North Sydney Girls High School

2025 **HSC TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8 – 15)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME:____________________________ **TEACHER:**____________________________

STUDENT NUMBER:

Question	1–10	11	12	13	14	15	16	Total
Mark	/10	/14	/15	/15	/16	/14	/16	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1−10.

---

1 Consider the statement:

"If you do not try, then you will not succeed."

Which of the following statements is logically equivalent to the statement above?

A. If you did not succeed, then you tried.

B. If you did not succeed, then you did not try.

C. If you succeeded, then you tried.

D. If you succeeded, then you did not try.

2 Which of the following vectors is perpendicular to the vector $2\underset{\sim}{i}+9\underset{\sim}{j}-3\underset{\sim}{k}$?

A. $3\underset{\sim}{i}+\underset{\sim}{j}-5\underset{\sim}{k}$

B. $6\underset{\sim}{i}-\underset{\sim}{j}-\underset{\sim}{k}$

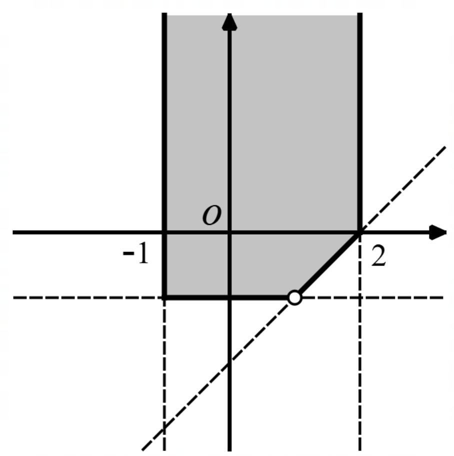
C. $-\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}$

D. $-3\underset{\sim}{i}+2\underset{\sim}{j}+4\underset{\sim}{k}$

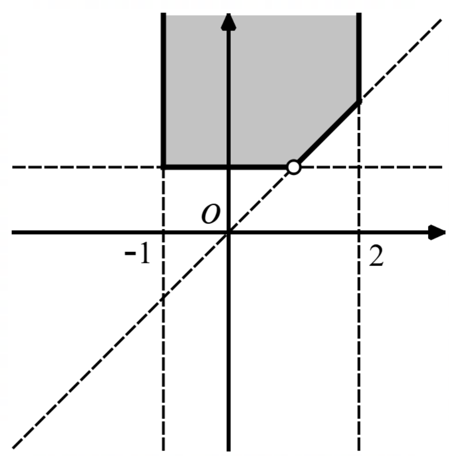
3 Let R be the region in the complex plane defined by $-2 \le z + \overline{z} \le 4$ and $\frac{\pi}{4} \le \operatorname{Arg}(z - 1 + i) \le \pi$.

Which diagram best represents the region R?

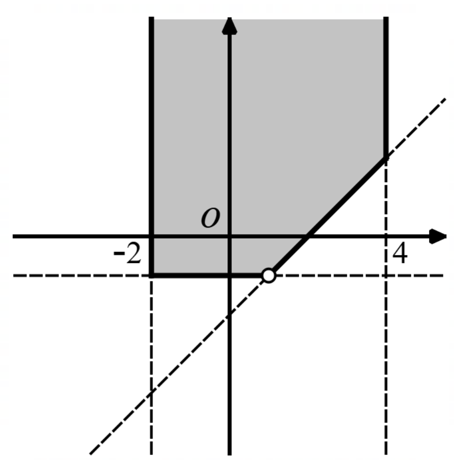
A.



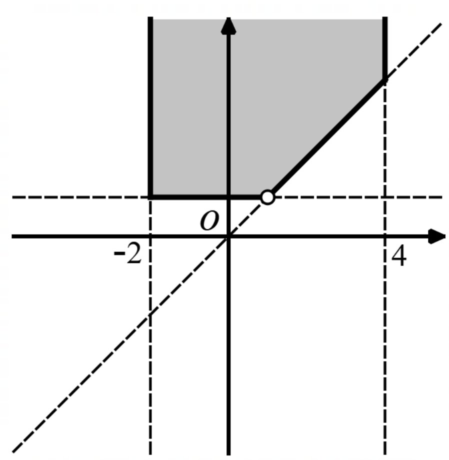
B.



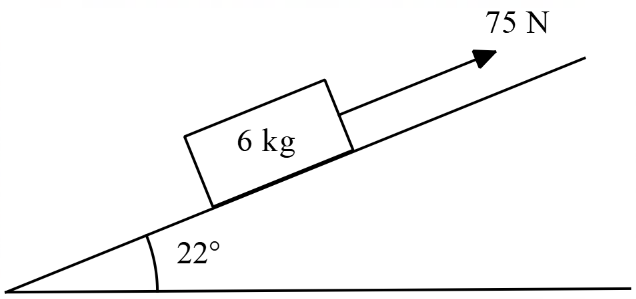
C.



D.



4 An object of mass 6 kg is placed on a smooth ramp that is inclined at an angle of $22°$ to the horizontal. It is pulled up the ramp by a force of 75 Newtons as shown in the diagram.



The acceleration due to gravity is 9.8 ms^{-2}.

Which of the following is closest to the acceleration of the block?

A. 3.4 ms^{-2}

B. 8.8 ms^{-2}

C. 16.2 ms^{-2}

D. 21.6 ms^{-2}

5 Which of the following is a vector equation of the line joining the points $A(3,-2,5)$ and $B(-1,4,9)$?

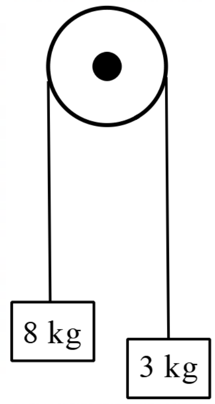
A. $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 4 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

B. $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 4 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$

C. $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 9 \end{pmatrix}$

D. $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

6 A light inextensible string passes over a smooth pulley. Attached to the ends of the string are masses of 8 kg and 3 kg as shown.



The acceleration due to gravity is 9.8 ms^{-2}.

Which of the following is closest to the acceleration of the masses?

A. 4.5 ms^{-2}

B. 6.1 ms^{-2}

C. 13.5 ms^{-2}

D. 21.6 ms^{-2}

7 Which of the following is equal to $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{6+\cos 2x}\,dx$?

A. $\displaystyle\int_0^1 \frac{1}{7+5t^2}\,dt$

B. $\displaystyle\int_0^1 \frac{1}{5+7t^2}\,dt$

C. $\displaystyle\frac{1}{3}\int_0^1 \frac{1}{3t^2+t+3}\,dt$

D. $\displaystyle\frac{1}{3}\int_0^1 \frac{1}{3t^2-t+3}\,dt$

8 A particle undergoes simple harmonic motion between $x = -3$ and 5 where x is the displacement measured in metres from a fixed point O. Given that its maximum acceleration is 20 ms^{-2}, what is the velocity of the particle at $x = 0$?

A. 8 ms^{-1}

B. $5\sqrt{3} \text{ ms}^{-1}$

C. $4\sqrt{5} \text{ ms}^{-1}$

D. $\dfrac{8\sqrt{15}}{3} \text{ ms}^{-1}$

9 Of the following expressions, which one need **NOT** contain a term involving arctan in its anti-derivative?

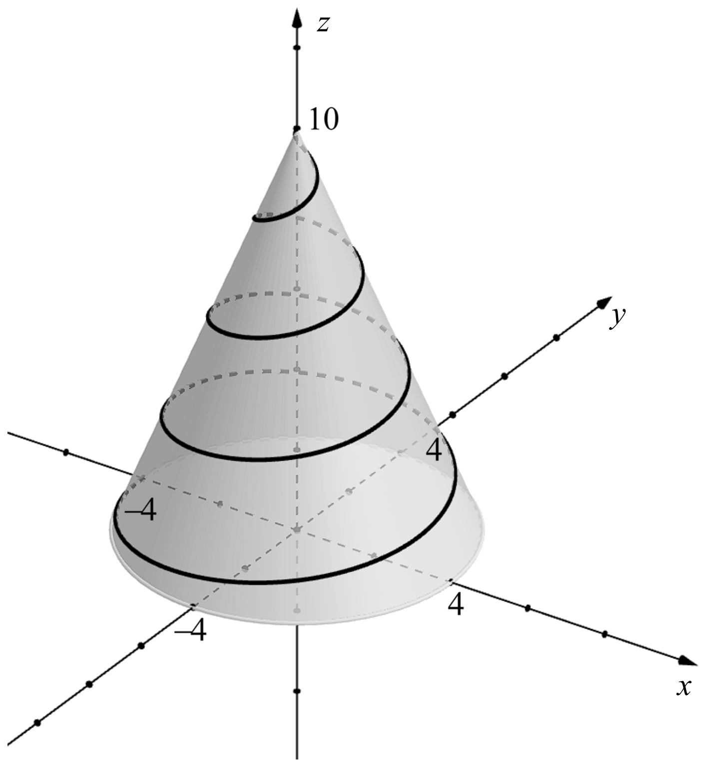
A. $\dfrac{2}{x^2 + 2x + 7}$

B. $\dfrac{e^x}{e^{2x} + 5e^x + 6}$

C. $\dfrac{1}{\sqrt{x}\left(x+3\right)}$

D. $\dfrac{2\sin x}{\cos 2x + 9}$

10 The diagram below shows a curve on the surface of a cone with radius 4 and height 10.



Which of the following position vectors can **NOT** describe the curve?

A. $\underset{\sim}{r}(t) = 2\left(t - \frac{19}{8}\right)\sin(-4\pi t)\underset{\sim}{i} + 2\left(t - \frac{19}{8}\right)\cos(4\pi t)\underset{\sim}{j} + 5\left(t - \frac{3}{8}\right)\underset{\sim}{k}$ for $\frac{3}{8} \le t \le \frac{19}{8}$

B. $\underset{\sim}{r}(t) = 2(t-2)\cos(4\pi t)\underset{\sim}{i} + 2(t-2)\sin(4\pi t)\underset{\sim}{j} + 5t\underset{\sim}{k}$ for $0 \le t \le 2$

C. $\underset{\sim}{r}(t) = 2(t-2)\cos(4\pi t)\underset{\sim}{i} + 2(t-2)\sin(-4\pi t)\underset{\sim}{j} + 5t\underset{\sim}{k}$ for $0 \le t \le 2$

D. $\underset{\sim}{r}(t) = 2(t-2)\cos(4\pi t)\underset{\sim}{i} + 2(2-t)\sin(-4\pi t)\underset{\sim}{j} + 5t\underset{\sim}{k}$ for $0 \le t \le 2$

Section II

90 marks
Attempt Questions 11−16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks**)** Use a SEPARATE writing booklet

(a) Consider the complex numbers $w = 1 - 2i$ and $z = 1 + 3i$.

(i) Express $w + \overline{z}$ in the form $x + iy$, where x and y are real numbers. **2**

(ii) Express $\dfrac{w}{z}$ in modulus-argument form. **2**

(b) Find $\displaystyle\int xe^{-x}dx$. **2**

(c) Find the angle between the vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$, giving the angle in degrees correct to 1 decimal place. **3**

(d) Given that $3 - \sqrt{3}i$ is a fourth root of $-72\left(1 + \sqrt{3}i\right)$, find the remaining 3 roots and express them in the form $re^{i\theta}$. **2**

(e) Consider the sequence u_n where $u_1 = 0,\ u_{n+1} = 2n - u_n$ for $n = 1,\ 2,\ \ldots$. **3**
Use mathematical induction to show that $2u_n = 2n - 1 + (-1)^n$ for $n = 1,\ 2,\ \ldots$.

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Consider the Statement A. **2**

Statement A: 'If the last digit of the integer n is 5 then n is divisible by 5.'

Prove by contradiction that Statement A is true.

(b) Use an appropriate substitution of the form $x = a + b\sin\theta$ to evaluate **3**

$$\int_7^{10} \sqrt{(x-4)(10-x)}\, dx.$$

(c) The velocity of a particle moving in a straight line is given by $v^2 = 128 + 32x - 16x^2$, where v is measured in ms^{-1} and x is the displacement measured in m from the origin.

(i) Show that the particle moves in simple harmonic motion with period $\frac{\pi}{2}$ s. **2**

(ii) What distance does the particle travel during a full period of its motion? **2**

(d) The position vectors of points A and B are $\begin{pmatrix} -7 \\ -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -1 \\ -8 \end{pmatrix}$ respectively. **3**

Find the vector equation of the sphere with AB as its diameter.

(e) (i) Prove that $\forall x, y \in \mathbb{R}^+$, $\frac{x+y}{2} \geq \sqrt{xy}$. **1**

(ii) Let a, b, c and d be positive real numbers. Use the result from (i) to prove that **2**

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 16.$$

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Evaluate $\displaystyle\int_0^{\pi} x\sin x\cos^4 x\,dx$ using the property $\displaystyle\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx$. **2**

(b) Consider the two lines in three dimensions given by **3**

$$\underset{\sim}{r} = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \text{ and } \underset{\sim}{r} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix}.$$

Find the point of intersection of the two lines.

(c) The acceleration of a particle moving in a straight line is given by $a = 4 - v^2$, where v is its velocity.

Initially, the particle has a velocity of $\dfrac{2}{3}$ ms^{-1} at $x = 0$.

(i) What is the displacement of the particle when it is travelling at a velocity of $1\ \text{ms}^{-1}$? **2**

(ii) What is the velocity of the particle at $t = \dfrac{1}{2}$? **4**

(d) The integral I_n is defined by $I_n = \displaystyle\int_0^{1} e^{-x}(1-x)^n\,dx$ for $n \geq 0$.

(i) Show that $I_n = 1 - nI_{n-1}$ for $n \geq 1$. **2**

(ii) Given that I_n can be expressed in the form $a + be^{-1}$ where a and b are integers, find an expression for b in terms of n. **2**

Question 14 (16 marks**)** Use a SEPARATE writing booklet

(a) Prove by mathematical induction that $n^3 - 1 > n^2 + n$, for integers $n \geq 2$. **3**

(b) Use the method of partial fractions to find the value of a, where $a > 0$, such that **4**

$$\int_0^a \frac{3-5x}{(1+x)\left(1-x+2x^2\right)}\,dx = 0.$$

(c) Let $w = \cos\theta + i\sin\theta$, where $0 < \theta < \pi$.

It is known that the complex number $w^2 + \frac{3\sqrt{2}}{w} - 3$ is purely imaginary.

(i) Show that $2\cos^2\theta + 3\sqrt{2}\cos\theta - 4 = 0$. **2**

(ii) Hence, or otherwise, find w. **2**

(iii) A and B are two points on an Argand diagram representing two distinct complex numbers z_1 and z_2 respectively. Suppose that $z_2 = wz_1$ and $|z_1| = 3$. Find the exact length of AB. **2**

(d) Consider $f(x) = x^m - mx$ where $m > 1$. **3**

By using calculus, or otherwise, show that $x^m - 1 \geq m(x-1)$ for all $x > 0$.

Question 15 (14 marks**)** Use a SEPARATE writing booklet

(a) Let z and w be non-zero complex numbers. Consider the statements below: **3**

$$P:\ z\overline{w}+\overline{z}w=0$$

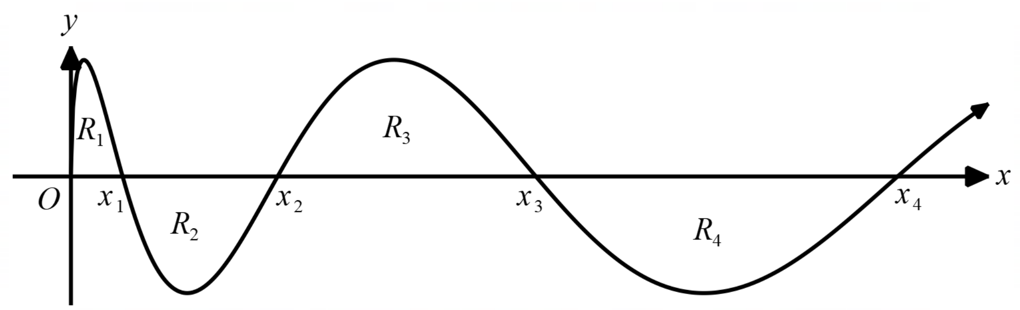
$$Q:\ \frac{z}{w}=ik,\ k\in\mathbb{R}$$

Prove that $P\Rightarrow Q$.

(b) (i) By using a suitable substitution, show that **3**

$$\int \sin\sqrt{x}\ dx = 2\sin\sqrt{x}-2\sqrt{x}\cos\sqrt{x}+C.$$

The diagram below shows part of the curve $y=\sin\sqrt{x}$ for $x\geq 0$.



The curve intersects the x-axis at $0, x_1, x_2, x_3, x_4, \ldots$.

The regions bounded by the curve and the x-axis are defined by $R_1, R_2, R_3, R_4, \ldots$ as shown in the diagram.

(ii) By first finding an expression for x_n, or otherwise, express the area of the region R_n in terms of n and π. **3**

Question 15 continues on page 13

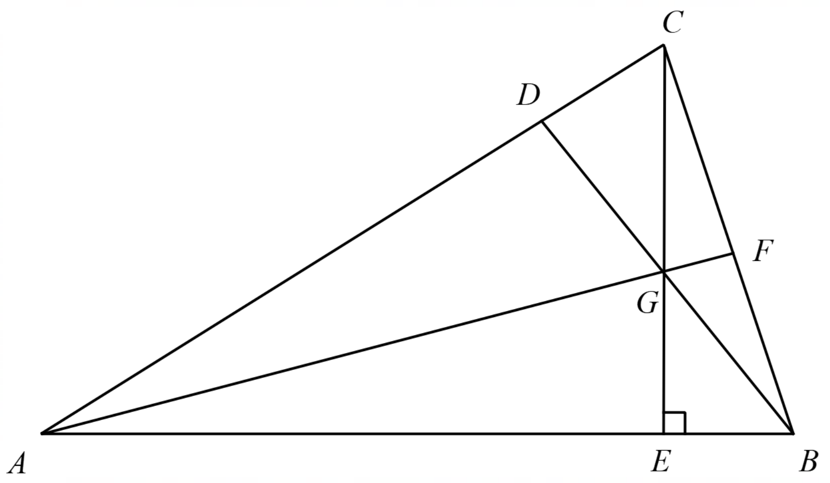
(c) The diagram shows triangle ABC.

The point E lies on AB so that $AE : EB = \lambda : 1$ and $CE \perp AB$.

The point G is the mid-point of CE.

BG is produced to meet AC at D and $AD : DC = r : 1$.

The point F is chosen on BC so that A, G, F are collinear.



NOT TO SCALE

Let $\overrightarrow{BE} = \underset{\sim}{u}$ and $\overrightarrow{EG} = \underset{\sim}{v}$.

(i) Express $\overrightarrow{BG}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **1**

(ii) By considering $\overrightarrow{BD}$, use vector methods to show that $r = \lambda + 1$. **4**

End of Question 15

Question 16 (16 marks) Use a SEPARATE writing booklet

(a) It is given that for positive numbers $a_1,\ a_2,\ a_3,\ ...,\ a_n$ with arithmetic mean A,

$$A^n \geq a_1 \times a_2 \times a_3 \times ... \times a_n. \qquad \text{(Do NOT prove this.)}$$

Let k be a positive integer.

(i) Prove that $k^k \geq 1 \times 3 \times 5 \times ... \times (2k-1)$. **2**

(ii) Hence prove that $\left(k^2+k\right)^k \geq (2k)!$. **2**

(b) Consider the equation $(z+2)^8 = z^8$.

(i) Show that the 6 non-real roots of the equation can be expressed in the form **4**

$$-1 - i\cot\left(\frac{k\pi}{8}\right),\ k = \pm 1,\ \pm 2,\ \pm 3.$$

(ii) Hence, find the value of $\sin^2\left(\frac{\pi}{8}\right)\sin^2\left(\frac{2\pi}{8}\right)\sin^2\left(\frac{3\pi}{8}\right)$ given that the product of roots of the equation is -16. **3**

Question 16 continues on page 15

Question 16 (continued)

(c) Let $z = \cos\theta + i\sin\theta$.

(i) Find the four values of z such that $\operatorname{Im}\left(z^2 + \overline{z}\right) = 0$. **2**

Let z_1 and z_2 be two of the values of z obtained in (i) such that $\operatorname{Im}\left(z_1\right) < 0 < \operatorname{Im}\left(z_2\right)$.

Also let $S_n = \sum_{r=1}^{n} w^r$, where $w = \frac{z_2}{z_1}$, for any positive integer n.

(ii) Find all positive integers k such that $\left(S_n\right)^k + \left(S_{n+1}\right)^k + \left(S_{n+2}\right)^k = 2$ for any positive integer n. **3**

End of paper

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NAME:____________________________ **TEACHER:**__________________________

STUDENT NUMBER:

Question	1–10	11	12	13	14	15	16	Total
Mark	/10	/14	/15	/15	/16	/14	/16	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1−10.

---

1 Consider the statement:

"If you do not try, then you will not succeed."

Which of the following statements is logically equivalent to the statement above?

A. If you did not succeed, then you tried.

B. If you did not succeed, then you did not try.

C. If you succeeded, then you tried.

D. If you succeeded, then you did not try.

Consider

P: Do not try
Q: Do not succeed

The statement is $P \Rightarrow Q$.

The contrapositive is logically equivalent i.e. $\sim Q \Rightarrow \sim P$

2 Which of the following vectors is perpendicular to the vector $2\underset{\sim}{i}+9\underset{\sim}{j}-3\underset{\sim}{k}$?

A. $3\underset{\sim}{i}+\underset{\sim}{j}-5\underset{\sim}{k}$

B. $6\underset{\sim}{i}-\underset{\sim}{j}-\underset{\sim}{k}$

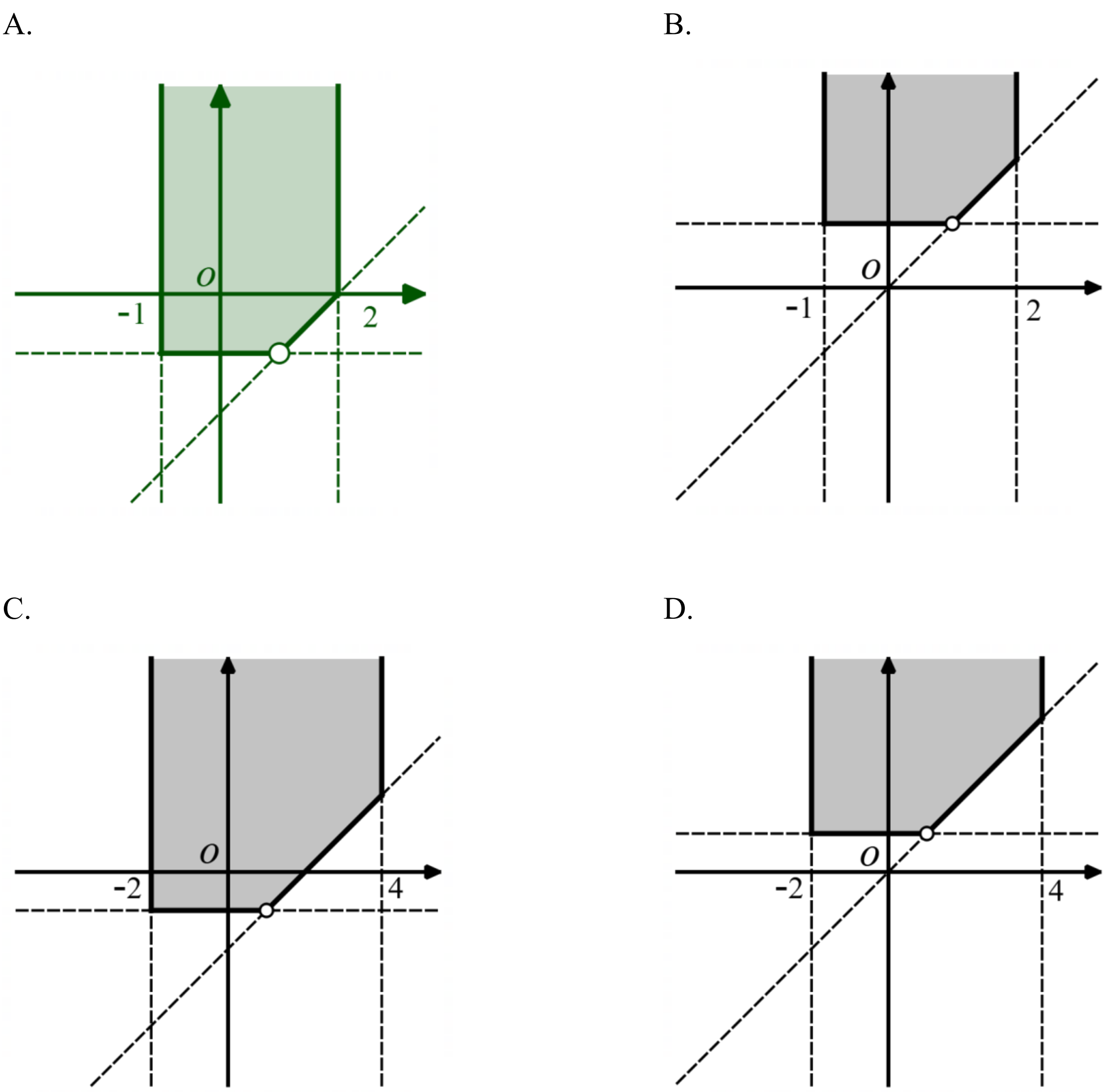
C. $-\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}$

D. $-3\underset{\sim}{i}+2\underset{\sim}{j}+4\underset{\sim}{k}$

$$\left(2\underset{\sim}{i}+9\underset{\sim}{j}-3\underset{\sim}{k}\right)\cdot\left(-3\underset{\sim}{i}+2\underset{\sim}{j}+4\underset{\sim}{k}\right)=2(-3)+9(2)-3(4)=0$$

3 Let R be the region in the complex plane defined by $-2 \le z+\bar{z} \le 4$ and $\frac{\pi}{4} \le \operatorname{Arg}(z-1+i) \le \pi$.

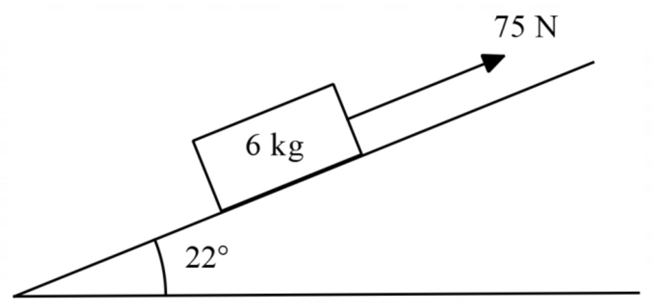
Which diagram best represents the region R?



$$-2 \le z+\bar{z} \le 4$$
$$-2 \le 2\operatorname{Re}(z) \le 4$$
$$-1 \le \operatorname{Re}(z) \le 2$$

$$\frac{\pi}{4} \le \operatorname{Arg}\left(z-(1-i)\right) \le \pi$$

4 An object of mass 6 kg is placed on a smooth ramp that is inclined at an angle of $22°$ to the horizontal. It is pulled up the ramp by a force of 75 Newtons as shown in the diagram.



The acceleration due to gravity is 9.8 ms^{-2}.

Which of the following is closest to the acceleration of the block?

A. 3.4 ms^{-2}

B. 8.8 ms^{-2}

C. 16.2 ms^{-2}

D. 21.6 ms^{-2}

$$\begin{aligned} F_{net} &= 75 - 6 \times 9.8 \times \sin 22° \\ 6a &= 75 - 6 \times 9.8 \times \sin 22° \\ a &= \frac{75 - 6 \times 9.8 \times \sin 22°}{6} \\ a &\approx 8.8 \end{aligned}$$

5 Which of the following is a vector equation of the line joining the points $A(3,-2,5)$ and $B(-1,4,9)$?

A. $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 4 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

B. $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 4 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$

C. $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 9 \end{pmatrix}$

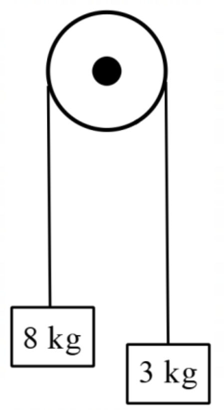
D. $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

Direction vector:

$$\begin{pmatrix} 3-(-1) \\ -2-4 \\ 5-9 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ -4 \end{pmatrix}$$

Direction vector can be any vector parallel to $\begin{pmatrix} 4 \\ -6 \\ -4 \end{pmatrix}$ e.g. $\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$

6 A light inextensible string passes over a smooth pulley. Attached to the ends of the string are masses of 8 kg and 3 kg as shown.



The acceleration due to gravity is $9.8\ \text{ms}^{-2}$.

Which of the following is closest to the acceleration of the masses?

A. $4.5\ \text{ms}^{-2}$

B. $6.1\ \text{ms}^{-2}$

C. $13.5\ \text{ms}^{-2}$

D. $21.6\ \text{ms}^{-2}$

For the 8 kg mass:

$$8a = 8g - T \quad (1)$$

For the 3 kg mass:

$$3a = T - 3g \quad (2)$$

$(1)+(2):$

$$11a = 5g$$
$$a = \frac{5g}{11}$$
$$a \approx 4.5$$

7 Which of the following is equal to $\int_0^{\frac{\pi}{4}} \frac{1}{6+\cos 2x}\, dx$?

A. $\int_0^1 \frac{1}{7+5t^2}\, dt$

B. $\int_0^1 \frac{1}{5+7t^2}\, dt$

C. $\frac{1}{3}\int_0^1 \frac{1}{3t^2+t+3}\, dt$

D. $\frac{1}{3}\int_0^1 \frac{1}{3t^2-t+3}\, dt$

Using the substitution $t = \tan x$

$$\frac{dt}{dx} = \sec^2 x$$

$$\frac{dt}{1+t^2} = dx$$

$$= \int_0^1 \frac{1}{6+\cos 2x}\, dx$$

$$= \int_0^1 \frac{1}{6+\frac{1-t^2}{1+t^2}}\, \frac{dt}{1+t^2}$$

$$= \int_0^1 \frac{1}{6\left(1+t^2\right)+1-t^2}\, dt$$

$$= \int_0^1 \frac{1}{7+5t^2}\, dt$$

8 A particle undergoes simple harmonic motion between $x = -3$ and 5 where x is the displacement measured in metres from a fixed point O. Given that its maximum acceleration is 20 ms^{-2}, what is the velocity of the particle at $x = 0$?

A. 8 ms^{-1}

B. $5\sqrt{3} \text{ ms}^{-1}$

C. $4\sqrt{5} \text{ ms}^{-1}$

D. $\dfrac{8\sqrt{15}}{3} \text{ ms}^{-1}$

Centre: $\dfrac{-3+5}{2} = 1$

Amplitude: $\dfrac{5-(-3)}{2} = 4$

$$\therefore \ddot{x} = -n^2(x-1)$$

Maximum acceleration occurs at left most extremity i.e. $x = -3$

$$20 = -n^2(-3-1)$$
$$n^2 = 5$$

$$v^2 = n^2\left(A^2 - (x-1)^2\right)$$

At $x = 0$

$$v^2 = 5\left(4^2 - (0-1)^2\right)$$
$$v^2 = 75$$
$$v = \pm 5\sqrt{3}$$

9 Of the following expressions, which one need **NOT** contain a term involving arctan in its anti-derivative?

A. $\dfrac{2}{x^2+2x+7}$

B. $\dfrac{e^x}{e^{2x}+5e^x+6}$

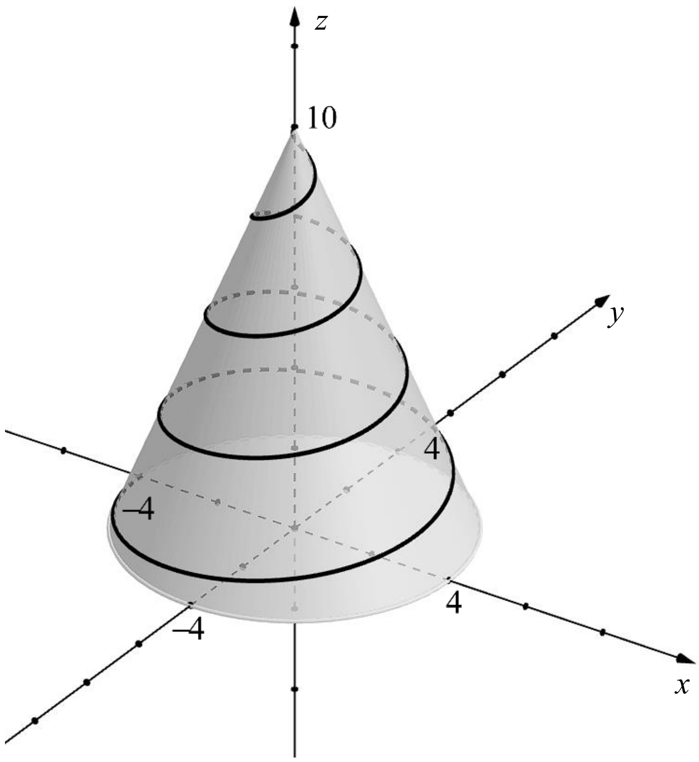
C. $\dfrac{1}{\sqrt{x}\left(x+3\right)}$

D. $\dfrac{2\sin x}{\cos 2x+9}$

$$\frac{e^x}{\left(e^x+2\right)\left(e^x+3\right)}=\frac{e^x}{e^{2x}+5e^2+6}=\frac{e^x}{e^{2x}+5e^x+\left(\frac{5}{2}\right)^2-\frac{1}{4}}=\frac{e^x}{\left(e^x+\frac{5}{2}\right)^2-\frac{1}{4}}$$

As the denominator is not in the form $\left[f\left(x\right)\right]^2+a^2$, it will not contain a term involving arctan.

10 The diagram below shows a curve on the surface of a cone with radius 4 and height 10.



Which of the following position vectors can **NOT** describe the curve?

A. $\underset{\sim}{r}(t) = 2\left(t - \frac{19}{8}\right)\sin(-4\pi t)\underset{\sim}{i} + 2\left(t - \frac{19}{8}\right)\cos(4\pi t)\underset{\sim}{j} + 5\left(t - \frac{3}{8}\right)\underset{\sim}{k}$ for $\frac{3}{8} \le t \le \frac{19}{8}$

B. $\underset{\sim}{r}(t) = 2(t-2)\cos(4\pi t)\underset{\sim}{i} + 2(t-2)\sin(4\pi t)\underset{\sim}{j} + 5t\underset{\sim}{k}$ for $0 \le t \le 2$

C. $\underset{\sim}{r}(t) = 2(t-2)\cos(4\pi t)\underset{\sim}{i} + 2(t-2)\sin(-4\pi t)\underset{\sim}{j} + 5t\underset{\sim}{k}$ for $0 \le t \le 2$

D. $\underset{\sim}{r}(t) = 2(t-2)\cos(4\pi t)\underset{\sim}{i} + 2(2-t)\sin(-4\pi t)\underset{\sim}{j} + 5t\underset{\sim}{k}$ for $0 \le t \le 2$

All options starts at $(-4, 0, 0)$ and ends at $(0, 0, 10)$, so these two points along cannot allow us to determine the answer.

Check for direction of "rotation" by checking location after $\frac{1}{4}$ of a cycle.

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$\frac{1}{4}T = \frac{1}{8}$$

For C, when $t = \frac{1}{8}$, $\underset{\sim}{j}$ component gives:

$$2\left(\frac{1}{8} - 2\right)\sin\left(-4\pi\left(\frac{1}{8}\right)\right) = 3.75 \not\le 0$$

$\because$ C is not correct.

Section II

90 marks
Attempt Questions 11−16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE writing booklet

(a) Consider the complex numbers $w = 1-2i$ and $z = 1+3i$.

(i) Express $w+\overline{z}$ in the form $x+iy$, where x and y are real numbers. **2**

$$1-2i+1-3i=2-5i$$

(ii) Express $\dfrac{w}{z}$ in modulus-argument form. **2**

$$\frac{1-2i}{1+3i}=\frac{1-2i}{1+3i}\times\frac{1-3i}{1-3i}=\frac{-5-5i}{10}=\frac{1}{2}(-1-i)=\frac{\sqrt{2}}{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)$$

(b) Find $\displaystyle\int xe^{-x}dx$. **2**

$$\int xe^{-x}dx \quad = -\int xd\left(e^{-x}\right) \quad = -\left(xe^{-x}-\int e^{-x}dx\right) = -xe^{-x}-e^{-x}+C$$

(c) Find the angle between the vectors $\underset{\sim}{u} = 2\underset{\sim}{i}-5\underset{\sim}{j}+\underset{\sim}{k}$ and $\underset{\sim}{v} = -\underset{\sim}{i}+4\underset{\sim}{j}+3\underset{\sim}{k}$, giving the angle in degrees correct to 1 decimal place. **3**

$$|\underset{\sim}{u}|=\sqrt{2^2+(-5)^2+1^2}=\sqrt{30}$$
$$|\underset{\sim}{v}|=\sqrt{(-1)^2+4^2+3^2}=\sqrt{26}$$
$$\underset{\sim}{u}\cdot\underset{\sim}{v}=(2)(-1)+(-5)(4)+(1)(3)=-19$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v}=|\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$$
$$-19=\sqrt{30}\sqrt{26}\cos\theta$$
$$\cos\theta=\frac{-19}{\sqrt{780}}$$
$$\theta\approx 132.9°$$

(d) Given that $3-\sqrt{3}i$ is a fourth root of $-72\left(1+\sqrt{3}i\right)$, find the remaining 3 roots and express them in the form $re^{i\theta}$. **2**

$$3-\sqrt{3}i=2\sqrt{3}e^{i\left(-\frac{\pi}{6}\right)}$$

Since the roots are evenly distributed around a circle, the angle between each root must be

$$\frac{2\pi}{4}=\frac{\pi}{2}$$

The remaining roots are

$$2\sqrt{3}e^{i\left(-\frac{\pi}{6}-\frac{\pi}{2}\right)},2\sqrt{3}e^{i\left(-\frac{\pi}{6}+\frac{\pi}{2}\right)},2\sqrt{3}e^{i\left(-\frac{\pi}{6}+\pi\right)}$$

$$=2\sqrt{3}e^{i\left(-\frac{2\pi}{3}\right)},2\sqrt{3}e^{i\left(\frac{\pi}{3}\right)},2\sqrt{3}e^{i\left(\frac{5\pi}{6}\right)}$$

(e) Consider the sequence u_n where $u_1=0,\ u_{n+1}=2n-u_n$ for $n=1,\ 2,\ \ldots$. **3**

Use mathematical induction to show that $2u_n=2n-1+\left(-1\right)^n$ for $n=1,\ 2,\ \ldots$.

Prove that it is true for $n=1$

When $n=1$,

Recurrence relation gives: $u_1=0$

Formula gives:

$$2u_1=2\left(1\right)-1+\left(-1\right)^1$$

$$2u_1=0$$

$$u_1=0$$

$$\therefore \text{ it is true for } n=1$$

Assume that it is true for $n=k$

$$2u_k=2k-1+(-1)^k$$

Prove that it is true for $n=k+1$

$$\text{i.e. RTP: } 2u_{k+1}=2\left(k+1\right)-1+\left(-1\right)^{k+1}$$

From recurrence relation:

$$u_{k+1}=2k-u_k \quad \left(\text{given recurrence relation}\right)$$

$$2u_{k+1}=4k-2u_k$$

$$2u_{k+1}=4k-\left(2k-1+\left(-1\right)^k\right) \quad \left(\text{by assumption}\right)$$

$$2u_{k+1}=4k-2k+1-\left(-1\right)^k$$

$$2u_{k+1}=2k+1+\left(-1\right)\left(-1\right)^k$$

$$2u_{k+1}=2\left(k+1\right)-1+\left(-1\right)^{k+1}$$

$$\therefore \text{ it is true for } n=k+1 \text{ if it is true for } n=k$$

Hence the formula is true by mathematical induction.

Question 12 (15 marks**)** Use a SEPARATE writing booklet

(a) Consider the Statement A. **2**

Statement A: 'If the last digit of the integer n is 5 then n is divisible by 5.'

Prove by contradiction that Statement A is true.

Assume that the last digit of the integer n is 5 and n is not divisible by 5.

As the last digit of n is 5, n can be expressed as:

$n = 10k + 5$ where k is an integer

$n = 5(2k+1)$

$n = 5A$ where $A = 2k+1$ is also an integer since k is an integer

$\therefore n$ is divisible by 5 which contradicts the assumption

Hence Statement A is true by contradiction.

(b) Use an appropriate substitution of the form $x = a + b\sin\theta$ to evaluate **3**

$$\int_7^{10} \sqrt{(x-4)(10-x)}\ dx.$$

$$\int_7^{10} \sqrt{(x-4)(10-x)}\ dx = \int_7^{10} \sqrt{-((x-7)+3)((x-7)-3)}\ dx = \int_7^{10} \sqrt{3^2-(x-7)^2}\ dx$$

Using the substitution $x = 7 + 3\sin\theta$

$$\frac{dx}{d\theta} = 3\cos\theta \qquad \text{when } x = 10,\ \theta = \frac{\pi}{2}$$

$$dx = 3\cos\theta\ d\theta \qquad \text{when } x = 7,\ \theta = 0$$

$$\int_0^{\frac{\pi}{2}} \sqrt{[(7+3\sin\theta)-4][10-(7+3\sin\theta)]}\ 3\cos\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 3\cos\theta\sqrt{(3+3\sin\theta)(3-3\sin\theta)}\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 3\cos\theta\sqrt{9-9\sin^2\theta}\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 9\cos^2\theta\ d\theta$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} 1+\cos 2\theta\ d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{9}{2}\left(\frac{\pi}{2} + \frac{1}{2}\sin\left(2\times\frac{\pi}{2}\right) - \left(0 + \frac{1}{2}\sin(2\times 0)\right)\right)$$

$$= \frac{9\pi}{4}$$

(c) The velocity of a particle moving in a straight line is given by $v^2 = 128 + 32x - 16x^2$, where v is measured in ms^{-1} and x is the displacement measured in m from the origin.

(i) Show that the particle moves in simple harmonic motion with period $\dfrac{\pi}{2}$ s. **2**

$$\frac{1}{2}v^2 = 64 + 16x - 8x^2$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 16 - 16x$$

$$a = -4^2(x-1)$$

$$\therefore n = 4$$

$$T = \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$\therefore$ motion is simple harmonic with period of $\dfrac{\pi}{2}$ and centre at $x = 1$.

(ii) What distance does the particle travel during a full period of its motion? **2**

At the extremities, $v = 0$

$$0 = 128 + 32x - 16x^2$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$\therefore$ extremities are at $x = -2,\ 4$

$\therefore\ A = 3$

The particle will travel a distance of $4A = 12\mathrm{m}$ during a full period of its motion.

(d) The position vectors of points A and B are $\begin{pmatrix}-7\\-3\\2\end{pmatrix}$ and $\begin{pmatrix}5\\-1\\-8\end{pmatrix}$ respectively. **3**

Find the vector equation of the sphere with AB as its diameter.

Position vector of centre: $\begin{pmatrix}\frac{-7+5}{2}\\\frac{-3+(-1)}{2}\\\frac{2+(-8)}{2}\end{pmatrix}=\begin{pmatrix}-1\\-2\\-3\end{pmatrix}$

$$\overrightarrow{AB}=\begin{pmatrix}5-(-7)\\-1-(-3)\\-8-2\end{pmatrix}=\begin{pmatrix}12\\2\\-10\end{pmatrix}$$

Radius of sphere: $\frac{1}{2}\left|\overrightarrow{AB}\right|=\frac{1}{2}\sqrt{12^2+2^2+(-10)^2}=\sqrt{62}$

Vector equation of sphere:

$$\left|\underset{\sim}{r}-\begin{pmatrix}-1\\-2\\-3\end{pmatrix}\right|=\sqrt{62}$$

(e) (i) Prove that $\forall x,y\in\mathbb{R}^+,\ \frac{x+y}{2}\geq\sqrt{xy}$. **1**

$$(a-b)^2\geq 0$$
$$a^2-2ab+b^2\geq 0$$
$$a^2+b^2\geq 2ab$$
$$\frac{a^2+b^2}{2}\geq ab$$

Sub $a=\sqrt{x},\ b=\sqrt{y}$

$$\frac{x+y}{2}\geq\sqrt{xy}$$

(ii) Let a, b, c and d be positive real numbers. Use the result from (i) to prove that **2**

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 16.$$

Method 1:
Using (i)

$$a+b\geq 2\sqrt{ab},\ c+d\geq 2\sqrt{cd},\ \frac{1}{a}+\frac{1}{b}\geq 2\sqrt{\frac{1}{ab}},\ \frac{1}{c}+\frac{1}{d}\geq 2\sqrt{\frac{1}{cd}}$$

$$\because 2\sqrt{ab},\ 2\sqrt{cd},\ \frac{2}{\sqrt{ab}},\ \frac{2}{\sqrt{cd}}>0$$

$$
\begin{aligned}
LHS &= (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \\
&\ge \left(2\sqrt{ab}+2\sqrt{cd}\right)\left(\frac{2}{\sqrt{ab}}+\frac{2}{\sqrt{cd}}\right) \\
&= 4+\frac{4\sqrt{ab}}{\sqrt{cd}}+\frac{4\sqrt{cd}}{\sqrt{ab}}+4 \\
&= 8+4\left(\frac{\sqrt{ab}}{\sqrt{cd}}+\frac{\sqrt{cd}}{\sqrt{ab}}\right) \\
&\ge 8+4\left(2\sqrt{\frac{\sqrt{abcd}}{\sqrt{abcd}}}\right) \\
&= 16
\end{aligned}
$$

Method 2:

$$
\begin{aligned}
LHS &= (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \\
&= 1+\frac{a}{b}+\frac{a}{c}+\frac{a}{d}+\frac{b}{a}+1+\frac{b}{c}+\frac{b}{d}+\frac{c}{a}+\frac{c}{b}+1+\frac{c}{d}+\frac{d}{a}+\frac{d}{b}+\frac{d}{c}+1 \\
&= 4+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{a}{d}+\frac{d}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{b}{d}+\frac{d}{b}\right)+\left(\frac{c}{d}+\frac{d}{c}\right) \\
&\ge 4+2\sqrt{\frac{ab}{ab}}+2\sqrt{\frac{ac}{ac}}+2\sqrt{\frac{ad}{ad}}+2\sqrt{\frac{bc}{bc}}+2\sqrt{\frac{bd}{bd}}+2\sqrt{\frac{cd}{cd}} \quad (\text{using (i)}) \\
&= 16
\end{aligned}
$$

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Evaluate $\int_0^{\pi} x\sin x\cos^4 x\ dx$ using the property $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$. **2**

$$\int_0^{\pi} x\sin x\cos^4 x\ dx$$

$$= \int_0^{\pi} (\pi - x)\sin(\pi - x)\cos^4(\pi - x)dx$$

$$= \int_0^{\pi} (\pi - x)\sin x(-\cos x)^4\ dx$$

$$= \int_0^{\pi} (\pi - x)\sin x\cos^4 xdx$$

$$= \pi\int_0^{\pi} \sin x\cos^4 xdx - \int_0^{\pi} x\sin x\cos^4 xdx$$

$$2\int_0^{\pi} x\sin x\cos^4 x\ dx \quad = \pi\int_0^{\pi} \sin x\cos^4 xdx$$

$$\int_0^{\pi} x\sin x\cos^4 x\ dx \quad = \frac{\pi}{2}\int_0^{\pi} \sin x\cos^4 xdx$$

$$\int_0^{\pi} x\sin x\cos^4 x\ dx$$

$$= \frac{\pi}{2}\left[\frac{-\cos^5 x}{5}\right]_0^{\pi}$$

$$= \frac{\pi}{2}\left(\frac{-\cos^5 \pi}{5} - \left(\frac{-\cos^5 0}{5}\right)\right)$$

$$= \frac{\pi}{5}$$

(b) Consider the two lines in three dimensions given by **3**

$$\underset{\sim}{r} = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \text{ and } \underset{\sim}{r} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix}.$$

Find the point of intersection of the two lines.

Equating $\underset{\sim}{i}$ components

$$-2+\lambda = 5+2\mu \qquad (1)$$

Equating $\underset{\sim}{j}$ components

$$5+3\lambda = 4-5\mu \qquad (2)$$

$(2)-3\times(1)$:

$$(5+3\lambda)-3(-2+\lambda) = (4-5\mu)-3(5+2\mu)$$
$$11 = -11-11\mu$$
$$\mu = -2$$

Sub back into (1)

$$\lambda = 3$$

Check using $\underset{\sim}{k}$ component:

$$1-6(3) = -17$$
$$-1+8(-2) = -17$$

Intersection point: $(-2+3,\ 5+3(3),\ -17) = (1,14,-17)$

(c) The acceleration of a particle moving in a straight line is given by $a = 4 - v^2$, where v is its velocity.

Initially, the particle has a velocity of $\frac{2}{3}$ ms^{-1} at $x = 0$.

(i) What is the displacement of the particle when it is travelling at a velocity of 1 ms^{-1}? **2**

Method 1 Indefinite integral approach:

$$v\frac{dv}{dx} = 4 - v^2$$

$$\int \frac{v\,dv}{4 - v^2} = \int dx$$

$$-\frac{1}{2}\ln\left(4 - v^2\right) = x + C$$

When $x = 0,\ v = \frac{2}{3}$

$$-\frac{1}{2}\ln\left(4 - \left(\frac{2}{3}\right)^2\right) = 0 + C$$

$$C = -\frac{1}{2}\ln\left(\frac{32}{9}\right)$$

$$-\frac{1}{2}\ln\left(4 - v^2\right) = x - \frac{1}{2}\ln\left(\frac{32}{9}\right)$$

When $v = 1$

$$-\frac{1}{2}\ln\left(4 - 1^2\right) = x - \frac{1}{2}\ln\left(\frac{32}{9}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{32}{9}\right) - \frac{1}{2}\ln 3$$

$$x = \frac{1}{2}\ln\frac{32}{27}$$

Method 2 Definite integral approach:

$$v\frac{dv}{dx} = 4 - v^2$$

$$\int_{\frac{2}{3}}^{1} \frac{v\,dv}{4 - v^2} = \int_0^x dx$$

$$\left[-\frac{1}{2}\ln\left(4 - v^2\right)\right]_{\frac{2}{3}}^{1} = \left[x\right]_0^x$$

$$-\frac{1}{2}\ln\left(4 - 1^2\right) + \frac{1}{2}\ln\left(4 - \left(\frac{2}{3}\right)^2\right) = x$$

$$x = \frac{1}{2}\left(\ln\frac{32}{9} - \ln 3\right)$$

$$x = \frac{1}{2}\ln\frac{32}{27}$$

(ii) What is the velocity of the particle at $t = \frac{1}{2}$? **4**

$$\frac{dv}{dt} = 4 - v^2$$

$$\int \frac{dv}{4 - v^2} = \int dt$$

$$\int \frac{dv}{(2-v)(2+v)} = \int dt$$

By splitting into partial fractions:

$$\int \frac{1}{4(2-v)} + \frac{1}{4(2+v)} dv = \int dt$$

$$\frac{1}{4}\left[-\ln(2-v) + \ln(2+v)\right] = t + C_1$$

$$\frac{1}{4}\left[-\ln(2-v) + \ln(2+v)\right] = t + C_1$$

$$\ln\left(\frac{2+v}{2-v}\right) = 4t + C_2$$

When $t = 0,\ v = \frac{2}{3}$

$$\ln\left(\frac{2+\frac{2}{3}}{2-\frac{2}{3}}\right) = 4(0) + C_2$$

$$C_2 = \ln 2$$

$$\ln\left(\frac{2+v}{2-v}\right) = 4t + \ln 2$$

$$\frac{2+v}{2-v} = 2e^{4t}$$

$$2 + v = (2-v)2e^{4t}$$

$$\left(1 + 2e^{4t}\right)v = 2\left(2e^{4t} - 1\right)$$

$$v = \frac{2\left(2e^{4t} - 1\right)}{2e^{4t} + 1}$$

When $t = \frac{1}{2}$

$$v = \frac{2\left(2e^{4\left(\frac{1}{2}\right)} - 1\right)}{2e^{4\left(\frac{1}{2}\right)} + 1} = \frac{2\left(2e^2 - 1\right)}{\left(2e^2 + 1\right)}\ \text{ms}^{-1}$$

(d) The integral I_n is defined by $I_n = \int_0^1 e^{-x}(1-x)^n \; dx$ for $n \geq 0$.

(i) Show that $I_n = 1 - nI_{n-1}$ for $n \geq 1$. **2**

By using integration by parts:

$$\begin{aligned} I_n &= \int_0^1 e^{-x}(1-x)^n \, dx \\ &= -\int_0^1 (1-x)^n \, d\left(e^{-x}\right) \\ &= -\left[\left[(1-x)^n e^{-x}\right]_0^1 - \int_0^1 e^{-x} d\left((1-x)^n\right)\right] \\ &= -\left[(1-1)^n e^{-1} - (1-0)^n e^{-0}\right]_0^1 + \int_0^1 e^{-x}\left(-n(1-x)^{n-1}\right)dx \\ &= 1 - n\int_0^1 e^{-x}(1-x)^{n-1} \, dx \\ &= 1 - nI_{n-1} \end{aligned}$$

(ii) Given that I_n can be expressed in the form $a + be^{-1}$ where a and b are integers, find an expression for b in terms of n. **2**

$$\begin{aligned} &I_0 = 1 - e^{-1} \\ &I_1 = e^{-1} \\ &I_2 = 1 - 2e^{-1} \\ &I_3 = 1 - 3\left(1 - 2e^{-1}\right) = -2 + 3 \times 2e^{-1} \\ &I_4 = 1 - 4\left(-2 + 3 \times 2e^{-1}\right) = 9 - 4 \times 3 \times 2e^{-1} \\ &\vdots \\ &I_n = a + \left[(-1)^{n+1} n!\right]e^{-1} \\ &\qquad \therefore b = (-1)^{n+1} n! \end{aligned}$$

Question 14 (16 marks**)** Use a SEPARATE writing booklet

(a) Prove by mathematical induction that $n^3 - 1 > n^2 + n$, for integers $n \geq 2$. **3**

Prove that it is true for $n = 2$

When $n = 2$,

$$LHS = 2^3 - 1 = 7$$
$$RHS = 2^2 + 2 = 6$$
$$\therefore LHS > RHS$$

$\therefore$ it is true for $n = 2$

Assume that it is true for $n = k$

$$k^3 - 1 > k^2 + k$$

Prove that it is true for $n = k + 1$

i.e. RTP: $(k+1)^3 - 1 > (k+1)^2 + (k+1)$

$$(k+1)^3 - 1 - (k+1)^2 - (k+1) > 0$$

$$\begin{aligned}
LHS &= (k+1)^3 - 1 - (k+1)^2 - (k+1) \\
&= k^3 + 3k^2 + 3k + 1 - 1 - (k^2 + 2k + 1) - k - 1 \\
&= (k^3 - 1) + 2k^2 - 1 \\
&> (k^2 + k) + 2k^2 - 1 \ (\text{by assumption}) \\
&= 3k^2 + k - 1 \\
&> k - 1 \ (\because 3k^2 > 0) \\
&> 1 \ (\because k \geq 2) \\
&> 0
\end{aligned}$$

$\therefore$ it is true for $n = k + 1$ if it is true for $n = k$

Hence the formula is true by mathematical induction.

(b) Use the method of partial fractions to find the value of a, where $a > 0$, such that **4**

$$\int_0^a \frac{3-5x}{(1+x)(1-x+2x^2)}\,dx = 0\,.$$

$$\frac{3-5x}{(1+x)(1-x+2x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1-x+2x^2}$$

$$3-5x = A(1-x+2x^2) + (Bx+C)(1+x)$$

Sub $x=-1$

$$3-5(-1) = A\left(1-(-1)+2(-1)^2\right)$$

$$A = 2$$

Sub $x=0$

$$3-5(0) = A + C$$

$$C = 1$$

Sub $x=1$

$$3-5(1) = A\left(1-1+2(1)^2\right) + (B+C)(1+1)$$

$$-2 = 2A + 2(B+C)$$

$$B = -4$$

$$\int_0^a \frac{3-5x}{(1+x)(1-x+2x^2)}\,dx = \int_0^a \frac{2}{1+x} + \frac{-4x+1}{1-x+2x^2}\,dx = 0$$

$$\int_0^a \frac{2}{1+x} + \frac{-4x+1}{1-x+2x^2}\,dx = 0$$

$$\left[2\ln|1+x| - \ln\left|1-x+2x^2\right|\right]_0^a = 0$$

$$\left(2\ln|1+a| - \ln\left|1-a+2a^2\right|\right) - \left(2\ln|1+0| - \ln\left|1-0+2\times 0^2\right|\right) = 0$$

$$2\ln|1+a| - \ln\left|1-a+2a^2\right| = 0$$

$$\ln\left|\frac{(1+a)^2}{1-a+2a^2}\right| = 0$$

$$\frac{(1+a)^2}{1-a+2a^2} = \pm 1$$

$$(1+a)^2 = 1-a+2a^2$$

$$1+2a+a^2 = 1-a+2a^2$$

$$a^2-3a = 0$$

$$a(a-3) = 0$$

$$a = 0,\ 3$$

$$(1+a)^2 = -\left(1-a+2a^2\right)$$

$$1+2a+a^2 = -1+a-2a^2$$

$$3a^2+a+2 = 0$$

no solution as $\Delta<0$

$$\because a > 0,\ a = 3$$

(c) Let $w = \cos\theta + i\sin\theta$, where $0 < \theta < \pi$.

It is known that the complex number $w^2 + \dfrac{3\sqrt{2}}{w} - 3$ is purely imaginary.

(i) Show that $2\cos^2\theta + 3\sqrt{2}\cos\theta - 4 = 0$. **2**

$$
\begin{aligned}
& w^2 + \frac{3\sqrt{2}}{w} - 3 \\
&= (\cos\theta + i\sin\theta)^2 + 3\sqrt{2}(\cos\theta + i\sin\theta)^{-1} - 3 \\
&= (\cos 2\theta + i\sin 2\theta) + 3\sqrt{2}(\cos(-\theta) + i\sin(-\theta)) - 3 \\
&= \left(\cos 2\theta + 3\sqrt{2}\cos\theta - 3\right) + i\left(\sin 2\theta - 3\sqrt{2}\sin\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Re}\left(w^2 + \frac{3\sqrt{2}}{w} - 3\right) &= 0 \\
\cos 2\theta + 3\sqrt{2}\cos\theta - 3 &= 0 \\
2\cos^2\theta - 1 + 3\sqrt{2}\cos\theta - 3 &= 0 \\
2\cos^2\theta + 3\sqrt{2}\cos\theta - 4 &= 0
\end{aligned}
$$

(ii) Hence, or otherwise, find w. **2**

$$
\begin{aligned}
& 2\cos^2\theta + 3\sqrt{2}\cos\theta - 4 = 0 \\
& \cos\theta = \frac{-3\sqrt{2} \pm \sqrt{\left(3\sqrt{2}\right)^2 - 4(2)(-4)}}{2(2)} \\
& \cos\theta = \frac{-3\sqrt{2} \pm 5\sqrt{2}}{4} \\
& \cos\theta = \frac{\sqrt{2}}{2},\ \cos\theta = -2\sqrt{2}
\end{aligned}
$$

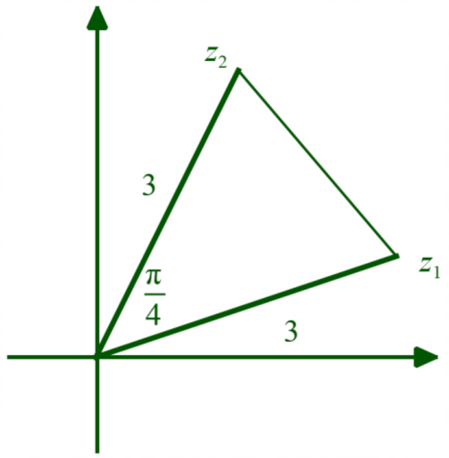
$\cos\theta = -2\sqrt{2}$ has no solution as $-2\sqrt{2} < -1$

$$
\begin{aligned}
\cos\theta &= \frac{\sqrt{2}}{2} \\
\theta &= \frac{\pi}{4} \text{ as } 0 < \theta < \pi \\
w &= \operatorname{cis}\frac{\pi}{4}
\end{aligned}
$$

(iii) A and B are two points on an Argand diagram representing two distinct complex numbers z_1 and z_2 respectively. Suppose that $z_2 = wz_1$ and $|z_1| = 3$. Find the exact length of AB. **2**

Method 1: Geometrical approach

$\because z_2 = wz_1$



$$AB^2 = 3^2 + 3^2 - 2\times3\times3\times\cos\frac{\pi}{4}$$

$$AB^2 = 18 - 9\sqrt{2}$$

$$AB = 3\sqrt{2-\sqrt{2}}$$

Method 2: Algebraic approach

$$\begin{aligned}
AB &= |z_2 - z_1| \\
&= |wz_1 - z_1| \\
&= |z_1(w-1)| \\
&= |z_1||w-1| \\
&= 3\left|\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - 1\right| \\
&= 3\left|\left(\cos\frac{\pi}{4} - 1\right) + i\sin\frac{\pi}{4}\right| \\
&= 3\sqrt{\left(\cos\frac{\pi}{4} - 1\right)^2 + \sin^2\frac{\pi}{4}} \\
&= 3\sqrt{-2\cos\frac{\pi}{4} + 1 + 1} \\
&= 3\sqrt{2 - 2\left(\frac{\sqrt{2}}{2}\right)} \\
&= 3\sqrt{2-\sqrt{2}}
\end{aligned}$$

(d) Consider $f(x)=x^m-mx$ where $m>1$. **3**

By using calculus, or otherwise, show that $x^m-1\geq m(x-1)$ for all $x>0$.

$$f'(x)=mx^{m-1}-m$$

When $f'(x)=0$

$$0=mx^{m-1}-m$$
$$m\left(x^{m-1}-1\right)=0$$
$$x^{m-1}=1$$
$$x=1\ (\because x>0)$$

$$f''(x)=m(m-1)x^{m-2}$$
$$f''(1)=m(m-1)>0\ (\because m>1)$$
$\therefore$ at $x=1$, there is a min turning point

$$f(1)=1^m-m(1)=1-m$$

$$f(x)>1-m\ (\because \text{ there are no other turning points})$$
$$x^m-mx>1-m$$
$$x^m-1>mx-m$$
$$x^m-1>m(x-1)$$

Question 15 (14 marks) Use a SEPARATE writing booklet

(a) Let z and w be non-zero complex numbers. Consider the statements below: **3**

$$P:\ z\overline{w}+\overline{z}w=0$$

$$Q:\ \frac{z}{w}=ik,\ k\in\mathbb{R}$$

Prove that $P\Rightarrow Q$.

Method 1:

Prove $P\Rightarrow Q$ i.e. if $z\overline{w}+\overline{z}w=0$, then $\dfrac{z}{w}=ik$

$$z\overline{w}+\overline{z}w=0$$

$$z\overline{w}+\overline{z\overline{w}}=0$$

$$2\operatorname{Re}(z\overline{w})=0$$

$$\therefore z\overline{w} \text{ is purely imaginary}$$

$$z\overline{w}=ia,\ a\in\mathbb{R}$$

$$\frac{z\overline{w}w}{w}=ia$$

$$\frac{z|w|^2}{w}=ia$$

$$\frac{z}{w}=i\frac{a}{|w|^2}$$

$$\frac{z}{w}=ik,\ k\in\mathbb{R}\ \left(\because a,\ |w|^2\in\mathbb{R}\right)$$

Method 2:

Let $z=a+bi,\ w=c+di$

$$z\overline{w}+\overline{z}w=0$$

$$(a+bi)(c-di)+(a-bi)(c+di)=0$$

$$ac+bd+i(bc-ad)+ac+bd+i(-bc+ad)=0$$

$$2(ac+bd)=0$$

$$ac+bd=0$$

z and w can also be considered as two vectors $\begin{pmatrix}a\\b\end{pmatrix}$, $\begin{pmatrix}c\\d\end{pmatrix}$, hence the above results is equivalent to

$\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}c\\d\end{pmatrix}=0$, i.e. $\begin{pmatrix}a\\b\end{pmatrix}\perp\begin{pmatrix}c\\d\end{pmatrix}$, this means z and w must be perpendicular on the argand diagram,

$$z=(ik)w,\ k\in\mathbb{R}$$

$$\frac{z}{w}=ik$$

(b) (i) By using a suitable substitution, show that **3**

$$\int \sin\sqrt{x}\ dx = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C.$$

$$\text{Let } u = \sqrt{x}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{1}{2u}$$

$$2u\ du = dx$$

$$\int \sin\sqrt{x}\ dx = \int \sin u\ (2u\ du) = 2\int u\sin u\ du$$

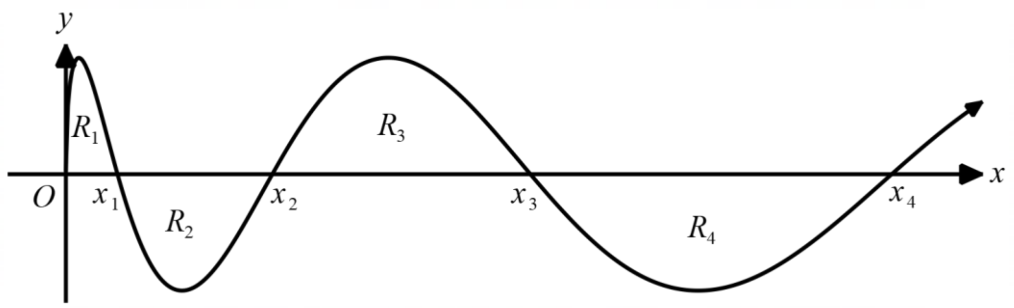
Using integration by parts:

$$2\int u\sin u\ du = -2\int u\ d(\cos u) = -2\left(u\cos u - \int \cos u\ \ du\right) = -2u\cos u + 2\sin u + C$$

Sub $u = \sqrt{x}$:

$$\int \sin\sqrt{x}\ dx \ = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

The diagram below shows part of the curve $y = \sin\sqrt{x}$ for $x \geq 0$.



The curve intersects the x-axis at $0, x_1, x_2, x_3, x_4, \ldots$.

The regions bounded by the curve and the x-axis are defined by $R_1, R_2, R_3, R_4, \ldots$ as shown in the diagram.

(ii) By first finding an expression for x_n, or otherwise, express the area of the region R_n in terms of n and π. **3**

$$\sin\sqrt{x} = 0$$

$$\sqrt{x} = 0,\ \pi,\ 2\pi,\ \ldots$$

$$\sqrt{x} = n\pi$$

$$x = n^2\pi^2$$

$$\begin{aligned}
R_n &= \left| \int_{x_{n-1}}^{x_n} \sin\sqrt{x} \quad dx \right| \\
&= (-1)^{n+1} \int_{(n-1)^2\pi^2}^{n^2\pi^2} \sin\sqrt{x} \quad dx \\
(-1)^{n+1} R_n &= \left[-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} \right]_{(n-1)^2\pi^2}^{n^2\pi^2} \\
&= \left(-2\sqrt{n^2\pi^2}\cos\sqrt{n^2\pi^2} + 2\sin\sqrt{n^2\pi^2} \right) - \left(-2\sqrt{(n-1)^2\pi^2}\cos\sqrt{(n-1)^2\pi^2} + 2\sin\sqrt{(n-1)^2\pi^2} \right) \\
&= -2n\pi\cos n\pi + 2\sin n\pi + 2(n-1)\pi\cos(n-1)\pi - 2\sin(n-1)\pi \\
&= -2n\pi\cos n\pi + 2(n-1)\pi\cos(n-1)\pi \quad (\because \sin n\pi = 0) \\
&= -2n\pi\cos n\pi + 2(n-1)\pi(\cos n\pi\cos\pi + \sin n\pi\sin\pi) \\
&= -2n\pi\cos n\pi - 2(n-1)\pi\cos n\pi \\
&= (-4n+2)\pi\cos n\pi \\
&= (-4n+2)\pi(-1)^n \quad (\because \cos n\pi = -1 \text{ if } n \text{ is odd and } \cos n\pi = 1 \text{ if } n \text{ is even}) \\
R_n &= -(-4n+2)\pi \\
&= (4n-2)\pi
\end{aligned}$$

Question 15 continues on page 13

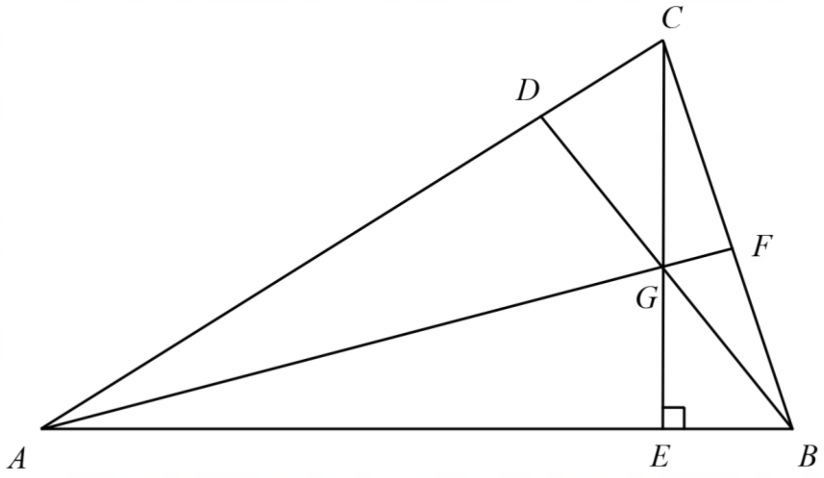
(c) The diagram shows triangle ABC.

The point E lies on AB so that $AE : EB = \lambda : 1$ and $CE \perp AB$.

The point G is the mid-point of CE.

BG is produced to meet AC at D and $AD : DC = r : 1$.

The point F is chosen on BC so that A, G, F are collinear.



NOT TO SCALE

Let $\overrightarrow{BE} = \underset{\sim}{u}$ and $\overrightarrow{EG} = \underset{\sim}{v}$.

(i) Express $\overrightarrow{BG}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **1**

$$\overrightarrow{BG} = \underset{\sim}{u} + \underset{\sim}{v}$$

(ii) By considering $\overrightarrow{BD}$, use vector methods to show that $r = \lambda + 1$. **4**

$$\overrightarrow{BD} = k \cdot \overrightarrow{BG} = k\left(\underset{\sim}{u} + \underset{\sim}{v}\right),\ k \in R \quad \left(\text{as } B, G, D \text{ collinear}\right) \ (1)$$

$$\begin{aligned}
\overrightarrow{BD} &= \overrightarrow{BA} + \overrightarrow{AD} \\
&= (\lambda+1)\overrightarrow{BE} + \frac{r}{r+1}\overrightarrow{AC} \\
&= (\lambda+1)\underset{\sim}{u} + \frac{r}{r+1}\left(\overrightarrow{AE} + \overrightarrow{EC}\right) \\
&= (\lambda+1)\underset{\sim}{u} + \frac{r}{r+1}\left(-\lambda\overrightarrow{BE} + 2\overrightarrow{EG}\right) \\
&= (\lambda+1)\underset{\sim}{u} + \frac{r}{r+1}(-\lambda\underset{\sim}{u} + 2v) \\
&= \left(\lambda + 1 - \frac{r\lambda}{r+1}\right)\underset{\sim}{u} + \frac{2r}{r+1}\underset{\sim}{v} \qquad (2)
\end{aligned}$$

Equating coefficients of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ in (1) and (2) [as $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are independent]:

$$\begin{aligned}
\lambda + 1 - \frac{r\lambda}{r+1} &= \frac{2r}{r+1} \quad (=k) \\
(\lambda+1)(r+1) - r\lambda &= 2r \\
\cancel{\lambda r} + \lambda + r + 1 - \cancel{r\lambda} &= 2r \\
r &= \lambda + 1
\end{aligned}$$

End of Question 15

Question 16 (16 marks) Use a SEPARATE writing booklet

(a) It is given that for positive numbers $a_1,\ a_2,\ a_3,\ ...,\ a_n$ with arithmetic mean A,

$$A^n \geq a_1 \times a_2 \times a_3 \times ... \times a_n. \qquad \text{(Do NOT prove this.)}$$

Let k be a positive integer.

(i) Prove that $k^k \geq 1\times 3\times 5\times ...\times (2k-1)$. **2**

Consider the positive odd numbers $1, 3, 5, ..., (2k-1)$

Using the results given:

$$\left[\frac{(1+3+5+...+(2k-1))}{k}\right]^k \geq 1\times 3\times 5 ...\times (2k-1)$$

Using sum of AP on the LHS

$$\left[\frac{\frac{k}{2}(1+(2k-1))}{k}\right]^k \geq 1\times 3\times 5 ...\times (2k-1)$$

$$\left[\frac{1}{2}(2k)\right]^k \geq 1\times 3\times 5 ...\times (2k-1)$$

$$k^k \geq 1\times 3\times 5 ...\times (2k-1)$$

(ii) Hence prove that $(k^2+k)^k \geq (2k)!$. **2**

Consider the positive even numbers $2, 4, 6, ..., 2k$

Using the results given:

$$\left[\frac{(2+4+6+...+2k)}{k}\right]^k \geq 2\times 4\times 6 ...\times 2k$$

Using sum of AP on the LHS

$$\left[\frac{\frac{k}{2}(2+2k)}{k}\right]^k \geq 2\times 4\times 6 ...\times 2k$$

$$\left[\frac{1}{2}(2k+2)\right]^k \geq 2\times 4\times 6 ...\times 2k$$

$$(k+1)^k \geq 2\times 4\times 6 ...\times 2k$$

As LHS and RHS of both inequalities are positive, multiplying the two inequalities give:

$$k^k (k+1)^k \geq 1\times 3\times 5 ...\times (2k-1)\times 2\times 4\times 6 ...\times 2k$$

$$(k(k+1))^k \geq 1\times 2\times 3\times ...\times (2k-1)\times 2k$$

$$(k^2+k)^k \geq (2k)!$$

(b) Consider the equation $(z+2)^8 = z^8$.

(i) Show that the 6 non-real roots of the equation can be expressed in the form **4**

$$-1 - i\cot\left(\frac{k\pi}{8}\right),\ k = \pm 1,\ \pm 2,\ \pm 3.$$

$$\left(\frac{z+2}{z}\right)^8 = 1$$

$$\left(\frac{z+2}{z}\right)^8 = \text{cis } 0$$

$$\frac{z+2}{z} = \text{cis}\left(\frac{2k\pi}{8}\right),\ k = \pm 1,\ \pm 2,\ \pm 3,\ \pm 4$$

When $k = \pm 4$, $\text{cis}\left(\dfrac{2(\pm 4)\pi}{8}\right) = \text{cis}(\pm\pi) = -1$ (which is a real root)

The 6 non-real roots must be

$$\frac{z+2}{z} = \text{cis}\left(\frac{2k\pi}{8}\right),\ k = \pm 1,\ \pm 2,\ \pm 3$$

$$z + 2 = z\ \text{cis}\left(\frac{2k\pi}{8}\right)$$

$$\left(\text{cis}\left(\frac{2k\pi}{8}\right) - 1\right)z = 2$$

$$z = \frac{2}{\text{cis}\left(\frac{2k\pi}{8}\right) - 1}$$

$$z=\frac{2}{\operatorname{cis}\left(\frac{2k\pi}{8}\right)-1}$$

$$z=\frac{2}{\left(\cos\left(\frac{2k\pi}{8}\right)-1\right)+i\sin\left(\frac{2k\pi}{8}\right)}$$

$$z=\frac{2\left(\left(\cos\left(\frac{2k\pi}{8}\right)-1\right)-i\sin\left(\frac{2k\pi}{8}\right)\right)}{\left(\cos\left(\frac{2k\pi}{8}\right)-1\right)^2+\sin^2\left(\frac{2k\pi}{8}\right)}$$

$$z=\frac{2\left(-\sin^2\left(\frac{k\pi}{8}\right)\right)-i\left(2\sin\left(\frac{k\pi}{8}\right)\cos\left(\frac{k\pi}{8}\right)\right)}{\cos^2\left(\frac{2k\pi}{8}\right)-2\cos\left(\frac{2k\pi}{8}\right)+1+\sin^2\left(\frac{2k\pi}{8}\right)}$$

$$z=\frac{-2\sin\left(\frac{k\pi}{8}\right)\left(\sin\left(\frac{k\pi}{8}\right)+i\cos\left(\frac{k\pi}{8}\right)\right)}{2-2\cos\left(\frac{2k\pi}{8}\right)}$$

$$z=\frac{-2\sin\left(\frac{k\pi}{8}\right)\left(\sin\left(\frac{k\pi}{8}\right)+i\cos\left(\frac{k\pi}{8}\right)\right)}{2\sin^2\left(\frac{k\pi}{8}\right)}$$

$$z=\frac{-\sin\left(\frac{k\pi}{8}\right)-i\cos\left(\frac{k\pi}{8}\right)}{\sin\left(\frac{k\pi}{8}\right)}$$

$$z=-1-i\cot\left(\frac{k\pi}{8}\right),\ k=\pm1,\ \pm2,\ \pm3$$

Alternate solution involves using t-substitution

(ii) Hence, find the value of $\sin^2\left(\frac{\pi}{8}\right)\sin^2\left(\frac{2\pi}{8}\right)\sin^2\left(\frac{3\pi}{8}\right)$ given that the product of roots of the equation is -16. **3**

$$\left(-1-i\cot\left(\frac{\pi}{8}\right)\right)\left(-1-i\cot\left(-\frac{\pi}{8}\right)\right)$$
$$=\left(-1-i\cot\left(\frac{\pi}{8}\right)\right)\left(-1+i\cot\left(\frac{\pi}{8}\right)\right)$$
$$=\left(1+\cot^2\left(\frac{\pi}{8}\right)\right)$$
$$=\csc^2\left(\frac{\pi}{8}\right)$$

Similarly:

$$\left(-1-i\cot\left(\frac{2\pi}{8}\right)\right)\left(-1-i\cot\left(-\frac{2\pi}{8}\right)\right)=\csc^2\left(\frac{2\pi}{8}\right)$$
$$\left(-1-i\cot\left(\frac{3\pi}{8}\right)\right)\left(-1-i\cot\left(\frac{-3\pi}{8}\right)\right)=\csc^2\left(\frac{3\pi}{8}\right)$$

7 roots in total: real root of -1 plus the 6 non-real roots in (i)

Product of roots:

$$\csc^2\left(\frac{\pi}{8}\right)\csc^2\left(\frac{2\pi}{8}\right)\csc^2\left(\frac{3\pi}{8}\right)(-1)=-16$$
$$\frac{1}{\sin^2\left(\frac{\pi}{8}\right)\sin^2\left(\frac{2\pi}{8}\right)\sin^2\left(\frac{3\pi}{8}\right)}=16$$
$$\sin^2\left(\frac{\pi}{8}\right)\sin^2\left(\frac{2\pi}{8}\right)\sin^2\left(\frac{3\pi}{8}\right)=\frac{1}{16}$$

Question 16 continues on page 15

Question 16 (continued)

(c) Let $z = \cos\theta + i\sin\theta$.

(i) Find the four values of z such that $\operatorname{Im}\left(z^2 + \overline{z}\right) = 0$. **2**

$$\begin{aligned} z^2 + \overline{z} &= \left(\cos\theta + i\sin\theta\right)^2 + \left(\cos\theta - i\sin\theta\right) \\ &= \left(\cos 2\theta + i\sin 2\theta\right) + \left(\cos\theta - i\sin\theta\right) \\ &= \left(\cos 2\theta + \cos\theta\right) + i\left(\sin 2\theta - \sin\theta\right) \end{aligned}$$

$$\begin{aligned} \operatorname{Im}\left(z^2 + \overline{z}\right) &= 0 \\ \sin 2\theta - \sin\theta &= 0 \\ 2\sin\theta\cos\theta - \sin\theta &= 0 \\ \sin\theta\left(2\cos\theta - 1\right) &= 0 \end{aligned}$$

$$\sin\theta = 0 \qquad \cos\theta = \frac{1}{2}$$

$$\theta = 0,\ \pi \qquad \theta = \frac{-\pi}{3}\ ,\ \frac{\pi}{3}$$

$$z = \text{cis } 0,\ \text{cis } \pi,\ \text{cis } \frac{-\pi}{3},\ \text{cis } \frac{\pi}{3}$$

$$z = 1,\ -1,\ \text{cis } \frac{-\pi}{3},\ \text{cis } \frac{\pi}{3}$$

Let z_1 and z_2 be two of the values of z obtained in (i) such that $\text{Im}(z_1) < 0 < \text{Im}(z_2)$.

Also let $S_n = \sum_{r=1}^{n} w^r$, where $w = \frac{z_2}{z_1}$, for any positive integer n.

(ii) Find all positive integers k such that $(S_n)^k + (S_{n+1})^k + (S_{n+2})^k = 2$ for any positive integer n. **3**

$$z_1 = \text{cis}\,\frac{-\pi}{3},\ z_2 = \text{cis}\,\frac{\pi}{3}$$

$$w = \frac{z_2}{z_1} = \frac{\text{cis}\,\frac{\pi}{3}}{\text{cis}\,\frac{-\pi}{3}} = \text{cis}\left(\frac{\pi}{3} - \left(\frac{-\pi}{3}\right)\right) = \text{cis}\,\frac{2\pi}{3}$$

Since $n,\ n+1,\ n+2$ are consecutive integers, one must be a multiple of 3, one must leave a remainder of 1 when divided by 3, one must leave a remainder of 2 when divided by 3

Assuming n is a multiple of 3, $n = 3A,\ A \in \mathbb{R}^+$

$$\text{Also } w = \text{cis}\,\frac{2\pi}{3} \quad \therefore w^3 = \left(\text{cis}\,\frac{2\pi}{3}\right)^3 = 1$$

$$w^3 = 1$$

$$w^3 - 1 = 0$$

$$(w-1)\left(w^2 + w + 1\right) = 0$$

$$\begin{aligned} S_n &= \sum_{k=1}^{n} w^k \\ &= w + w^2 + w^3 + \ldots + w^n \\ &= \frac{w\left(w^n - 1\right)}{w-1} \\ &= \frac{w\left(w^{3A} - 1\right)}{w-1} \\ &= \frac{w\left(\left(w^3\right)^A - 1\right)}{w-1} \\ &= \frac{w\left(1^A - 1\right)}{w-1} \quad \left(\because w^3 = 1\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} S_{n+1} &= \sum_{k=1}^{n+1} w^k \\ &= \sum_{k=1}^{n} w^k + w^{n+1} \\ &= 0 + w^{n+1} \\ &= w^{3A+1} \\ &= w^{3A} \times w \\ &= \left(w^3\right)^A \times w \\ &= 1^A \times w \\ &= w \end{aligned}$$

$$\begin{aligned} S_{n+2} &= \sum_{k=1}^{n+2} w^k \\ &= \sum_{k=1}^{n+1} w^k + w^{n+2} \\ &= w + w^{n+2} \\ &= w + w^{3A+2} \\ &= w + w^{3A} \times w^2 \\ &= w + \left(w^3\right)^A \times w^2 \\ &= w + w^2 \\ &= -1 \quad \left(\because 1 + w + w^2 = 0\right) \end{aligned}$$

$$(S_n)^k + (S_{n+1})^k + (S_{n+2})^k = 2$$

$$0^k + w^k + (-1)^k = 2$$

Now consider cases when k is odd and when k is even

When k is odd,

$$w^k - 1 = 2$$
$$w^k = 3$$
$$\text{No solution as } |w| = 1$$

When k is even,

$$w^k + 1 = 2$$
$$w^k = 1$$
$$\because w^3 = 1,\ k \text{ must be an even multiple of 3 i.e. } k \text{ is a multiple of 6 or } k = 6B,\ B \in \mathbb{R}^+$$

End of paper