Name:..................................

Student Number

Teacher:................................

Pymble Ladies' College

Mathematics Extension 2

HSC Trial Examination

Term 3 2025

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning or calculations

Total marks 100

Section 1 – 10 marks (pages 1-5)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7-13)

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section
- Answer each question in the appropriate space in the Answer Booklet. Extra writing pages are included at the end of each question.

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Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. Consider the proposition:

"Only the diagonals of rectangles are equal".

Which of the following statements is a counterexample to this proposition?

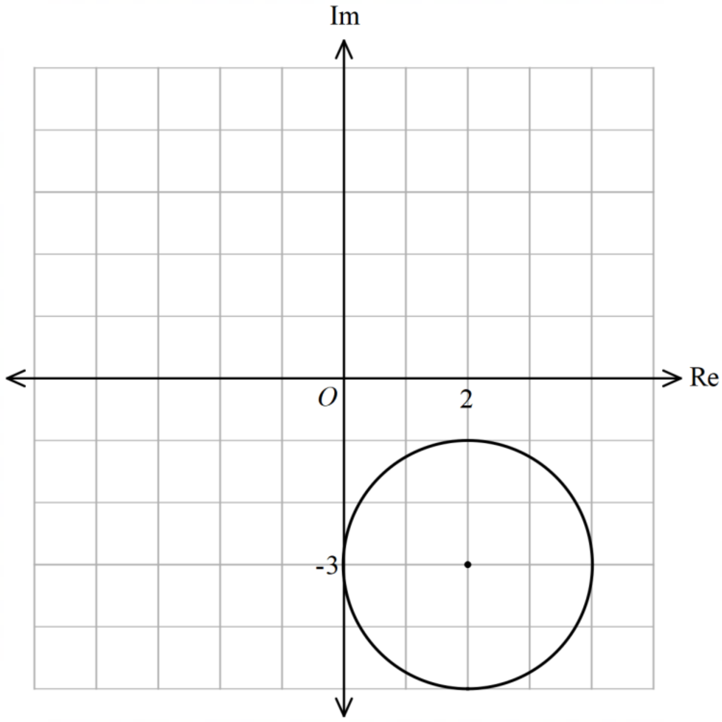
(A) The diagonals of a rhombus are equal.

(B) The diagonals of a kite are equal.

(C) The diagonals of a hexagon are equal.

(D) The diagonals of an isosceles trapezium are equal.

2. The diagram below shows a locus on an Argand diagram.



Which of the equations below represents the locus shown above?

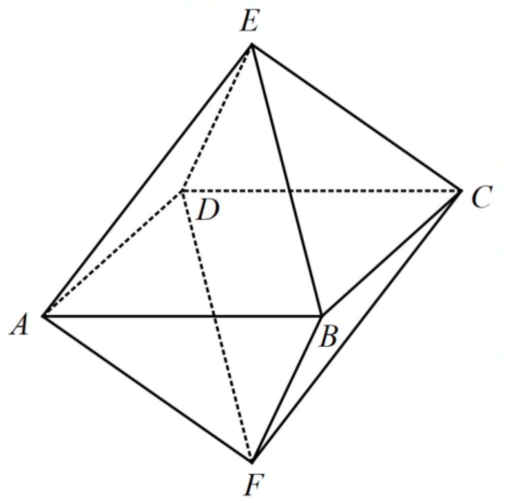
(A) $|z-2+3i|=2$

(B) $|z+2-3i|=2$

(C) $|z-2+3i|=4$

(D) $|z+2-3i|=4$

3. A regular octahedron is shown in the diagram below. All edges are of equal length.



Which of the following vectors is equal to $\overrightarrow{EC}+\overrightarrow{FC}$?

(A) $\overrightarrow{AC}$

(B) $\overrightarrow{CA}$

(C) $\overrightarrow{EF}$

(D) $\overrightarrow{FE}$

4. The sum of the non-real complex numbers z and w is zero.

Which of the following statements is FALSE?

(A) $\dfrac{w}{z}=-1$

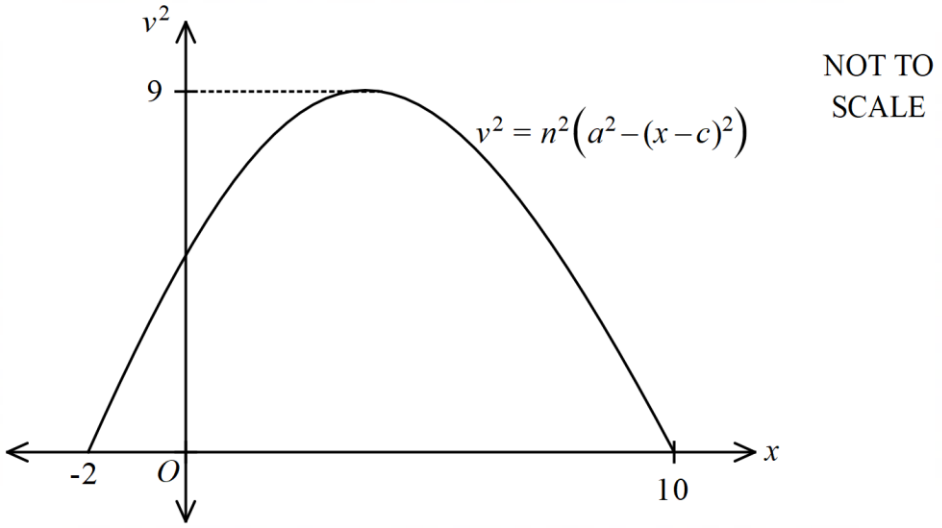
(B) $z-w=2z$

(C) $w-z=2w$

(D) $z+\overline{w}=0$

5. A particle is moving along the x-axis in simple harmonic motion. The displacement of the particle is x metres and its velocity is v ms^{-1}.

The graph of $v^2 = n^2\left(a^2 - (x-c)^2\right)$, where a, c and n are positive constants, is shown.



What are the values of a, c and n?

(A) $a = 4$, $c = 6$ and $n = \frac{1}{2}$

(B) $a = 4$, $c = 6$ and $n = \frac{3}{2}$

(C) $a = 6$, $c = 4$ and $n = \frac{1}{2}$

(D) $a = 6$, $c = 4$ and $n = \frac{3}{2}$

6. Which of the following is an expression for $\int \cos^3 x \, dx$?

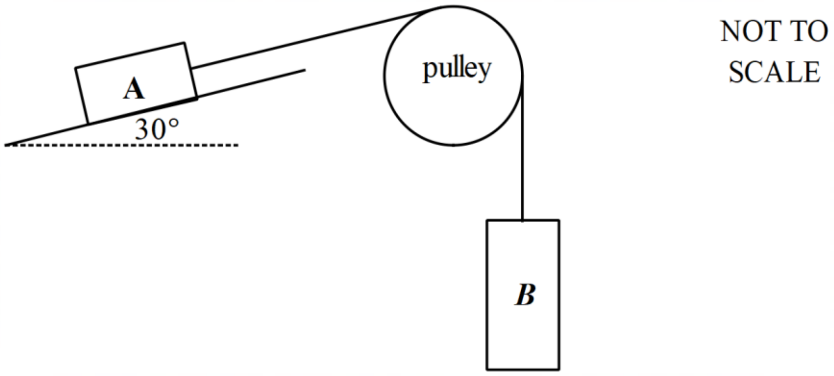
(A) $\sin x \cos^2 x + \frac{1}{3}\sin^3 x + C$

(B) $\sin x \cos^2 x - \frac{1}{3}\sin^3 x + C$

(C) $\sin x + \frac{1}{3}\sin^3 x + C$

(D) $\sin x - \frac{1}{3}\sin^3 x + C$

7. Particle A and particle B, of masses M_A and M_B respectively, are connected by a light inextensible string passing over a frictionless pulley as shown.



Particle A lies on a frictionless surface inclined at $30°$ to the horizontal while particle B hangs vertically from the string.

The particles are initially at rest, and when they are released neither particle moves.

Which expression shows the relationship between M_A and M_B?

(A) $M_A = M_B$

(B) $M_A = \sqrt{3}M_B$

(C) $M_A = 2M_B$

(D) $M_A = 2\sqrt{3}M_B$

8. Consider the statement:

$$\forall n \in Z,\ \text{if } n^2 \text{ is odd then } n \text{ is odd.}$$

Which of the following is the contrapositive of this statement?

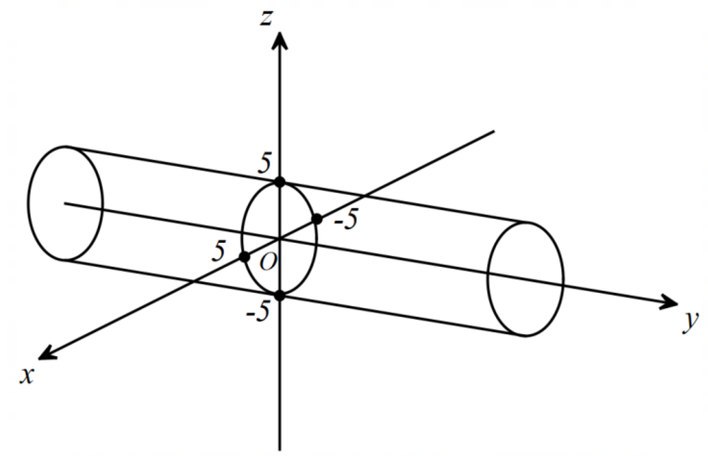
(A) $\forall n \in Z$, if n is not odd then n^2 is not odd.

(B) $\exists n \in Z$, if n is not odd then n^2 is not odd.

(C) $\forall n \in Z$, if n is not odd then n^2 is odd.

(D) $\exists n \in Z$, if n is not odd then n^2 is odd.

9. The diagram below is of a hollow cylinder of infinite height in three dimensions, centred about the y-axis.



What is the Cartesian equation that describes this cylinder?

(A) $y^2 + z^2 = 25$

(B) $x^2 + z^2 = 25$

(C) $x^2 + z^2 = (5 - y)^2$

(D) $y^2 + z^2 = (5 - x)^2$

10. It is given that $f(a) = -2$, $f(b) = 8$, $g(a) = 1$ and $g(b) = 2$.

What is the value of $\int_a^b \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,dx$?

(A) 6

(B) 2

(C) -6

(D) -18

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Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer these questions in the Answer Book provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) (i) Express $-5-5i$ in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$. **2**

(ii) The point on an Argand diagram that represents $-5-5i$ is one of the vertices of an equilateral triangle whose centre is at the origin. **2**

Find the complex numbers represented by the other two vertices of the triangle.

Give your answers in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$.

(b) Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$.

(i) Find $|\underset{\sim}{a}|$. **1**

(ii) Find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, in degrees correct to 1 decimal place. **2**

(c) Find $\displaystyle\int \frac{4x}{\sqrt[3]{1-x^2}}\,dx$. **2**

(d) A particle is moving along the x-axis. At time t seconds it has displacement x metres from the origin O, velocity v m/s and acceleration a m/s^2 given by $a = x - 10$. Initially the particle is 1 metre to the right of O and moving away from O with a speed of 9 m/s.

(i) Find an expression for v in terms of x. **2**

(ii) Find an expression for x in terms of t. **2**

(e) Prove algebraically that the difference between two consecutive square numbers is odd. **2**

End of Question 11

Question 12 (17 marks) **Marks**

(a) (i) The complex number z is such that **2**

$$z = \frac{1+i}{1-ki} \text{ where } k \text{ is a real number.}$$

Find the real part of z and the imaginary part of z, giving your answers in terms of k.

(ii) In the case where $k = \sqrt{3}$, use part (i) to show that **3**

$$\cos\frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4}.$$

(b) Relative to a fixed origin O

- the point A has position vector $5\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$.
- the point B has position vector $2\underset{\sim}{i} + 4\underset{\sim}{j} + a\underset{\sim}{k}$.

where a is a positive integer.

(i) Show that $\left|\overrightarrow{OA}\right| = \sqrt{38}$. **1**

(ii) Find the smallest value of a for which **2**

$$\left|\overrightarrow{OB}\right| > \left|\overrightarrow{OA}\right|$$

(c) (i) Find the values of a, b and c where $\dfrac{2x+1}{\left(x^2+1\right)\left(x+1\right)} \equiv \dfrac{ax+b}{x^2+1} + \dfrac{c}{x+1}$. **3**

(ii) Hence, or otherwise find $\displaystyle\int_0^1 \frac{2x+1}{\left(x^2+1\right)\left(x+1\right)}\,dx$. **3**

(d) Let ℓ_1 be the straight-line segment with equation $\underset{\sim}{r}_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $-3 \le \lambda \le 1$. **3**

The straight-line segment ℓ_2 is perpendicular to ℓ_1, and the endpoints of ℓ_2 are also its x and y intercepts.

Given that the endpoint of the position vector $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ lies on both ℓ_1 and ℓ_2, find the vector equation of ℓ_2.

End of Question 12

Question 13 (14 marks) **Marks**

(a) In the set of real numbers, let P be the proposition.

"If $a+b$ is irrational then at least one of a and b is irrational"

(i) State the contrapositive of the proposition P. **1**

(ii) State the negation of the proportion P. **1**

(iii) Write the converse of the proposition P, and state, with reasons, whether this converse is true or false. **2**

(b) (i) The region R on an Argand diagram satisfies both $|z+2i| \leq 3$ and $-\frac{\pi}{6} \leq \arg(z) \leq \frac{\pi}{2}$. **2**

Sketch R on the Argand diagram given in your Answer booklet.

(ii) Find the maximum value of $|z|$ in the region R, giving your answer in exact form. **2**

(c) Use mathematical induction to prove that ${}^{2n}C_n < 2^{2n-2}$ for all positive integers $n \geq 5$. **3**

(d) Use integration by parts to find in simplest exact form $\int_1^e 2^{\ln x}\, dx$. **3**

End of Question 13

Question 14 (16 marks) **Marks**

(a) A particle of mass m kilograms is initially at the origin, moving to the right along the x-axis at a velocity of u ms^{-1} where $u > 0$, until it comes to rest.

The only force acting on the particle is a resistance with magnitude $m(1+v)$ Newtons acting against the direction of the motion.

(i) Show that $\ddot{x} = -(1+v)$. **1**

(ii) Show that $x = u - v + \ln\left(\dfrac{1+v}{1+u}\right)$ metres. **3**

(iii) Show that the distance travelled before coming to rest is **2**

$$x = \ln\left(\frac{e^u}{1+u}\right) \text{ metres.}$$

(b) (i) Show that $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx.$ **2**

(ii) Hence, evaluate $\displaystyle\int_{-1}^{1} \frac{x^5 + x^2 - \tan x}{x^2+1}\,dx.$ **2**

(c) As a particle undergoes simple harmonic motion, its displacement, x metres, at any time, t seconds, is given by the equation

$$x(t) = 6.766\sin(2t - 0.224)$$

(i) Find the range of the particle's motion, correct to two decimal places. **2**

(ii) Find the maximum speed of the particle, correct to two decimal places. **2**

(iii) Find the distance travelled by the particle from its initial position to when it first comes to rest, correct to two decimal places. **2**

End of Question 14

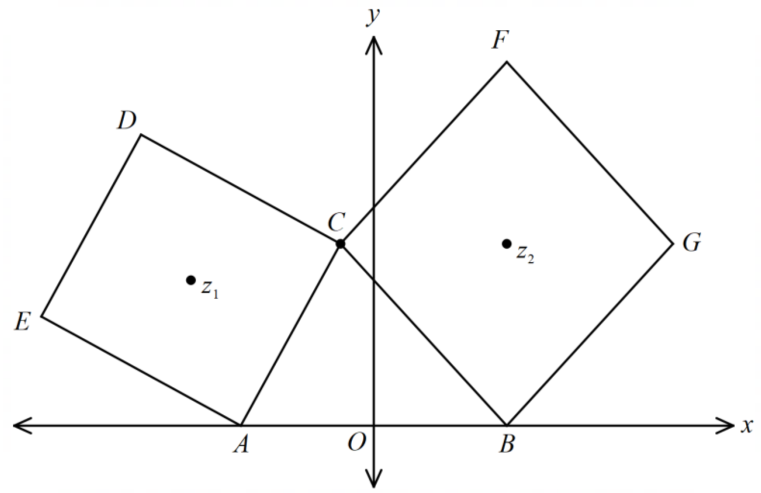
Question 15 (15 marks) **Marks**

(a) Using the substitution $u = \sin x + 1$, or otherwise, find **2**

$$\int \frac{\cos x}{\sin^2 x + 2\sin x + 5}\,dx.$$

(b) The Argand diagram below shows a triangle with vertices *A, B* and *C* representing the complex numbers -3, 3 and $-1+5i$, respectively.

z_1 is the centre of square *ACDE* and z_2 is the centre of square *BCFG.*



(i) Find the complex number represented by *E*. **1**

(ii) Hence, or otherwise, find the complex numbers z_1 and z_2. **3**

(iii) Show that the vectors representing z_1 and z_2 are perpendicular. **1**

(iv) Prove that, if *C* is moved to any complex number $a+ib$, where $-3 \le a \le 3$ and $b > 0$, then the vectors representing z_1 and z_2 are always perpendicular. **3**

(c) Let $I_n = \displaystyle\int_1^e \frac{1}{x^2}(\ln x)^n\,dx$ for all integers $n \ge 0$.

(i) Show that $I_n = nI_{n-1} - \dfrac{1}{e}$ for all integers $n \ge 1$. **2**

(ii) Hence show that **3**

$$I_n = n! - \frac{1}{e}\left({}^nP_0 + {}^nP_1 + {}^nP_2 + \ldots {}^nP_n\right)$$

End of Question 15

Question 16 (13 marks) **Marks**

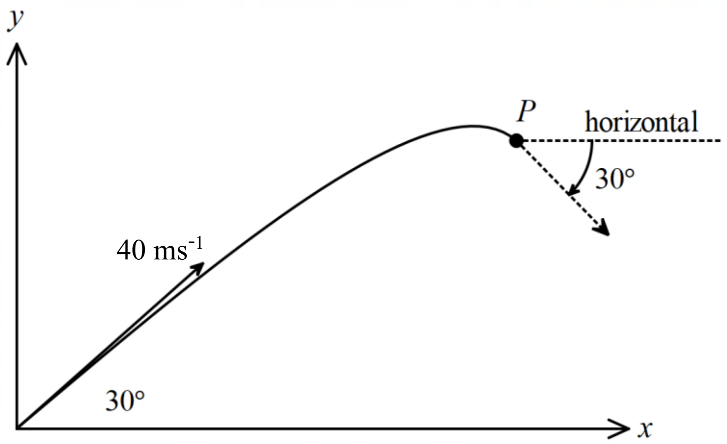
(a) A particle of mass 1 kilogram is projected from the origin with speed 40 ms^{-1} at an angle of $30°$ to the horizontal plane as shown.

The position vector of the particle, at time t seconds after the particle was projected, is given by

$$\underset{\sim}{r}(t)=\begin{pmatrix} 5\sqrt{3}\left(1-e^{-4t}\right) \\ \frac{45}{8}\left(1-e^{-4t}\right)-\frac{5}{2}t \end{pmatrix}. \quad \text{(Do NOT prove this)}$$

(i) Find the velocity vector $\underset{\sim}{v}(t)$, of the particle t seconds after it was projected. **1**

(ii) At point P the particle is descending at $30°$ to the horizontal. **2**



NOT TO SCALE

Show that the time taken for the particle to reach P is $\log_e \sqrt[4]{17}$ seconds.

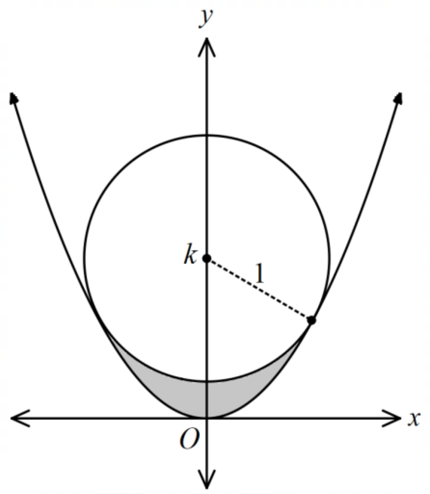
(b) (i) It is known that $|\underset{\sim}{a}|=1$, $|\underset{\sim}{b}|=2$, $|\underset{\sim}{c}|=4$, $\underset{\sim}{a}\cdot\underset{\sim}{b}=-2$ and $\underset{\sim}{c}\cdot\left(\underset{\sim}{a}+\underset{\sim}{b}\right)=-4$. **2**

Using dot products, show that $|\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}|^2=9$.

(ii) By considering the angles between the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, or otherwise, show that $|\underset{\sim}{a}-\underset{\sim}{b}+\underset{\sim}{c}|=7$. **3**

Question 16 continues on page 13

(c) The diagram shows a circle of radius 1, with its centre on the y-axis, that is tangent to the parabola $y = x^2$ at two distinct points. This circle has centre $(0, k)$ for some value $k > 0$ therefore has equation $x^2 + (y-k)^2 = 1$. (Do NOT prove this) **5**



By showing the x-values of the points of contact are $x = \pm\dfrac{\sqrt{3}}{2}$, find the exact area between the circle and the parabola.

END OF PAPER

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NSW Education Standards Authority

HIGHER SCHOOL CERTFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

$$S = \frac{a}{1 - r}, |r| < 1$$

Logarithmic and Exponential Functions

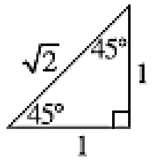
$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$



$A = \dfrac{1}{2}ab\sin C$

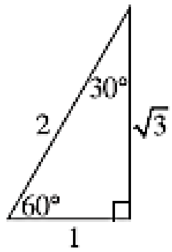
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$\cos A = \dfrac{1-t^2}{1+t^2}$

$\tan A = \dfrac{2t}{1-t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

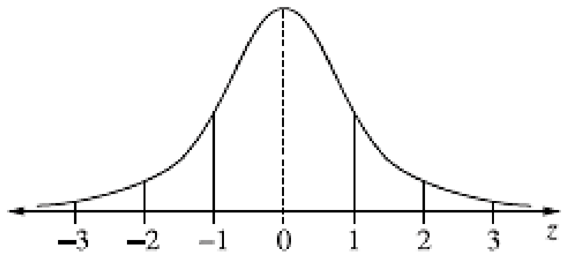
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq r) = \displaystyle\int_a^r f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

Name:……………………………

Student Number

Teacher:…………………………..

Pymble Ladies' College

Mathematics Extension 2
HSC Trial Examination
Term 3 2025

ANSWER BOOKLET

- **Put your name, Teacher's name and Student Number on this Booklet.**
- For Questions 1-10 use the Multiple-Choice Answer Sheet provided.
- Answer Questions 11-16 in the spaces allocated for each question as these spaces provide guidance for the expected length of your response.
- Write using a non-erasable black or blue pen. Black is preferred.
- Extra writing space is provided.
- Calculators approved by NESA may be used.

Mathematics Extension 2

HSC Trial Examination

2025

Name:......................................

Student Number:

Teacher's Name:............................

Section I – Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ○ B ● C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.

correct

A ● B ● C ○ D ○

1	A ○	B ○	C ○	D ●
2	A ●	B ○	C ○	D ○
3	A ●	B ○	C ○	D ○
4	A ○	B ○	C ○	D ●
5	A ○	B ○	C ●	D ○
6	A ○	B ○	C ○	D ●
7	A ○	B ○	C ●	D ○
8	A ●	B ○	C ○	D ○
9	A ○	B ●	C ○	D ○
10	A ●	B ○	C ○	D ○

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section.

Answer these questions in the spaces provided.
Extra writing space is available at the end of each question.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) 2

(i)

let $\omega = -5-5i$

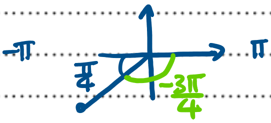
$|\omega| = \sqrt{(-5)^2+(-5)^2}$

$= \sqrt{50}$

$= 5\sqrt{2}$ ✓

rel. $\theta = \tan^{-1}(1)$

$= \frac{\pi}{4}$

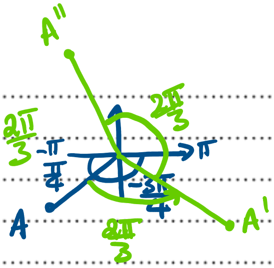


$\text{Arg}(\omega) = -\frac{3\pi}{4}$

$\therefore \omega = 5\sqrt{2}e^{-\frac{3\pi}{4}i}$

2	correct
1	either $-\frac{3\pi}{4}$ or $5\sqrt{2}$

(ii) 2



$2\pi \div 3 = \frac{2\pi}{3}$

$-\frac{3\pi}{4} + \frac{2\pi}{3} = -\frac{\pi}{12}$

$-\frac{\pi}{12} + \frac{2\pi}{3} = \frac{7\pi}{12}$

$A' = 5\sqrt{2}e^{-\frac{\pi}{12}i}$

$A'' = 5\sqrt{2}e^{\frac{7\pi}{12}i}$

$\therefore$ other two vertices are

$5\sqrt{2}e^{-\frac{\pi}{12}i}$, $5\sqrt{2}e^{\frac{7\pi}{12}i}$

equilateral triangle so vertices are equidistant from the centre (origin)

2	correct
1	progress incl. equidistant and angle at centre $\frac{2\pi}{3}$, argument of one vertex within domain

Question 11 continued

Marks

(b)

(i) $|\underset{\sim}{a}| = \sqrt{2^2+3^2+1^2}$ **1**

$= \sqrt{14}$

R|W

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ **2**

$= -2+3+4$

$= 5$

$|\underset{\sim}{b}| = \sqrt{1+1+16}$

$= \sqrt{18}$

$= 3\sqrt{2}$

2 | correct (incl. rounding)

1 | progress

$\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$= \dfrac{5}{3\sqrt{2} \times \sqrt{14}}$

$\therefore \theta = 71.64097347...$

$= 71.6^\circ$

Question 11 continued **Marks**

(c) **2**

$$\int \frac{4x}{\sqrt[3]{1-x^2}}\,dx$$

$$= \int \frac{-2x-2x}{\sqrt[3]{1-x^2}}\,dx$$

$$= -2\int -2x(1-x^2)^{-\frac{1}{3}}\,dx$$

$$= -2\left[\frac{-2x\,(1-x^2)^{\frac{2}{3}}}{\frac{2}{3}}\right] + c$$

$$= -3(1-x^2)^{\frac{2}{3}} + C$$

2	correct
1	writing in reverse chain (or equiv.) form.

Question 11 continued **Marks**

(d)

(i) 2

$a = x - 10$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x - 10$

$\frac{1}{2}v^2 = \int (x-10)\, dx$

$\therefore \frac{1}{2}v^2 = \frac{x^2}{2} - 10x + C$

when $t=0$, $x=1$, $v=9$

$\therefore \frac{1}{2} \times 81 = \frac{1}{2} - 10 + C$

$\therefore C = 50$

$\therefore \frac{1}{2}v^2 = \frac{x^2}{2} - 10x + 50$

$v^2 = x^2 - 20x + 100$

$= (x-10)^2$

$\therefore v = \pm(x-10)$

when $x=1$, $v=9$

$\therefore v = -(x-10)$

$= 10 - x$

OR $v\frac{dv}{dx} = x - 10$

$\int v \, dv = \int (x-10)\, dx$

2 | $v = x - 10$ incl. remove.
1 | v^2 w const

(ii) 2

$\frac{dx}{dt} = 10 - x$

$\int \frac{-1}{10-x}\, dx = -\int dt$

$\ln|10-x| = -t + D$

$10 - x = Ae^{-t}$, where $A = \pm e^{D}$

When $t=0$, $x=1$

$10 - 1 = A$

$\therefore A = 9$

$\therefore 10 - x = 9e^{-t}$

$\therefore x = 10 - 9e^{-t}$

2 | correct incl. remove ||
1 | separate variables & integrate.

 Marks

(e) let n and $n+1$ be the two consecutive numbers, **2**
$n \in \mathbb{Z}$.

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2$$
$$= 2n+1$$

As $n \in \mathbb{Z}$,
$2n$ is an even number,
and $2n+1$ is an odd number
$\therefore$ the difference of two consecutive square numbers is odd.

2	correct
1	good progress

End of Question 11

Question 12 (17 marks) **Marks**

(a)

(i) 2

$$z = \frac{1+i}{1-ki} \times \frac{1+ki}{1+ki}$$

$$= \frac{1-k+i+ki}{1+k^2}$$

Real part $= \dfrac{1-k}{1+k^2}$

& Imaginary part $= \dfrac{1+k}{1+k^2}$

1 for imaginary part
& 1 for real part

(ii) 3

When $k=\sqrt{3}$, $z = \dfrac{1+i}{1-\sqrt{3}i}$.

$$|z| = \frac{|1+i|}{|1-\sqrt{3}i|}$$

$$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\arg z = \arg(1+i) - \arg(1-\sqrt{3}i)$$

$$= \frac{\pi}{4} - \left(\frac{-\pi}{3}\right)$$

$$= \frac{7\pi}{12}$$

$$\text{Re } z = \frac{1}{\sqrt{2}} \cos\frac{7\pi}{12}$$

From (i), $\text{Re } z = \dfrac{1-\sqrt{3}}{4}$

So $\dfrac{1}{\sqrt{2}} \cos\dfrac{7\pi}{12} = \dfrac{1-\sqrt{3}}{4}$

$$\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{4} \times \sqrt{2}$$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

1 for $|z|$
1 for $\arg z$
1 for final prove.

 Marks

(b)

(i) $|\overrightarrow{OA}| = \sqrt{5^2 + 3^2 + 2^2}$ **1** R/W

$= \sqrt{38}$

(ii) $|\overrightarrow{OB}| = \sqrt{2^2 + 4^2 + a^2}$ **2**

$= \sqrt{20 + a^2}$

$|\overrightarrow{OB}| > |\overrightarrow{OA}|$

$\Rightarrow \sqrt{20 + a^2} > \sqrt{38}$

$a^2 + 20 > 38$

$a^2 > 18$

Since a is a positive,

$a > \sqrt{18} = 3\sqrt{2}$

and smallest value of a as a positive integer is 5.

1 for $a^2 + 20 > 38$

1 for final answer

(c)

(i) **3**

$2x+1 \equiv (ax+b)(x+1)+c(x^2+1)$

When $x=-1$; $c = \frac{-1}{2}$

When $x=0$; $b+c = 1$

$b = 1+\frac{1}{2}$

$= \frac{3}{2}$

When $x=1$; $2(a+b)+2c = 3$

$2(a+1\frac{1}{2}) = 4$

$a = \frac{1}{2}$

$\therefore a = \frac{1}{2}$, $b = 1\frac{1}{2}$ and $c = \frac{-1}{2}$.

1 for each a, b

(ii) **3**

$\int_0^1 \frac{2x+1}{(x^2+1)(x+1)}\,dx$

$= \frac{1}{2}\int_0^1 \left(\frac{x+3}{x^2+1} - \frac{1}{x+1}\right)dx$

$= \frac{1}{2}\int_0^1 \left[\frac{1}{2}\left(\frac{2x}{x^2+1}\right) + \frac{3}{x^2+1} - \frac{1}{x+1}\right]dx$

$= \frac{1}{2}\left[\frac{1}{2}\ln|x^2+1| + 3\tan^{-1}x - \ln|x+1|\right]_0^1$

$= \frac{1}{4}\ln 2 + 3\tan^{-1}1 - \ln 2 - \ln 1 - 3\tan^{-1}0 - \ln$

$= \frac{-1}{4}\ln 2 + \frac{3\pi}{8}$

1 for attempt to get to
1 for correct integrat
1 for final answer

Question 12 continued **Marks**

(d) **3**

$\underset{\sim}{r}_1 = \begin{bmatrix} 6 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Since $l_2 \perp l_1$, direction vector of $l_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$,

$\underset{\sim}{r} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ lies on both l_1 and l_2,

So $\underset{\sim}{r}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 2 - \mu \\ 2 + 2\mu \end{bmatrix}$

When $x = 0$, $2 - \mu = 0$

$\mu = 2$

When $y = 0$, $2 + 2\mu = 0$

$\mu = -1$

$\therefore$ Equation of l_2 is

$\underset{\sim}{r}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

where $-1 \leq \mu \leq 2$.

1 for correct direction

1 for equation of l_2 with find limits of μ

1 for correct answer

End of Question 12

Question 13 (14 marks) **Marks**

(a)

(i) If a and b are rational, then $a+b$ is rational. **1**

R/W

(ii) $(a+b)$ is irrational, and neither a or b is irrational. **1**

R/W

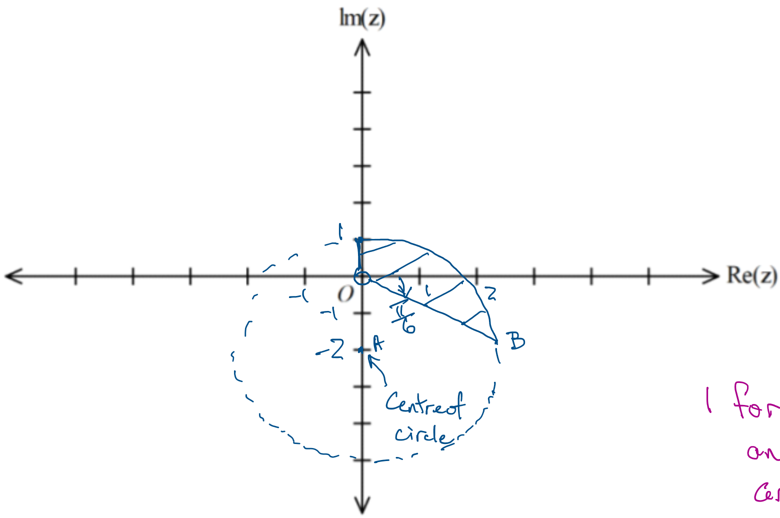
(iii) If at least one of a and b is irrational, then $a+b$ is irrational. **2**

Let $a = \sqrt{2}$ and $b = -\sqrt{2}$; i.e. at least one of a and b is irrational, but $a+b = \sqrt{2} + (-\sqrt{2}) = 0$ which is rational so the converse is false.

1 for statement
1 for reason

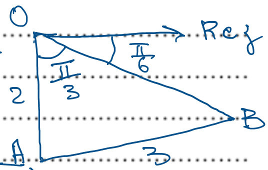
(b)

(i) **2**



1 for drawing c
and attempting
arg z
2 for all corre

(ii) **2**



$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos\frac{\pi}{3}$

$3^2 = 2^2 + b^2 - 2(2)(b)(\frac{1}{2})$

$b^2 - 2b - 5 = 0$

$b = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2(1)}$; let $b = OB$

$= \frac{2 \pm \sqrt{24}}{2}$

$= 1 \pm \sqrt{6}$

Since b is a length, $b \neq 1 - \sqrt{6} < 0$, $b = 1 + \sqrt{6}$ and maximum value of $|z| = 1 + \sqrt{6}$.

1 for attempting to f
the correct leng
1 for final answ

(c) **3**

Let the statement be ${}^{2n}C_n < 2^{2n-2}$, $n \geq 5$.

Step 1: Test statement is true for $n = 5$,

LHS $= {}^{10}C_5 = 252 < 256$

RHS $= 2^8 = 256$

Step 2: Assume statement is true for $n = k$ where $k \geq 5$,

i.e. ${}^{2k}C_k < 2^{2k-2}$.

Step 3: Prove statement is true for $n = k+1$,

i.e. ${}^{2(k+1)}C_{k+1} < 2^{2(k+1)-2} = 2^{2k}$.

$$\begin{aligned}
\text{LHS} &= {}^{2k+2}C_{k+1} \\
&= \frac{(2k+2)!}{(k+1)!\,(2k+2-k-1)!} \\
&= \frac{(2k+2)!}{(k+1)!\,(k+1)!} \\
&= \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!\,(k+1)k!} \\
&= \frac{2(k+1)(2k+1)}{(k+1)^2} \times \frac{(2k)!}{k!\,k!} \\
&= \frac{2(2k+1)}{k+1} \times {}^{2k}C_k \\
&< \frac{2^2(k+\frac{1}{2})}{k+1} \times 2^{2k-2} \quad \text{, from Step 2} \\
&= \frac{k+\frac{1}{2}}{k+1} \times 2^{2k} \\
&< 2^{k} = \text{RHS}
\end{aligned}$$

since $k \geq 5$, $k + \frac{1}{2} < k+1$.

so $\frac{k+\frac{1}{2}}{k+1} < 1$

Step 4: Since the statement is true for $n = 5$, assumed true for $n = k$ and prove true for $n = k+1$, by mathematical induction the statement is true for $n = 5+1 = 6$, $n = 6+1 = 7$ and so on, so the statement ${}^{2n}C_n < 2^{2n-2}$ is true for all positive integers of n where $n \geq 5$.

1 for steps 1, 2, setting up of 3 a

1 for attempting to break down ^{2k}

1 for final correct proof

 Marks

(d) $\int_1^e 2^{\ln x} \, dx$ **3**

$= \left[x \, 2^{\ln x} \right]_1^e - \int_1^e \left[x (\ln 2) \, 2^{\ln x} \left(\frac{1}{x}\right) \right] dx$

$= e \cdot 2^{\ln e} - 2^{\ln 1} - \int_1^e (\ln 2) \cdot 2^{\ln x} \, dx$

$= (2e - 1) - \ln 2 \int_1^e 2^{\ln x} \, dx$

$\Rightarrow (1 + \ln 2) \int_1^e 2^{\ln x} \, dx = 2e - 1$

$\int_1^e 2^{\ln x} \, dx = \dfrac{2e - 1}{1 + \ln 2}$

1 for attempting to integrate by parts with the introduction of x
1 for integrating correctly
1 for final answer

End of Question 13

Question 14 (16 marks) **Marks**

(a)

(i) 1

$F = ma$

$m\ddot{x} = -m(1+v)$ since it is a resistance force

$\ddot{x} = -(1+v)$

(ii) 3

$\ddot{x} = -(1+v)$

$v\dfrac{dv}{dx} = -(1+v)$

$\displaystyle\int_0^x (-1)\,dx = \int_u^v \frac{v}{1+v}\,dv$

$\displaystyle\left[-x\right]_0^x = \int_u^v \left[\frac{1+v}{1+v} - \frac{1}{1+v}\right] dv$

$\displaystyle -x+0 = \int_u^v \left(1 - \frac{1}{1+v}\right) dv$

$-x = \left[v - \ln|1+v|\right]_u^v$

$= v - \ln|1+v| - u + \ln|1+u|$

$x = u - v + \ln(1+v) - \ln(1+u)$; since the p... moves to th... both u & ...

$x = u - v + \ln\left(\dfrac{1+v}{1+u}\right)$

1 for putting into integra... in terms of x & v

1 for correct integrati...

1 for getting rid of ab... value with reason and logs together

Question 14 continued **Marks**

Part (a)

(iii) **2**

When $v = 0$:

$$x = u - 0 + \ln\left(\frac{1+0}{1+u}\right)$$

$$= u + \ln\left(\frac{1}{1+u}\right)$$

$$= \ln e^{u} + \ln\left(\frac{1}{1+u}\right)$$

$$= \ln\left(\frac{e^{u}}{1+u}\right)$$

1 for correct substituti
of $v = 0$

1 for getting to final ans

Marks

(b) (i) **2**

LHS $= \int_{-a}^{a} f(x)\,dx$

$= \int_{0}^{a} f(x)\,dx + \int_{-a}^{0} f(x)\,dx$

$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx$; from ✱

$=$ RHS

$\int_{-a}^{0} f(x)\,dx$; let $x = -u$

$dx = -du$

$= \int_{a}^{0} -f(-u)\,du$ when $x = 0$, $u = 0$

$= \int_{0}^{a} f(-u)\,du$ when $x = -a$, $u = a$

$= \int_{0}^{a} f(-x)\,dx$

So $\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-x)\,dx$ — ✱

1 for attempting to get to ✱

1 for bringing things together

Question 14 continued **Marks**

Part (b)

(ii) **2**

$$\int_{-1}^{1} \frac{x^5+x^2-\tan x}{x^2+1}\,dx$$

$$= \int_0^1 \frac{x^5+x^2-\tan x}{x^2+1}\,dx + \int_0^1 \frac{(-x)^5+(-x)^2-\tan(-x)}{(-x)^2+1}\,dx \quad ; \text{ from (i}$$

$$= \int_0^1 \frac{x^5+x^2-\tan x - x^5 + x^2 + \tan x}{x^2+1}\,dx \quad ; \ \tan(-x) = -\tan x$$

$$= \int_0^1 \frac{2x^2}{x^2+1}\,dx$$

$$= \int_0^1 \frac{2x^2+2-2}{x^2+1}\,dx$$

$$= \int_0^1 \left(2 - \frac{2}{x^2+1}\right) dx$$

$$= \left[2x - 2\tan^{-1} x\right]_0^1$$

$$= 2 - 2\tan^{-1} 1 - 0 + 2\tan^{-1} 0$$

$$= 2 - \frac{\pi}{2}$$

1 for using (i) correctly and reduce integral to

1 for final answer

 Marks

(c)

(i) **2**

$x(t) = 6.766\sin(2t - 0.224)$

Range : $-6.766 \le x(t) \le 6.766$

1 for giving amplitude

(ii) **2**

$v(t) = 2 \times 6.766\cos(2t - 0.224)$

$= 13.532\cos(2t - 0.224)$

Maximum speed $= 13.532$

$= 13.53\ ms^{-1}$

(iii) **2**

First come to rest when $v(t) = 0$;

when $\cos(2t - 0.224) = 0$; i.e. $2t - 0.224 = \frac{\pi}{2}$

$x(t) = 6.766\sin\frac{\pi}{2} = 6.766$

When $t = 0$; $x(0) = 6.766\sin(0 - 0.224)$

$= -1.5029\ldots$

$\therefore$ Distance travelled

$= 6.766 + 1.5029\ldots$

$= 8.2689\ldots$

$= 8.27$ m

End of Question 14

Question 15 (15 marks) **Marks**

(a) **2**

$$\int \frac{\cos x \, dx}{\sin^2 x + 2\sin x + 5}$$

$$= \int \frac{\cos x \, dx}{\sin^2 x + 2\sin x + 1 + 4}$$

$$= \int \frac{\cos x \, dx}{(\sin x + 1)^2 + 4}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\sin x + 1}{2}\right) + C$$

1 — progress

2 — correct solutions

Question 15 continued **Marks**

(b)

(i) **1**

$\overrightarrow{AC}$ rotates $\frac{\pi}{2}$ becomes $\overrightarrow{AE}$.

$\overrightarrow{AE} = (2+5i)\,i = -5+2i$. 1 – correct solution

$\therefore \overrightarrow{OE} = -3+(-5+2i)$

$= -8+2i$.

(ii) **3**

Z_1 is the midpoint of OE.

$z_1 = \frac{-8-1}{2} + i\left(\frac{2+5}{2}\right)$

$= -\frac{9}{2} + \frac{7}{2}i$. 1 – Getting z_1

$\overrightarrow{BC}$ rotates $(-\frac{\pi}{2})$ becomes $\overrightarrow{BG}$ 2 – progress towards z_2

$\overrightarrow{BG} = (-4+5i)(-i) = 5+4i$

$\overrightarrow{OG} = 3+(5+4i) = 8+4i$. 3 – correct solutions.

z_2 is the midpoint of OG:

$z_2 = \left(\frac{-1+8}{2}\right) + i\left(\frac{5+4}{2}\right)$

$= \frac{7}{2} + \frac{9}{2}i$.

(iii) **1**

$z_2 i = \left(\frac{7}{2}+\frac{9}{2}i\right)(i)$ 1 – correct solutions

$= -\frac{9}{2}+\frac{7}{2}i$

$= z_1$,

Hence $\overrightarrow{OZ_1}$ is $\overrightarrow{OZ_2}$ rotates $\frac{\pi}{2}$.

$\therefore OZ_1$ & OZ_2 are perpendicular.

Question 15 continued **Marks**

Part (b)

(iv) **3**

$c = a + ib$, $\therefore \vec{AC} = (a+3) + ib$.

$\vec{AE} = (a+3+ib)i$

$= -b + (a+3)i$. $\therefore \vec{OE} = (-b-3) + (a+3)i$.

$z_1 = \left(\frac{a-b-3}{2}\right) + \left(\frac{a+b+3}{2}\right)i$.

$\vec{BC} = (a-3) + ib$.

$\vec{BG} = (a-3+ib)(-i)$

$= b - (a-3)i$. $\therefore \vec{OG} = (b+3) - (a-3)i$.

$z_2 = \left(\frac{a+b+3}{2}\right) + \left(\frac{-a+b+3}{2}\right)i$.

Now $z_2 i = \left(\left(\frac{a+b+3}{2}\right) + \left(\frac{-a+b+3}{2}\right)i\right)i$

$= \frac{a-b-3}{2} + \left(\frac{a+b+3}{2}\right)i$

$= z_1$

Hence $\vec{OZ_1}$ is $\vec{OZ_2}$ rotates $\frac{\pi}{2}$.

$\therefore \vec{OZ_1}$ is perpendicular to $\vec{OZ_2}$ for the given conditions.

1 — Getting z_1

2 — Getting z_2 & progress towards

3 — correct solutions.

(c)

(i) **2**

$I_n = \int_1^e \frac{1}{x^2}(\ln x)^n \, dx.$

$= \left[-\frac{1}{x}(\ln x)^n\right]_1^e + n\int_1^e \frac{1}{x} \cdot \frac{1}{x}(\ln x)^{n-1} dx.$ (Integration by parts)

$= -\frac{1}{e}(1)^n - 0 + n \int_1^e \frac{1}{x^2}(\ln x)^{n-1} dx$

$= -\frac{1}{e} + n I_{n-1}$

1 – progress

2 – correct solutions

Marks

(ii) $I_n = nI_{n-1} - \frac{1}{e}$ **3**

$= n(\,(n-1)I_{n-2} - \frac{1}{e}) - \frac{1}{e}$

$= n(n-1)I_{n-2} - \frac{n}{e} - \frac{1}{e}$

$= n(n-1)(n-2)I_{n-3} - \frac{n(n-1)}{e} - \frac{n}{e} - \frac{1}{e}$

$\vdots$

$= n(n-1)(n-2)\cdots(1)I_0 - \left(\frac{n(n-1)(n-2)\cdots(2)}{e} + \frac{n(n-1)\cdots(3)}{e}\right.$

$\left. + \cdots \frac{n}{e} + \frac{1}{e}\right)$

$I_0 = \int_1^e \frac{1}{x^2}\,dx$ $\quad {}^nP_0 = \frac{n!}{n!} = 1, \quad {}^nP_1 = \frac{n!}{(n-1)!} = n.$

$= \left[-\frac{1}{x}\right]_1^e$ $\quad \vdots$

$\quad {}^nP_{n-2} = \frac{n!}{(n-(n-2))!} = n(n-1)\cdots(3)$

$= 1 - \frac{1}{e}$ $\quad {}^nP_{n-1} = \frac{n!}{(n-(n-1))!} = n(n-1)\cdots(2).$

$I_n = n!(1-\frac{1}{e}) - \frac{1}{e}({}^nP_{n-1} + {}^nP_{n-2} + \cdots + {}^nP_1 + {}^nP_0)$

$= n! - \frac{1}{e}({}^nP_n) - \frac{1}{e}({}^nP_{n-1} + {}^nP_{n-2} + \cdots + {}^nP_1 + {}^nP_0)$

$\therefore I_n = n! - \frac{1}{e}({}^nP_0 + {}^nP_1 + {}^nP_2 + \cdots + {}^nP_n)$

1 — get the expression of I_n in terms of I_{n-3}

2 — getting I_0 & progress towards I_n

3 — correct solutions.

End of Question 15

Question 16 (13 marks) **Marks**

(a)

(i) 1

$$x(t) = \begin{pmatrix} 20\sqrt{3}e^{-4t} \\ \frac{45}{2}e^{-4t} - \frac{5}{2} \end{pmatrix}$$

1 – correct & simplified solutions.

(ii) 2

$$\tan\theta = \frac{\frac{45}{2}e^{-4t} - \frac{5}{2}}{20\sqrt{3}e^{-4t}}.$$

$$= \tan(-30^\circ).$$

$$-\frac{1}{\sqrt{3}} = \frac{45e^{-4t} - 5}{40\sqrt{3}e^{-4t}}$$

$\times \frac{\sqrt{3}}{5}$

$$-8e^{-4t} = 9e^{-4t} - 1.$$

$$17e^{-4t} = 1.$$

$$e^{-4t} = \frac{1}{17}.$$

$$-4t = \ln\frac{1}{17}$$

$$t = -\frac{1}{4}\ln\frac{1}{17}$$

$$= \ln\sqrt[4]{17} \quad \text{(log laws)}.$$

1 – progress

2 – correct solutions.

Question 16 continued **Marks**

(b)

(i) $|\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}|^2 = (\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})\cdot(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})$ **2**

$= \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{c} + 2\underset{\sim}{a}\cdot\underset{\sim}{b}$

$+ 2\underset{\sim}{b}\cdot\underset{\sim}{c} + 2\underset{\sim}{c}\cdot\underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + 2(\underset{\sim}{a}\cdot\underset{\sim}{b})$

$+ 2\underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b})$

$= 1 + 2^2 + 4^2 + 2(-2) + 2(-4)$

$= 9.$

1 – progress

2 – correct solutions.

Part (b)

(ii) Let θ be the angle between $\underset{\sim}{a}$ & $\underset{\sim}{b}$. **3**

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{-2}{2}$$

$$= -1.$$

$\therefore \theta = \pi$, $\underset{\sim}{a}$ & $\underset{\sim}{b}$ are opposite to each other.

since $|\underset{\sim}{a}| = 1$ & $|\underset{\sim}{b}| = 2$, $\therefore \underset{\sim}{b} = -2\underset{\sim}{a}$.

$\underset{\sim}{c} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = -4.$

$\underset{\sim}{c} \cdot (-\underset{\sim}{a}) = -4.$

$\therefore \underset{\sim}{a} \cdot \underset{\sim}{c} = 4.$

$$|\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c}|^2 = |\underset{\sim}{a} - (-2\underset{\sim}{a}) + \underset{\sim}{c}|^2$$
$$= |3\underset{\sim}{a} + \underset{\sim}{c}|^2$$
$$= (3\underset{\sim}{a} + \underset{\sim}{c}) \cdot (3\underset{\sim}{a} + \underset{\sim}{c})$$
$$= 9|\underset{\sim}{a}|^2 + |\underset{\sim}{c}|^2 + 6\,\underset{\sim}{a} \cdot \underset{\sim}{c}$$
$$= 9 + 16 + 6(4)$$
$$= 49$$

$\therefore |\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c}| = 7.$

or

let α be the angle between $\underset{\sim}{a}$ & $\underset{\sim}{c}$.

$$\cos\alpha = \frac{\underset{\sim}{a} \cdot \underset{\sim}{c}}{|\underset{\sim}{a}||\underset{\sim}{c}|}$$
$$= \frac{4}{4}$$
$$= 1.$$

$\therefore \alpha = 0$, & $\underset{\sim}{c} = 4\underset{\sim}{a}$

since $|\underset{\sim}{a}| = 1$ & $|\underset{\sim}{c}| = 4$

$|\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c}|$

$= |\underset{\sim}{a} - (-2\underset{\sim}{a}) + 4\underset{\sim}{a}|$

$= |7\underset{\sim}{a}|$

$= 7|\underset{\sim}{a}|$

$= 7.$

1 — find θ.

2 — progress towards by applying θ.

3 — correct solutions.

Question 16 continued **Marks**

(c) **5**

For points of contact: solve $y=x^2$ & $x^2+(y-k)^2=1$.

$y+(y-k)^2=1$.

$y+y^2-2yk+k^2-1=0$.

$y^2+(1-2k)y+(k^2-1)=0$.

Since the parabola is tangent to the circle, $\Delta=0$.

$(1-2k)^2-4(k^2-1)=0$.

$4k^2-4k+1-4k^2+4=0$.

$4k=5$

$k=\frac{5}{4}$.

$y^2+(1-\frac{5}{2})y+(\frac{25}{16}-1)=0$.

$y^2-\frac{3}{2}y+\frac{9}{16}=0$.

$(y-\frac{3}{4})^2=0$.

$y=\frac{3}{4}$ $\therefore$ $x=\pm\frac{\sqrt{3}}{2}$. (since $y=x^2$).

$x^2+(y-\frac{5}{4})^2=1$.

$(y-\frac{5}{4})^2=1-x^2$.

$y=\frac{5}{4}\pm\sqrt{1-x^2}$

$\therefore y=\frac{5}{4}-\sqrt{1-x^2}$ (using the lower half of the circle).

$A=2\int_0^{\frac{\sqrt{3}}{2}}(\frac{5}{4}-\sqrt{1-x^2}-x^2)\,dx$.

$=2\left[\frac{5}{4}x-\frac{x^3}{3}\right]_0^{\frac{\sqrt{3}}{2}}-2\int_0^{\frac{\sqrt{3}}{2}}\sqrt{1-x^2}\,dx$.

$=2(\frac{5\sqrt{3}}{8}-\frac{\sqrt{3}}{8})-2I$ where $(I=\int_0^{\frac{\sqrt{3}}{2}}\sqrt{1-x^2})$.

$\frac{1}{2}A=A_{trapezium}-A_{sector}-$

$A_T=(\frac{5}{4}+\frac{3}{4})(\frac{\sqrt{3}}{2})\cdot\frac{1}{2}$.

$=\frac{\sqrt{3}}{2}$.

$A_{sector}=\frac{1}{2}(1)^2\theta$.

$\sin\theta=\frac{\sqrt{3}}{2}\div 1$ $\theta=\frac{\pi}{3}$.

$A_{sector}=\frac{1}{2}\cdot\frac{\pi}{3}=\frac{\pi}{6}$.

$A_1=\int_0^{\frac{\sqrt{3}}{2}}x^2\,dx$

$=\frac{1}{3}\left[x^3\right]_0^{\frac{\sqrt{3}}{2}}=\frac{\sqrt{3}}{8}$.

$\frac{1}{2}A=\frac{\sqrt{3}}{2}-\frac{\pi}{12}-\frac{\sqrt{3}}{8}$.

$=\frac{3\sqrt{3}}{8}-\frac{\pi}{6}$.

$\therefore A=(\frac{3\sqrt{3}}{4}-\frac{\pi}{3})$ units

$= \sqrt{3} - 2I.$

$I = \int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\, dx$ let $x = \sin\theta$. $dx = \cos\theta\, d\theta$. $x = 0,\ \theta = 0$ $x = \frac{\sqrt{3}}{2},\ \theta = \frac{\pi}{3}$.

$= \int_0^{\frac{\pi}{3}} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta.$

$= \int_0^{\frac{\pi}{3}} \cos^2\theta\, d\theta.$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (1+\cos 2\theta)\, d\theta.$

$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\left(\frac{\pi}{3} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right).$

$\therefore A = \sqrt{3} - 2\left(\frac{1}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)\right).$

$= \left(\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right)$ unit^2

1 – Applying $\Delta = 0$ in the simultaneous equation.

2 – show that $x = \pm\frac{\sqrt{3}}{2}$

3 – find the correct integral for A.

4 – progress towards in integration

5 – correct solutions.

END OF PAPER