Name:....................................

Student Number

Teacher:.................................

Pymble Ladies' College

Mathematics Extension 2
HSC Trial Examination
Term 3 2024

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning or calculations

Total marks 100

Section 1 – 10 marks (pages 1-4)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 5-12)

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section
- Answer each question in the appropriate space in the Answer Booklet. Extra writing pages are included at the end of each question.

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Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. The velocity of a particle moving in simple harmonic motion along the x-axis is given by $V^2 = 60 + 8x - 4x^2$.

What is the centre of motion?

(A) $x = -5$

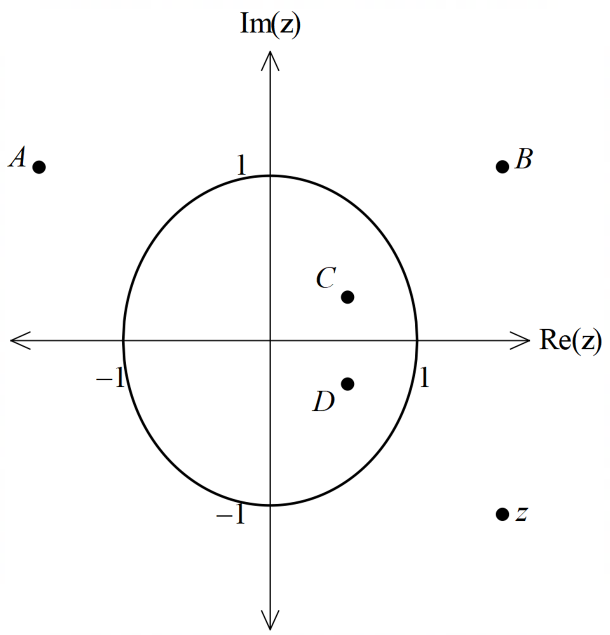
(B) $x = 3$

(C) $x = 1$

(D) $x = 60$

2. The diagram shows the complex number z in the fourth quadrant on the Argand diagram. The modulus of z is 2.

Which of the points marked A, B, C or D best shows the position of $\frac{1}{z}$?



(A) Point A

(B) Point B

(C) Point C

(D) Point D

3. Which expression is equal to $\int \tan^3 x \sec^2 x \, dx$?

(A) $\dfrac{\tan^4 x}{4} + c$

(B) $\tan^4 x + c$

(C) $\dfrac{\tan^5 x}{5} + \dfrac{\tan^3 x}{3} + c$

(D) $\tan^5 x + \tan^3 x + c$

4. What value of z satisfies $z^2 = 7 + 24i$?

(A) $4 + 3i$

(B) $-4 + 3i$

(C) $-3 + 4i$

(D) $3 + 4i$

5. Consider the complex, non-real cube roots of unity w and w^2.
What is the value of $\left(1 - w + w^2\right)\left(1 + w - w^2\right)$?

(A) 0

(B) 1

(C) 2

(D) 4

6. What is the size of the acute angle θ between the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{k}$?

(A) $\theta = \dfrac{\pi}{6}$

(B) $\theta = \dfrac{\pi}{5}$

(C) $\theta = \dfrac{\pi}{4}$

(D) $\theta = \dfrac{\pi}{3}$

7. A particle has initial velocity $2\underset{\sim}{i}$ ms^{-1}. At time t seconds it has acceleration $(2t-3)\underset{\sim}{i}+\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\underset{\sim}{j}$ ms^{-2}.

When is the particle at rest?

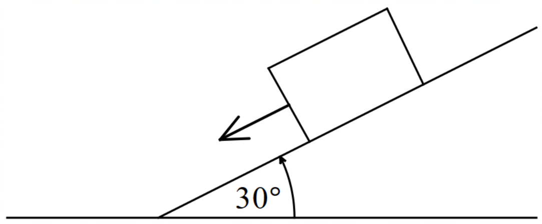
(A) Never.

(B) At time $t=1$ second only.

(C) At time $t=2$ seconds only.

(D) At time $t=1$ and $t=2$ seconds.

8. A 10 kg box on a plane inclined at an angle of $30°$ to the horizontal is undergoing uniform acceleration of 1.5 m/s^2 down the plane.
Take the acceleration g due to gravity to be 9.8 m/s^2.



What is the magnitude of the frictional force resisting the motion of the box?

(A) 34 N

(B) 64 N

(C) 70 N

(D) 100 N

9. The points *A*, *B* and *C* are collinear where $\overrightarrow{OA}=\underset{\sim}{i}-\underset{\sim}{j}$, $\overrightarrow{OB}=-3\underset{\sim}{j}-\underset{\sim}{k}$ and $\overrightarrow{OC}=2\underset{\sim}{i}+a\underset{\sim}{j}+b\underset{\sim}{k}$ for some constants a and b. What are the values of a and b?

(A) $a=-1$ and $b=-1$

(B) $a=-1$ and $b=1$

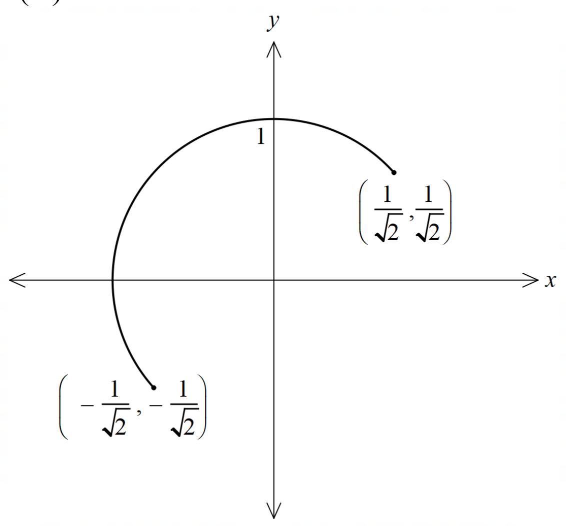
(C) $a=1$ and $b=-1$

(D) $a=1$ and $b=1$

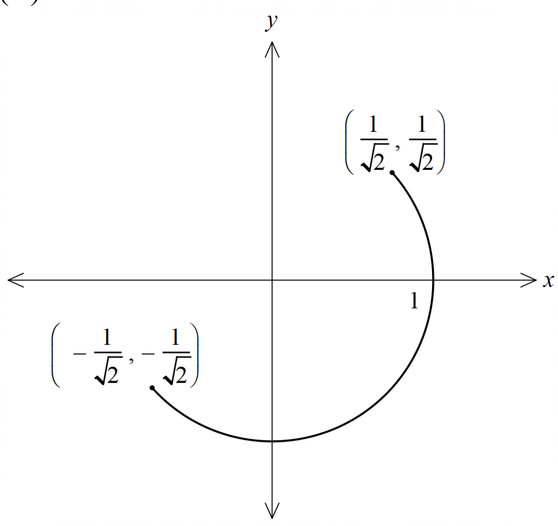
10. Which diagram best shows the curve described by the position vector

$$\underset{\sim}{r}(t) = \sin(t)\,\underset{\sim}{i} - \cos(t)\,\underset{\sim}{j} \text{ for } \frac{\pi}{4} \le t \le \frac{5\pi}{4}?$$

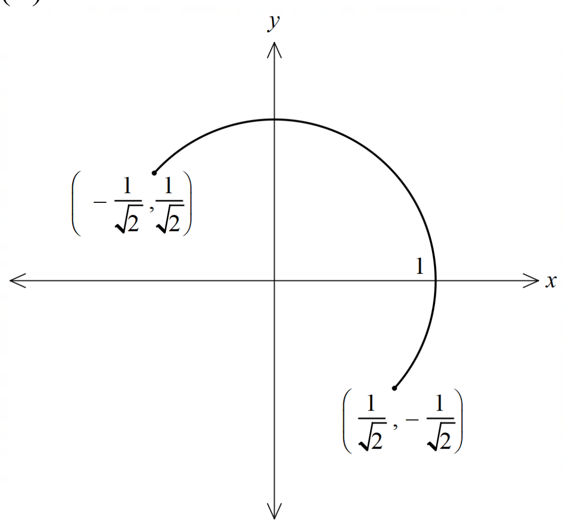
(A)



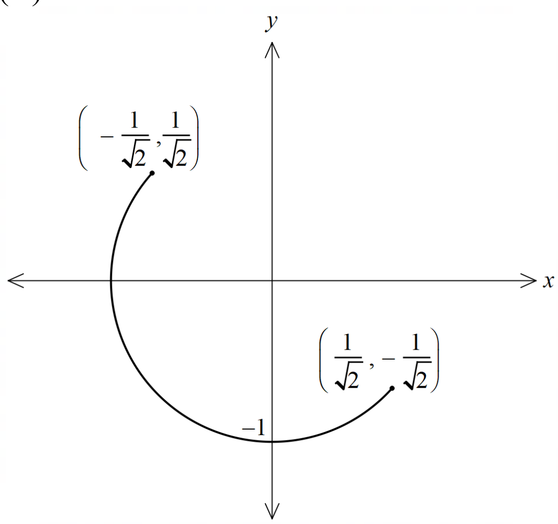
(B)



(C)



(D)



Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer these questions in the Answer Book provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (17 marks) **Marks**

(a) Express $2\sqrt{2}\,e^{\frac{-3\pi}{4}i}$ in the form $x+iy$. **2**

(b) A particle starts at rest from the origin and moves in a straight line such that its displacement x metres at time t seconds is determined by the equation $\ddot{x}=-25x$. **2**

How long will it take for the particle to first return to the origin?

(c) Consider the two points $A(3,\,3,\,3)$ and $B(3,\,-3,\,3)$.

(i) Find $\overrightarrow{AB}$. **1**

(ii) Find $\left|\overrightarrow{AB}\right|$. **1**

(iii) Find $\angle AOB$. Give your answer correct to the nearest degree. **2**

(d) The function $f(x)$ is defined by

$$f(x)=\frac{6x+4}{(x-1)(x+1)(2x+3)}.$$

(i) Express $f(x)$ in terms of partial fractions. **2**

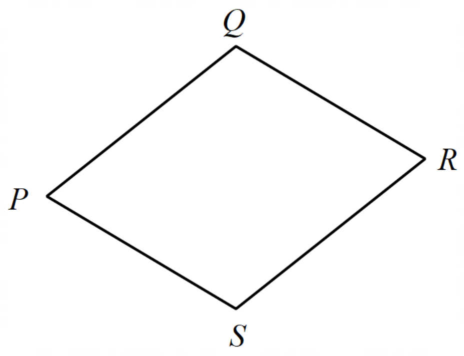
(ii) Hence, or otherwise, find $\displaystyle\int\frac{3x+2}{(x-1)(x+1)(2x+3)}\,dx$. **2**

(e) (i) Find the two square roots of $2i$, giving the answers in the form $x+iy$, where x and y are real numbers. **2**

(ii) Hence, or otherwise, solve $2z^2+2\sqrt{2}\,z+1-i=0$, leaving answers in the form $x+iy$. **3**

End of Question 11

Question 12 (15 marks) **Marks**



(a) *PQRS* is a parallelogram. Given that $\overrightarrow{PQ} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\overrightarrow{QR} = 5\underset{\sim}{i} - 2\underset{\sim}{k}$

(i) Show that *PQRS* is a rhombus. **2**

(ii) Find the exact area of *PQRS*. **3**

(b) Evaluate $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1-x^2}}\, dx$. **4**

(c) Prove by mathematical induction that $3n^2 - 3n \le 2^n - 1$ for $n \ge 7$. **3**

(d) Use the *t*-formula to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3 - \cos x}$. **3**

End of Question 12

Question 13 (14 marks) **Marks**

(a) A particle is initially 2 metres to the left of the origin, moving to the left at a speed of 3 m/s. The acceleration of the particle is given by $a = v + v^3$, where v is the velocity of the particle. **3**
Find an expression for x, the displacement of the particle, in terms of v.

(b) Consider the statement "If Sally studies, she will pass the test."

(i) Write down the converse of the statement. **1**

(ii) Write down the contrapositive of the statement. **1**

(c) (i) The complex number z is such that $\arg(z+2i) = \tan^{-1}\frac{1}{2}$. **2**
It is given that $z = x + iy$, where x and y are real and $x > 0$.
Find an equation for y in terms of x.

(ii) The complex number z satisfies $\frac{-\pi}{2} \le \arg(z+2i) \le \tan^{-1}\frac{1}{2}$. **2**
The region R is the locus of z. In your answer booklets, sketch the region R on the given Argand diagram.

(iii) z_1 is the point in R at which $|z|$ is minimum. Calculate the exact value of $|z_1|$. **1**

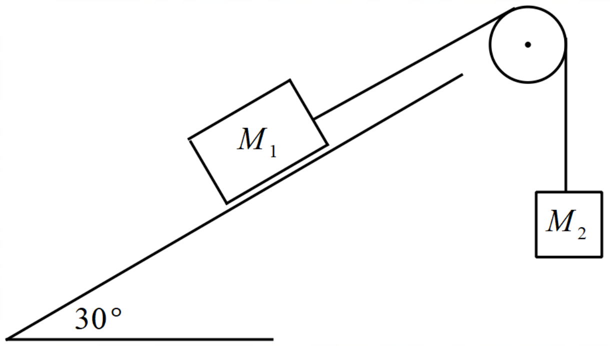
(iv) Express z_1 in the form $a + ib$ where a and b are real. **1**

(d) The equations of two lines are $\underset{\sim}{r} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k} + \lambda\left(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$ and **3**
$\underset{\sim}{u} = \underset{\sim}{i} + 11\underset{\sim}{j} - 4\underset{\sim}{k} + \mu\left(-\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$.
Show that the lines intersect, stating the point of intersection.

End of Question 13

Question 14 (16 marks) **Marks**

(a) Two masses, M_1 kilograms and M_2 kilograms, are attached by a light inextensible string. The string is placed over a smooth pulley, and the mass M_1 rests on a smooth frictionless plane inclined at 30 degrees to the horizontal as shown. **3**
The mass M_2 is suspended vertically by the string.



NOT TO SCALE

The mass M_1 accelerates down the inclined plane at 2 m/s^2, and $g = 10$ m/s^2.

Given $M_1 = kM_2$, find the value of k.

(b) Evaluate $\int_0^4 \frac{1}{\sqrt{1+\sqrt{x}}}\, dx$. **3**

Question 14 continues on page 9

(c) A missile is fired from point O with speed V m/s at an angle α above the horizontal where $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$. The missile moves in a vertical plane under gravity where the acceleration due to gravity is g m/s^2. At time t seconds the position vector of the missile relative to O is $\underset{\sim}{r}(t) = (Vt\cos\alpha)\underset{\sim}{i} + \left(Vt\sin\alpha - \frac{1}{2}gt^2\right)\underset{\sim}{j}$.

(i) Show that the greatest height reached by the missile is $H = \dfrac{V^2\sin^2\alpha}{2g}$ metres and that when the missile is at its greatest height its angle of elevation from O is $\tan^{-1}\left(\frac{1}{2}\tan\alpha\right)$. **3**

(ii) When the missile reaches a height of h metres its angle of inclination to the horizontal falls to $\frac{\pi}{4}$. Show that $\frac{h}{H} = 2 - \operatorname{cosec}^2\alpha$. **3**

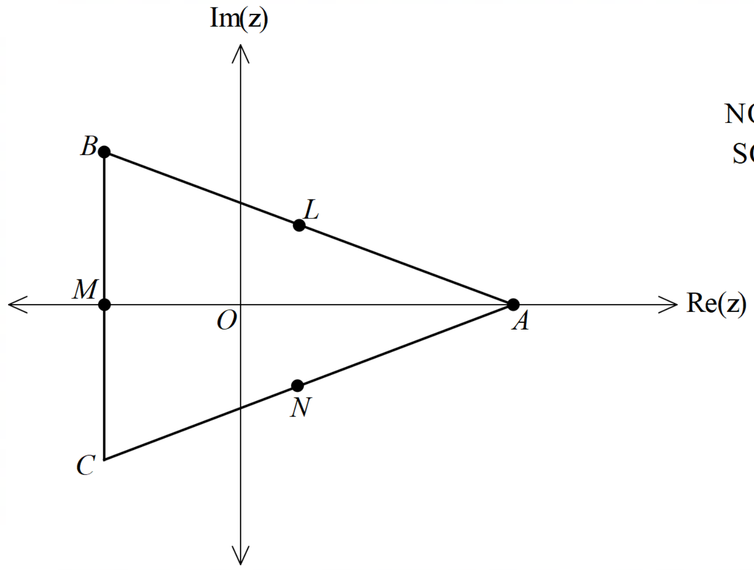
(d) (i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ **1**

(ii) Hence or otherwise, to find the value of $\displaystyle\int_0^{\pi} \frac{\sin x}{e^{\frac{\pi}{2}} + e^x}\,dx$. **3**

End of Question 14

Question 15 (15 marks) **Marks**

(a) The cube roots of unity are represented on the Argand diagram below by the points *A, B* and *C*. **3**



The points *L, M* and *N* are the midpoints of the line segments *AB, BC* and *CA* respectively. Determine a degree 6 polynomial equation with integer coefficient whose roots are the complex numbers represented by the points *A, B, C, L, M* and *N*.

(b) (i) Differentiate $\left(16+t^2\right)^{\frac{3}{2}}$ with respect to *t*. **1**

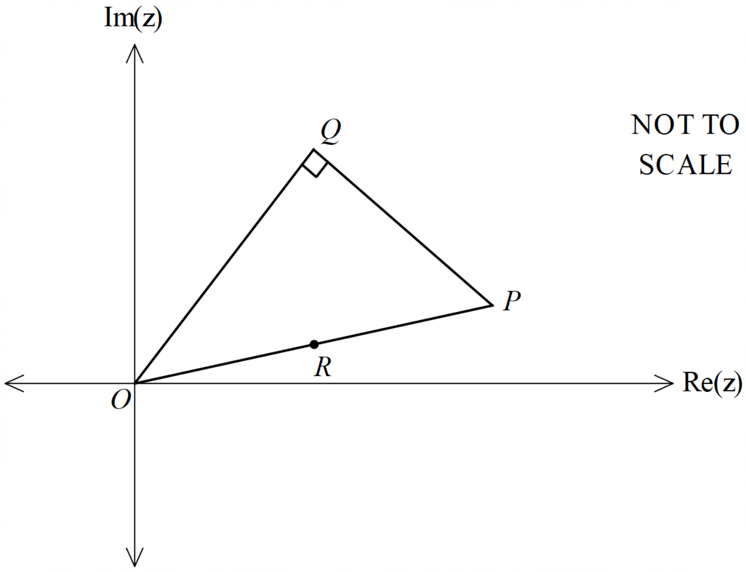
(ii) Let $I_n = \int_0^3 t^n \sqrt{16+t^2}\, dt$ for integers $n \geq 1$. Show that, for $n \geq 3$, **3**

$$(n+2) I_n = 125 \times 3^{n-1} - 16(n-1) I_{n-2}.$$

(c) A particle of mass *m* is projected vertically upward under gravity with speed *u* in a medium in which the resistance is *mk* times the speed, where *k* is a positive constant. **4**
If the particle reaches its greatest height *H* in time *T*, show that $u = gT + kH$.

Question 15 continues on page 11

(d) In the Argand diagram below, OPQ is a triangle which is right-angled at Q.
The point R is the midpoint of OP.



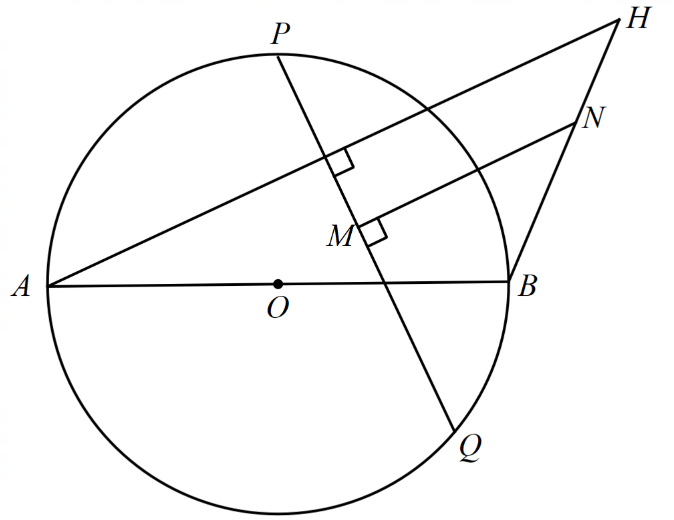
(i) If $\overrightarrow{OP} = z$ and $\overrightarrow{OQ} = w$ show that $\overrightarrow{OR} = \frac{1}{2}(1 - ki)w$ where k is a constant, $k > 0$. **2**

(ii) Express $\overrightarrow{RQ}$ in terms of w and hence show $\left|\overrightarrow{OR}\right| = \left|\overrightarrow{RQ}\right|$. **2**

End of Question 15

Question 16 (13 marks) **Marks**

(a)



NOT TO SCALE

In the diagram, AB is a diameter of a circle with centre O and PQ is a chord of the circle that is not perpendicular to AB. The perpendicular from A to PQ is produced to the point H outside the circle. M is the midpoint of PQ and N is the point on BH such that $MN \perp PQ$. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OP} = \underset{\sim}{p}$, $\overrightarrow{OQ} = \underset{\sim}{q}$ and $\overrightarrow{AH} = \underset{\sim}{h}$.

(i) By writing $\overrightarrow{OM}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$, show that the points O, M and N are collinear. **2**

(ii) If $\overrightarrow{BN} = \lambda \overrightarrow{BH}$ for some scaler λ, express $\overrightarrow{ON}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{h}$ and λ. **2**

(iii) Hence, show that N is the midpoint of BH. **3**

(b) (i) By considering the graph of $y = \dfrac{1}{x\sqrt{x}}$, or otherwise, show that for all positive integers $k \geq 1$, $\dfrac{1}{(k+1)\sqrt{k+1}} < \dfrac{2}{\sqrt{k}} - \dfrac{2}{\sqrt{k+1}}$. **3**

(ii) Hence, use Mathematical Induction to show that for all positive integers $n \geq 2$, **3**

$$\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \ldots + \frac{1}{n\sqrt{n}} < 3 - \frac{2}{\sqrt{n}}.$$

END OF PAPER

NSW Education Standards Authority

2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$$

$$S = \frac{a}{1 - r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

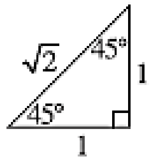
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

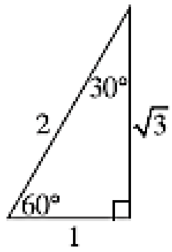


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

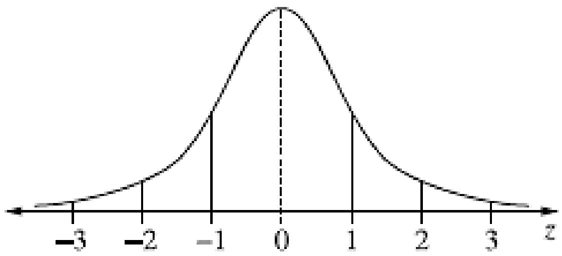
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X-\mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1-p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

Name: Solutions.

Student Number

Teacher:

Pymble Ladies' College

Mathematics Extension 2
HSC Trial Examination
Term 3 2024

ANSWER BOOKLET

- **Put your name, Teacher's name and Student Number on this Booklet.**
- For Questions 1-10 use the Multiple-Choice Answer Sheet provided.
- Answer Questions 11-16 in the spaces allocated for each question as these spaces provide guidance for the expected length of your response.
- Write using a non-erasable black or blue pen. Black is preferred.
- Extra writing space is provided.
- Calculators approved by NESA may be used.

Name:......................................

Student Number:

Teacher's Name:...........................

Mathematics Extension 2
HSC Trial Examination

Section I – Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

1	A ○	B ○	C ●	D ○
2	A ○	B ○	C ●	D ○
3	A ●	B ○	C ○	D ○
4	A ●	B ○	C ○	D ○
5	A ○	B ○	C ○	D ●
6	A ●	B ○	C ○	D ○
7	A ○	B ○	C ●	D ○
8	A ●	B ○	C ○	D ○
9	A ○	B ○	C ○	D ●
10	A ○	B ○	C ●	D ○

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is available at the end of each question.

Question 11 (17 marks) **Marks**

(a) $2\sqrt{2}\, e^{-\frac{3\pi}{4}} = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$ **2**

$= -2 - 2i$

2	Correct
1	One error

(b) $\ddot{x} = -5^2 x \quad \therefore n = 5 \quad T = \frac{2\pi}{5}$ **2**

$\therefore$ Particle returns to origin in half a period $\therefore \frac{\pi}{5}$ sec

2	Correct
1	Finds the period

Question 11 continues page 2

Question 11 continued. **Marks**

(c)

(i) $\overrightarrow{AB} = 0\underset{\sim}{i} - 6\underset{\sim}{j} + 0\underset{\sim}{k}$ 1 r/w

(ii) $|\overrightarrow{AB}| = \sqrt{(-6)^2} = 6$ 2 1 r/w

(iii) $\cos\angle AOB = \dfrac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}| \times |\overrightarrow{OB}|}$ 2

2	correct
1	one error

$= \dfrac{9 - 9 + 9}{\sqrt{3^2 + 3^2 + 3^2}\sqrt{3^2 + 3^2 + 3^2}}$

$= \dfrac{9}{27}$

Add lines

$\cos\angle AOB = \dfrac{1}{3}$

$\angle AOB = 70°\,31'\,43.61''$

$\doteqdot 71°$

Question 11 continues page 3

Question 11 continued. **Marks**

(d)

(i) $\dfrac{6x+4}{(x-1)(x+1)(2x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{2x+3}$ **2**

$6x+4 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1)$

when $x=1$ | when $x=-1$ | $x=-3/2$

$10 = 10A$ | $-2 = -2B$ | $-5 = \frac{5}{4}C$

$\therefore A = 1$ | $\therefore B = 1$ | $\therefore C = -4$

$\therefore \dfrac{6x+4}{(x-1)(x+1)(2x+3)} = \dfrac{1}{x-1} + \dfrac{1}{x+1} - \dfrac{4}{2x+3}$

2	Correct
1	Two of the 3 parts correct

(ii) $\displaystyle\int \frac{3x+2}{(x-1)(x+1)(2x+3)}\,dx = \frac{1}{2}\int\left(\frac{1}{x-1} + \frac{1}{x+1} - \frac{4}{2x+3}\right)dx$ **2**

$= \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| - \ln|2x+3| + C$

2	Correct
1	one error

Question 11 continues page 4

Question 11 continued. **Marks**

(e)

(i) 2

$(a+ib)^2 = 2i$

$a^2 - b^2 + 2abi = 2i$ Roots are

$a^2 - b^2 = 0$ $1+i, \; -1-i$

$2ab = 2$

$ab = 1$

$\therefore a = \pm 1$

$b = \pm 1$

2 | correct
1 | one root correct

(ii) 2 3

$$z = \frac{-2\sqrt{2} \pm \sqrt{8 - 4(2)(1-i)}}{4}$$

$$= \frac{-2\sqrt{2} \pm \sqrt{8i}}{4}$$

$$= \frac{-2\sqrt{2} \pm 2\sqrt{2i}}{4}$$

$$= \frac{-\sqrt{2} \pm (1+i)}{2}$$

$$= \frac{-\sqrt{2}+1}{2} + \frac{i}{2}, \quad \frac{-\sqrt{2}-1}{2} - \frac{i}{2}$$

3 | correct
2 | one error
1 | Just quadratic formula

End of Question 11

Question 12 (15 marks) **Marks**

(a)

(i) $|\overrightarrow{PQ}| = \sqrt{2^2 + 3^2 + (-4)^2}$ 2

$= \sqrt{29}$

$|\overrightarrow{QR}| = \sqrt{5^2 + (-2)^2}$

$= \sqrt{29}$

Since $|\overrightarrow{PQ}| = |\overrightarrow{QR}|$, adjacent sides of a parallelogram are equal in length, so all four sides are equal in length, $\therefore PQRS$ is a rhombus

2	correct
1	Finds one modulus

(ii) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = 7\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$ 3

$\overrightarrow{QS} = \overrightarrow{PS} - \overrightarrow{PQ} = 3\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

Area $PQRS = \frac{1}{2} \times |\overrightarrow{PR}| \times |\overrightarrow{QS}|$

$= \frac{1}{2}\sqrt{49+9+36} \times \sqrt{9+9+4} = \sqrt{517}$ units^2

3	correct
2	one error
1	Finds diagonal lengths

(b) 4

$$\int_0^{1/\sqrt{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx$$

Let $x = \sin\theta$

when $x = 1/\sqrt{2}$ $\theta = \pi/4$

$x = 0$ $\theta = 0$

$dx = \cos\theta\, d\theta$

$$= \int_0^{\pi/4} \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta\, d\theta$$

$$= \int_0^{\pi/4} \sin^3\theta\, d\theta$$

$$= \int_0^{\pi/4} \sin\theta\,(1-\cos^2\theta)\,d\theta$$

$$= \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi/4}$$

$$= -\cos\pi/4 + \frac{1}{3}\cos^3\pi/4 - \left(-\cos 0 + \frac{1}{3}\cos^3 0\right)$$

$$= -\frac{1}{\sqrt{2}} + \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + 1 - \frac{1}{3}$$

$$= \frac{8-5\sqrt{2}}{12}$$

4	Correct
3	Correct strategy one error
2	Correct substitution and some progress
1	Correct substitution

Question 12 continues on page 8

(c) 3

Let the statement be $3n^2-3n \le 2^n-1$ for $n \ge 7$

when $n=7$

LHS : $3\times 7^2 - 3\times 7$ RHS : 2^7-1

$= 126$ $= 127$

$\therefore$ Statement is true for $n=7$

Assume the statement true for $n=k$

i.e $3k^2-3k \le 2^k-1$ $k \ge 7$

RTP the statement true for $n=k+1$

i.e $3(k+1)^2-3(k+1) \le 2^{k+1}-1$ $k \ge 7$

Now from assumption

$$3k^2-3k \le 2^k-1$$

$$6k^2-6k \le 2^{k+1}-2$$

$$6k^2-6k+1 \le 2^{k+1}-1$$

$$3k^2+3k+3k^2-9k+1 \le 2^{k+1}-1$$

$3k^2-9k+1 > 6$ when $k \ge 7$ since $a>0$ and $k = \frac{9\pm\sqrt{69}}{6}$ are the roots

3	correct
2	Some progress with set up
1	$n=7$ and set up and conclusion

Question 12 continues on page 9

$$\therefore 3k(k+1) < 3k(k+1) + (3k^2-9k+1) \le 2^{k+1}-1$$

$$\therefore 3(k+1)^2-3(k+1) \le 2^{k+1}-1$$

Hence, since the statement is true for $n=7$, assumed true for $n=k$ and proved true for $n=k+1$, so the statement is true for $n=7+1=8$ and so on $\forall n$ by the principle of mathematical induction.

Question 12 continued. **Marks**

(d) 3

$$\int_0^{\pi/2} \frac{dx}{3-\cos x}$$

when $x = \pi/2$ $t = \tan \pi/4$
$= 1$

when $x = 0$ $t = \tan 0$
$= 0$

$$= \int_0^1 \frac{1}{3-\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{1}{1+t^2}\, dt$$

$$= \int_0^1 \frac{1}{\frac{3+3t^2-1+t^2}{1+t^2}} \times \frac{1}{1+t^2}\, dt$$

$$= \int_0^1 \frac{dt}{2+4t^2}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{1+2t^2}$$

3	correct
2	one error
1	Just the substitution for all parts

$$= \frac{1}{2\sqrt{2}}\left[\tan^{-1}\sqrt{2}t\right]_0^1$$

$$= \frac{1}{2\sqrt{2}}\left\{\tan^{-1}\sqrt{2} - \tan^{-1}0\right\}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\sqrt{2}$$

End of Question 12

Question 13 (14 marks) **Marks**

(a) 3

when $t = 0$, $x = -2$, $v = -3$ m/s

$v \frac{dv}{dx} = v + v^3$ $\therefore x = \tan^{-1} v - 2 + \tan^{-1}(3)$

$\frac{dv}{1+v^2} = dx$

$x = \int \frac{dv}{1+v^2}$ — $*$

$x = \tan^{-1} v + c$

$-2 = \tan^{-1}(-3) + c$

$c = -2 + \tan^{-1}(3)$

3	correct
2	one error
1	progress to $*$

(b)

(i) If she passes the test, Sally studied 1

(ii) If Sally does not pass the test, then she did not study. 1

Question 13 continues on page 14

Question 13 continued. **Marks**

(c)

(i) **2**

Let $z = x + iy$

$\arg(z + 2i) = \tan^{-1}(\tfrac{1}{2})$

$\arg(x + (y+2)i) = \tan^{-1} \tfrac{1}{2}$

$\therefore \tan^{-1} \frac{y+2}{x} = \tan^{-1} \frac{1}{2}$

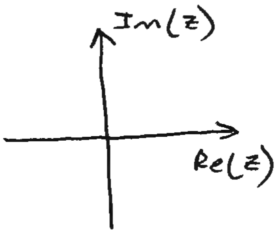
$\therefore \frac{y+2}{x} = \frac{1}{2}$ $x - 2y - 4 = 0$

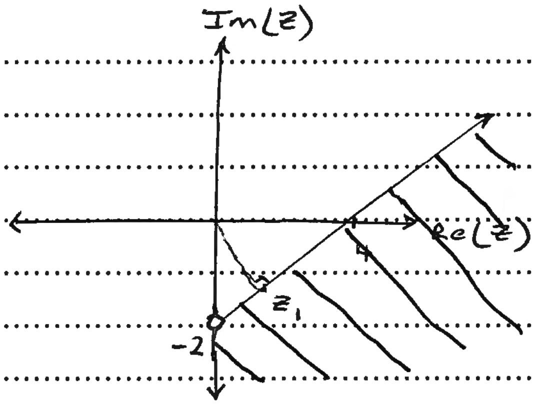
$y = \frac{x}{2} - 2$

2	correct
1	one error

(ii) **2**

Should have





2	correct
1	one error

(iii) **1** r/w

$|z_1| = \frac{|0 - 0 - 4|}{\sqrt{1^2 + (-2)^2}}$

$= \frac{4}{\sqrt{5}}$

(iv) **1** r/w

$z_1 = \frac{4}{5} - \frac{8}{5} i$

$y = \frac{x}{2} - 2$

$y = -2x$

$-2x = \frac{x}{2} - 2$

$x = 4/5, \; y = -8/5$

Question 13 continues on page 15

Question 13 continued. **Marks**

(d) **3**

$2+\lambda = 1-\mu \Rightarrow \mu+\lambda = -1$ — ①

$3-2\lambda = 11+3\mu \Rightarrow 3\mu+2\lambda = -8$ — ②

$3+\lambda = -4-2\mu$ — ③

① × 2 − ② $\quad -\mu = 6$

$\mu = -6$

From ① $\quad -6+\lambda = -1$

$\lambda = 5$

Check when $\lambda = 5$ and $\mu = -6$.

③ $\Rightarrow$ LHS $= 3+5 = 8$ and RHS $= -4-2(-6) = 8 =$ LHS

$\underset{\sim}{r} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k} + 5\underset{\sim}{i} - 10\underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{r} = 7\underset{\sim}{i} - 7\underset{\sim}{j} + 8\underset{\sim}{k}$ $\quad \therefore$ Point of intersection

$\underset{\sim}{\mu} = 7\underset{\sim}{i} - 7\underset{\sim}{j} + 8\underset{\sim}{k}$ $\quad$ is $(7, -7, 8)$

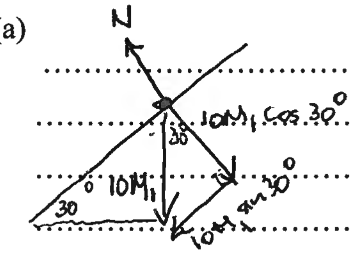
$\therefore$ the lines intersect.

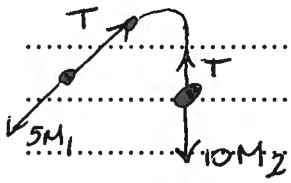
3	Collect
2	One error
1	Setting up the simultaneous eq^ns.

End of Question 13

Question 14 (16 marks) **Marks**

(a) 3





$(M_1+M_2)\times 2 = (5M_1 - T) + (T - 10M_2)$

$2M_1 + 2M_2 = 5M_1 - 10M_2$

$12M_2 = 3M_1$

$M_1 = 4M_2$

$\therefore K = 4$

<u>OR</u> $\left[\begin{array}{l} M_1 g \sin 30^\circ - T = 2M_1 \\ T - M_2 g = 2M_2 \end{array}\right.$

3	collect
2	Equating the forces on M_1 and M_2
1	Some attempt at creating the force on M_1

(b) 3

$\int_0^4 \frac{1}{\sqrt{1+\sqrt{x}}}\,dx$

Let $u^2 = 1+\sqrt{x}$

$2u\,du = \frac{1}{2\sqrt{x}}\,dx$

$4u\sqrt{x}\,du = dx$

$dx = 4u(u^2-1)$

$= \int_1^{\sqrt{3}} \frac{1}{u} \times 4u(u^2-1)\,du$

$= 4\int_1^{\sqrt{3}} (u^2-1)\,du$

When $x = 4$ $u^2 = 1+\sqrt{4}$

$u^2 = 3$

$\therefore u = \sqrt{3}$

$= 4\left[\frac{u^3}{3} - u\right]_1^{\sqrt{3}}$

When $x = 0$ $u^2 = 1$

$u = 1$

$= 4\left\{\frac{3\sqrt{3}}{3} - \sqrt{3} - \frac{1}{3} + 1\right\}$

$= \frac{8}{3}$

3	Correct
2	integrate correctly from Correct substitution
1	Correct and relevant Substitution

Question 14 continues on page 20

Question 14 continued. **Marks**

(c)

(i) 3

$\underset{\sim}{x}(t) = Vt\cos\alpha\, \underset{\sim}{i} + (Vt\sin\alpha - \frac{1}{2}gt^2)\, \underset{\sim}{j}$

$\dot{\underset{\sim}{x}}(t) = V\cos\alpha\, \underset{\sim}{i} + (V\sin\alpha - gt)\, \underset{\sim}{j}$

When $\dot{y} = 0$, $V\sin\alpha - gt = 0$.

$t = \frac{V\sin\alpha}{g}$.

Max height $H = V\left(\frac{V\sin\alpha}{g}\right) - \frac{1}{2}\left(\frac{V\sin\alpha}{g}\right)^2$

$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$

$= \frac{V^2\sin^2\alpha}{2g}$.

Horizontal displacement at max height $X = V\left(\frac{V\sin\alpha}{g}\right)\cos\alpha$

$= \frac{V^2\sin\alpha\cos\alpha}{g}$.

$\tan\theta = \frac{H}{X}$.

$= \frac{V^2\sin^2\alpha}{2g} \div \frac{V^2\sin\alpha\cos\alpha}{g}$

$= \frac{\sin\alpha}{2\cos\alpha}$

$= \frac{1}{2}\tan\alpha$

$\therefore \theta = \tan^{-1}\left(\frac{1}{2}\tan\alpha\right)$.

1: max height H.

2: X & some progress on $\tan\theta$

3: correct solutions.

Question 14 continues on page 21

Question 14 continued. **Marks**

(c)

(ii) Let β be the angle of inclination to the horizontal: **3**

$$\tan\beta = \frac{\dot{y}}{\dot{x}} = \frac{V\sin\alpha - gt}{V\cos\alpha}$$

since $\beta = \frac{\pi}{4}$, $\frac{V\sin\alpha - gt}{V\cos\alpha} = 1.$

$$V\sin\alpha - gt = V\cos\alpha \quad ①$$

for $y = h$, $Vt\sin\alpha - \frac{1}{2}gt^2 = h \quad ②$

From ①, $t = \frac{V(\sin\alpha - \cos\alpha)}{g}$

① $\times t$ − ② $\times 2$. $Vt\sin\alpha - 2Vt\sin\alpha = Vt\cos\alpha - 2h$.

$$-Vt\sin\alpha = Vt\cos\alpha - 2h.$$

$$h = \frac{Vt}{2}(\cos\alpha + \sin\alpha).$$

sub $t = \frac{V(\sin\alpha - \cos\alpha)}{g}$. $h = \frac{V^2}{2g}(\sin^2\alpha - \cos^2\alpha)$.

$H = \frac{V^2\sin^2\alpha}{2g}$. $\frac{h}{H} = \frac{\sin^2\alpha - \cos^2\alpha}{\sin^2\alpha}$

$$= 1 - \frac{\cos^2\alpha}{\sin^2\alpha}$$

$$= 1 - \cot^2\alpha$$

$$= 1 - (\operatorname{cosec}^2\alpha - 1)$$

$$= 2 - \operatorname{cosec}^2\alpha.$$

1 – apply $\tan\beta = 1$

2 – significant progress

3 – correct solutions.

Question 14 continues on page 22

Question 14 continued. **Marks**

(d)

(i) 1 r/w

Let $a - x = M$ When $x = 0$ $M = a$

$x = a$ $M = 0$

RHS : $\int_0^a f(a-x)\,dx = \int_a^0 f(M)\,(-dM)$

$= -\int_a^0 f(M)\,dM$

$= \int_0^a f(M)\,dM$

but $\int_0^a f(x)\,dx = \int_0^a f(M)\,dM$

$\therefore = LHS$.

(ii) 3

$$I = \int_0^{\pi} \frac{\sin x}{e^{\pi/2} + e^{x}}\,dx$$

$$= \int_0^{\pi} \frac{\sin(\pi - x)}{e^{\pi/2} + e^{(\pi - x)}}\,dx$$

$$I = \int_0^{\pi} \frac{\sin x\, e^{-\pi/2} e^{x}}{e^{\pi/2} + e^{x}}\,dx$$

$$e^{\pi/2} I = \int_0^{\pi} \frac{e^{x} \sin x}{e^{\pi/2} + e^{x}}$$

3	Collect
2	one error
1	Replacement of x by $\pi - x$ and attempt at process

$$2e^{\pi/2} I = \int_0^{\pi} \frac{e^{\pi/2}\sin x\,dx}{e^{\pi/2} + e^{x}} + \int_0^{\pi} \frac{e^{x} \sin x}{e^{\pi/2} + e^{x}}\,dx$$

$$= \int_0^{\pi} \frac{(e^{\pi/2} + e^{x})\sin x}{e^{\pi/2} + x}\,dx$$

$$= \int_0^{\pi} \sin x\,dx$$

$$2e^{\pi/2} I = -\left[\cos x\right]_0^{\pi}$$

$$= -(-1-1)$$

$$\therefore I = e^{-\pi/2}$$

End of Question 14

Question 15 (15 marks) **Marks**

(a) 3

Vertices of ABC satisfy $z^3 - 1 = 0$

Complex number represented by $M = -\frac{1}{2}$

Vertices of LMN satisfy $8z^3 + 1 = 0$

$(z^3-1)(8z^3+1) = 0$

$8z^6 - 7z^3 - 1 = 0$

OR $A(1)$ $B\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $C\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$L\left(\frac{1}{4} + \frac{\sqrt{3}}{4}i\right)$ $N\left(\frac{1}{4} - \frac{\sqrt{3}}{4}i\right)$ $M\left(-\frac{1}{2}\right)$

Quadratic satisfying BC $z^2 + z + 1 = 0$

LN $4z^2 - 2z + 1 = 0$

AM $2z^2 - z - 1 = 0$

All 6 pts

$(z^2+z+1)(4z^2-2z+1)(2z^2-z-1) = 0$

$8z^6 - 7z^3 - 1 = 0$

Question 15 continues on page 26

3	Correct
2	Some progress using either method
1	Just the coordinates of the 6 pts.

(b)

(i) $\frac{d}{dt}\left(16+t^2\right)^{3/2} = \frac{3}{2}\left(16+t^2\right)^{1/2} \times 2t$ **1** r/w

$= 3t\sqrt{16+t^2}$

(ii) $I_n = \int_0^3 t^{n-1} t\sqrt{16+t^2}\,dt$ **3**

$* \quad = \left[\frac{1}{3}t^{n-1}\left(16+t^2\right)^{3/2}\right]_0^3 - \frac{1}{3}\int_0^3 (n-1)t^{n-2}\left(16+t^2\right)^{3/2}\,dt$

$3I_n = \left[t^{n-1}\left(16+t^2\right)^{3/2}\right]_0^3 - (n-1)\int_0^3 t^{n-2}\left(16+t^2\right)\sqrt{16+t^2}\,dt$

$= 3^{n-1}(16+9)^{3/2} - (n-1)\left(16I_{n-2} + I_n\right)$

$(n+2)I_n = 125 \times 3^{n-1} - 16(n-1)I_{n-2}$ As req'd

3	Collect
2	Significant process
1	First working line *

Question 15 continues on page 27

Question 15 continued. **Marks**

(c) **4**

$F = ma = -mg - mkV$

$a = -g - kv.$

$\frac{dv}{dt} = -g - kv$

$\frac{dv}{g+kv} = -dt$

$\int_u^0 \frac{dv}{g+kv} = \int_0^T - dt$

$\frac{1}{k}\left[\ln|g+kv|\right]_u^0 = -T$

$\frac{1}{k}(\ln(g+ku) - \ln(g)) = T.$

$T = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right).$

$v\frac{dv}{dx} = -g - kv.$

$\int_u^0 \frac{v\,dv}{g+kv} = \int_0^H - dx.$

$LHS = \int_u^0 \frac{\frac{1}{k}(g+kv-g)}{g+kv}\,dv$ $RHS = -H$

$= \int_u^0 \left(\frac{\frac{1}{k}(g+kv)}{g+kv} - \frac{g}{k}\frac{1}{g+kv}\right)dv.$

$= \frac{1}{k}\left[v - \frac{g}{k}\ln|g+kv|\right]_u^0$

$= \frac{1}{k}\left[\left(0 - \frac{g}{k}\ln g\right) - \left(u - \frac{g}{k}\ln(g+ku)\right)\right]$

$= \frac{1}{k}\left(\frac{g}{k}\ln\left(\frac{g+ku}{g}\right) - u\right).$

$\therefore H = -\frac{1}{k}\left(\frac{g}{k}\ln\left(\frac{g+ku}{g}\right) - u\right)$

$= -\frac{1}{k}(gT - u)$

$kH = -gT + u.$ $\therefore u = gT + kH.$

1 – progress to T or H.

2 – progress to both T & H

3 – Get to T & H

4 – Correct solutions.

Question 15 continues on page 30

Question 15 continued. **Marks**

(d)

(i) **2**

$\overrightarrow{QP}$ is a 90° clockwise rotation of $\overrightarrow{OQ}$ multiplied by some scale factor

i.e $\overrightarrow{QP} = -ki\,\overrightarrow{OQ}$

$\therefore \overrightarrow{QP} = -ki\omega$

$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$

$= \omega - ki\omega$

$\overrightarrow{OR} = \frac{1}{2}\overrightarrow{OP}$

$= \frac{1}{2}(\omega - ik\omega)$

$= \frac{1}{2}(1 - ik)\omega$

2	Correctly show
1	Not fully explained

(ii) **2**

$\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$

$= \omega - \frac{1}{2}(1-ik)\omega$

$= \frac{1}{2}\omega - \frac{1}{2}ik\omega$

$\overrightarrow{OR}$ and $\overrightarrow{RQ}$ are conjugates, hence their moduli are equal

2	Correct
1	Finding $\overrightarrow{RQ}$

End of Question 15

If you use this space, clearly indicate which question you are answering.

$$t = \frac{1}{k}\ln\left(\frac{g+ku}{g+kv}\right).$$

$$e^{kt} = \frac{g+ku}{g+kv}.$$

$$ge^{kt} + kve^{kt} = g+ku.$$

$$v = \frac{1}{ke^{kt}}\left(g+ku-ge^{kt}\right).$$

$$= \frac{(g+ku)e^{-kt}-g}{k}.$$

$$\int_0^H dx = \int_0^T \frac{(g+ku)e^{-kt}-g}{k}\,dt.$$

$$H = \left[\frac{-\frac{1}{k}(g+ku)e^{-kt}-gt}{k}\right]_0^T$$

$$= \left[\frac{-\frac{1}{k}(g+ku)e^{-kT}-gT}{k} - \frac{-\frac{1}{k}(g+ku)}{k}\right].$$

$$= \frac{-(g+ku)e^{-kT}+(g+ku)}{k^2} - \frac{gT}{k}.$$

Question 16 (13 marks) **Marks**

(a)

(i) 2

$\overrightarrow{OM} = \frac{1}{2}(\underset{\sim}{p} + \underset{\sim}{q})$

$\overrightarrow{OM} \cdot \overrightarrow{PQ} = \frac{1}{2}(\underset{\sim}{p} + \underset{\sim}{q}) \cdot (\underset{\sim}{q} - \underset{\sim}{p})$

$= \frac{1}{2}(\underset{\sim}{q} \cdot \underset{\sim}{q} - \underset{\sim}{p} \cdot \underset{\sim}{p}) = 0$

since $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$ (radii)

$\angle QMN + \angle OMQ = 90^\circ + 90^\circ = 180^\circ$

$\therefore$ O, M, N are collinear

2	correct
1	one part of proof only

(ii) 2

$\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN}$

$= -\underset{\sim}{a} + \lambda \overrightarrow{BH}$

$= -\underset{\sim}{a} + \lambda(\overrightarrow{BA} + \overrightarrow{AH})$

$= -\underset{\sim}{a} + \lambda(2\underset{\sim}{a} + \underset{\sim}{h})$

$\therefore \overrightarrow{ON} = (2\lambda - 1)\underset{\sim}{a} + \lambda \underset{\sim}{h}$

2	correct
1	Some progress

Question 16 continues on page 32

Question 16 continued. **Marks**
Part (a)

(iii) $AH \perp PQ \quad \therefore \underset{\sim}{h} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) = 0$ and O, M, N are collinear 3

$\therefore \vec{ON} \cdot \vec{PQ} = 0 \quad MN \perp PQ$

3	Correct
2	Significant progress
1	To *

$* \quad \{(2\lambda - 1)\underset{\sim}{a} + \lambda \underset{\sim}{h}\} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) = 0$

$(2\lambda - 1)\underset{\sim}{a} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) = 0$

But AB and PQ are not perpendicular

$\therefore \underset{\sim}{a} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) \neq 0$ Hence $2\lambda - 1 = 0$

$\therefore \lambda = \frac{1}{2}$ and

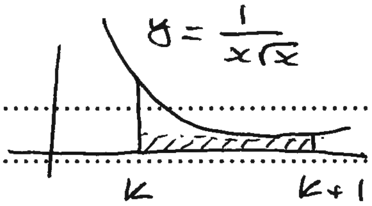
$BN = \frac{1}{2} BH$

$\therefore H$ is the midpoint of BH

3

(b)

(i) 2



Area of shaded rectangle $= \dfrac{1}{(k+1)\sqrt{k+1}}$

$$* \quad \therefore \int_k^{k+1} \frac{1}{x\sqrt{x}}\,dx > \frac{1}{(k+1)\sqrt{k+1}}$$

3	Correct
2	Set up and method correct, one error
1	To *

$$\left[-\frac{2}{\sqrt{x}}\right]_k^{k+1} > \frac{1}{(k+1)\sqrt{k+1}}$$

$$-\frac{2}{\sqrt{k+1}} + \frac{2}{\sqrt{k}} > \frac{1}{(k+1)\sqrt{k+1}}$$

$$\therefore \frac{1}{(k+1)\sqrt{k+1}} < \frac{2}{\sqrt{k}} - \frac{2}{\sqrt{k+1}}$$

Question 16 continues page 33

Question 16 continued. **Marks**

Part (b)

(ii) **3**

Let the statement be $\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \cdots + \frac{1}{n\sqrt{n}}$

$< 3 - \frac{2}{\sqrt{n}} \quad n \geq 2$

3	Correct
2	Correct set up and some progress
1	Just n = 2 and set u[p]

When $n = 2$:

LHS: $\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}}$ RHS: $3 - \frac{2}{\sqrt{2}}$

$= \frac{2\sqrt{2}+1}{2\sqrt{2}}$ $= \frac{3\sqrt{2}-2}{\sqrt{2}}$

$= \frac{6\sqrt{2}-4}{2\sqrt{2}}$

$\therefore$ LHS < RHS.

Assume the statement is true for $n = k$

i.e $\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \cdots + \frac{1}{k\sqrt{k}} < 3 - \frac{2}{\sqrt{k}} \quad k \geq 2$

RTP the statement is true for $n = k+1$

i.e $\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \cdots + \frac{1}{k\sqrt{k}} + \frac{1}{(k+1)\sqrt{k+1}} < 3 - \frac{2}{\sqrt{k+1}}$

Now

$\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \cdots + \frac{1}{k\sqrt{k}} + \frac{1}{(k+1)\sqrt{k+1}} < 3 - \frac{2}{\sqrt{k}} + \frac{1}{(k+1)\sqrt{k+1}}$

$< 3 - \frac{2}{\sqrt{k}} + \frac{2}{\sqrt{k}} - \frac{2}{\sqrt{k+1}}$

from 1)

$< 3 - \frac{2}{\sqrt{k+1}}$ As req'd

Hence since the statement is true for $n = 2$ it is true for $n = 2+1 = 3$ and so one for all $n \geq 2$ by principle of mathematical induction

END OF PAPER