Student Number: ____________________ Teacher Name: ______________________

PENRITH SELECTIVE HIGH SCHOOL
MATHEMATICS DEPARTMENT

HSC Trial Examination 2025
Year 12 Mathematics Extension 2

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided with this examination paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- No correction tape/white out allowed.

Total marks: 100

Section I – 10 marks (pages 2–6)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–15)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

	Multiple Choice	Q11	Q12	Q13	Q14	Q15	Q16	TOTAL
Complex Numbers	6 /1	a, e /8	c /3	b /4		c /5	b /5	/26
Vectors	4, 5 /2	d /3	e /3		c /3	a /5		/16
Proof	1, 3 /2	b /2	a, d /6		a /3			/13
Calculus	7, 9 /2		b /3	a /7	b, d /8	b /5		/25
Mechanics	2, 8, 10 /3	c /2		c /4			a, c /11	/20
TOTAL	/10	/15	/15	/15	/14	/15	/16	/100
								%

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Consider the statement

"for any integers m and n, if $m + n \geq 7$ then $m \geq 4$ or $n \geq 4$"

The **contrapositive** of this statement is

A. If $m < 4$ or $n < 4$, then $m + n < 7$.

B. If $m \geq 4$ or $n \geq 4$, then $m + n \geq 7$.

C. If $m < 4$ and $n < 4$, then $m + n < 7$.

D. If $m \leq 4$ or $n \leq 4$, then $m + n < 7$.

2 Which of the following functions does not describe simple harmonic motion?

A. $x = \sin^2 3t - \cos^2 3t$

B. $x = 5\cos\left(2t + \frac{\pi}{6}\right)$

C. $x = 2\sin 4t - 4\cos 2t$

D. $x = 2\sin^2 2t - 5\cos 4t + 2$

3 Consider the statement:

"Whichever positive number r you pick, it is possible to find a number x greater than zero such that $\frac{1}{x^2} < r$."

When this statement is written in a formal language of proof, which of the following is obtained?

A. $\forall r > 0, \exists x > 0$ such that $\frac{1}{x^2} < r$

B. $\exists r > 0, \exists x > 0$ such that $\frac{1}{x^2} < r$

C. $\forall r > 0, \forall x > 0$ such that $\frac{1}{x^2} < r$

D. $\exists r > 0, \forall x > 0$ such that $\frac{1}{x^2} < r$

4 Which of the following is a vector equation of the line joining the points $P(1, -3, 4)$ and $Q(5, 0, -2)$?

A. $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$, λ is a scalar parameter.

B. $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -3 \\ 6 \end{pmatrix}$, λ is a scalar parameter.

C. $\mathbf{r} = \begin{pmatrix} 2 \\ -1.5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ -6 \end{pmatrix}$, λ is a scalar parameter.

D. $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$, λ is a scalar parameter.

5 The position of a particle at a time t is given by $\mathbf{r}(t) = \begin{pmatrix} \sqrt{t-2} \\ t \end{pmatrix}$ for $t \geq 2$.

The Cartesian equation of the path of the particle is given by

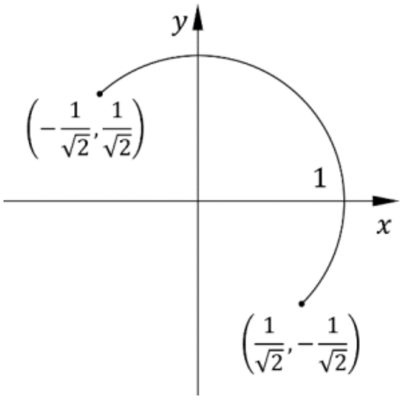
A. $y = x^2 - 2$ for $x \geq 0$

B. $y = x^2 + 2$ for $x \geq 2$

C. $y = x^2 + 2$ for $x \geq 0$

D. $y = 2x^2 + 2$ for $x \geq 2$

6 Which of the following curve describes the diagram given below?



A. $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$ for $\frac{\pi}{4} \leq t \leq \frac{5\pi}{4}$

B. $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ for $\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$

C. $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$ for $\frac{3\pi}{4} \leq t \leq \frac{5\pi}{4}$

D. $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ for $\frac{\pi}{4} \leq t \leq \frac{5\pi}{4}$

7 Which expression is equal to $\int \frac{dx}{x^2+4x+6}$?

A. $\tan^{-1}\frac{x+2}{2}+c$

B. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+2}{\sqrt{2}}+c$

C. $\tan^{-1}\frac{x+2}{\sqrt{2}}+c$

D. $\frac{1}{2}\tan^{-1}\frac{x+2}{2}+c$

8 A particle moves on the x-axis with velocity **v**. The particle is initally at rest at $x=2$. Its acceleration is given by $\ddot{x}=x-5$. The speed at $x=1$ is given by

A. $|v|=7m/s$

B. $|v|=\frac{7}{2}m/s$

C. $|v|=\sqrt{7}m/s$

D. $|v|=\frac{\sqrt{7}}{2}m/s$

9 Let $I_n=\int_0^{\frac{\pi}{4}} \tan^n x\,dx$, then I_n+I_{n+2} is equal to

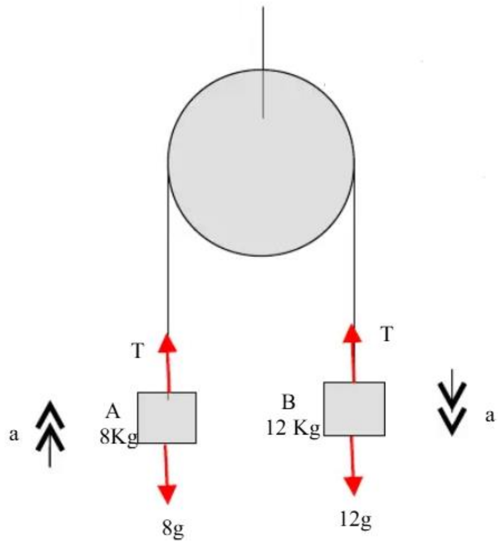
A. $\frac{1}{n-1}$

B. $\frac{1}{n-2}$

C. $\frac{1}{n+1}$

D. $\frac{1}{n+2}$

10 Two masses of 8 kg and 12 kg are hanging by a light and inextensible string over a frictionless pulley as shown in the digram given below.



The tension *T* in the string is given by

A. $\frac{20g}{5}$ N

B. $\frac{24g}{5}$ N

C. $\frac{4g}{5}$ N

D. $\frac{48g}{5}$ N

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate Writing Booklet for Question 11.

(a) Given $z = 1 + i$, find $z^2 + \frac{1}{z^2}$ in the form of $a + ib$, where a and b are real numbers. **3**

(b) Suppose that $a, b \in \mathbb{Z}$. If ab and $a - b$ are both even, prove that a and b are both even. **2**

(c) A particle moves in simple harmonic motion described by the differential equation **2**
$\ddot{x} = -16(x + 2)$.
Find the period of motion and the centre of motion about which the motion occurs.

(d) Show that the pair of lines $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $L_2 : \mathbf{s} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ are **3**
perpendicular to each other, where λ and μ are scalar parameters.

Also, show that they intersect at $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(e) i. Show that for any integer n, $e^{in\theta} + e^{-in\theta} = 2 \cos n\theta$. **2**

ii. By expanding $\left(e^{i\theta} + e^{-i\theta}\right)^3$, show that $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$. **3**

End of Question 11

Question 12 (15 marks) Use a separate Writing Booklet for Question 12.

(a) Prove by contradiction that $\log_4 9$ is an irrational number. **3**

(b) Evaluate $\displaystyle\int \frac{dx}{1 - \cos x + \sin x}$. **3**

(c) The complex number $3 - i$ is a zero of a cubic polynomial $Q(z) = z^3 + pz^2 + qz + r$ where p, q, r are real numbers. **3**

Given that $z = -1$ is also a zero of $Q(z)$, find the vales of p, q, r and then write the complete factorisation of $Q(z)$.

(d) For all real numbers x and y, with $x^2 + y^2 \neq 0$,

i. prove that $\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = 1.$ **1**

ii. Hence, or otherwise, prove that $\left|\frac{x}{\sqrt{x^2+y^2}}\right| \leq 1$ and $\left|\frac{y}{\sqrt{x^2+y^2}}\right| \leq 1.$ **2**

(e) With respect to the origin O, the lines L_1 and L_2 are given by the equations: **3**

$$L_1 \colon r_1 = \begin{pmatrix} 10 \\ 0 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$L_2 \colon r_1 = \begin{pmatrix} 17 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$$

where λ and μ are scalar parameters.

Show that the shortest distance between the two lines is $\frac{2}{\sqrt{11}}$ units.

End of Question 12

Question 13 (15 marks) Use a separate Writing Booklet for Question 13.

(a) i. Express $\frac{x^2+1}{(x+1)(x-2)(x^2+x+1)}$ as the sum of partial fractions. **4**

ii. Hence, or otherwise, show that **3**

$$\int \frac{x^2+1}{(x+1)(x-2)(x^2+x+1)}\,dx$$

$$= -\frac{2}{3}\ln|x+1| + \frac{5}{21}\ln|x-2| + \frac{3}{14}\ln|x^2+x+1| + \frac{1}{7\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

where C is the constant of integration.

(b) Let $z \in \mathbb{C}$ and define $A = 1 + i$ and $B = -1 + i$.

Find the set of points satisfying the condition $|z - A| = 2|z - B|$.

i. Express this as a Cartesian equation. **3**

ii. Interpret the result geometrically. **1**

(c) A crate of mass 15 kg is being pulled along a smooth horizontal surface by a rope inclined at 30° to the horizontal. ($g = 10\ m/s^2$)

The tension in the rope is 18 N.

i. Draw a force diagram for this situation. **1**

ii. Find the acceleration of the box. **1**

iii. Find the normal force between the box and the floor. **2**

End of Question 13

Question 14 (14 marks) Use a separate Writing Booklet for Question 14.

(a) Prove that a three-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3. **3**

(b) Evaluate **3**

$$\int_{-0.5}^{0.5} \sqrt{\frac{1-x}{1+x}}\,dx.$$

(c) A sphere has a centre at $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and its radius is 4 units.

A line has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ where μ is a scalar parameter.

i. Write down the vector equation of the sphere. **1**

ii. Determine whether the line is a tangent to the sphere, clearly justifying your conclusion. **2**

(d) i. Show that for $k \geq 0$, **2**

$$\int_0^{\pi} \sin^{2k}\theta \cos 2\theta \, d\theta = -\frac{k}{k+1}\int_0^{\pi} \sin^{2k}\theta \, d\theta \,.$$

ii. Hence prove by mathematical induction that **3**

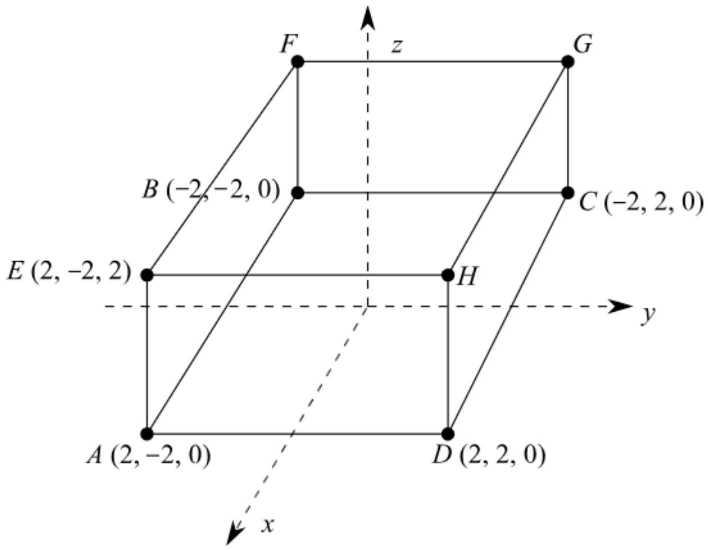
$$\int_0^{\pi} \sin^{2n}\theta \, d\theta = \frac{(2n)!}{n!^2} \times \frac{\pi}{2^{2n}}$$

for all integers $n \geq 0$.

End of Question 14

Question 15 (15 marks) Use a separate Writing Booklet for Question 15.

(a)



For the rectangular prism given above

i. Show that the internal diagonals AG and DF bisect each other. **2**

ii. Calculate the acute angle, θ, between the diagonals, to the nearest degree. **3**

(b) If $I_n = \displaystyle\int_0^1 \frac{x^{n-1}\,dx}{(1+x^2)^n}$ for $n \geq 1$

i. Show that $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \sin^{n-1}\theta\ \cos^{n-1}\theta\ d\theta$ for $n \geq 1$. **2**

ii. Let $K_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta\ d\theta$, show that $I_n = \dfrac{1}{2^n}K_{n-1}$. **2**

iii. Hence show that $I_n = 2^{-\frac{n(n+1)}{2}}\left(\dfrac{\pi}{2}\right)$. **1**

Question 15 continues to the next page ……………………….

Question 15 (15 marks) Use a separate Writing Booklet for Question 15.

(c) Let $\omega = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$.

i. Show that ω^k is a solution of $z^9 - 1 = 0$, where k is an integer. **1**

ii. Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0$. **2**

iii. Hence, show that $8\cos\frac{\pi}{9} \times \cos\frac{2\pi}{9} \times \cos\frac{4\pi}{9} - 1 = 0$. **2**

End of Question 15

Question 16 (16 marks) Use a separate Writing Booklet for Question 16.

(a) The point P is 5 metres to the right of the origin O on a straight line. **3**

A particle is released from rest at P and moves along the straight line in a simple harmonic motion about O, with a period 4π seconds.

After π seconds, another particle is released from rest at P and moves along this straight line in simple harmonic motion about O, with period 4π seconds.

Find when and where the two particles first collide.

(b) Let $z = e^{\frac{2\pi}{5}i}$ and $w = e^{\frac{4\pi}{5}i}$.

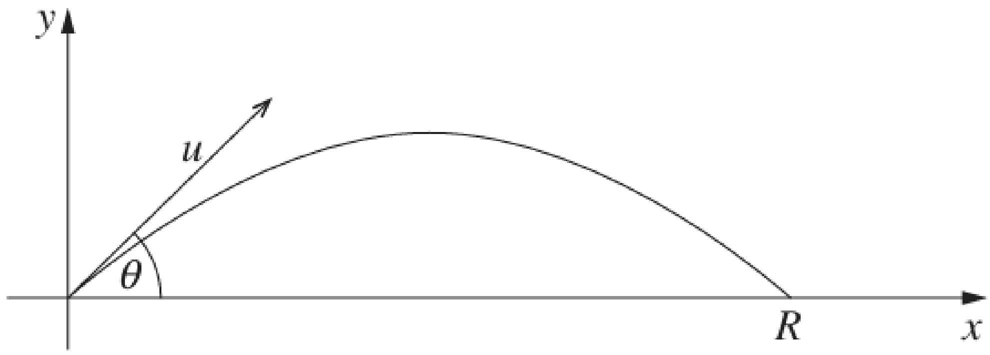
i. Express z and w in polar form. **1**

ii. Hence, show that $|z + w| = \sqrt{2\left(1 + \cos\frac{2\pi}{5}\right)}$. **2**

iii. In the complex plane, vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ represent the complex numbers z, w and $z + w$ respectively, where O is the origin. Show that $\angle AOC = \frac{\pi}{5}$. **2**

Question 16 continues to the next page ....

(c) A particle is projected upward from the origin with an initial velocity u m/s at an angle θ above the horizontal as shown in the diagram given below. Let the position vector of the particle at time t be $\mathbf{r}\ (t)$ and define the total time of flight as T. $(g = 10\ m/s^2)$.



The particle moves under the constant acceleration due to gravity, $\ddot{\mathbf{r}} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$. The particle lands on the x-axis after time T, when $x = R$.

i. Show that the range R of the projectile is given by **3**

$$R = \frac{u^2 \sin 2\theta}{g}.$$

ii. A garden sprinkler sprays water symmetrically about the vertical axis with a constant speed of u metres per second. The initial direction of the spray varies continuously between angles 10° and 75° to the horizontal. **1**

Prove that the sprinkler will wet the surface of the region on the ground that forms an annular area with the centre at the sprinkler's position and with its internal and external radii given by

$\dfrac{u^2 \sin(2\theta_{\min})}{g}$ and $\dfrac{u^2 \sin(2\theta_{\max})}{g}$ respectively.

Question 16 (c) continues to the next page ……………………….

(c) iii. A rectangular garden bed has dimensions 8 metres by 5 metres. Assume that the sprinkler is to be placed in a fixed position, such that the entire garden bed must be watered. **3**

Determine the minimum velocity u required for the sprinkler to cover the entire garden bed. Express the condition for u in terms of the acceleration due to gravity g ensuring that the sprinkler's maximum range at $\theta_{max} = 75°$ is sufficient to cover the diagonal of the rectangular diagonal bed.

iv. To prevent water wastage and ensure the bed is adequately watered, it is decided that the sprinkler should be positioned at a minimum possible distance from the garden bed, while still covering all of it. **1**

What is the required value of u, ensuring it satisfies the condition that

$$\frac{u^2}{g} \geq 8 + \sqrt{13},$$

where u is the speed of the water jet?

END OF PAPER

2025 Mathematics Extension 2 Trial Examination Solutions

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 Consider the statement

"for any integers m and n, if $m + n \geq 7$ then $m \geq 4$ or $n \geq 4$"

The **contrapositive** of this statement is

A. If $m < 4$ or $n < 4$, then $m + n < 7$.

B. If $m \geq 4$ or $n \geq 4$, then $m + n \geq 7$.

C. If $m < 4$ and $n < 4$, then $m + n < 7$.

D. If $m \leq 4$ or $n \leq 4$, then $m + n < 7$.

2 Which of the following functions does not describe simple harmonic motion?

A. $x = \sin^2 3t - \cos^2 3t$

B. $x = 5\cos\left(2t + \frac{\pi}{6}\right)$

C. $x = 2\sin 4t - 4\cos 2t$

D. $x = 2\sin^2 2t - 5\cos 4t + 2$

3 Consider the statement:

"Whichever positive number r you pick, it is possible to find a number x greater than zero such that $\frac{1}{x^2} < r$."

When this statement is written in a formal language of proof, which of the following is obtained?

A. $\forall r > 0, \exists x > 0$ such that $\frac{1}{x^2} < r$

B. $\exists r > 0, \exists x > 0$ such that $\frac{1}{x^2} < r$

C. $\forall r > 0, \forall x > 0$ such that $\frac{1}{x^2} < r$

D. $\exists r > 0, \forall x > 0$ such that $\frac{1}{x^2} < r$

4 Which of the following is a vector equation of the line joining the points $P(1, -3, 4)$ and $Q(5, 0, -2)$?

A. $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$, λ is a scalar parameter.

B. $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -3 \\ 6 \end{pmatrix}$, λ is a scalar parameter.

C. $\mathbf{r} = \begin{pmatrix} 2 \\ -1.5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ -6 \end{pmatrix}$, λ is a scalar parameter.

D. $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$, λ is a scalar parameter.

5 The position of a particle at a time t is given by $\mathbf{r}(t) = \begin{pmatrix} \sqrt{t-2} \\ t \end{pmatrix}$ for $t \geq 2$.

The Cartesian equation of the path of the particle is given by

A. $y = x^2 - 2$ for $x \geq 0$

B. $y = x^2 + 2$ for $x \geq 2$

C. $y = x^2 + 2$ for $x \geq 0$

D. $y = 2x^2 + 2$ for $x \geq 2$

$\left.\begin{array}{l} x = \sqrt{t-2} \\ y = t \end{array}\right\}$

$x^2 = t - 2$

$x^2 = y - 2$

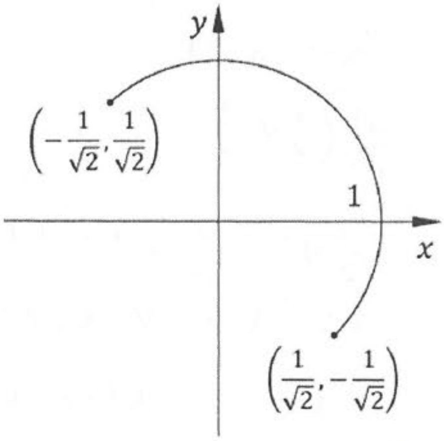
$y = x^2 + 2$

when $t \geq 2$,

$x \geq 0$

$\therefore$ C

6 Which of the following curve describes the diagram given below?



A. $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$ for $\frac{\pi}{4} \leq t \leq \frac{5\pi}{4}$

B. $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ for $\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$

C. $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$ for $\frac{3\pi}{4} \leq t \leq \frac{5\pi}{4}$

D. $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ for $\frac{\pi}{4} \leq t \leq \frac{5\pi}{4}$

consider A:

t		$\frac{\pi}{4}$	$\frac{5\pi}{4}$
x	sin t	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
y	-cos t	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

which works

7 Which expression is equal to $\int \frac{dx}{x^2+4x+6}$?

A. $\tan^{-1}\frac{x+2}{2}+c$

B. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+2}{\sqrt{2}}+c$

C. $\tan^{-1}\frac{x+2}{\sqrt{2}}+c$

D. $\frac{1}{2}\tan^{-1}\frac{x+2}{2}+c$

$= \int \frac{dx}{(x+2)^2+\sqrt{2}^2}$

$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{x+2}{\sqrt{2}} + c$

8 A particle moves on the x-axis with velocity $\mathbf{v}$. The particle is initally at rest at $x=2$. Its acceleration is given by $\ddot{x}=x-5$. The speed at $x=1$ is given by

A. $|v|=7m/s$

B. $|v|=\frac{7}{2}m/s$

C. $|v|=\sqrt{7}m/s$

D. $|v|=\frac{\sqrt{7}}{2}m/s$

$\ddot{x} = x-5$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x-5$

$\frac{1}{2}v^2 = \frac{x^2}{2} - 5x + C$

when $t=0,\ v=0,\ x=2$

$0 = \frac{2^2}{2} - 5(2) + C$

$C = 8$

$\frac{1}{2}v^2 = \frac{x^2}{2} - 5x + 8$

when $x=1$

$\frac{1}{2}v^2 = \frac{1}{2} - 5 + 8$

$v^2 = 7$

$|v| = \sqrt{7}\ ms^{-1}$

9 Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$, then $I_n + I_{n+2}$ is equal to

A. $\frac{1}{n-1}$

B. $\frac{1}{n-2}$

C. $\frac{1}{n+1}$

D. $\frac{1}{n+2}$

$I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$

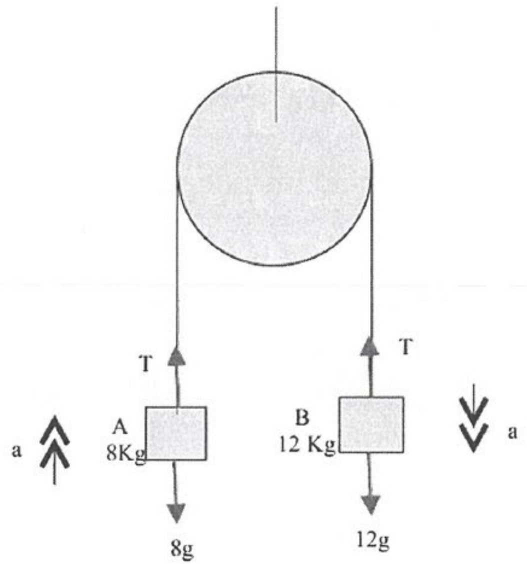
$I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^{n+2} x\, dx$

$I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n x\,(1+\tan^2 x)\, dx$

$= \int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x\, dx$

$= \left.\frac{\tan^{n+1} x}{n+1}\right]_0^{\frac{\pi}{4}} = \frac{1}{n+1}$

10 Two masses of 8 kg and 12 kg are hanging by a light and inextensible string over a frictionless pulley as shown in the digram given below.



The tension T in the string is given by

A. $\frac{20g}{5}$ N

B. $\frac{24g}{5}$ N

C. $\frac{4g}{5}$ N

D. $\frac{48g}{5}$ N

$T - 8g = 8a \rightarrow a = \frac{T-8g}{8}$

$12g - T = 12a$

$12g - T = \cancel{12}^{3}\left(\frac{T-8g}{\cancel{8}_{2}}\right)$

$24g - 2T = 3T - 24g$

$48g = 5T$

$\therefore T = \frac{48g}{5}$

Question 11 (15 marks) Use a separate Writing Booklet for Question 11.

(a) Given $z = 1 + i$, find $z^2 + \frac{1}{z^2}$ in the form of $a + ib$, where a and b are real numbers. **3**

11a) $z = 1+i$

$z^2 = (1+i)^2$

$= 1 + 2i - 1$

$= 2i$

$\frac{1}{z^2} = \frac{1}{2i} \frac{\times i}{\times i}$

$= \frac{i}{-2}$

$\therefore z^2 + \frac{1}{z^2} = 2i - \frac{i}{2}$

$= \frac{3}{2}i$

$\therefore a = 0, \ b = \frac{3}{2}$

Question was done well, nearly all students got full marks.

(b) Suppose that $a, b \in \mathbb{Z}$. If ab and $a - b$ are both even, prove that a and b are both even. **2**

b) Proof by contrapositive

Case 1: One odd and one even

Let $a = 2m+1$

$b = 2n$

Then $ab = (2m+1)(2n)$

$= 4mn + 2n$

$= 2(2mn+n)$, which is even

$a-b = (2m+1) - 2n$

$= 2m - 2n + 1$

$= 2(m-n) + 1$, which is odd

Case 2: Both odd

$ab = (2m+1)(2n+1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn+m+n) + 1$, odd

$\therefore$ this contrdicts the original condition.

(c) A particle moves in simple harmonic motion described by the differential equation **2**

$\ddot{x} = -16(x+2)$.

Find the period of motion and the centre of motion about which the motion occurs.

c) $\ddot{x} = -16(x+2)$

In the form

$\ddot{x} = -n^2(x-c)$

$\therefore\ n = 4$

$c = -2$

period : $\frac{2\pi}{4}$ $= \frac{\pi}{2}$

centre of motion $= -2$

Question was done well, nearly all students got full marks.

(d) Show that the pair of lines $L_1\colon \mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}$ and $L_2\colon \mathbf{s} = \begin{pmatrix}3\\1\\3\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\0\end{pmatrix}$ are perpendicular to each other, where λ and μ are scalar parameters. **3**

Also, show that they intersect at $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

d) $\begin{pmatrix}-1\\-2\\1\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$= -2 + 2 + 0$

$= 0$, $\therefore\ \underset{\sim}{r} \cdot \underset{\sim}{s} = 0$ and the two are perpendicular

when $\lambda = 0$

$\underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + 0\begin{pmatrix}-1\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

for $\underset{\sim}{s}$

$\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3\\1\\3\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\0\end{pmatrix}$

$1 = 3 + 2\mu$ (1)

$2 = 1 - \mu$ (2)

from (2)

$\mu = -1$

sub into (1)

$1 = 3 + 2(-1)$

$= 1$

$\therefore\ \underset{\sim}{r}$ and $\underset{\sim}{s}$ intersect at $\begin{pmatrix}1\\2\\3\end{pmatrix}$

(e) i. Show that for any integer n, $e^{in\theta} + e^{-in\theta} = 2\ \cos n\theta$. **2**

ii. By expanding $\left(e^{i\theta} + e^{-i\theta}\right)^3$, show that $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$. **3**

11e)

i) $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$e^{ix} + e^{-ix} = \cos x + \cancel{i\sin x} + \cos x - \cancel{i\sin x}$

$= 2\cos x$

let $x = n\theta$

$e^{in\theta} + e^{-in\theta} = 2\cos n\theta$

ii) From part i)

$(e^{i\theta} + e^{-i\theta})^3$

$= (2\cos\theta)^3$

$= 8\cos^3\theta$

$(e^{i\theta} + e^{-i\theta})^3 = (e^{i\theta} + e^{-i\theta})^2(e^{i\theta} + e^{-i\theta})$

$= e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}$

$= e^{3i\theta} + e^{-3i\theta} + 3(e^{i\theta} + e^{-i\theta})$

$= 2\cos 3\theta + 3(2\cos\theta)$

$= 2\cos 3\theta + 6\cos\theta$

$\therefore\ 8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$

$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

Question 12 (15 marks) Use a separate Writing Booklet for Question 12.

(a) Prove by contradiction that $\log_4 9$ is an irrational number. **3**

Assume $\log_4 9$ is rational.

i.e. $\exists\, p, q \in \mathbb{Z}$, $q \neq 0$ such that

$\log_4 9 = \frac{p}{q}$ where p, q have no common factors

$4^{\frac{p}{q}} = 9$

$4^p = 9^q$

Now, since 4 is even then 4^p is also even.

Also since 9 is odd then 9^q is also odd.

$\therefore 4^p \neq 9^q$ which is a contradiction since an even number cannot be equal to an odd number.

$\therefore$ the original assumption is incorrect.

$\therefore \log_4 9$ is irrational.

Mostly well done.
Common errors:
Students saying that 4 and 9 have no common factors, when obviously 1 is a common factor of 4 and 9.

(b) Evaluate $\displaystyle\int \frac{dx}{1 - \cos x + \sin x}$. **3**

$I = \displaystyle\int \frac{dx}{1 - \cos x + \sin x}$

let $t = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$dx = \frac{2}{1+t^2}\,dt$

$\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$\therefore I = \displaystyle\int \frac{\frac{2\,dt}{1+t^2}}{1 - \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}}$

$= \displaystyle\int \frac{2\,dt}{1+t^2 - 1 + t^2 + 2t}$

$= \displaystyle\int \frac{\cancel{2}\,dt}{\cancel{2}t(t+1)} = \int \left(\frac{A}{t} + \frac{B}{t+1}\right) dt$

$1 = A(t+1) + Bt$

when $t = 0$, $A = 1$

when $t = -1$, $B = -1$

$I = \displaystyle\int \frac{1}{t} - \frac{1}{t+1}\, dt$

$= \ln|t| - \ln|t+1| + C$

$= \ln\left|\frac{t}{t+1}\right| + C$

$= \ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2} + 1}\right| + C$

Common errors:
incorrect dx and
incorrect integration of $\displaystyle\int \frac{dt}{t(t+1)}$

(c) The complex number $3 - i$ is a zero of a cubic polynomial $Q(z) = z^3 + pz^2 + qz + r$ where p, q, r are real numbers. **3**

Given that $z = -1$ is also a zero of $Q(z)$, find the vales of p, q, r and then write the complete factorisation of $Q(z)$.

Since $3-i$ is a zero of $Q(z)$

$\Rightarrow$ $3+i$ is also a zero since coefficients are $\mathbb{R}$

$\therefore$ $z^2-6z+10$ is a factor of $Q(z)$

Since $z=-1$ is also a zero,

$\Rightarrow$ $z+1$ is a factor of $Q(z)$

$$\therefore Q(z) = (z^2-6z+10)(z+1)$$
$$= z^3-6z^2+10z+z^2-6z+10$$
$$= z^3-5z^2+4z+10 \equiv z^3+pz^2+qz+r$$

$$\therefore p=-5,\ q=4,\ r=10$$

$$\therefore Q(z) = (z-3+i)(z-3-i)(z+1)$$

Common errors:
Students writing $r = -10$ after using sum & product of the roots rather than correspondence.
Students forgetting to write the complete factorisation of $Q(z)$.

(d) For all real numbers x and y, with $x^2 + y^2 \neq 0$,

i. prove that $\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = 1.$ **1**

$$\text{LHS} = \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2$$
$$= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}$$
$$= \frac{x^2+y^2}{x^2+y^2}$$
$$= 1 = \text{RHS}$$

This question was done very well.

ii. Hence, or otherwise, prove that $\left|\frac{x}{\sqrt{x^2+y^2}}\right| \leq 1$ and $\left|\frac{y}{\sqrt{x^2+y^2}}\right| \leq 1$. **2**

From (i) $\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = 1$

Since both squared terms are non-negative, each term must be less than or equal to 1.

ie, $\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 \leq 1$ Similarly, $\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 \leq 1$

$\Rightarrow \left|\frac{x}{\sqrt{x^2+y^2}}\right| \leq 1$ $\Rightarrow \left|\frac{y}{\sqrt{x^2+y^2}}\right| \leq 1$

Common errors:
Students lost a mark because they did not state adequate reasons.

(e) With respect to the origin O, the lines L_1 and L_2 are given by the equations: **3**

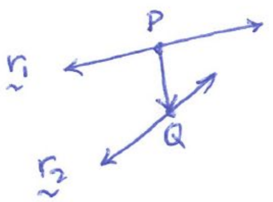
$$L_1: r_1 = \begin{pmatrix} 10 \\ 0 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$L_2: r_1 = \begin{pmatrix} 17 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$$

where λ and μ are scalar parameters.

Show that the shortest distance between the two lines is $\frac{2}{\sqrt{11}}$ units.

Assume $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ be skew lines.



let $\overrightarrow{PQ}$ be $\perp$ to both $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$.

$\vec{P} = (10-\lambda, \lambda, -9+2\lambda)$

$\vec{Q} = (17+5\mu, 1-\mu, 3+2\mu)$

$\overrightarrow{PQ} = (7+5\mu+\lambda, 1-\mu-\lambda, 12+2\mu-2\lambda)$

$\overrightarrow{PQ} \cdot (-1,1,2) = 0$

$-7-5\mu-\lambda+1-\mu-\lambda+24+4\mu-4\lambda = 0$

$18-2\mu-6\lambda = 0$

$\mu+3\lambda = 9 \rightarrow (1) \Rightarrow \mu = 9-3\lambda$

$\overrightarrow{PQ} \cdot (5,-1,2) = 0$

$35+25\mu+5\lambda-1+\mu+\lambda+24+4\mu-4\lambda = 0$

$58+30\mu+2\lambda = 0$

$15\mu+\lambda = -29 \rightarrow (2)$

$15(9-3\lambda)+\lambda = -29$

$135-45\lambda+\lambda = -29$

$\lambda = \frac{41}{11}$

$\mu = \frac{-24}{11}$

$\therefore \overrightarrow{PQ} = \left(\frac{-2}{11}, \frac{-6}{11}, \frac{2}{11}\right)$

$|\overrightarrow{PQ}| = \sqrt{\left(\frac{-2}{11}\right)^2+\left(\frac{-6}{11}\right)^2+\left(\frac{2}{11}\right)^2}$

$= \sqrt{\frac{4}{11}}$

$= \frac{2}{\sqrt{11}}$

A slight error will result in a deduction of 1 mark. The method shown is the most common way of solving. Although a couple of students used the projection method which is not shown in the sample solution. Students who achieved full marks made no mistake in the process.

Question 13 (15 marks) Use a separate Writing Booklet for Question 13.

(a) i. Express $\dfrac{x^2+1}{(x+1)(x-2)(x^2+x+1)}$ as the sum of partial fractions. **4**

ii. Hence, or otherwise, show that **3**

$$\int \frac{x^2+1}{(x+1)(x-2)(x^2+x+1)}\,dx$$

$$= -\frac{2}{3}\ln|x+1| + \frac{5}{21}\ln|x-2| + \frac{3}{14}\ln|x^2+x+1| + \frac{1}{7\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

where C is the constant of integration.

Question 13

a) i)

$$\frac{x^2+1}{(x+1)(x-2)(x^2+x+1)} \equiv \frac{A}{(x+1)} + \frac{B}{x-2} + \frac{Cx+D}{x^2+x+1}$$

$$x^2+1 \equiv A(x-2)(x^2+x+1) + B(x+1)(x^2+x+1) + (Cx+D)(x+1)(x-2)$$

Sub $x=2$

$5 = B(3)(7)$

$5 = 21B$

$\therefore B = \frac{5}{21}$

Sub $x=-1$

$2 = A(-3)(1)$

$\therefore A = \frac{-2}{3}$

Sub $x=0$

$1 = -2A + B - 2D$

$1 = -2\left(\frac{-2}{3}\right) + \frac{5}{21} - 2D$

$2D = \frac{4}{3} + \frac{5}{21} - 1$

$= \frac{4}{7}$

$D = \frac{2}{7}$

Sub $x=1$

$2 = -3A + 6B - 2C - 2D$

$2C = -3\left(\frac{-2}{3}\right) + 6\left(\frac{5}{21}\right) - 2\left(\frac{2}{7}\right) - 2$

$= 2 + \frac{30}{21} - \frac{4}{7} - 2$

$= \frac{6}{7}$

$C = \frac{3}{7}$

$$\therefore \frac{x^2+1}{(x+1)(x-2)(x^2+x+1)} \equiv -\frac{2}{3(x+1)} + \frac{5}{21(x-2)} + \frac{3x+2}{7(x^2+x+1)}$$

A handful of students did not solve by observation and substitution but created simultaneous equations.

ii) From part i)

$$\frac{x^2+1}{(x+1)(x-2)(x^2+x+1)} = -\frac{2}{3(x+1)} + \frac{5}{21(x-2)} + \frac{3x+2}{7(x^2+x+1)}$$

$$\int \frac{x^2+1}{(x+1)(x-2)(x^2+x+1)}\,dx = \frac{-2}{3}\ln|x+1| + \frac{5}{21}\ln|x-2| + \int \frac{\frac{3x}{7}+\frac{2}{7}}{x^2+x+1}\,dx + C$$

$$= \frac{-2}{3}\ln|x+1| + \frac{5}{21}\ln|x-2| + \frac{3}{14}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{\cancel{3}}{14}\int \frac{1\,dx}{\cancel{3}(x^2+x+1)} + C$$

$$= \frac{-2}{3}\ln|x+1| + \frac{5}{21}\ln|x-2| + \frac{3}{14}\ln|x^2+x+1| + \frac{1}{14}\int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + C$$

$$= \frac{-2}{3}\ln|x+1| + \frac{5}{21}\ln|x-2| + \frac{3}{14}\ln|x^2+x+1| + \frac{1}{14}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{-2}{3}\ln|x+1| + \frac{5}{21}\ln|x-2| + \frac{3}{14}\ln|x^2+x+1| + \frac{1}{7\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

Alternate way to change:
(more elegant)

$$\int \frac{3x/7 + 2/7}{x^2+x+1}\,dx$$

$$= \frac{1}{7}\int \frac{3x+2}{x^2+x+1}\,dx$$

$$= \frac{1}{7}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{7}\int \frac{x+1}{x^2+x+1}\,dx$$

$$= \frac{1}{7}\ln|x^2+x+1| + \frac{1}{14}\int \frac{2x+1\,dx}{x^2+x+1} - \frac{1}{14}\int \frac{1}{x^2+x+1}\,dx$$

$$= \frac{1}{7}\ln|x^2+x+1| + \frac{1}{14}\ln|x^2+x+1| - \frac{1}{14}\int \frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \frac{3}{14}\ln|x^2+x+1| - \frac{1}{14}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right) + C$$

$$= \frac{3}{14}\ln|x^2+x+1| - \frac{1}{7\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

(b) Let $z \in \mathbb{C}$ and define $A = 1 + i$ and $B = -1 + i$.

Find the set of points satisfying the condition $|z - A| = 2|z - B|$.

i. Express this as a Cartesian equation. **3**

ii. Interpret the result geometrically. **1**

i) let $z = x + iy$

$A = 1+i, \quad B = -1+i$

$$|z-A| = 2|z-B|$$
$$|(x-1)+i(y-1)| = 2|(x+1)+i(y-1)|$$

$$\sqrt{(x-1)^2+(y-1)^2} = 2\sqrt{(x+1)^2+(y-1)^2}$$

Square both sides

$$(x-1)^2+(y-1)^2 = 4(x+1)^2+4(y-1)^2$$
$$0 = 4(x+1)^2-(x-1)^2+4(y-1)^2-(y-1)^2$$
$$0 = 4(x^2+2x+1)-(x^2-2x+1)+3(y-1)^2$$
$$0 = 4x^2+8x+4-x^2+2x-1+3(y-1)^2$$
$$0 = 3x^2+10x+3+3(y-1)^2$$
$$0 = x^2+\frac{10x}{3}+1+(y-1)^2$$
$$-1+\left(\frac{5}{3}\right)^2 = x^2+\frac{10}{3}x+\left(\frac{5}{3}\right)^2+(y-1)^2$$
$$\frac{16}{9} = \left(x+\frac{5}{3}\right)^2+(y-1)^2$$

ii) this is an equation of a circle with centre $\left(-\frac{5}{3}, 1\right)$

and radius $\frac{4}{3}$

(c) A crate of mass 15 kg is being pulled along a smooth horizontal surface by a rope inclined at 30° to the horizontal. ($g = 10\ m/s^2$)

The tension in the rope is 18 N.

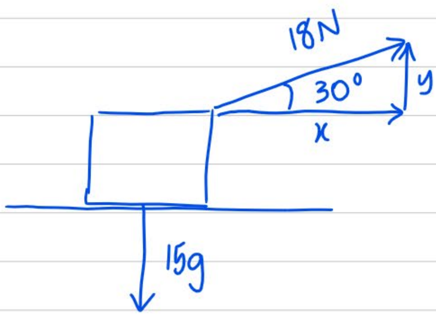
i. Draw a force diagram for this situation. **1**

ii. Find the acceleration of the box. **1**

iii. Find the normal force between the box and the floor. **2**

c)

i)



$\sin 30 = \frac{y}{18} \rightarrow y = 18\sin 30$

$\cos 30 = \frac{x}{18} \rightarrow x = 18\cos 30$

ii) Since there is no friction, net force is only in the horizontal direction.

$$F = ma$$
$$18\cos 30 = 15a$$
$$\therefore a = \frac{18\left(\frac{\sqrt{3}}{2}\right)}{15}$$
$$= \frac{6}{5}\,\frac{\sqrt{3}}{2}$$
$$a = \frac{3\sqrt{3}}{5}\ m/s^2$$

iii) Normal force is balanced by vertical force

$$N + 18\sin 30 = 15(10)$$
$$N = 150 - 18\left(\frac{1}{2}\right)$$
$$N = 141\ N$$

Question 14 (14 marks) Use a separate Writing Booklet for Question 14.

(a) Prove that a three-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3. **3**

let h = hundred's digit
t = ten's digit
u = unit's digit

$\therefore$ the 3-digit no. can be represented by
$N = 100h + 10t + u$

$N = 99h + 9t + (h + t + u)$

If the sum of the digits, $h+t+u$ is divisible by 3,
then $h + t + u = 3m$, $m \in \mathbb{Z}$

$\therefore N = 99h + 9t + 3m$
$= 3(33h + 3t + m)$

$\therefore$ N is divisible by 3 since $33h+3t+m$ is an integer.

Also, if N is divisible by 3

$\Rightarrow N = 3k$, $k \in \mathbb{Z}$

then $3k = 99h + 9t + (h + t + u)$

$\Rightarrow h + t + u = 3k - 99h - 9t$
$= 3(k - 33h - 3t)$

$\therefore$ $h+t+u$ is divisible by 3 since $k-33h-3t$ is an integer

$\therefore$ Sum of its digits is divisible by 3.

This question was done very well for students who recognise how to set-up a three-digit number. Since this is an IFF question, both the statement and its converse need to be proven.

(b) Evaluate **3**

$$\int_{-0.5}^{0.5} \sqrt{\frac{1-x}{1+x}}\,dx.$$

Method 1:

$$\int_{-0.5}^{0.5} \sqrt{\frac{1-x}{1+x}}\,dx = \int_{-0.5}^{0.5} \frac{\sqrt{1-x}}{\sqrt{1+x}} \cdot \frac{\sqrt{1-x}}{\sqrt{1-x}}\,dx$$

$$= \int_{-0.5}^{0.5} \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int_{-0.5}^{0.5} \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}\int_{-0.5}^{0.5} \frac{-2x\,dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x + \frac{1}{2}\,\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\Bigg]_{-0.5}^{0.5}$$

$$= \sin^{-1}x + \sqrt{1-x^2}\Big]_{-0.5}^{0.5}$$

$$= \left(\sin^{-1}0.5 + \cancel{\sqrt{1-(0.5)^2}}\right) - \left(\sin^{-1}(-0.5) + \cancel{\sqrt{1-(-0.5)^2}}\right)$$

$$= \frac{1}{6}\pi - \left(-\frac{1}{6}\pi\right) = \frac{1}{3}\pi$$

Method 2:

let $x = \sin\theta$ when $x = -0.5, \theta = \frac{-\pi}{6}$

$dx = \cos\theta\, d\theta$

when $x = 0.5, \theta = \frac{\pi}{6}$

$$\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \frac{1-\sin\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta\, d\theta$$

$$= \int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} (1-\sin\theta)\,d\theta$$

$$= \theta + \cos\theta\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6} + \cos\frac{\pi}{6} - \left[-\frac{\pi}{6} + \cos\left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{\pi}{3}$$

This question was **not** done very well. Most common error is when students tried to rationalise the denominator instead of the numerator. Another common error is using the wrong substitution.

(c) A sphere has a centre at $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and its radius is 4 units.

A line has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ where μ is a scalar parameter.

i. Write down the vector equation of the sphere. **1**

equation of a sphere:

$$|\underset{\sim}{v} - \underset{\sim}{c}| = r$$

$$\left|\underset{\sim}{v} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}\right| = 4$$

Common error:
Students writing the Cartesian equation of the sphere instead of its vector equation. No marks awarded for students who wrote the Cartesian equation.

ii. Determine whether the line is a tangent to the sphere, clearly justifying your conclusion. **2**

Method 1:

let $\underset{\sim}{p}$ be a vector from the centre of the sphere to the line

$$\underset{\sim}{p} = \begin{pmatrix} 2+2\mu-3 \\ 3+\mu-2 \\ 5-\mu+1 \end{pmatrix} = \begin{pmatrix} 2\mu-1 \\ \mu+1 \\ -\mu+6 \end{pmatrix}$$

If the line is tangent to the sphere, then

$$\left|\begin{pmatrix} 2\mu-1 \\ \mu+1 \\ -\mu+6 \end{pmatrix}\right| = 4$$

$$(2\mu-1)^2 + (\mu+1)^2 + (6-\mu)^2 = 4^2$$

$$4\mu^2 - 4\mu + 1 + \mu^2 + 2\mu + 1 + 36 - 12\mu + \mu^2 = 16$$

$$6\mu^2 - 14\mu + 22 = 0$$

$$\Delta = (-14)^2 - 4(6)(22) = -332$$

$\therefore$ no real values for μ

$\therefore$ the line is not tangent to the sphere as they don't touch.

Common error:
Students messed up in their algebra, thus, getting the incorrect discriminant.

Method 2:

let $\underset{\sim}{p}$ be a vector from the centre of the sphere to the line

$$\underset{\sim}{p} = \begin{pmatrix} 2+2\mu-3 \\ 3+\mu-2 \\ 5-\mu+1 \end{pmatrix} = \begin{pmatrix} 2\mu-1 \\ \mu+1 \\ -\mu+6 \end{pmatrix}$$

For perpendicular vectors, dot product $= 0$

$$\begin{pmatrix} 2\mu-1 \\ \mu+1 \\ -\mu+6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$4\mu-2+\mu+1+\mu-6=0$$

$$6\mu-7=0$$

$$\mu=\frac{7}{6}$$

Sub $\mu=\frac{7}{6}$ into $\underset{\sim}{p}$.

$$\underset{\sim}{p} = \left(2\times\frac{7}{6}-1, \frac{7}{6}+1, -\frac{7}{6}+6\right)$$

$$= \left(\frac{4}{3}, \frac{13}{6}, \frac{29}{6}\right)$$

$$|\underset{\sim}{p}| = \sqrt{\left(\frac{4}{3}\right)^2+\left(\frac{13}{6}\right)^2+\left(\frac{29}{6}\right)^2} = \sqrt{\frac{179}{6}} \approx 5.46 > 4$$

$\therefore$ the line is not tangent to the sphere.

(d) i. Show that for $k \geq 0$, **2**

$$\int_0^{\pi} \sin^{2k}\theta \cos 2\theta \, d\theta = -\frac{k}{k+1}\int_0^{\pi} \sin^{2k}\theta \, d\theta .$$

for $k \geq 0$

$$I = \int_0^{\pi} \sin^{2k}\theta \cos 2\theta \, d\theta = -\frac{k}{k+1}\int_0^{\pi} \sin^{2k}\theta \, d\theta$$

let $u = \sin^{2k}\theta$ | $dv = \cos 2\theta \, d\theta$

$du = 2k\sin^{2k-1}\theta \cos\theta \, d\theta$ | $v = \frac{\sin 2\theta}{2}$

by parts:

$$\left[\sin^{2k}\theta \, \frac{\sin 2\theta}{2}\right]_0^{\pi} - \int_0^{\pi} \cancel{2}k\sin^{2k-1}\theta \cos\theta \, \frac{\sin 2\theta}{\cancel{2}}$$

$$I = 0-0-k\int_0^{\pi} \sin^{2k-1}\theta \left(2\sin\theta\cos^2\theta\right)d\theta$$

$$I = -2k\int_0^{\pi} \sin^{2k}\theta\cos^2\theta \, d\theta$$

$$I = -\cancel{2}k\int_0^{\pi} \sin^{2k}\theta \left(\frac{1+\cos 2\theta}{\cancel{2}}\right)d\theta$$

$$I = -k\int_0^{\pi} \sin^{2k}\theta \, d\theta - k\int_0^{\pi} \sin^{2k}\theta\cos 2\theta \, d\theta$$

$$I = -k\int_0^{\pi} \sin^{2k}\theta \, d\theta - kI$$

$$kI + I = -k\int_0^{\pi} \sin^{2k}\theta \, d\theta$$

$$I(k+1) = -k\int_0^{\pi} \sin^{2k}\theta \, d\theta$$

$$I = \frac{-k}{k+1}\int_0^{\pi} \sin^{2k}\theta \, d\theta$$

Common error:
Students need to show <u>integration by parts</u> in this show question in order to achieve at least one mark.
This question was **<u>not</u>** done very well.

ii. Hence prove by mathematical induction that **3**

$$\int_0^{\pi} \sin^{2n}\theta \, d\theta = \frac{(2n)!}{n!^2} \times \frac{\pi}{2^{2n}}$$

for all integers $n \geq 0$.

$$I_n = \int_0^{\pi} \sin^{2n}\theta \, d\theta = \frac{(2n)!}{n!^2} \times \frac{\pi}{2^{2n}} \quad , \forall n \geq 0, n \in \mathbb{Z}$$

For $n=0$

LHS: $I_0 = \int_0^{\pi} (\sin\theta)^0 \, d\theta$
$= \theta \Big]_0^{\pi}$
$= \pi$

RHS: $\dfrac{(2\times 0)!}{0!^2} \times \dfrac{\pi}{2^{2(0)}}$
$= \dfrac{1 \times \pi}{1 \times 1}$
$= \pi$

$\therefore$ LHS = RHS

$\therefore$ true for $n=0$

Assume true for $n=k$, $k \geq 0$, $k \in \mathbb{Z}$

i.e. $\displaystyle\int_0^{\pi} \sin^{2k}\theta \, d\theta = \frac{(2k)!}{k!^2} \times \frac{\pi}{2^{2k}}$

Show that it's true for $n = k+1$

i.e $\displaystyle\int_0^{\pi} \sin^{2(k+1)}\theta \, d\theta = \frac{[2(k+1)]!}{(k+1)!^2} \times \frac{\pi}{2^{2(k+1)}}$

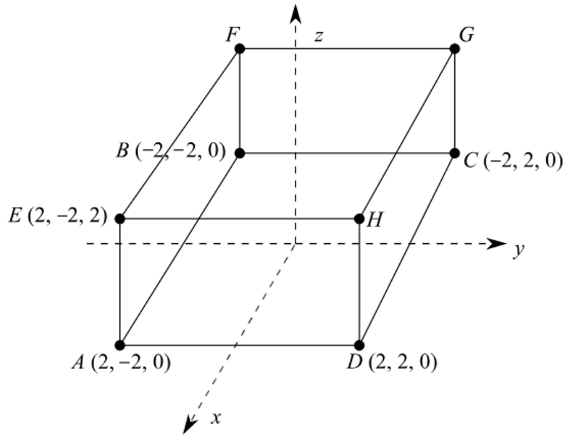
$$\begin{aligned}
\text{LHS} &= \int_0^{\pi} \sin^{2k}\theta \sin^2\theta \, d\theta \\
&= \int_0^{\pi} \sin^{2k}\theta \left(\frac{1-\cos 2\theta}{2}\right) d\theta \\
&= \frac{1}{2}\int_0^{\pi} \sin^{2k}\theta \, d\theta - \frac{1}{2}\int_0^{\pi} \sin^{2k}\theta \cos 2\theta \, d\theta \\
&= \frac{1}{2}\int_0^{\pi} \sin^{2k}\theta \, d\theta - \frac{1}{2}\left(\frac{-k}{k+1}\right)\int_0^{\pi} \sin^{2k}\theta \, d\theta \quad \rightarrow \text{from (i)} \\
&= \left(\frac{1}{2} + \frac{k}{2(k+1)}\right)\int_0^{\pi} \sin^{2k}\theta \, d\theta \\
&= \frac{2k+1}{2(k+1)}\int_0^{\pi} \sin^{2k}\theta \, d\theta \\
&= \frac{2k+1}{2(k+1)} \times \frac{(2k)!}{k!^2} \times \frac{\pi}{2^{2k}} \quad (\text{using assumption}) \\
&= \frac{2k+2}{2k+2} \times \frac{2k+1}{2(k+1)} \times \frac{(2k)!}{k!^2} \times \frac{\pi}{2^{2k}} \\
&= \frac{(2k+2)(2k+1)(2k)!}{2(k+1) \times 2(k+1)\, k!^2} \times \frac{\pi}{2^{2k}} \\
&= \frac{(2k+2)!}{(k+1)!^2} \times \frac{\pi}{2^2 \times 2^{2k}} \\
&= \frac{[2(k+1)]!}{(k+1)!^2} \times \frac{\pi}{2^{2(k+1)}}
\end{aligned}$$

Hence proved by mathematical induction for all integers $n \geq 0$.

This question was not done well at all. Most common error is using $n = 1$ as their base case when the question mentions "for all integers $n \geq 0$. A thorough presentation of mathematical induction is required to achieve full marks.

Question 15 (15 marks) Use a separate Writing Booklet for Question 15.

(a)



For the rectangular prism given above

i. Show that the internal diagonals AG and DF bisect each other. **2**

Q15 a)

Show midpoints equal.

i) Midpoint (AG) $= \frac{1}{2}((2,-2,0)+(-2,2,2))$

$= \frac{1}{2}(0,0,2)$

$= (0,0,1)$

Midpoint (DF) $= \frac{1}{2}((2,2,0)+(-2,-2,2))$

$= \frac{1}{2}(0,0,2)$

$= (0,0,1)$

= Midpoint AG, since same the diagonals bisect.

ii. Calculate the acute angle, θ, between the diagonals, to the nearest degree. **3**

ii) $\vec{AG} = (-2,2,2) - (2,-2,0)$
$= (-4,4,2)$
$\vec{DF} = (-2,-2,2) - (2,2,0)$
$= (-4,-4,2)$

$$\cos\theta = \frac{\vec{AG}\cdot\vec{DF}}{|\vec{AG}||\vec{DF}|}$$
$$= \frac{(-4)(-4) + (4)(-4) + (2)(2)}{\sqrt{(-4)^2+4^2+(2)^2} \times \sqrt{4^2+4^2+(-2)^2}}$$
$$= \frac{4}{6\times 6}$$
$$= \frac{4}{36}$$
$\theta = 83^\circ.62$
$= 84^\circ$ (nearest degree)

(b) If $I_n = \displaystyle\int_0^1 \frac{x^{n-1}\,dx}{(1+x^2)^n}$ for $n \geq 1$

i. Show that $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \sin^{n-1}\theta\ \cos^{n-1}\theta\ d\theta$ for $n \geq 1$. **2**

ii. Let $K_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta\ d\theta$, show that $I_n = \dfrac{1}{2^n}K_{n-1}$. **2**

iii. Hence show that $I_n = 2^{-\frac{n(n+1)}{2}}\left(\dfrac{\pi}{2}\right)$. **1**

15b) $I_n = \int_0^1 \frac{x^{n-1}}{(1+x^2)^n}\, dx$

i)

let $x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta\, d\theta$

when $x = 1$, $\theta = \pi/4$

when $x = 0$, $\theta = 0$

$1+x^2 = 1+\tan^2\theta = \sec^2\theta$

$$I_n = \int_0^{\pi/4} \frac{\tan^{n-1}\theta}{\sec^{2n}\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-1}\theta \cdot \sec^{-2(n-1)}\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{\sin^{n-1}\theta}{\cancel{\cos^{n-1}}\theta} \cdot \cos^{\cancel{2}(n-1)}\theta\, d\theta$$

$$= \int_0^{\pi/4} \sin^{n-1}\theta \cdot \cos^{n-1}\theta\, d\theta$$

ii) From part i)

$$I_n = \int_0^{\pi/4} \sin^{n-1}\theta \cdot \cos^{n-1}\theta\, d\theta$$

let $u = 2\theta$

$\theta = \frac{u}{2}$

$\frac{d\theta}{du} = \frac{1}{2} \rightarrow d\theta = \frac{du}{2}$

$\theta = 0$, $u = 0$

$\theta = \frac{\pi}{4}$, $u = \frac{\pi}{2}$

$$I_n = \int_0^{\pi/2} \sin^{n-1}\left(\frac{u}{2}\right) \cos^{n-1}\left(\frac{u}{2}\right) \frac{du}{2}$$

$$= \int_0^{\pi/2} \left(\frac{1}{2}\right)^{n-1} \sin^{n-1} u\ \frac{du}{2}$$

$$= \frac{1}{2^n} \int_0^{\pi/2} \sin^{n-1} u\, du$$

$$= \frac{1}{2^n} K_{n-1}$$

from double angle identity.

$\sin^{n-1}\left(\frac{u}{2}\right) \cos^{n-1}\left(\frac{u}{2}\right) = \left(\frac{1}{2}\right)^{n-1} \sin^{n-1} u$

Part ii)

Poorly done.

iii) $I_n = \frac{1}{2^n} \cdot \frac{1}{2^{n-1}} \cdot \frac{1}{2^{n-2}} \cdots \frac{1}{2^1} \cdot K_0$

$$K_0 = \int_0^{\pi/2} d\theta$$

$$= \left[\theta\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

$$I_n = \frac{1}{2^{1+2+3\cdots+n}} \cdot \frac{\pi}{2}$$

sum of AP

$$= 2^{-\frac{n(n+1)}{2}} \cdot \frac{\pi}{2}$$

Part iii)

Most students did not get up to this part.

(c) Let $\omega = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$.

i. Show that ω^k is a solution of $z^9 - 1 = 0$, where k is an integer. **1**

ii. Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0$. **2**

iii. Hence, show that $8\cos\frac{\pi}{9} \times \cos\frac{2\pi}{9} \times \cos\frac{4\pi}{9} - 1 = 0$. **2**

Q15c)

i) $\omega = cis\left(\frac{2\pi}{9}\right)$

$\omega^9 = cis\left(9 \cdot \frac{2\pi}{9}\right)$

$= cis(2\pi)$

$= 1$

$\therefore (\omega^k)^9 = \omega^{9k}$, where k is an integer.

$= 1$

$\therefore z = \omega^k$ satisfies $z^9 - 1 = 0$.

ii) From part i)

$z^9 - 1 = (z-1)(z^8 + z^7 + \ldots + z + 1)$

sub $z = \omega$

$(\omega - 1)(\omega^8 + \omega^7 + \omega^6 + \ldots + \omega + 1) = 0$, $\omega \neq 1$.

$\therefore 1 + \omega + \omega^2 + \ldots + \omega^7 + \omega^8 = 0$

Alternate solution:

since $\omega \neq 1$ but $\omega^9 = 1$

$1 + \omega + \omega^2 + \ldots + \omega^8 = \frac{1 - \omega^9}{1 - \omega}$ sum of G.P

$a = 1$

$r = \omega$

$= \frac{1 - 1}{1 - \omega}$

$= 0$

iii) From part ii)

$$1+\omega+\omega^2+\ldots+\omega^8=0$$

$$\omega+\omega^2+\ldots+\omega^8=-1$$

$$(\omega+\omega^8)+(\omega^2+\omega^7)+(\omega^3+\omega^6)+(\omega^4+\omega^5)=-1$$

$$2\left(\cos\frac{2\pi}{9}\right)+2\left(\cos\frac{4\pi}{9}\right)+\cancel{2\left(\cos\frac{6\pi}{9}\right)}+2\cos\left(\frac{8\pi}{9}\right)=\cancel{-1}$$

$$\cos\frac{6\pi}{9}=\frac{-1}{2}$$

$$2\cos\frac{2\pi}{9}+2\cos\frac{4\pi}{9}-2\cos\frac{\pi}{9}=0$$

$$2\cos\frac{2\pi}{9}+2\cos\frac{4\pi}{9}=2\cos\frac{\pi}{9}$$

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}=\cos\frac{\pi}{9}$$

$$(1+\omega)(1+\omega^2)(1+\omega^3)\cdots(1+\omega^8)=1$$

Group conjugate pairs

$$(1+\omega)(1+\omega^8)=1+\omega+\omega^8+\omega^9$$
$$=2+(\omega+\omega^8)$$
$$=2+2\cos\frac{2\pi}{9}$$

$$(1+\omega^2)(1+\omega^7)=2+(\omega^2+\omega^7)$$
$$=2+2\cos\frac{4\pi}{9}$$

$$(1+\omega^3)(1+\omega^6)=2+(\omega^3+\omega^6)$$
$$=2+2\cos\frac{6\pi}{9}$$

$$(1+\omega^4)(1+\omega^5)=2+(\omega^4+\omega^5)$$
$$=2+2\cos\frac{8\pi}{9}$$

multiply all groups of conjugate pairs

$$\left(2+2\cos\frac{2\pi}{9}\right)\left(2+2\cos\frac{4\pi}{9}\right)\left(2+2\cos\frac{6\pi}{9}\right)\left(2+2\cos\frac{8\pi}{9}\right)=1$$

Using $2+2\cos2\theta=4\cos^2\theta$

$$4\cos^2\frac{\pi}{9}\cdot4\cos^2\frac{2\pi}{9}\cdot4\cos^2\frac{3\pi}{9}\cdot4\cos^2\frac{4\pi}{9}=1$$

$$64\left(\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\right)^2=1$$

$$8\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}-1=0$$

Question 16 (16 marks) Use a separate Writing Booklet for Question 16.

(a) The point P is 5 metres to the right of the origin O on a straight line. **3**

A particle is released from rest at P and moves along the straight line in a simple harmonic motion about O, with a period 4π seconds.

After π seconds, another particle is released from rest at P and moves along this straight line in simple harmonic motion about O, with period 4π seconds.

Find when and where the two particles first collide.

This question was not done well at all. Most common error is not recognising that the answer has to be greater than π seconds. Some students also forgot to calculate where the two particles first collide, i.e. solving for x.

$x = a\cos(nt + \alpha)$

where $a = 5$

$n = \frac{2\pi}{T} = \frac{2\pi}{4\pi} = \frac{1}{2}$

$\alpha = 0$

$\therefore x_1 = 5\cos\frac{1}{2}t$ (particle 1)

For particle 2.

$x_2 = 5\cos\frac{1}{2}(t-\pi)$, where $t \geq \pi$

Find the first time $t > \pi$ when $x_1 = x_2$

i.e. $5\cos\frac{1}{2}t = 5\cos\frac{1}{2}(t-\pi)$

$\cos\frac{1}{2}t = \cos\frac{1}{2}(t-\pi)$

Use the identity:

$\cos A = \cos B \iff A = B + 2n\pi$ or $A = -B + 2n\pi$

Case 1: $\frac{1}{2}t = \frac{1}{2}t - \frac{\pi}{2} + 2n\pi$

$\frac{\pi}{2} = 2n\pi$

$n = \frac{1}{4}$ (not valid since $n \notin \mathbb{Z}$)

Case 2: $\frac{1}{2}t = -\left(\frac{1}{2}t - \frac{\pi}{2}\right) + 2n\pi$

$\frac{1}{2}t = -\frac{1}{2}t + \frac{\pi}{2} + 2n\pi$

$t = \frac{\pi}{2} + 2n\pi$

when $n = 1$:

$t = \frac{\pi}{2} + 2(1)\pi = \frac{5\pi}{2} > \pi$

So, first collision happens when $t = \frac{5\pi}{2}$ sec. or 7.85 s

Sub $t = \frac{5\pi}{2}$ into $x_1 = 5\cos\frac{1}{2}t$

$x_1 = 5\cos\frac{1}{2}\left(\frac{5\pi}{2}\right)$

$= 5\cos\frac{5\pi}{4}$

$= 5 \times -\frac{\sqrt{2}}{2}$

$= -\frac{5\sqrt{2}}{2}$ m or -3.54 m

(b) Let $z = e^{\frac{2\pi}{5}i}$ and $w = e^{\frac{4\pi}{5}i}$.

i. Express z and w in polar form. **1**

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$z = \cos\tfrac{2}{5}\pi + i\sin\tfrac{2}{5}\pi$$
$$w = \cos\tfrac{4}{5}\pi + i\sin\tfrac{4}{5}\pi$$

This question was done very well.

ii. Hence, show that $|z + w| = \sqrt{2\left(1 + \cos\frac{2\pi}{5}\right)}$. **2**

Method 1:

$$z+w = \left(\cos\tfrac{2}{5}\pi + \cos\tfrac{4}{5}\pi\right) + i\left(\sin\tfrac{2}{5}\pi + \sin\tfrac{4}{5}\pi\right)$$
$$|z+w|^2 = \left(\cos\tfrac{2}{5}\pi + \cos\tfrac{4}{5}\pi\right)^2 + \left(\sin\tfrac{2}{5}\pi + \sin\tfrac{4}{5}\pi\right)^2$$
$$= \cos^2\tfrac{2}{5}\pi + 2\cos\tfrac{2}{5}\pi\cos\tfrac{4}{5}\pi + \cos^2\tfrac{4}{5}\pi$$
$$+ \sin^2\tfrac{2}{5}\pi + 2\sin\tfrac{2}{5}\pi\sin\tfrac{4}{5}\pi + \sin^2\tfrac{4}{5}\pi$$
$$= 1 + 2\left(\cos\tfrac{4}{5}\pi\cos\tfrac{2}{5}\pi + \sin\tfrac{4}{5}\pi\sin\tfrac{2}{5}\pi\right) + 1$$
$$= 2 + 2\cos\left(\tfrac{4}{5}\pi - \tfrac{2}{5}\pi\right)$$
$$= 2\left(1 + \cos\tfrac{2}{5}\pi\right)$$
$$\therefore |z+w| = \sqrt{2\left(1+\cos\tfrac{2}{5}\pi\right)}$$

This question was done well. Method 1 is the most common solution presented. Students may have lost mark for not showing a clear and presentable solution.

Method 2:

$$z+w = \left(\cos\tfrac{2}{5}\pi + \cos\tfrac{4}{5}\pi\right) + i\left(\sin\tfrac{2}{5}\pi + \sin\tfrac{4}{5}\pi\right)$$
$$\cos A + \cos B = 2\cos\left(\tfrac{A+B}{2}\right)\cos\left(\tfrac{A-B}{2}\right)$$
$$\cos\tfrac{4}{5}\pi + \cos\tfrac{2}{5}\pi = 2\cos\tfrac{3}{5}\pi\cos\tfrac{1}{5}\pi$$
$$\sin A + \sin B = 2\sin\left(\tfrac{A+B}{2}\right)\cos\left(\tfrac{A-B}{2}\right)$$
$$\sin\tfrac{4}{5}\pi + \sin\tfrac{2}{5}\pi = 2\sin\tfrac{3}{5}\pi\cos\tfrac{1}{5}\pi$$

\+

$$\therefore z+w = 2\cos\tfrac{1}{5}\pi\left(\cos\tfrac{3}{5}\pi + i\sin\tfrac{3}{5}\pi\right)$$
$$|z+w|^2 = 4\cos^2\tfrac{\pi}{5}\left(\overbrace{\cancel{\cos^2\tfrac{3}{5}\pi + \sin^2\tfrac{3}{5}\pi}}^{1}\right) = 4\cos^2\tfrac{\pi}{5}$$
$$\therefore |z+w| = 2\cos\tfrac{\pi}{5}$$

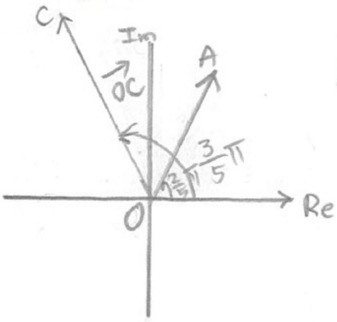
using the identity:

$$\cos\tfrac{\pi}{5} = \sqrt{\frac{1+\cos\frac{2\pi}{5}}{2}}$$
$$\therefore 2\cos\tfrac{\pi}{5} = \sqrt{\frac{4\left(1+\cos\frac{2\pi}{5}\right)}{2}}$$
$$\therefore |z+w| = \sqrt{2\left(1+\cos\tfrac{2\pi}{5}\right)}.$$

iii. In the complex plane, vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ represent the complex numbers z, w and $z + w$ respectively, where O is the origin. Show that $\angle AOC = \frac{\pi}{5}$. **2**

$\overrightarrow{OA} = z$, $\overrightarrow{OB} = w$, $\overrightarrow{OC} = z + w$

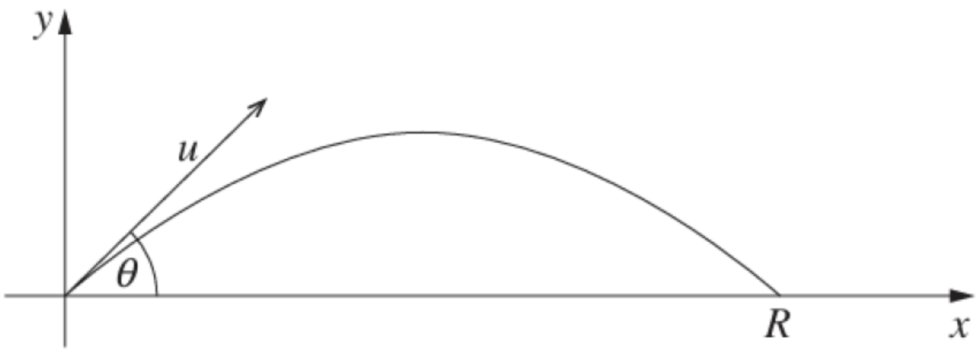
$\arg(z) = \frac{2}{5}\pi$ $\quad \arg(z+w) = \frac{3}{5}\pi$ (from ii)



$$\begin{aligned}\angle AOC &= \arg(z+w) - \arg(z) \\ &= \frac{3}{5}\pi - \frac{2}{5}\pi \\ &= \frac{\pi}{5}\end{aligned}$$

Since there was an error in the original question "Show that $\angle AOC = \frac{2\pi}{5}$", everyone was awarded 2 marks.

(c) A particle is projected upward from the origin with an initial velocity u m/s at an angle θ above the horizontal as shown in the diagram given below. Let the position vector of the particle at time t be $\mathbf{r}\,(t)$ and define the total time of flight as T. ($g = 10\ m/s^2$).



The particle moves under the constant acceleration due to gravity, $\ddot{\mathbf{r}} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$. The particle lands on the x-axis after time T, when $x = R$.

i. Show that the range R of the projectile is given by **3**

$$R = \frac{u^2 \sin 2\theta}{g}.$$

$\ddot{x} = 0$

$\dot{x} = C_1$

at $t=0$, $\dot{x} = u\cos\theta$ $\Rightarrow C_1 = u\cos\theta$

$\therefore \dot{x} = u\cos\theta$

$x = ut\cos\theta + C_2$

at $t=0$, $x=0$, $\Rightarrow C_2 = 0$

$\therefore x = ut\cos\theta$

$\ddot{y} = -g$

$\dot{y} = -gt + C_3$

at $t=0$, $\dot{y} = u\sin\theta$

$\Rightarrow C_3 = u\sin\theta$

$\therefore \dot{y} = -gt + u\sin\theta$

$y = -\frac{gt^2}{2} + ut\sin\theta + C_4$

at $t=0$, $y=0$, $\Rightarrow C_4 = 0$

$\therefore y = -\frac{1}{2}gt^2 + ut\sin\theta$

when $x=R$, $y=0$

$0 = t\left(-\frac{1}{2}gt + u\sin\theta\right)$

$\therefore t=0$, $-\frac{1}{2}gt + u\sin\theta = 0$

$\Rightarrow x=0$ $\quad t = \frac{2u\sin\theta}{g}$

$$\therefore R = ut\cos\theta = u\left(\frac{2u\sin\theta}{g}\right)\cos\theta = \frac{u^2(2\sin\theta\cos\theta)}{g} = \frac{u^2\sin 2\theta}{g}$$

A clear presentable solution needs to be demonstrated in order to achieve full marks.
This question was mostly well done.

ii. A garden sprinkler sprays water symmetrically about the vertical axis with a constant speed of u metres per second. The initial direction of the spray varies continuously between angles 10° and 75° to the horizontal. **1**

Prove that the sprinkler will wet the surface of the region on the ground that forms an annular area with the centre at the sprinkler's position and with its internal and external radii given by

$$\frac{u^2\sin(2\theta_{\min})}{g} \quad \text{and} \quad \frac{u^2\sin(2\theta_{\max})}{g} \text{ respectively.}$$

Since the sprinkler rotates continuously between θ_{min} and θ_{max},

$\therefore$ minimum radius (ie inner edge of annulus)

$R_{min} = \frac{u^2\sin(2\theta_{min})}{g}$

maximum radius (ie outer edge of annulus)

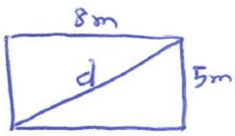
$R_{max} = \frac{u^2\sin(2\theta_{max})}{g}$

$\therefore$ the ground area wetted by the sprinkler is an annular region with radii R_{min} and R_{max}.

A certain explanation should be provided to achieve a mark for this question.

(c) iii. A rectangular garden bed has dimensions 8 metres by 5 metres. Assume that the sprinkler is to be placed in a fixed position, such that the entire garden bed must be watered. **3**

Determine the minimum velocity u required for the sprinkler to cover the entire garden bed. Express the condition for u in terms of the acceleration due to gravity g ensuring that the sprinkler's maximum range at $\theta_{max} = 75°$ is sufficient to cover the diagonal of the rectangular diagonal bed.



$d = \sqrt{8^2+5^2} = \sqrt{89}$ m

To ensure entire bed is watered, the water must reach the farthest point of the diagonal corner.

Also $\theta_{max} = 75°$

$$R_{max} = \frac{u^2 \sin 150°}{g} \quad \text{where } \sin 150° = \frac{1}{2}$$

$$= \frac{u^2}{2g}$$

To cover the diagonal,

$$\frac{u^2}{2g} \geq \sqrt{89}$$

$$u^2 \geq 2g\sqrt{89}$$

$$u \geq \sqrt{2g\sqrt{89}} \quad \text{where } g = 10\ ms^{-2}$$

$$\therefore U_{min} = \sqrt{20\sqrt{89}} = 13.74\ ms^{-1}$$

Very few students solved this problem. For those that did, students tried to put the sprinkler at the centre, which resulted in not getting full marks.

iv. To prevent water wastage and ensure the bed is adequately watered, it is decided that the sprinkler should be positioned at a minimum possible distance from the garden bed, while still covering all of it. **1**

What is the required value of u, ensuring it satisfies the condition that

$$\frac{u^2}{g} \geq 8 + \sqrt{13},$$

where u is the speed of the water jet?

$\frac{u^2}{g} \geq 8 + \sqrt{13}$ (given)

$u^2 \geq g(8+\sqrt{13})$

$u^2 \geq 10(8+\sqrt{13})$

$u^2 \geq 116.0555...$

$u \geq 10.77 \text{ ms}^{-1}$

$\therefore$ to guarantee full coverage, $\frac{u^2}{2g}$ is just slightly insufficient, so sprinkler must either increase the velocity or move slightly closer to the bed.

Since this is the last question, in order for a student to achieve a mark, students need to state a reason or a recommendation and a conclusion. Simply solving for u is not enough to get a mark.