightarrow{AL}$

$= \underset{\sim}{a} + \dfrac{1}{k+1}(\overrightarrow{AB})$

$= \underset{\sim}{a} + \dfrac{1}{k+1}(\underset{\sim}{b} - \underset{\sim}{a}) = \underset{\sim}{a} + \dfrac{1}{k+1}\underset{\sim}{b} - \dfrac{1}{k+1}\underset{\sim}{a}$

$= \dfrac{(k+1)\underset{\sim}{a} - \underset{\sim}{a}}{k+1} + \dfrac{1}{k+1}\underset{\sim}{b}$

$= \dfrac{k}{k+1}\underset{\sim}{a} + \dfrac{1}{k+1}\underset{\sim}{b}$ as required.

(ii) Let A' and B' be points on OA and OB such that $\overrightarrow{OA'} = \hat{\underset{\sim}{a}}$ and $\overrightarrow{OB'} = \hat{\underset{\sim}{b}}$

Let $\overrightarrow{OA'} + \overrightarrow{OB'} = \overrightarrow{OM}$ so that $\overrightarrow{OM} = \hat{\underset{\sim}{a}} + \hat{\underset{\sim}{b}}$

$OA'B'M$ is a rhombus, so $\overrightarrow{OM}$ bisects $\angle BOA$ (diagonals of rhombus bisect vertex angles).

To show that $\overrightarrow{OL}$ bisects $\angle BOA$ we need to show that $\overrightarrow{OL}$ is a scalar multiple of $\overrightarrow{OM}$.

$\overrightarrow{OM} = \hat{\underset{\sim}{a}} + \hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|} + \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$ Now $\dfrac{|\underset{\sim}{a}|}{|\underset{\sim}{b}|} = \dfrac{1}{k}$ so $|\underset{\sim}{a}| = \dfrac{|\underset{\sim}{b}|}{k}$

$= k\dfrac{\underset{\sim}{a}}{|\underset{\sim}{b}|} + \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$

$= \dfrac{k+1}{|\underset{\sim}{b}|}\left(\dfrac{k}{k+1}\underset{\sim}{a} + \dfrac{1}{k+1}\underset{\sim}{b}\right)$

$= \dfrac{k+1}{|\underset{\sim}{b}|}\overrightarrow{OL}$ using (i)

$\overrightarrow{OM}$ is a scalar multiple of $\overrightarrow{OL}$ so, O, L, M are collinear

As $\overrightarrow{OM}$ bisects $\angle BOA$ then $\overrightarrow{OL}$ bisects $\angle BOA$

Question 14 (continued)

Alternate solution

(d) (ii) we need to show that $A\hat{O}L = B\hat{O}L$

so $\cos A\hat{O}L = \cos B\hat{O}L$ (angle in 1^{st} or 2^{nd} quad)

$$\cos A\hat{O}L = \frac{\underset{\sim}{a} \cdot \vec{OL}}{|\underset{\sim}{a}||\vec{OL}|} \qquad \cos B\hat{O}L = \frac{\underset{\sim}{b} \cdot \vec{OL}}{|\underset{\sim}{b}||\vec{OL}|}$$

RTP: $$\frac{\underset{\sim}{a} \cdot \vec{OL}}{|\underset{\sim}{a}|\cancel{|\vec{OL}|}} = \frac{\underset{\sim}{b} \cdot \vec{OL}}{|\underset{\sim}{b}|\cancel{|\vec{OL}|}}$$

$$LHS = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} \cdot \left(\frac{k}{k+1}\underset{\sim}{a} + \frac{1}{k+1}\underset{\sim}{b}\right) \quad \text{using (i)}$$

$$= \left(\frac{k}{k+1}\frac{|\underset{\sim}{a}|^2}{|\underset{\sim}{a}|} + \frac{1}{k+1}\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|}\right) \quad \text{as } \underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$$

$$= \frac{k}{k+1}|\underset{\sim}{a}| + \frac{1}{k+1}\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|} \quad \text{But } \frac{|\underset{\sim}{a}|}{|\underset{\sim}{b}|} = \frac{1}{k} \Rightarrow |\underset{\sim}{a}| = \frac{|\underset{\sim}{b}|}{k}$$

$$= \frac{\cancel{k}}{k+1}\frac{|\underset{\sim}{b}|}{\cancel{k}} + \frac{k}{k+1}\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{1}{k+1}|\underset{\sim}{b}| + \frac{k}{k+1}\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$RHS = \frac{\underset{\sim}{b} \cdot \vec{OL}}{|\underset{\sim}{b}|}$$

$$= \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}\left(\frac{k}{k+1}\underset{\sim}{a} + \frac{1}{k+1}\underset{\sim}{b}\right)$$

$$= \frac{k}{k+1}\frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{b}|} + \frac{1}{k+1}\frac{|\underset{\sim}{b}|^2}{|\underset{\sim}{b}|} \quad \text{as } \underset{\sim}{b} \cdot \underset{\sim}{b} = |\underset{\sim}{b}|^2$$

$$= \frac{1}{k+1}|\underset{\sim}{b}| + \frac{k}{k+1}\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|} \quad \text{as } \underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$$

LHS = RHS $\therefore \cos A\hat{O}L = \cos B\hat{O}L$

Then $A\hat{O}L = B\hat{O}L$

$\therefore$ OL is the angle bisector of $A\hat{O}B$

Question 15

a) $I_n = \int_0^1 x^n\sqrt{1-x^2}\,dx$

$u = x^{n-1}$ $\quad v' = x\sqrt{1-x^2}$

$u' = (n-1)x^{n-2}$ $\quad v = -\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3}(1-x^2)^{3/2}$

$= -\frac{1}{3}(1-x^2)^{3/2}$

$= \int_0^1 x^{n-1}\; x\sqrt{1-x^2}\,dx$

$= -\frac{1}{3}\left[x^{n-1}(1-x^2)^{3/2}\right]_0^1 - \int_0^1 (n-1)x^{n-2} \times -\frac{1}{3}(1-x^2)^{3/2}\,dx$

$= 0 + \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2}\,dx$

$= \frac{n-1}{3}\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx - \frac{n-1}{3}\int_0^1 x^n\sqrt{1-x^2}\,dx$

$I_n = \frac{n-1}{3}I_{n-2} - \frac{n-1}{3}I_n$

$\left(1 + \frac{n-1}{3}\right)I_n = \frac{n-1}{3}I_{n-2}$

$\frac{3+n-1}{\cancel{3}}I_n = \frac{n-1}{\cancel{3}}I_{n-2}$

$(n+2)I_n = (n-1)I_{n-2}$

$I_n = \frac{n-1}{n+2}I_{n-2}$ as required.

Question 15 (continued)

(b) (i) $$\sum_{r=1}^{n} \cos 2r\theta = \frac{\sin(2n+1)\theta - \sin\theta}{2\sin\theta}$$

Test $n=1$

LHS $= \cos 2\theta$

$$\text{RHS} = \frac{\sin 3\theta - \sin\theta}{2\sin\theta}$$

$$= \frac{3\sin\theta - 4\sin^3\theta - \sin\theta}{2\sin\theta}$$

$$= \frac{\cancel{2\sin\theta}(1-2\sin^2\theta)}{\cancel{2\sin\theta}}$$

$$= 1 - 2\sin^2\theta$$

$$= \cos 2\theta$$

LHS = RHS $\therefore$ true for $n=1$

Assume true for $n=k$ ie. $\displaystyle\sum_{r=1}^{k} \cos 2r\theta = \frac{\sin(2k+1)\theta - \sin\theta}{2\sin\theta}$

Prove true for $n=k+1$ RTP: $\displaystyle\sum_{r=1}^{k+1} \cos 2r\theta = \frac{\sin(2k+3)\theta - \sin\theta}{2\sin\theta}$

$$\text{LHS} = \sum_{r=1}^{k+1} \cos 2r\theta$$

$$= \sum_{r=1}^{k} \cos 2r\theta + \cos(2k+2)\theta$$

$$= \frac{\sin(2k+1)\theta - \sin\theta}{2\sin\theta} + \cos(2k+2)\theta \quad \text{by assumption}$$

$$= \frac{\sin(2k+1)\theta - \sin\theta + 2\sin\theta\cos(2k+2)\theta}{2\sin\theta}$$

$$= \frac{\sin(2k+1)\theta - \sin\theta + \sin[(2k+2)\theta + \theta] + \sin[(2k+2)\theta - \theta]}{2\sin\theta}$$

using products to sums

$$= \frac{\cancel{\sin(2k+1)\theta} - \sin\theta + \sin(2k+3)\theta - \cancel{\sin(2k+1)\theta}}{2\sin\theta}$$

$$= \frac{\sin(2k+3)\theta - \sin\theta}{2\sin\theta} = \text{RHS}$$

$\therefore$ true for $n=k+1$ when true for $n=k$

$\therefore$ stmt is true by Mathematical Induction

Question 15

(b) (ii) $\sum_{r=1}^{n} \cos^2\left(\frac{r\pi}{5}\right) = \frac{1}{2} \sum_{r=1}^{n} \left(1 + \cos \frac{2r\pi}{5}\right)$

$= \frac{1}{2}\left[\sum_{r=1}^{n} 1 + \sum_{r=1}^{n} \cos\left(\frac{2r\pi}{5}\right)\right]$

$= \frac{1}{2}\left[n + \frac{\sin (2n+1)\frac{\pi}{5} - \sin\frac{\pi}{5}}{2\sin\frac{\pi}{5}}\right]$

$\therefore \sum_{r=1}^{32} \cos^2\left(\frac{r\pi}{5}\right) = \frac{1}{2}\left[32 + \frac{\sin 13\pi - \sin\frac{\pi}{5}}{2\sin\frac{\pi}{5}}\right]$

$= \frac{1}{2}\left[32 + \frac{0 - \cancel{\sin \pi/5}}{2\cancel{\sin \pi/5}}\right]$

$= \frac{1}{2}\left[32 - \frac{1}{2}\right]$

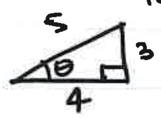
$= \frac{63}{4}$

Question 15 (continued)

(c) (i) At Q:

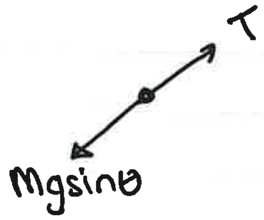
$$2M\ddot{x} = 2Mg - T \quad (1)$$

$\tan\theta = \frac{3}{4}$



$\sin\theta = \frac{3}{5}$ $\quad \cos\theta = \frac{4}{5}$

At P:



$$M\ddot{x} = T - Mg\sin\theta \quad (2)$$

$$(1)+(2): \quad 3\cancel{M}\ddot{x} = 2\cancel{M}g - \cancel{M}g\sin\theta$$

$$3\ddot{x} = 2g - g \times \frac{3}{5}$$

$$= \frac{7g}{5}$$

$$\ddot{x} = \frac{7g}{15}$$

(ii) $\ddot{x} = \frac{7g}{15}$ Integrate wrt t

$\dot{x} = \frac{7g}{15}t + C$ At $t=0$ $\dot{x}=0 \Rightarrow C=0$

$\dot{x} = \frac{7g}{15}t$ Integrate wrt t

$x = \frac{7g}{15}\frac{t^2}{2} + C_1$ At $t=0, x=0 \Rightarrow C_1=0$

$x = \frac{7g}{30}t^2$

when $x=2$ $\quad t^2 = \frac{2\times 30}{7g}$

$$t = \sqrt{\frac{60}{7g}}, \quad t>0$$

$$= \sqrt{\frac{6}{7}}$$

Question 15 (continued)

(c) (iii) When Q reaches the ground, the string will go slack so there is no tension.

At $t = \sqrt{\frac{6}{7}}$, $v = \frac{79}{15}\sqrt{\frac{6}{7}} = \frac{2}{3}\sqrt{42}$ from (ii)

Reset the clock and the origin.

At $t = 0$, $x = 0$, $v_0 = \frac{2}{3}\sqrt{42}$

At Q:

$$\cancel{M}\ddot{x} = -\cancel{M}g\sin\theta$$

$$\ddot{x} = -\frac{3g}{5} = -6 \qquad (g=10)$$

$$v\frac{dv}{dx} = -6$$

$$v\,dv = -6\,dx$$

$$\int_{v_0}^{0} v\,dv = -6\int_0^{D} dx$$

$$\left[\frac{v^2}{2}\right]_{v_0}^{0} = -6\left[x\right]_0^{D}$$

$$0 - \frac{v_0^2}{2} = -6(D-0)$$

$$\cancel{-}\frac{1}{\cancel{2}} \times \frac{\cancel{4}^{2}}{\cancel{9}_{3}} \times \cancel{42}^{14} = \cancel{-}6D$$

$$D = \frac{28}{18}$$

$$= 1.56 \text{ m (2dp)}$$

Question 16

(a) (i) $\omega^3 - 1 = 0$ as ω is a root of $x^3 - 1 = 0$

$(\omega - 1)(\omega^2 + \omega + 1) = 0$ difference of cubes

$\omega = 1$ or $\omega^2 + \omega + 1 = 0$

As ω is non-real then $\omega^2 + \omega + 1 = 0$

(ii) RTP: If n is divisible by 3 then $(x+1)^{2n} + x^{2n} + 1$ is not divisible by $x^2 + x + 1$

For the sake of contradiction, assume that n is divisible by 3 <u>and</u> $(x+1)^{2n} + x^{2n} + 1$ is divisible by $x^2 + x + 1$

Then

$(x+1)^{2n} + x^{2n} + 1 = (x^2 + x + 1)\,Q(x)$, $Q(x)$ polynomial

Sub $x = \omega$

$$(\omega+1)^{2n} + \omega^{2n} + 1 = (\omega^2 + \omega + 1)\,Q(\omega)$$
$$= 0 \quad \text{as } \omega^2 + \omega + 1 = 0$$
$$(-\omega^2)^{2n} + \omega^{2n} + 1 = 0 \quad \text{Now } n = 3k,\ k \in \mathbb{Z}$$
$$(-\omega^2)^{6k} + \omega^{6k} + 1 = 0$$
$$(-1)^{6k}(\omega^3)^{4k} + (\omega^3)^{2k} + 1 = 0 \quad \omega^3 = 1 \ (\text{root of } x^3 - 1 = 0)$$
$$1 + 1 + 1 = 0$$
$$3 = 0 \quad \text{which is absurd}$$

Hence, our assumption is incorrect.

$\therefore$ if n is divisible by 3, $(x+1)^{2n} + x^{2n} + 1$ is not divisible by $x^2 + x + 1$

Question 16 (continued)

(b) (i) $CP \perp AB$

vector eqn of line $\vec{AB}$: $\underset{\sim}{r} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix}$

Point M on $\vec{AB}$ = $\begin{pmatrix} 6-6\lambda \\ 2\lambda \\ 0 \end{pmatrix}$

$\vec{CM} = \begin{pmatrix} 6-6\lambda \\ 2\lambda \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6-6\lambda \\ 2\lambda \\ -3 \end{pmatrix}$

CM is perpendicular to AB

$\therefore \begin{pmatrix} 6-6\lambda \\ 2\lambda \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix} = 0$

$-6(6-6\lambda) + 2(2\lambda) = 0$

$40\lambda = 36$

$\lambda = \frac{36}{40} = \frac{9}{10}$

$\therefore \vec{CM} = \begin{pmatrix} 6-6\lambda \\ 2\lambda \\ -3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 9/5 \\ -3 \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$

$\hat{CP} = \dfrac{\cancel{3/5}\,(\underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k})}{\cancel{3/5} \times \sqrt{35}}$

$= \frac{1}{\sqrt{35}}\underset{\sim}{i} + \frac{3}{\sqrt{35}}\underset{\sim}{j} - \frac{5}{\sqrt{35}}\underset{\sim}{k}$

(ii) acceleration vector is $\begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix}$. we need the projection of this vector in the direction of $\vec{CP}$

acceleration down the plane $= \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{35} \\ 3/\sqrt{35} \\ -5/\sqrt{35} \end{pmatrix} \hat{CP}$

$= \frac{50}{\sqrt{35}}\left(\frac{1}{\sqrt{35}}\underset{\sim}{i} + \frac{3}{\sqrt{35}}\underset{\sim}{j} - \frac{5}{\sqrt{35}}\underset{\sim}{k}\right)$

$= \frac{50}{35}\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} = \frac{10}{7}\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$

Question 16 (continued)

(b) (iii) $\ddot{x} = \frac{10}{7}\begin{pmatrix}1\\3\\-5\end{pmatrix}$ from (ii)

Integrate wrt t

$\dot{\underset{\sim}{r}} = \frac{10}{7}\begin{pmatrix}t\\3t\\-5t\end{pmatrix} + \underset{\sim}{c}$ At $t=0$ $\dot{\underset{\sim}{r}} = \underset{\sim}{0}$ $\therefore \underset{\sim}{c} = 0$

$\dot{\underset{\sim}{r}} = \frac{10t}{7}\begin{pmatrix}1\\3\\-5\end{pmatrix}$

Integrate wrt t

$\underset{\sim}{r} = \frac{\cancel{10}^{5}}{7}\frac{t^2}{\cancel{2}}\begin{pmatrix}1\\3\\-5\end{pmatrix} + \underset{\sim}{D}$

$= \frac{5t^2}{7}\begin{pmatrix}1\\3\\-5\end{pmatrix} + \underset{\sim}{D}$ At $t=0$ $\underset{\sim}{r} = \begin{pmatrix}0\\0\\3\end{pmatrix}$ $\therefore \underset{\sim}{D} = \begin{pmatrix}0\\0\\3\end{pmatrix}$

$\underset{\sim}{r} = \frac{5t^2}{7}\begin{pmatrix}1\\3\\-5\end{pmatrix} + \begin{pmatrix}0\\0\\3\end{pmatrix}$

Sub $t = 0.5$

$\underset{\sim}{r} = \frac{5}{7}\times\frac{1}{4}\begin{pmatrix}1\\3\\-5\end{pmatrix} + \begin{pmatrix}0\\0\\3\end{pmatrix}$

$= \frac{5}{28}\underset{\sim}{i} + \frac{15}{28}\underset{\sim}{j} + \frac{59}{28}\underset{\sim}{k}$

Location of the ball is $\left(\frac{5}{28}, \frac{15}{28}, \frac{59}{28}\right)$

Question 16 (continued)

(c) (i) $\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} {}^nC_k\left(\frac{1}{n}\right)^k$

$${}^nC_k\left(\frac{1}{n}\right)^k = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!\quad n^k}$$

$$= \frac{1}{k!} \times \frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots \times \frac{n-k+1}{n}$$

$$< \frac{1}{k!} \quad \text{as } \frac{n-j}{n} < 1$$

$$\therefore \left(1+\frac{1}{n}\right)^n < \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$< 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$$

$$\therefore \left(1+\frac{1}{n}\right)^n < e$$

(ii) $P(n) = \left(\frac{3}{2} \times \frac{5}{4} \times \frac{9}{8} \times \cdots \times \frac{2^n+1}{2^n}\right)$

$$< \left[\frac{\left(\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\cdots+\frac{2^n+1}{2^n}\right)}{n}\right]^n \quad \text{using AM-GM}$$

$$= \left[\frac{1}{n}\left[\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{4}\right)+\left(1+\frac{1}{8}\right)+\cdots+\left(1+\frac{1}{2^n}\right)\right]\right]^n$$

$$= \left[\frac{1}{n}\left(\underbrace{1+1+\cdots+1}_{n \text{ times}} + \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}\right)\right)\right]^n$$

$$= \left[\frac{1}{n}\left(n + \frac{1}{2}\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}\right)\right]^n$$

$$= \left[\frac{1}{n}\left(n+1-\frac{1}{2^n}\right)\right]^n$$

$$= \left(1+\frac{1}{n}-\frac{1}{n2^n}\right)^n$$

$$< \left(1+\frac{1}{n}\right)^n < e \quad \text{from (i)} \qquad \therefore P(n) < e$$